A Short Proof of Seymour's 6-Flow Theorem

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Abstract

We give a compact variation of Seymour's proof that every 2-edge-connected graph has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow.

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All graphs are finite; loops and multiple edges are allowed. For notation not defined here we use [3]. Let $G = (V, E)$ be a directed graph, A an additively-written abelian group, and $f : E \to A$ a function. We say f is an A-flow whenever $\sum_{e \in \delta^+(v)} f(e) = \sum_{e \in \delta^-(v)} f(e)$ holds for every $v \in V$, where $\delta^+(v)$ ($\delta^-(v)$) is the set of edges whose initial (terminal) vertex is v. If $0 \notin f(E)$, then we say f is *nowhere-zero*. If f is a $\mathbb{Z}$-flow with $f(E) \subseteq \{0, \pm 1, \pm 2, \ldots, \pm(k-1)\}$, then we say f is a *k-flow*. Note that reversing an edge e and replacing $f(e)$ with its negation preserves all of the aforementioned properties; accordingly the existence of a nowhere-zero A-flow or k-flow depends only on the underlying graph.

A famous conjecture of Tutte [6] asserts that every 2-edge-connected graph has a nowhere-zero 5-flow. This conjecture remains open with the best result due to Seymour [5] who proved that such graphs have nowhere-zero 6-flows. His argument involves a standard reduction due to Tutte equating the existence of a nowhere-zero k-flow and a nowhere-zero A-flow whenever $|A| = k$, together with the following central result.

Theorem 1 (Seymour)**.** *Every 2-edge-connected digraph has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_3$-flow.*

We give a compact version of Seymour's proof using a slightly stronger inductive statement and simple contraction-based arguments. The value of our proof is its simplicity: our proof uses the same graph structure that Seymour discovered. There are other proofs of the 6-flow theorem, for example [4, 2, 1] whose arguments offer a different perspective, however our proof is not a simplification of any of those.

Our proof of Theorem 2 relies on contracting a set of edges S, finding a flow f in the smaller graph, then uncontracting S and extending the domain of f to include S. Observe that it is always possible to extend the domain while maintaining that f is a flow; and if

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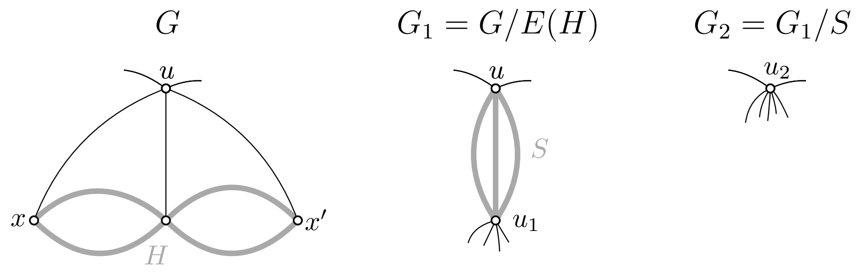
S is a set of at least two parallel edges, and the abelian group has size at least 3, then it is possible to extend the domain so that additionally $0 \notin f(S)$.

In the following we use G/S to denote the graph obtained from G by contracting the set of edges $S \subseteq E$, and $\delta(u)$ is the set of edges incident to vertex u.

Theorem 2. *If $G = (V, E)$ is a 2-edge-connected digraph, and $u \in V$, then G has a nowhere-zero flow $f_2 \times f_3 : E \to \mathbb{Z}_2 \times \mathbb{Z}_3$ so that $\delta(u) \cap \mathrm{supp}(f_2) = \emptyset$.*

Proof. We proceed by induction on $|V|$, with the base case $|V| = 1$ holding trivially. First, suppose $G - u$ has a 1-edge-cut $E(V_1, V_2) = \{e\}$. Choose a partition $\{E_1, E_2\}$ of $E \setminus \{e\}$ so that for $i = 1, 2$ the edges in E_i have ends in $V_i \cup \{u\}$. Let $G_i = G/E_i$. By induction, G_i has a nowhere-zero flow $f_2^i \times f_3^i : E(G_i) \to \mathbb{Z}_2 \times \mathbb{Z}_3$ so the edges incident to the contracted vertex are not in the support of f_2^i. By possibly replacing f_3^1 with its negation, we may assume that $f_3^1(e) = f_3^2(e)$ and then the $\mathbb{Z}_2 \times \mathbb{Z}_3$-flows in each G_i combine to give the desired flow in G. Thus we may assume $G - u$ has no cut-edge.



Choose distinct edges ux and ux' so that x and x' are in the same component of $G - u$ (possibly $x = x'$). By our assumptions, we may choose two edge-disjoint paths $P_1, P_2 \subseteq G - u$ from x to x'. Set $H = P_1 \cup P_2$, $S = E(u, V(H))$, $G_1 = G/E(H)$ with contracted vertex u_1, and $G_2 = G_1/S$ with contracted vertex u_2. By induction, G_2 has a flow $f_2 \times f_3 : E(G_2) \to \mathbb{Z}_2 \times \mathbb{Z}_3$ so that $\delta(u_2) \cap \mathrm{supp}(f_2) = \emptyset$. Because S is a set of at least two parallel edges, we may extend f_3 to $E(G_1)$ so that it remains a flow and $f_3(e) \neq 0$ for all $e \in S$. Because $\delta(u_2) \cap \mathrm{supp}(f_2) = \emptyset$, setting $f_2(e) = 0$ for all $e \in S$ extends f_2 to $E(G_1)$ keeping it a flow. Note that $\delta_{G_1}(u) \cap \mathrm{supp}(f_2) = \emptyset = \delta_{G_1}(u_1) \cap \mathrm{supp}(f_2)$. Now, further extend f_3 to $E(G)$ so that it remains a flow. Because $\delta(u_1) \cap \mathrm{supp}(f_2) = \emptyset$, and every vertex of H has even degree, we may extend f_2 to $E(G)$ by setting $f_2(e) = 1$ for all $e \in E(H)$, keeping it a flow. Now $S \subseteq \mathrm{supp}(f_3), E(H) \subseteq \mathrm{supp}(f_2)$ and so $f_2 \times f_3$ is as desired. $\square$

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