Degree-Similar Graphs

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Abstract

The *degree matrix* of a graph is the diagonal matrix with diagonal entries equal to the degrees of the vertices of X. If X_1 and X_2 are graphs with respective adjacency matrices A_1 and A_2 and degree matrices D_1 and D_2, we say that X_1 and X_2 are *degree similar* if there is an invertible real matrix M such that $M^{-1}A_1M = A_2$ and $M^{-1}D_1M = D_2$. If graphs X_1 and X_2 are degree similar, then their adjacency matrices, Laplacian matrices, unsigned Laplacian matrices and normalized Laplacian matrices are similar. We first show that the converse is not true. Then, we provide a number of constructions of degree-similar graphs. Finally, we show that the matrices $A_1 - \mu D_1$ and $A_2 - \mu D_2$ are similar over the field of rational functions $\mathbb{Q}(\mu)$ if and only if the Smith normal forms of the matrices $tI - (A_1 - \mu D_1)$ and $tI - (A_2 - \mu D_2)$ are equal.

Mathematics Subject Classifications: 05C50

1 Introduction

Let X be a graph with vertex set $V(X)$ and edge set $E(X)$. We use $A = A(X)$ to denote the *adjacency matrix* of X and $D = D(X)$ to denote the *degree matrix*, the diagonal matrix with $D_{i,i}$ equals to the valency of the vertex i in X. If X_1 and X_2 are graphs with respective adjacency matrices A_1 and A_2 and degree matrices D_1 and D_2, we say that X_1 and X_2 are *degree similar* if there is an invertible real matrix M such that

$$M^{-1}A_1M = A_2, \quad M^{-1}D_1M = D_2. \tag{1}$$

Clearly, if X_1 and X_2 are degree similar, then their adjacency matrices, Laplacians $D-A$, unsigned Laplacians $D + A$, and their normalized Laplacians $D^{-1/2}AD^{-1/2}$ are similar. (When using the normalized Laplacian, we assume the underlying graph has no isolated vertices.) Thus we have a hierarchy of conditions on a pair of graphs:

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(a) They are degree similar.

(b) Their adjacency matrices and the three Laplacians are similar.

(c) Their adjacency matrices are similar.

We note that these three conditions are equivalent for regular graphs.

We discuss some related earlier work. Butler et al. [1] constructed graphs that are cospectral with respect to adjacency, Laplacian, unsigned Laplacian and normalized Laplacian matrices. In [14], by using local switching, Wang et al. gave a construction of pairs of degree-similar graphs. Guo et al. [6] derived six reduction procedures on the Laplacian, unsigned Laplacian and normalized Laplacian characteristic polynomials of a graph which can be used to construct larger Laplacian, unsigned Laplacian and normalized Laplacian cospectral graphs, respectively.

Tutte [13] defined the *idiosyncratic polynomial* of a graph to be

$$p(x, \alpha) := \det(A + \alpha(J - I - A) - xI),$$

where I is the identity matrix and J is the all 1s matrix of appropriate size. van Dam and Haemers [11] worked with the *generalized adjacency matrix* $sI+tA+\mu J$, the characteristic polynomial of this matrix is a form of the idiosyncratic polynomial.

Wang et al. [14] defined the generalized characteristic polynomial $\psi(X, t, \mu)$ of a graph X as follows:

$$\psi(X, t, \mu) := \det(tI - (A - \mu D)).$$

Note that if $\psi(X_1, t, \mu) = \psi(X_2, t, \mu)$, the adjacency matrices of X_1 and X_2 are similar, along with the three Laplacians. They observed that if X_1 and X_2 are degree similar, then $\psi(X_1, t, \mu) = \psi(X_2, t, \mu)$, and asked if the converse was true. Our results in Section 3 show that it is not. Hence if the adjacency and the three Laplacian matrices of two graphs are similar, it does not follow that the graphs are degree similar.

In Section 4, we prove that if X and Y are connected graphs and are degree similar, then their complements $\overline{X}$ and $\overline{Y}$ are degree similar. In Sections 5-8, we provide a number of constructions of pairs of (non-isomorphic) degree-similar graphs, for example, graph products, adding or deleting vertices and so on. In Section 9, we study the relation between similarity and Smith normal forms of matrices. In the last section, we provide some further discussions.

2 Ihara zeta function

In this section, we note one further consequence of degree similarity.

A walk in a graph X is *reduced* if it does not contain any subsequence of the form uvu; such walks may also be called *non-backtracking*. If $|V(X)| = n$, then $p_r(A)$ denotes the $n \times n$ matrix where $(p_r(A))_{u,v}$ is the number of reduced walks in X from u to v. So

$$p_0(A) = I, \ p_1(A) = A, \ p_2(A) = A^2 - D.$$

When $r \geqslant 3$, we have the recurrence

$$Ap_r(A) = p_{r+1}(A) + (D - I)p_{r-1}(A),$$

from which it follows that $p_r(A)$ is a polynomial in A and D. These observations are due to Biggs. For details, and for the following theorem, see Chan and Godsil [2].

Theorem 1. *For any connected graph on at least two vertices,*

$$\sum_{r\geqslant 0} t^r p_r(A) = (1 - t^2)(I - tA + t^2(D - I))^{-1}.$$

The determinant of the generating function on the left in this identity is the *Ihara zeta function* of the graph, and therefore if X_1 and X_2 are degree similar and have no isolated vertices, their Ihara zeta functions are equal. (This fact was noted by Wang et al. in [14], with a sketch of a proof. For more on Ihara zeta functions, see [12].)

3 Trees

Firstly, we describe some notation. For a graph X and a vertex $u \in V(X)$, we use $d_X(u)$ to denote the degree of u in X. For a vertex subset $U \subseteq V(X)$, denote the induced subgraph of X on U by $X[U]$, and the induced subgraph of X on $V(X) \setminus U$ by $X\backslash U$. For an $n \times n$ matrix M and a set $U \subset \{1, \dots, n\}$, we use $M(U)$ to denote a matrix obtained from M by deleting the rows in U and the columns in U. When $U = \{u\}$, we use $X\backslash u$ and $M(u)$ instead. If $\mathbf{x}$ and $\mathbf{y}$ are two column vectors, we use $[M|\mathbf{x}, \mathbf{y}]$ to denote the bordered matrix

$$\begin{pmatrix} 0 & \mathbf{x}^T \\ \mathbf{y} & M \end{pmatrix}.$$

Assume that S is a graph and T is a rooted tree. The *coalescence* $S \bullet T$ is the graph formed by identifying the root of T and a vertex of S.

Let T_1 and T_2 be the rooted trees shown in Figure 1, whose roots are v and w respectively. Clearly, T_1 and T_2 are two isomorphic trees with different roots. In 1977, McKay [9] showed that for any tree S with at least two vertices, $S \bullet T_1$ and $S \bullet T_2$ are not isomorphic, but they are cospectral with respect to several graph matrices (in particular, adjacency matrix, Laplacian matrix and unsigned Laplacian matrix). Osborne [10] showed that their normalized Laplacian matrices are also similar.

In fact, we can prove a more general result: for any graph S with at least two vertices, $\psi(S \bullet T_1, t, \mu) = \psi(S \bullet T_2, t, \mu)$, i.e., $S \bullet T_1$ and $S \bullet T_2$ are cospectral with respect to the matrix $A - \mu D$. For convenience, put $A_\mu(X) := A(X) - \mu D(X)$.

Lemma 2. *Let T_1 and T_2 be the two trees depicted in Figure 1, and let $S_i = S \bullet T_i$ for $i = 1, 2$, where S is a non-trivial graph. Then $\psi(S_1, t, \mu) = \psi(S_2, t, \mu)$.*

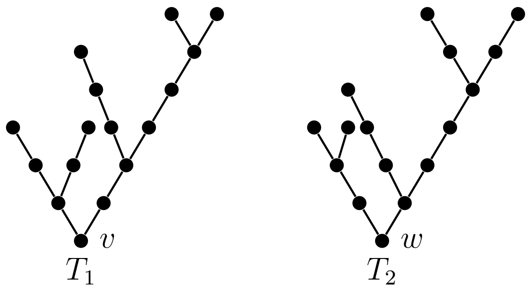

Figure 1: T_1 and T_2.

Proof. Without loss of generality, assume that S_i is obtained by identifying the root of T_i and a vertex r of S, here $i = 1, 2$. It is routine to check that

$$A_\mu(S_1) = \begin{pmatrix} -\mu(d_S(r)+2) & \mathbf{x}_{T_1}^T & \mathbf{y}_S^T \\ \mathbf{x}_{T_1} & A_\mu(T_1)(v) & \mathbf{0} \\ \mathbf{y}_S & \mathbf{0} & A_\mu(S)(r) \end{pmatrix}$$

for some column vectors $\mathbf{x}_{T_1}$ and $\mathbf{y}_S$. For convenience, denote $\phi(M) := \det(tI - M)$. Based on [9, Lemma 2.2(i)], one has

$$\begin{aligned} \psi(S_1,t,\mu) =& \phi(A_\mu(T_1)(v))\phi([A_\mu(S)(r)|\mathbf{y}_S,\mathbf{y}_S]) + \phi(A_\mu(S)(r))\phi([A_\mu(T_1)(v)|\mathbf{x}_{T_1},\mathbf{x}_{T_1}]) \\ &- (t-\mu(d_S(r)+2))\phi(A_\mu(T_1)(v))\phi(A_\mu(S)(r)). \end{aligned}$$

Similarly, one may write $\psi(S_2, t, \mu)$ as follows:

$$\begin{aligned} \psi(S_2,t,\mu) =& \phi(A_\mu(T_2)(w))\phi([A_\mu(S)(r)|\mathbf{y}_S,\mathbf{y}_S]) + \phi(A_\mu(S)(r))\phi([A_\mu(T_2)(w)|\mathbf{x}_{T_2},\mathbf{x}_{T_2}]) \\ &- (t-\mu(d_S(r)+2))\phi(A_\mu(T_2)(w))\phi(A_\mu(S)(r)). \end{aligned}$$

By a direct calculation, we obtain

$$\phi(A_\mu(T_1)(v)) = \phi(A_\mu(T_2)(w))$$

and

$$\phi([A_\mu(T_1)(v)|\mathbf{x}_{T_1},\mathbf{x}_{T_1}]) = \phi([A_\mu(T_2)(w)|\mathbf{x}_{T_2},\mathbf{x}_{T_2}]).$$

It follows that $\psi(S_1, t, \mu) = \psi(S_2, t, \mu)$. □

Obviously, if X_1 and X_2 are degree similar, then $\psi(X_1, t, \mu) = \psi(X_2, t, \mu)$. Wang et al. [14] proposed a problem: Is the converse true? Next, we give some examples to show that it is not.

The following result is a reformulation of [9, Theorem 5.3].

Theorem 3. *Two trees are degree similar if and only if they are isomorphic.*

Combining Lemma 2 with Theorem 3, we have the following corollary.

Corollary 4. *For any tree S with at least two vertices, we have $\psi(S \bullet T_1, t, \mu) = \psi(S \bullet T_2, t, \mu)$, but $S \bullet T_1$ and $S \bullet T_2$ are not degree similar.*

4 Subgraphs and complements

In this section, we study the subgraphs and complements of degree-similar graphs. Recalling the definition of degree-similar graphs, we first present a basic property of the invertible real matrix M in (1).

Lemma 5. *Let X_1 and X_2 be graphs with degree matrices D_1 and D_2 respectively. If there is an invertible real matrix M such that*

$$M^{-1}D_1M = D_2.$$

Then M is block diagonal.

Proof. Assume that $d_1, \ldots, d_t$ are all distinct vertex degrees of X_1. Partition the vertex set of X_1 as follows: $V(X_1) = V_1 \cup V_2 \cup \cdots \cup V_t$, where $V_i = \{w : d_{X_1}(w) = d_i\}$ for $i \in \{1, \ldots, t\}$. Since D_1 is a diagonal matrix, after reordering the vertices of X_1, we can write D_1 as follows:

$$D_1 = \begin{pmatrix} d_1 I_{|V_1|} & & & \\ & d_2 I_{|V_2|} & & \\ & & \ddots & \\ & & & d_t I_{|V_t|} \end{pmatrix}.$$

Notice that D_1 and D_2 are diagonal matrices. Together with $M^{-1}D_1M = D_2$, there exists a permutation matrix P such that $P^TM^{-1}D_1MP = P^TD_2P = D_1$. Let $Q = MP$. Then $Q^{-1}D_1Q = D_1$, i.e., $D_1Q = QD_1$. Therefore, Q is a block diagonal matrix with respect to the partition $V_1 \cup V_2 \cup \cdots \cup V_t$, which implies that M is block diagonal. □

The following result is an immediate consequence of Lemma 5, which gives some cospectral graphs with respect to the adjacency matrix.

Lemma 6. *Let X_1 and X_2 be two degree-similar graphs, and let d be the degree of some vertex in X_1. Assume that $V_i = \{w : d_{X_i}(w) = d\}$ for $i \in \{1, 2\}$. Then the induced subgraphs $X_1[V_1]$ and $X_2[V_2]$ are adjacency cospectral.*

Remark 7. In fact, Lemma 6 is more useful in determining two graphs that are not degree similar. For example, let X be a strongly regular graph with parameters SRG$(25, 12, 5, 6)$. In fact, there are exactly 15 non-isomorphic strongly regular graphs with such parameters. Here, we assume the adjacency matrix of X is the first one described in Spence's website: http://www.maths.gla.ac.uk/~es/srgraphs.php.

Assume that the first two rows of $A(X)$ are indexed by u and v respectively. One may check

$$\begin{aligned} &\det(xI_{12} - A(X\backslash N_X[u])) - \det(xI_{12} - A(X\backslash N_X[v])) \\ =& -2x^9 + 2x^8 + 64x^7 + 39x^6 - 372x^5 - 135x^4 + 648x^3 - 324x^2, \end{aligned}$$

here $N_X[u]$ denotes the closed neighborhood of u in X. This implies that $X\backslash N_X[u]$ and $X\backslash N_X[v]$ are not adjacency cospectral. Together with Lemma 6, we know $X\backslash u$ and $X\backslash v$ are not degree similar. □

Next, we show that degree similar is preserved under taking the complement of the underlying graphs. Let $\overline{X}$ denote the complement of a graph X.

Lemma 8. *If X is connected, X and Y are degree-similar, then their complements are degree similar.*

Proof. Assume that X and Y have n vertices. Since X and Y are degree similar, there exists an invertible real matrix M such that

$$M^{-1}A(X)M = A(Y), \quad M^{-1}D(X)M = D(Y).$$

Since the Laplacians of X and Y are cospectral, Y is connected. Then there is a polynomial p, determined by the spectrum of $D(X) - A(X)$, such that $p(D(X) - A(X)) = J_n$. Therefore $p(D(Y) - A(Y)) = J_n$. Consequently,

$$M^{-1}J_nM = M^{-1}p(D(X) - A(X))M = p(D(Y) - A(Y)) = J_n, \tag{2}$$

from which it follows that $J_n - I_n - A(X)$ and $J_n - I_n - A(Y)$ are cospectral.

Notice that $A(\overline{X}) = J_n - I_n - A(X)$ and $D(\overline{X}) = (n-1)I_n - D(X)$. Then,

$$\begin{aligned} M^{-1}A(\overline{X})M &= M^{-1}(J_n - I_n - A(X))M = J_n - I_n - A(Y) = A(\overline{Y}), \\ M^{-1}D(\overline{X})M &= M^{-1}((n-1)I_n - D(X))M = (n-1)I_n - D(Y) = D(\overline{Y}). \end{aligned}$$

It follows that $\overline{X}$ and $\overline{Y}$ are degree similar. □

5 Local switching

There is a powerful and productive method called *local switching* [4], which can produce numerous pairs of cospectral graphs. Wang et al. [14] constructed a family of degree-similar graphs by using local switching. In this section, we generalize their result, and use local switching to construct a large family of degree-similar graphs. Firstly, we describe local switching.

Local switching. Let X be a graph and let $\pi := C_1 \cup C_2 \cup \cdots \cup C_k \cup C$ be a partition of $V(X)$. Suppose that, whenever $1 \leqslant i, j \leqslant k$ and $v \in C$, we have

(a) any two vertices in C_i have the same number of neighbors in C_j, and

(b) v has either 0, $\frac{|C_i|}{2}$ or $|C_i|$ neighbors in C_i.

The graph X^π formed by *local switching* in X with respect to π is obtained from X as follows. For each $v \in C$ and $1 \leqslant i \leqslant k$ such that v has $\frac{|C_i|}{2}$ neighbors in C_i, delete those $\frac{|C_i|}{2}$ edges and join v instead to the other $\frac{|C_i|}{2}$ vertices in C_i.

Godsil and McKay [4] showed that if X^π is the graph formed by local switching in X with respect to a partition π, then X and X^π are cospectral, with cospectral complements. Now, we use local switching to construct a large family of degree-similar graphs.

Lemma 9. *Let X be a graph and let $\pi := C_1 \cup C_2 \cup \cdots \cup C_k \cup C$ be a partition of $V(X)$. Put $c := |C|$ and $c_i := |C_i|$ for $1 \leqslant i \leqslant k$. Suppose that, whenever $1 \leqslant i, j \leqslant k$, we have*

(i) *every vertex in C_i has d_{ij} neighbors in C_j,*

(ii) *for any $u \in C$, u has $\frac{c_i}{2}$ neighbors in C_i, and*

(iii) *for any $v \in C_i$, v has $\frac{c}{2}$ neighbors in C.*

If X^π is formed by local switching in X with respect to π, then X and X^π are degree similar.

Proof. Assume that the vertices of X are labelled in an order consistent with π. For $i \in \{1, \ldots, k\}$, let $Q_i = \frac{2}{c_i} J_{c_i} - I_{c_i}$. Clearly, Q_i is an orthogonal matrix and $Q_i = Q_i^T = Q_i^{-1}$. Define an orthogonal matrix Q as follows:

$$Q = \begin{pmatrix} Q_1 & & & & \\ & Q_2 & & & \\ & & \ddots & & \\ & & & Q_k & \\ & & & & I_c \end{pmatrix}. \tag{3}$$

According to the proof of [4, Theorem 2.2], we know $Q^{-1}A(X)Q = A(X^\pi)$.

On the other hand, based on the partition π, we can write the degree matrices of X and X^π as follows:

$$D(X) = D(X^\pi) = \begin{pmatrix} g_1 I_{c_1} & & & & \\ & g_2 I_{c_2} & & & \\ & & \ddots & & \\ & & & g_k I_{c_k} & \\ & & & & D(X[C]) + (\sum_{i=1}^k \frac{c_i}{2}) I_c \end{pmatrix},$$

where $g_i = \sum_{j=1}^k d_{ij} + \frac{c}{2}$ for $i \in \{1, 2, \ldots, k\}$. It is routine to check that $Q^{-1}D(X)Q = D(X^\pi)$. Thus, X and X^π are degree similar. □

Example 10. In Figure 2, $X_{1,2}$ can be obtained from $X_{1,1}$ by using local switching. Let

$$R_1 = \begin{pmatrix} \frac{1}{2} J_4 - I_4 & \\ & I_6 \end{pmatrix}.$$

Then $R_1^{-1} A(X_{1,1}) R_1 = A(X_{1,2})$ and $R_1^{-1} D(X_{1,1}) R_1 = D(X_{1,2})$. Hence $X_{1,1}$ and $X_{1,2}$ are degree similar.

□

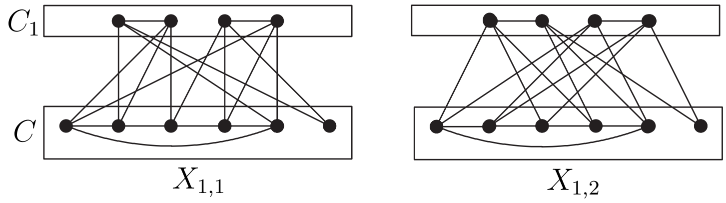

Figure 2: Local switching.

6 Joins and Products

In this section, we will investigate how other standard graph theoretic operations can be used to get examples of degree-similar graphs.

Let X be a graph with two induced subgraphs Y and Z such that $V(X)$ is the disjoint union of $V(Y)$ and $V(Z)$, $E(X)$ is the disjoint union of $E(Y)$ and $E(Z)$. We will say that X is the *union* of Y and Z, and denote it by $Y \cup Z$. The *join* of Y and Z, written as $X \vee Y$, is the graph obtained from $Y \cup Z$ by joining each vertex in Y to each vertex in Z.

Now, we construct degree-similar graphs by using union and join operations. The following results can be proved directly by the definition of degree-similar graphs.

Lemma 11. *Let X and Y be two connected degree-similar graphs. For any graph H, the following hold.*

(i) *$X \cup H$ and $Y \cup H$ are degree similar;*

(ii) *If X is regular, then $X \vee H$ and $Y \vee H$ are degree similar.*

Lemma 12. *Let X and X^π be graphs defined in Lemma 9 with vertex partition $\pi := C_1 \cup C_2 \cup \cdots \cup C_k \cup C$, and let Y be any graph. For convenience, put $C_{k+1} := C$. In X and X^π, if we add all edges between Y and $C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_l}$, where $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant k+1$, then the two graphs obtained are degree similar.*

Proof. Denote by Γ_1 and Γ_2 the graphs obtained from X and X^π by adding all edges between Y and $C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_l}$ respectively. Assume $|V(X)| = n$ and $|V(Y)| = m$. Let Q be the matrix defined in (3). Based on the proof of Lemma 9, we know

$$Q^{-1}A(X)Q = A(X^\pi), \quad Q^{-1}D(X)Q = D(X^\pi).$$

Notice that

$$A(\Gamma_1) = \begin{pmatrix} A(X) & B \\ B^T & A(Y) \end{pmatrix}, \quad D(\Gamma_1) = \begin{pmatrix} D(X) + mC & \\ & D(Y) + (\sum_{j=1}^{l} |C_{i_j}|)I_m \end{pmatrix}.$$

where $B = (b_{uv})_{n\times m}$ with $b_{uv} = 1$ if $u \in C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_l}$ and $b_{uv} = 0$ otherwise; $C = (c_{uv})_{n\times n}$ is a diagonal matrix with $c_{uu} = 1$ if $u \in C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_l}$ and $c_{uu} = 0$ otherwise.

Now, we define an orthogonal matrix as follows:

$$R = \begin{pmatrix} Q & \\ & I_m \end{pmatrix}.$$

Clearly, $R^T = R^{-1} = R$. By a direct calculation, one has $(\frac{2}{c_i}J_{c_i} - I_{c_i})\mathbf{1}_{c_i} = \mathbf{1}_{c_i}$ for all $i \in \{1, \ldots, k\}$, where $\mathbf{1}_{c_i}$ denotes the all 1s column vector of order c_i. Therefore, $QB = B$. Hence

$$R^{-1}A(\Gamma_1)R = \begin{pmatrix} A(X^\pi) & QB \\ B^TQ & A(Y) \end{pmatrix} = \begin{pmatrix} A(X^\pi) & B \\ B^T & A(Y) \end{pmatrix} = A(\Gamma_2),$$
$$R^{-1}D(\Gamma_1)R = \begin{pmatrix} D(X^\pi) + mC & \\ & D(Y) + (\sum_{j=1}^{l} |C_{i_j}|)I_m \end{pmatrix} = D(\Gamma_2).$$

That is, Γ_1 and Γ_2 are degree similar. □

Example 13. In Figure 3, $X_{2,i}$ is a graph obtained from $X_{1,i}$ by adding a path P_3 and join it to all vertices of C_1 in $X_{1,i}$ for $i \in \{1, 2\}$.

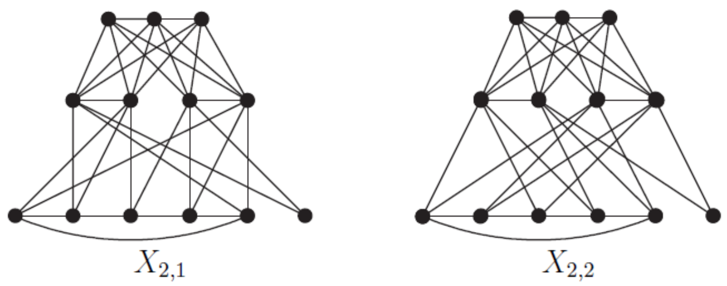

Figure 3: Joins of P_3 with a vertex subset C_1 of $X_{1,i}$.

Define

$$R_2 = \begin{pmatrix} \frac{1}{2}J_4 - I_4 & & \\ & I_6 & \\ & & I_3 \end{pmatrix}.$$

It is routine to check that $R_2^{-1}A(X_{2,1})R_2 = A(X_{2,2})$ and $R_2^{-1}D(X_{2,1})R_2 = D(X_{2,2})$. Therefore, $X_{2,1}$ and $X_{2,2}$ are degree similar. □

Recall that $\psi(X, t, \mu) = \psi(Y, t, \mu)$ is a necessary condition for two graphs X and Y being degree similar. In fact, there is more we can say about $\psi(X, t, \mu)$. The following theorem is a generalization of Johnson and Newman's result in [7]. For the detailed proof, one may see [5].

Theorem 14. *Assume X and Y are graphs with $\psi(X, t, \mu) = \psi(Y, t, \mu)$ and $\psi(\overline{X}, t, \mu) = \psi(\overline{Y}, t, \mu)$, then there is an orthogonal matrix Q such that*

$$Q^T A_\mu(X)Q = A_\mu(Y) \quad \textit{and} \quad Q\mathbf{1} = \mathbf{1}.$$

Based on the above theorem, by using join operator, we may obtain a family of cospectral graphs with respect to the $A - \mu D$ matrix.

Lemma 15. *If X and Y are graphs such taht $\psi(X,t,\mu) = \psi(Y,t,\mu)$ and $\psi(\overline{X},t,\mu) = \psi(\overline{Y},t,\mu)$, then $\psi(X \vee H,t,\mu) = \psi(Y \vee H,t,\mu)$ and $\psi(\overline{X \vee H},t,\mu) = \psi(\overline{Y \vee H},t,\mu)$, where H is any graph.*

Proof. Assume X and Y have n vertices, H has m vertices. In view of Theorem 14, there is an orthogonal matrix Q such that

$$Q^T A_\mu(X) Q = A_\mu(Y) \quad \text{and} \quad Q\mathbf{1}_n = \mathbf{1}_n.$$

Then $Q^T\mathbf{1}_n = \mathbf{1}_n$, which implies $Q^T J_{n\times m} = J_{n\times m}$. Clearly,

$$A_\mu(X \vee H) = \begin{pmatrix} A_\mu(X) - \mu n I_m & J_{n\times m} \\ J_{m\times n} & A_\mu(H) - \mu m I_n \end{pmatrix}.$$

It is routine to check that

$$\begin{pmatrix} Q^T & \\ & I \end{pmatrix} A_\mu(X \vee H) \begin{pmatrix} Q & \\ & I \end{pmatrix} = A_\mu(Y \vee H).$$

Hence $A_\mu(X\vee H)$ and $A_\mu(Y \vee H)$ are similar. It follows that $\psi(X\vee H,t,\mu) = \psi(Y \vee H,t,\mu)$.

Obviously, $\overline{X \vee H} = \overline{X} \cup \overline{H}$ and $\overline{Y \vee H} = \overline{Y} \cup \overline{H}$. Together with the fact that $\psi(\overline{X},t,\mu) = \psi(\overline{Y},t,\mu)$, then $\psi(\overline{X \vee H},t,\mu) = \psi(\overline{Y \vee H},t,\mu)$ holds obviously. ☐

Our next contribution involves constructions of degree-similar graphs using alternative graph products, namely Cartesian product, tensor product, strong product and lexicographic product.

Let A and B be matrices of order $m\times n$ and $p\times q$, respectively. The *Kronecker product* of two matrices A and B, denoted $A \otimes B$, is the $mp \times nq$ block matrix $[a_{ij}B]$. It can be verified from the definition that

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

Let X and Y be graphs with vertex sets $V(X)$ and $V(Y)$, respectively. Put $n := |V(X)|$ and $m := |V(Y)|$. The *Cartesian product* of X and Y, denoted by $X\Box Y$, is the graph defined as follows. The vertex set of $X\Box Y$ is $V(X) \times V(Y)$. The vertices (u,v) and (u',v') are adjacent if either $u = u'$ and v is adjacent to v' in Y, or $v = v'$ and u is adjacent to u' in X. It is well known that

$$A(X\Box Y) = (A(X) \otimes I_m) + (I_n \otimes A(Y)),\ D(X\Box Y) = (D(X) \otimes I_m) + (I_n \otimes D(Y)).$$

The *tensor product* of X and Y, denoted by $X \otimes Y$, is the graph defined as follows. The vertex set of $X \otimes Y$ is $V(X) \times V(Y)$. The vertices (u,v) and (u',v') are adjacent if u is adjacent to u' in X and v is adjacent to v' in Y. Notice that

$$A(X \otimes Y) = A(X) \otimes A(Y), \quad D(X \otimes Y) = D(X) \otimes D(Y).$$

The *strong product* of X and Y, denoted by $X \boxtimes Y$, is the graph defined as follows. The vertex set of $X \boxtimes Y$ is $V(X) \times V(Y)$. The vertices (u, v) and (u', v') are adjacent if either $u = u'$ and v is adjacent to v' in Y, or $v = v'$ and u is adjacent to u' in X, or u is adjacent to u' in X and v is adjacent to v' in Y. Then

$$\begin{aligned} A(X \boxtimes Y) &= (A(X) \otimes I_m) + (I_n \otimes A(Y)) + A(X) \otimes A(Y), \\ D(X \boxtimes Y) &= (D(X) \otimes I_m) + (I_n \otimes D(Y)) + D(X) \otimes D(Y). \end{aligned}$$

The *lexicographic product* of X and Y, denoted by $X \odot Y$, is the graph defined as follows. The vertex set of $X \odot Y$ is $V(X) \times V(Y)$. The vertices (u, v) and (u', v') are adjacent if either u is adjacent to u' in X, or $u = u'$ and v is adjacent to v' in Y. It is easy to obtain

$$\begin{aligned} A(X \odot Y) &= (A(X) \otimes J_m) + (I_n \otimes A(Y)), \\ D(X \odot Y) &= (mD(X) \otimes I_m) + (I_n \otimes D(Y)). \end{aligned}$$

Lemma 16. *If X_1 and X_2 are two degree-similar graphs with order n, Y_1 and Y_2 are two degree-similar graphs with order m, then $X_1 \Box Y_1$ and $X_2 \Box Y_2$ (resp. $X_1 \otimes Y_1$ and $X_2 \otimes Y_2$, $X_1 \boxtimes Y_1$ and $X_2 \boxtimes Y_2$) are degree similar. Furthermore, if Y_1 is connected, then $X_1 \odot Y_1$ and $X_2 \odot Y_2$ are also degree similar.*

Proof. Here, we only prove $X_1 \Box Y_1$ and $X_2 \Box Y_2$ are degree similar, the remaining cases can be proved similarly. By assumption, there exist two invertible real matrices M_1 and M_2 such that

$$M_1^{-1} A(X_1) M_1 = A(X_2), \quad M_1^{-1} D(X_1) M_1 = D(X_2),$$

and

$$M_2^{-1} A(Y_1) M_2 = A(Y_2), \quad M_2^{-1} D(Y_1) M_2 = D(Y_2).$$

Let $M = M_1 \otimes M_2$. By applying the properties of Kronecker product, one has

$$\begin{aligned} M^{-1} A(X_1 \Box Y_1) M &= (M_1 \otimes M_2)^{-1}((A(X_1) \otimes I_m) + (I_n \otimes A(Y_1)))(M_1 \otimes M_2) \\ &= (M_1^{-1} A(X_1) M_1 \otimes M_2^{-1} I_m M_2) + (M_1^{-1} I_n M_1 \otimes M_2^{-1} A(Y_1) M_2) \\ &= (A(X_2) \otimes I_m) + (I_n \otimes A(Y_2)) \\ &= A(X_2 \Box Y_2), \end{aligned}$$

and

$$\begin{aligned} M^{-1} D(X_1 \Box Y_1) M &= (M_1 \otimes M_2)^{-1}((D(X_1) \otimes I_m) + (I_n \otimes D(Y_1)))(M_1 \otimes M_2) \\ &= (M_1^{-1} D(X_1) M_1 \otimes M_2^{-1} I_m M_2) + (M_1^{-1} I_n M_1 \otimes M_2^{-1} D(Y_1) M_2) \\ &= (D(X_2) \otimes I_m) + (I_n \otimes D(Y_2)) \\ &= D(X_2 \Box Y_2). \end{aligned}$$

It follows that $X_1 \Box Y_1$ and $X_2 \Box Y_2$ are degree similar. □

7 k-sum and rooted product

In this section, we show how to use k-sum and rooted product of graphs to build families of degree-similar graphs. The *k-sum* of graphs X and Y is obtained by merging k distinct vertices in X with k distinct vertices in Y.

Lemma 17. *Let X_1 and X_2 be two degree-similar graphs, and Y be an n-vertex graph with $\{w_1, \dots, w_k\} \subseteq V(Y)$. Choose $u_1, \dots, u_k \in V(X_1)$ and $v_1, \dots, v_k \in V(X_2)$ such that for $i \in \{1, \dots, k\}$,*

(i) *the degree of u_i (resp. v_i) is different with that of all other vertices in X_1 (resp. X_2);*

(ii) $d_{X_1}(u_i) = d_{X_2}(v_i)$*;*

(iii) $X_1[\{u_1, \dots, u_k\}]$ *is connected.*

Denote by Γ_1 (resp. Γ_2) the k-sum of X_1 (resp. X_2) and Y, which is obtained by merging $\{u_1, \dots, u_k\}$ of X_1 (resp. $\{v_1, \dots, v_k\}$ of X_2) with $\{w_1, \dots, w_k\}$ of Y in order. Then Γ_1 and Γ_2 are degree similar.

Proof. Since X_1 and X_2 are two degree-similar graphs, there exists an invertible matrix M such that

$$M^{-1}A(X_1)M = A(X_2), \quad M^{-1}D(X_1)M = D(X_2).$$

In view of Lemma 5, M is block diagonal with respect to the partition $(V(X) \setminus \{u_1\}) \cup \{u_1\} \cup \cdots \cup \{u_k\}$. Therefore, M can be written as

$$M = \begin{pmatrix} M_1 & & & \\ & a_1 & & \\ & & \ddots & \\ & & & a_k \end{pmatrix}$$

for some invertible matrix M_1 and nonzero real numbers $a_1, \dots, a_k$. The adjacency matrix of Γ_1 is

$$A(\Gamma_1) = \begin{pmatrix} A(X_1\backslash\{u_1,\dots,u_k\}) & \mathbf{x_1} & \mathbf{x_2} & \cdots & \mathbf{x_k} & \mathbf{0} \\ \mathbf{x_1}^T & 0 & a_{12} & \cdots & a_{1k} & \mathbf{y_1}^T \\ \mathbf{x_2}^T & a_{21} & 0 & \cdots & a_{2k} & \mathbf{y_2}^T \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{x_k}^T & a_{k1} & a_{k2} & \cdots & 0 & \mathbf{y_k}^T \\ \mathbf{0} & \mathbf{y_1} & \mathbf{y_2} & \cdots & \mathbf{y_k} & A(Y\backslash\{w_1,\dots,w_k\}) \end{pmatrix},$$

for some column vectors $\mathbf{x_i}$, $\mathbf{y_i}$ and $a_{ij} \in \{0,1\}$ for $1 \leqslant i,j \leqslant k$. The degree matrix of Γ_1 is

$$D(\Gamma_1) = \begin{pmatrix} D(X_1)(\{u_1,\ldots,u_k\}) & & & & & \\ & b_1 & & & & \\ & & b_2 & & & \\ & & & \ddots & & \\ & & & & b_k & \\ & & & & & D(Y)(\{w_1,\ldots,w_k\}) \end{pmatrix},$$

where $b_i = d_{X_1}(u_i) + d_Y(w_i) - |\{j : u_iu_j \in E(X_1) \text{ and } w_iw_j \in E(Y)\}|$. Furthermore, the adjacency matrix and degree matrix of Γ_2 have the similar forms.

Recall that $M^{-1}A(X_1)M = A(X_2)$. Then $a_i^{-1}a_{ij}a_j = 0$ if $a_{ij} = 0$, and $a_i^{-1}a_{ij}a_j = 1$ if $a_{ij} = 1$. Therefore, $X_1[\{u_1,\ldots,u_k\}] \cong X_2[\{v_1,\ldots,v_k\}]$. Together with $X_1[\{u_1,\ldots,u_k\}]$ is connected, one has $a_1 = \cdots = a_k$. Let

$$Q = \begin{pmatrix} M_1 & & \\ & a_1I_k & \\ & & a_1I_{n-k} \end{pmatrix}.$$

It is straightforward to check that

$$Q^{-1}A(\Gamma_1)Q = A(\Gamma_2), \quad Q^{-1}D(\Gamma_1)Q = D(\Gamma_2),$$

i.e., Γ_1 and Γ_2 are degree similar. □

Taking a base graph X and a sequence $\mathcal{Y}$ of rooted graphs $Y_1,\ldots,Y_k$, and then merge the k roots of graphs in $\mathcal{Y}$ with k distinct vertices $u_1,\ldots,u_k$ of X. We refer to it as the *rooted product* of X with $\mathcal{Y}$ at $u_1,\ldots,u_k$. By a similar discussion as Lemma 17, we can get the following result.

Lemma 18. *Let X_1 and X_2 be two degree-similar graphs, and let $\mathcal{Y} = (Y_1,\ldots,Y_k)$ be a sequence of rooted graphs. Choose $u_1,\ldots,u_k \in V(X_1)$ and $v_1,\ldots,v_k \in V(X_2)$ such that for $i \in \{1,\ldots,k\}$,*

(i) *the degree of u_i (resp. v_i) is different with that of all other vertices in X_1 (resp. X_2);*

(ii) *$d_{X_1}(u_i) = d_{X_2}(v_i)$.*

Then the rooted product of X_1 with $\mathcal{Y}$ at $u_1,\ldots,u_k$ and the rooted product of X_2 with $\mathcal{Y}$ at $v_1,\ldots,v_k$ are degree similar.

Example 19. In Figure 4, for $i \in \{1,2\}$, $X_{3,i}$ is the 2-sum of $X_{1,i} + uv$ and a cycle C_3. Notice that the degree of u (resp. v) is different with that of all other vertices in $X_{1,i}+uv$. Let

$$R_3 = \begin{pmatrix} \frac{1}{2}J_4 - I_4 & & \\ & I_6 & \\ & & 1 \end{pmatrix}.$$

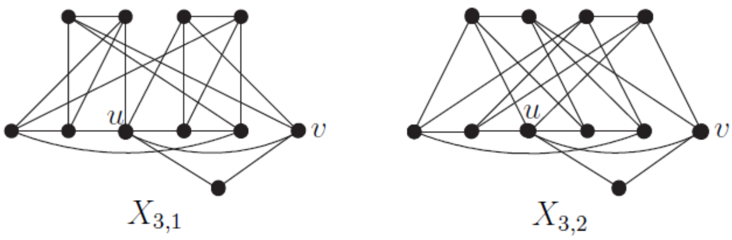

Figure 4: 2-sum.

One may check that $R_3^{-1}A(X_{3,1})R_3 = A(X_{3,2})$ and $R_3^{-1}D(X_{3,1})R_3 = D(X_{3,2})$. Hence $X_{3,1}$ and $X_{3,2}$ are degree similar.

In Figure 5, for $i \in \{1,2\}$, $X_{4,i}$ is the rooted product of $X_{1,i} + uv$ with (P_3, C_3) at u, v. Let

$$R_4 = \begin{pmatrix} \frac{1}{2}J_4 - I_4 & & & \\ & I_6 & & \\ & & I_2 & \\ & & & I_2 \end{pmatrix}.$$

By a direct calculation, one has $R_4^{-1}A(X_{4,1})R_4 = A(X_{4,2})$ and $R_4^{-1}D(X_{4,1})R_4 = D(X_{4,2})$. Thus, $X_{4,1}$ and $X_{4,2}$ are degree similar.

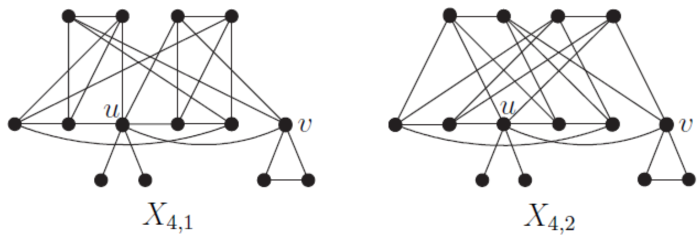

Figure 5: Rooted product.

□

8 Adding or deleting vertices

Butler et al. [1] showed that: if X_1 and X_2 are two graphs with $\psi(X_1, t, \mu) = \psi(X_2, t, \mu)$, then the graphs resulting from attaching an arbitrary rooted graph Y to each vertex of X_1 and each vertex of X_2 will be cospectral with respect to the matrix $A - \mu D$. Motivated by this, in this section, we construct degree-similar graphs by adding or deleting vertices.

Lemma 20. *Let X_1 and X_2 be two degree-similar graphs. Assume $d_1, \ldots, d_k$ are all distinct vertex degrees in X_1. For each $i \in \{1, \ldots, k\}$, attach s_i pendant vertices to each vertex with degree d_i in X_1 and X_2 respectively, then the two graphs obtained are degree similar.*

Proof. For convenience, denote by Γ_1 and Γ_2 the two graphs obtained from X_1 and X_2 respectively. Firstly, we reorder the vertices of X_1 and X_2 such that their degree matrices can be written as

$$D(X_1) = D(X_2) = \begin{pmatrix} d_1 I_{n_1} & & & \\ & d_2 I_{n_2} & & \\ & & \ddots & \\ & & & d_k I_{n_k} \end{pmatrix},$$

where n_i is the number of vertices in $V_i := \{v : d_{X_1}(v) = d_i\}$ for $i \in \{1, \ldots, k\}$.

Since X_1 and X_2 are degree similar, there exists an invertible real matrix M such that

$$M^{-1}A(X_1)M = A(X_2), \quad M^{-1}D(X_1)M = D(X_2).$$

In view of Lemma 5, M is block diagonal with respect to the partition $V_1 \cup \cdots \cup V_k$. That is,

$$M = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_k \end{pmatrix}$$

for some invertible matrices $M_1, \ldots, M_k$. For $i \in \{1, \ldots, k\}$, assume

$$V_i = \{v_{n_1+\cdots+n_{i-1}+1}, v_{n_1+\cdots+n_{i-1}+2}, \ldots, v_{n_1+\cdots+n_{i-1}+n_i}\},$$

and for $j \in \{1, \ldots, n_i\}$, assume $\{u^i_{(j-1)s_i+1}, \ldots, u^i_{js_i}\}$ are all pendant vertices attached to $v_{n_1+\cdots+n_{i-1}+j}$. Next, partition the vertex set of Γ_1 as follows:

$$\begin{aligned} &\{u^1_1, u^1_{s_1+1}, \ldots, u^1_{(n_1-1)s_1+1}\} \cup \{u^1_2, u^1_{s_1+2}, \ldots, u^1_{(n_1-1)s_1+2}\} \cup \cdots \cup \{u^1_{s_1}, u^1_{2s_1}, \ldots, u^1_{n_1 s_1}\} \\ &\cup \{u^2_1, u^2_{s_2+1}, \ldots, u^2_{(n_2-1)s_2+1}\} \cup \{u^2_2, u^2_{s_2+2}, \ldots, u^2_{(n_2-1)s_2+2}\} \cup \cdots \cup \{u^2_{s_2}, u^2_{2s_2}, \ldots, u^2_{n_2 s_2}\} \\ &\cup \cdots \\ &\cup \{u^k_1, u^k_{s_k+1}, \ldots, u^k_{(n_k-1)s_k+1}\} \cup \{u^k_2, u^k_{s_k+2}, \ldots, u^k_{(n_k-1)s_k+2}\} \cup \cdots \cup \{u^k_{s_k}, u^k_{2s_k}, \ldots, u^k_{n_k s_k}\} \\ &\cup V_1 \cup V_2 \cup \cdots \cup V_k. \end{aligned}$$

It is routine to check that for $i \in \{1, 2\}$, the adjacency matrix of Γ_i is

$$A(\Gamma_i) = \left(\begin{array}{cc} \mathbf{0} & \begin{matrix} \mathbf{1}_{s_1} \otimes I_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{s_2} \otimes I_{n_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_{s_k} \otimes I_{n_k} \end{matrix} \\ \begin{matrix} \mathbf{1}^T_{s_1} \otimes I_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}^T_{s_2} \otimes I_{n_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}^T_{s_k} \otimes I_{n_k} \end{matrix} & A(X_i) \end{array}\right),$$

the degree matrix of Γ_i is

$$D(\Gamma_i) = \begin{pmatrix} I_{n_1s_1+\cdots+n_ks_k} & & & & \\ & (d_1+s_1)I_{n_1} & & & \\ & & (d_2+s_2)I_{n_2} & & \\ & & & \ddots & \\ & & & & (d_k+s_k)I_{n_k} \end{pmatrix}.$$

Let Q be a matrix defined by

$$Q = \begin{pmatrix} I_{s_1}\otimes M_1 & & & & \\ & I_{s_2}\otimes M_2 & & & \\ & & \ddots & & \\ & & & I_{s_k}\otimes M_k & \\ & & & & M \end{pmatrix}.$$

By a direct calculation, one has

$$Q^{-1}A(\Gamma_1)Q = A(\Gamma_2), \quad Q^{-1}D(\Gamma_1)Q = D(\Gamma_2),$$

i.e., Γ_1 and Γ_2 are degree similar. □

Lemma 21. *Let X_1 and X_2 be two degree-similar graphs. Assume $d_1,\ldots,d_k$ are all distinct vertex degrees in X_1, and $V_i = \{v : d_{X_1}(v) = d_i\}$ with cardinality n_i. For each $i \in \{1,\ldots,l\}$ with $l \leqslant k$, add n_i isolated vertices and join each of them to all vertices with degree d_i in X_1 and X_2 respectively, then the obtained graphs are degree similar.*

Proof. Denote by Γ_1 and Γ_2 the two graphs obtained from X_1 and X_2. Firstly, we reorder the vertices of X_1 and X_2 such that their degree matrices can be written as

$$D(X_1) = D(X_2) = \begin{pmatrix} d_1I_{n_1} & & & \\ & d_2I_{n_2} & & \\ & & \ddots & \\ & & & d_kI_{n_k} \end{pmatrix}.$$

Since X_1 and X_2 are degree similar, there exists an invertible real matrix M such that

$$M^{-1}A(X_1)M = A(X_2), \quad M^{-1}D(X_1)M = D(X_2).$$

In view of Lemma 5, M is block diagonal with respect to the partition $V_1 \cup \cdots \cup V_k$. That is,

$$M = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_k \end{pmatrix}$$

for some invertible matrices $M_1, \ldots, M_k$. For $i \in \{1, \ldots, l\}$, assume

$$V_i = \{v_{n_1+\cdots+n_{i-1}+1}, v_{n_1+\cdots+n_{i-1}+2}, \ldots, v_{n_1+\cdots+n_{i-1}+n_i}\},$$

and assume $\{u_1^i, \ldots, u_{n_i}^i\}$ are all added vertices that are adjacent to vertices in V_i. Partition the vertex set of Γ_1 as follows:

$$\{u_1^1, \ldots, u_{n_1}^1\} \cup \{u_1^2, \ldots, u_{n_2}^2\} \cup \cdots \cup \{u_1^l, \ldots, u_{n_l}^l\} \cup V_1 \cup V_2 \cup \cdots \cup V_l \cup (V_{l+1} \cup \cdots \cup V_k).$$

For $i \in \{1, 2\}$, it is routine to check that the adjacency matrix of Γ_i is

$$A(\Gamma_i) = \begin{pmatrix} & \mathbf{0} & & J_{n_1} & & \mathbf{0} \\ & & & & \ddots & \vdots \\ & & & & J_{n_l} & \mathbf{0} \\ J_{n_1} & & & & & \\ & \ddots & & & A(X_i) & \\ & & J_{n_l} & & & \\ \mathbf{0} & \cdots & \mathbf{0} & & & \end{pmatrix},$$

and the degree matrix of Γ_i is

$$D(\Gamma_i) = \begin{pmatrix} n_1 I_{n_1} & & & & & & & & \\ & \ddots & & & & & & & \\ & & n_l I_{n_l} & & & & & & \\ & & & (d_1+n_1)I_{n_1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & (d_l+n_l)I_{n_l} & & & \\ & & & & & & d_{l+1}I_{n_{l+1}} & & \\ & & & & & & & \ddots & \\ & & & & & & & & d_k I_{n_k} \end{pmatrix}.$$

Let

$$Q = \begin{pmatrix} M_1 & & & \\ & \ddots & & \\ & & M_l & \\ & & & M \end{pmatrix}.$$

Together with (2), and by a direct calculation, one has

$$Q^{-1}A(\Gamma_1)Q = A(\Gamma_2), \quad Q^{-1}D(\Gamma_1)Q = D(\Gamma_2),$$

i.e., Γ_1 and Γ_2 are degree similar. $\square$

Finally, we construct degree-similar graphs by deleting a vertex.

Lemma 22. *Let X_1 and X_2 be two degree-similar graphs. For $i \in \{1,2\}$, choose $u_i \in V(X_i)$ such that*

(i) *the degree of u_i in X_i is different with that of all other vertices in X_i;*

(ii) $d_{X_1}(u_1) = d_{X_2}(u_2)$*;*

(iii) $w \in N_{X_1}(u_1)$ *implies* $\{w' : d_{X_1}(w') = d_{X_1}(w)\} \subseteq N_{X_1}(u_1)$.

Then $X_1\backslash u_1$ and $X_2\backslash u_2$ are degree similar.

Proof. Since X_1 and X_2 are two degree-similar graphs, there exists an invertible real matrix M such that

$$M^{-1}A(X_1)M = A(X_2), \quad M^{-1}D(X_1)M = D(X_2).$$

In view of Lemma 5, M is block diagonal with respect to the partition $\{u_1\} \cup N_{X_1}(u_1) \cup (V(X_1) \setminus N_{X_1}[u_1])$. That is, M can be written as follows:

$$M = \begin{pmatrix} a & & \\ & M_1 & \\ & & M_2 \end{pmatrix},$$

for some invertible matrices M_1, M_2 and a nonzero real number a. Furthermore, we can partition $A(X_1)$ and $D(X_1)$ as follows:

$$A(X_1) = \begin{pmatrix} 0 & \mathbf{1}^T & \mathbf{0} \\ \mathbf{1} & A_{11} & A_{12} \\ \mathbf{0} & A_{21} & A_{22} \end{pmatrix}, \quad D(X_1) = \begin{pmatrix} d_{X_1}(u_1) & & \\ & D_1 & \\ & & D_2 \end{pmatrix}.$$

Notice that the adjacency matrix and the degree matrix of $X_1\backslash u_1$ are

$$A(X_1\backslash u_1) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad D(X_1\backslash u_1) = \begin{pmatrix} D_1 - I & \\ & D_2 \end{pmatrix}.$$

Recall that $M^{-1}A(X_1)M = A(X_2)$ and $M^{-1}D(X_1)M = D(X_2)$, then

$$A(X_2) = \begin{pmatrix} 0 & a^{-1}\mathbf{1}^T M_1 & \mathbf{0} \\ M_1^{-1}\mathbf{1}a & M_1^{-1}A_{11}M_1 & M_1^{-1}A_{12}M_2 \\ \mathbf{0} & M_2^{-1}A_{21}M_1 & M_2^{-1}A_{22}M_2 \end{pmatrix},$$

and

$$D(X_2) = \begin{pmatrix} d_{X_1}(u_1) & & \\ & M_1^{-1}D_1M_1 & \\ & & M_2^{-1}D_2M_2 \end{pmatrix}.$$

Clearly, the first row of $D(X_2)$ is indexed by u_2. Together with Item (ii), we know each entry of $a^{-1}\mathbf{1}^T M_1$ is equal to 1. Thus, $A(X_2)$ is partitioned according to $\{u_2\} \cup N_{X_2}(u_2) \cup (V(X_2) \setminus N_{X_2}[u_2])$. Let $Q = \operatorname{diag}(M_1, M_2)$. Then

$$Q^{-1}A(X_1\backslash u_1)Q = A(X_2\backslash u_2), \quad Q^{-1}D(X_1\backslash u_1)Q = D(X_2\backslash u_2),$$

i.e., $X_1\backslash u_1$ and $X_2\backslash u_2$ are degree similar. ☐

Example 23. In Figure 6, for $i \in \{1,2\}$, $X_{5,i}$ is a graph obtained from $X_{1,i} + uv$ by adding one pendant vertex to each vertex with degree 4, adding two pendant vertices to u and adding three pendant vertices to v. Let

$$R_5 = \begin{pmatrix} \frac{1}{2}J_4 - I_4 & & & & & \\ & I_4 & & & & \\ & & I_2 & & & \\ & & & I_3 & & \\ & & & & \frac{1}{2}J_4 - I_4 & \\ & & & & & I_6 \end{pmatrix}.$$

It is straightforward to check that $R_5^{-1}A(X_{5,1})R_5 = A(X_{5,2})$ and $R_5^{-1}D(X_{5,1})R_5 = D(X_{5,2})$. Hence $X_{5,1}$ and $X_{5,2}$ are degree similar.

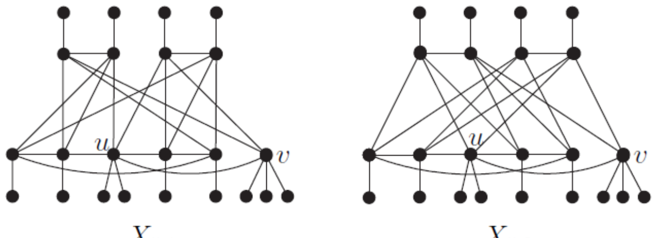

Figure 6: Adding pendant vertices.

In Figure 7, by using Lemma 9, we know $X_{6,1}\backslash\{w_1, w_2\}$ and $X_{6,2}\backslash\{w_1, w_2\}$ are degree similar. Let

$$R_6 = \begin{pmatrix} I_2 & & \\ & \frac{1}{2}J_4 - I_4 & \\ & & I_6 \end{pmatrix}.$$

By a direct calculation, one has $R_6^{-1}A(X_{6,1})R_6 = A(X_{6,2})$ and $R_6^{-1}D(X_{6,1})R_6 = D(X_{6,2})$. Then $X_{6,1}$ and $X_{6,2}$ are degree similar.

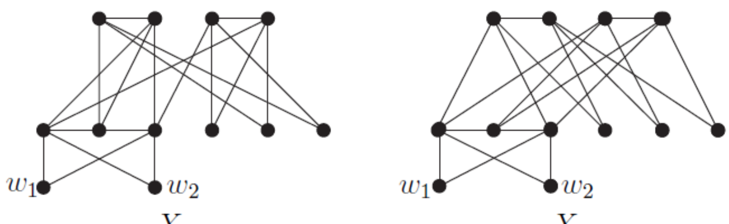

Figure 7: Adding complete graphs.

In Figure 8, for $i \in \{1,2\}$, $X_{7,i}$ can be viewed as a graph obtained from $X_{2,i}$ by deleting the unique vertex with degree 6. Notice that the neighborhood of this vertex contains all vertices with degree 5 and 7 in $X_{2,i}$. Let

$$R_7 = \begin{pmatrix} \frac{1}{2}J_4 - I_4 & & \\ & I_6 & \\ & & I_2 \end{pmatrix}.$$

Then $R_7^{-1}A(X_{7,1})R_7 = A(X_{7,2})$ and $R_7^{-1}D(X_{7,1})R_7 = D(X_{7,2})$. Hence $X_{7,1}$ and $X_{7,2}$ are degree similar. □

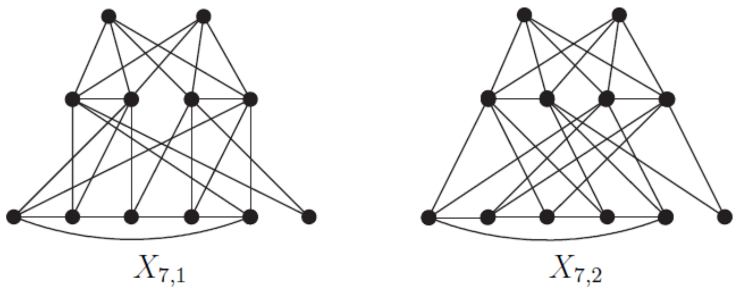

Figure 8: Deleting a vertex.

9 Similarity

Recall that

$$\psi(X, t, \mu) = \det(tI - (A - \mu D)).$$

Here we view $A - \mu D$ as a matrix over the field of rational functions $\mathbb{Q}(\mu)$, and then $\psi(X, t, \mu)$ is the characteristic polynomial of $A - \mu D$.

We are accustomed to the fact that symmetric matrices over $\mathbb{R}$ are similar if and only if their characteristic polynomials are equal. In general though, equality of characteristic polynomials is not enough to ensure similarity. Instead we have the following. (For a proof and more details, see Lancaster and Tismenetsky [8, Theorem 7.6.1] or Friedland [3, Theorem 2.1.4].)

Theorem 24. *Two matrices B_1 and B_2 over the field $\mathbb{F}$ are similar if and only if the matrices $tI - B_1$ and $tI - B_2$ have the same Smith normal form.* □

If A and D are integer matrices, then the Smith normal form of $tI - (A - \mu D)$ is determined by the determinants of submatrices of $tI - (A - \mu D)$, and by the greatest common divisors of sets of these polynomials. Thus all calculations are carried out in the principal ideal domain $\mathbb{Q}(\mu)[t]$. This yields the following:

Lemma 25. *Matrices $A_1 - \mu D_1$ and $A_2 - \mu D_2$ are similar over $\mathbb{R}(\mu)$ if and only if they are similar over $\mathbb{Q}(\mu)$.* □

If X_1 and X_2 are degree-similar graphs with adjacency matrices A_1, A_2 and degree matrices D_1, D_2, then in view of Theorem 24, the Smith normal forms of $tI-(A_1-\mu D_1)$ and $tI-(A_2-\mu D_2)$ are equal. For the converse, if $tI-(A_1-\mu D_1)$ and $tI-(A_2-\mu D_2)$ have the same Smith normal form, then $A_1-\mu D_1$ and $A_2-\mu D_2$ are similar over $\mathbb{Q}(\mu)$. Furthermore, we have the following result.

Lemma 26. *Let X_1 and X_2 be graphs with adjacency matrices A_1, A_2 and degree matrices D_1, D_2 respectively. If $A_1-\mu D_1$ and $A_2-\mu D_2$ are similar over $\mathbb{Q}(\mu)$, then A_1 and A_2 are similar over $\mathbb{Q}$, as are D_1 and D_2.*

Proof. Assume $A_1-\mu D_1$ and $A-\mu D_2$ are similar over $\mathbb{Q}(\mu)$. Then there is a matrix $M(\mu)$, with entries rational functions in μ, such that

$$M(\mu)^{-1}(A_1+\mu D_1)M(\mu)=A_2+\mu D_2. \tag{4}$$

The set of poles of the entries of M is finite, and therefore for all sufficiently large rational numbers γ, the real matrix $M(\gamma)$ is invertible. Hence it follows from Equation (4) that D_1 and D_2 are similar.

Next, there is a sequence of rational numbers $(\gamma_i)_{i\geqslant 0}$ converging to zero such that $M(\gamma_i)$ is defined and invertible, and it follows that A_1 and A_2 are similar. □

10 Problems

In Sections 4-8, we provide a number of constructions of pairs of (non-isomorphic) degree-similar graphs. It will be interesting to get more degree-similar graphs. In particular, the result in [9, Theorem 5.3] implies that two trees are degree similar if and only if they are isomorphic. A *unicyclic graph* can be viewed as a graph obtained from a tree by adding one edge. So, we present the first problem:

Problem 27. Find more degree-similar graphs. In particular, are there non-isomorphic degree-similar unicyclic graphs?

Based on Lemma 26, we know that if $tI-(A_1-\mu D_1)$ and $tI-(A_2-\mu D_2)$ have the same Smith normal form, then A_1 and A_2 are similar over $\mathbb{Q}$, as are D_1 and D_2. Then, a natural problem arises.

Problem 28. Let X and Y be two graphs. Assume that $tI-(A(X)-\mu D(X))$ and $tI-(A(Y)-\mu D(Y))$ have the same Smith normal form. Are X and Y degree similar?

For a graph X and an edge $e\in E(X)$, denote by $X\backslash e$ the graph obtained from X by deleting the edge e. In [5], the authors showed that if X is a strongly regular graph, then for any two edges e and f of X, the graphs $X\backslash e$ and $X\backslash f$ are cospectral with cospectral complements, with respect to the adjacency, Laplacian, unsigned Laplacian and normalized Laplacian matrices. Motivated by this, we consider a more general problem.

Problem 29. Let X be a strongly regular graph with two different edges e and f. Are $X\backslash e$ and $X\backslash f$ degree similar?

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