Strong Odd Colorings in Graph Classes of Bounded Expansion

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Abstract

We prove that for every $d \in \mathbb{N}$ and every graph class of bounded expansion $\mathscr{C}$, there exists some $c \in \mathbb{N}$ so that every graph from $\mathscr{C}$ admits a proper coloring with at most c colors satisfying the following condition: in every ball of radius d, every color appears either zero times or an odd number of times. For $d = 1$, this provides a positive answer to a question raised by Goetze, Klute, Knauer, Parada, Peña, and Ueckerdt [ArXiv 2505.02736] about the boundedness of the strong odd chromatic number in graph classes of bounded expansion. The key technical ingredient towards the result is a proof that the strong odd coloring number of a sets system can be bounded in terms of its semi-ladder index, 2VC dimension, and the maximum subchromatic number among induced subsystems.

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1 Introduction

Recall that a *proper coloring* of a graph G is a function ϕ mapping vertices of G to some palette of colors, say C, so that for every two vertices u, v that are adjacent in G, we have $\phi(u) \neq \phi(v)$. Recently, Kwon and Park [10] introduced the concept of a *strong odd coloring* of a graph, which is a proper coloring that satisfies the following additional condition: for every vertex u and every color $i \in C$, the number of vertices of color i among the neighbors of u is either zero or odd. This is a strengthening of the notion of an *odd coloring*, where we only require that the coloring is proper and for every vertex u with a nonempty neighborhood, there exists a color that is featured an odd number of times among the neighbors of u. The *(strong) odd chromatic number* of a graph G is defined as the minimum number of colors needed for a (strong) odd coloring of G.

Kwon and Park [10] proved several results relating the strong odd chromatic number of a graph and its maximum degree. They also asked whether the strong odd chromatic number is universally bounded on the class of planar graphs. This question was answered in positive by Caro et al. [2], who proved that the strong odd chromatic number of every

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planar graph is at most 388. Very recently, Goetze et al. [8] generalized this result by proving that every proper minor-closed class has bounded strong odd chromatic number. They also asked whether this conclusion is true even in the larger generality of graph classes of bounded expansion, which is a concept of uniform, local sparsity in graphs that underlies the theory of Sparsity of Nešetřil and Ossona de Mendez; see [13] for an introduction. Note that, as proved by Hickingbotham [9] and by Liu [11], graph classes of bounded expansion have bounded odd chromatic number. However, the odd chromatic number is known to be bounded in a much larger generality; see the work of Davies, Hatzel, Knauer, McCarty, and Ueckerdt [3] for recent improvements.

In this paper we answer the question of Goetze et al. [8] in positive.

Theorem 1. *Every graph class of bounded expansion has bounded strong odd chromatic number.*

In fact, we prove a more general statement that speaks about balls of any fixed radius d, instead of just neighborhoods.

Theorem 2. *Let $d \in \mathbb{N}$ and $\mathscr{C}$ be a graph class of bounded expansion. Then there exists $c \in \mathbb{N}$ such that for every graph $G \in \mathscr{C}$ there exists a proper coloring of G with c colors satisfying the following condition: for every vertex u of G and color i, the number of vertices of color i that are at distance at most d from u is either zero or odd.*

Note that Theorem 1 follows from Theorem 2 for $d = 1$. Indeed, if ϕ is a proper coloring of a graph G such that in every radius-1 ball, say around a vertex u, every color is featured zero times or an odd number, then this is also true for the (open) neighborhood $N_G(u)$, because the color of u is not featured among the colors present in $N_G(u)$ (due to ϕ being proper).

The core element of our proof is the following statement about strong odd colorings of set systems; that is, colorings of the universe of the set system so that every set in the family contains every color zero times or an odd number of times. Precisely, we bound the odd chromatic number of a set system in terms of three parameters: the semi-ladder index, the 2VC dimension, and the maximum subchromatic number among the induced subsystems (see Sections 2 and 3 for definitions).

Theorem 3. *There is a function $f \colon \mathbb{N}^3 \to \mathbb{N}$ such that the following holds. Suppose $\mathcal{S}$ is a set system with semi-ladder index at most ℓ, 2VC dimension at most h, and whose every induced subsystem has subchromatic number at most s, for some $\ell, h, s \in \mathbb{N}$. Then the strong odd chromatic number of $\mathcal{S}$ is at most $f(\ell, h, s)$.*

The proof of Theorem 3, presented in Section 4, proceeds by induction on the semi-ladder index. The boundedness of the 2VC dimension is exploited through a duality result of Ding et al. [4], whereas the boundedness of the subchromatic index is used to design the induction so that the semi-ladder index drops with every induction step.

To derive Theorem 2 from Theorem 3, it suffices to show that for every $d \in \mathbb{N}$ and graph class of bounded expansion $\mathscr{C}$, the class of set systems of radius-d balls in graphs

from $\mathscr{C}$ has all the relevant parameters bounded. This was explicitly proved for the semi-ladder index by Fabiański et al. [5], for the 2VC dimension by Pilipczuk and Siebertz [17], and for the subchromatic number by Nešetřil et al. [14]. However, the last proof applies only to the whole set system of balls, and not to its induced subsystems. Hence, in Section 3 we propose a new argument that suitably strengthens the result of [14].

2   Preliminaries

By $\mathbb{N}$ we denote the set of nonnegative integers. For a positive integer p, we define $[p] := \{1, \ldots, p\}$.

Graphs. We use standard graph notation. All graphs in this paper are finite, undirected, and simple, that is, without loops or parallel edges. For a graph G, by $V(G)$ we denote the vertex set of G. For $A \subseteq V(G)$, $G[A]$ denotes the subgraph induced by A, which is the graph on vertex set A where any two vertices are adjacent if and only if they are adjacent in G.

A *graph parameter* is a mapping that assigns each graph a nonnegative integer. A *graph class* is simply a set of graphs, typically infinite. For a graph parameter π and a graph class $\mathscr{C}$, we say that $\mathscr{C}$ has *bounded* π if there is a constant $c \in \mathbb{N}$ such that $\pi(G) \leqslant c$ for all $G \in \mathscr{C}$.

A *coloring* of a graph is just a mapping $\phi \colon V(G) \to C$ where C is a finite set of colors. In the context of colorings, we will often call the set C the *palette*. We will often assume $C = [k]$ where $k \in \mathbb{N}$ is the number of colors used by ϕ, but it will sometimes be convenient to use other palettes. Proper colorings and strong odd colorings of graphs have already been defined in Section 1. Recall that the *chromatic number* of a graph G is the least number of colors needed for a proper coloring of G.

For $d \in \mathbb{N}$, we say that a graph H is a *depth-d minor* of a graph G if one can map every vertex u of H to a connected subgraph $\eta(u)$ of G of radius at most d so that the subgraphs $\{\eta(u) \colon u \in V(H)\}$ are pairwise vertex-disjoint, and for every edge uv of H there is an edge in G with one endpoint in $\eta(u)$ and the other in $\eta(v)$. We have the following parameters:

- $\nabla_d(G)$ is the largest average degree among depth-d minors of G; and

- $\omega_d(G)$ is the largest t such that K_t — the complete graph on t vertices — is a depth-d minor of G.

We say that a class of graphs $\mathscr{C}$ has *bounded expansion* if $\mathscr{C}$ has bounded ∇_d for each $d \in \mathbb{N}$. Further, we say that $\mathscr{C}$ is *nowhere dense* if $\mathscr{C}$ has bounded ω_d, for each $d \in \mathbb{N}$. Clearly, every graph class of bounded expansion is also nowhere dense. It is well-known that every proper minor-closed class has bounded expansion, and every class of bounded expansion has bounded chromatic number. A more thorough introduction to the theory of Sparsity, centered around the notions of nowhere denseness and bounded expansion, can be found in [13, 15].

Set systems and their parameters. A *set system* is a pair $\mathcal{S} = (U, \mathcal{F})$ consisting of a set U, called the *universe*, and a family $\mathcal{F}$ of subsets of U. The *Gaifman graph* of $\mathcal{S}$, denoted $\mathsf{Gaif}(\mathcal{S})$, is the graph on the vertex set U where elements $x, y \in U$ are adjacent if and only if there exists $F \in \mathcal{F}$ such that $x, y \in F$. In other words, every set from $\mathcal{F}$ induces a clique in $\mathsf{Gaif}(\mathcal{S})$.

Let $\mathcal{S} = (U, \mathcal{F})$ be a set system. For a subset of the universe $W \subseteq U$ and a subfamily $\mathcal{Q} \subseteq \mathcal{F}$, we define the *induced subsystem* $\mathcal{S}[W, \mathcal{Q}]$ as follows:

$$\mathcal{S}[W, \mathcal{Q}] := (W, \{F \cap W : F \in \mathcal{Q}\}).$$

In other words, the universe of $\mathcal{S}[W, \mathcal{Q}]$ is W, and we keep only the sets belonging $\mathcal{Q}$, each trimmed to W.

Similarly as for graphs, a *class of set systems* is simply a set of set systems. Parameters of set systems and their boundedness on classes are defined analogously to graphs. We call a class of set systems $\mathscr{C}$ *hereditary* if for every $\mathcal{S} = (U, \mathcal{F}) \in \mathscr{C}$, $W \subseteq U$, and $\mathcal{Q} \subseteq \mathcal{F}$, we also have $\mathcal{S}[W, \mathcal{Q}] \in \mathscr{C}$. The *hereditary closure* of a class $\mathscr{C}$ consists of all set systems $\mathcal{S}[W, \mathcal{Q}]$ for $\mathcal{S} = (U, \mathcal{F}) \in \mathscr{C}$, $W \subseteq U$, and $\mathcal{Q} \subseteq \mathcal{F}$. We remark that if one thinks of a set system $(U, \mathcal{F})$ as of a bipartite graph with sides U and $\mathcal{F}$ and adjacency defined by membership, then a class of set systems is hereditary if and only if the corresponding class of bipartite graphs is hereditary, in the usual graph-theoretic sense of being closed under taking induced subgraphs.

A *coloring* of a set system $\mathcal{S} = (U, \mathcal{F})$ is just a mapping $\phi \colon U \to C$ for some palette of colors C. We say that ϕ is *strong odd* if for each $F \in \mathcal{F}$ and every color $i \in C$, the set $F \cap \phi^{-1}(i)$ is either empty or of odd cardinality.

We will exploit several parameters of set systems. In this section we discuss four of them: the semi-ladder index, the co-matching index, the ladder index, and the 2VC dimension. All of them are defined as the maximum size of a certain obstruction that can be found in a set system.

Definition 4. Let $\mathcal{S} = (U, \mathcal{F})$ be a set system, and consider a sequence of elements $u_1, u_2, \ldots, u_\ell \in U$ and a sequence of sets $F_1, F_2, \ldots, F_\ell \in \mathcal{F}$. We call this pair of sequences (see Fig. 1):

- a *semi-ladder* if we have that $u_i \notin F_i$ for each $i \in [\ell]$, and $u_i \in F_j$ for all $i, j \in [\ell]$ with $i < j$;

- a *co-matching* if we have that $u_i \notin F_i$ for each $i \in [\ell]$, and $u_i \in F_j$ for all $i, j \in [\ell]$ with $i \neq j$; and

- a *ladder* if we have that $u_i \notin F_j$ for all $i, j \in [\ell]$ with $i \geqslant j$, and $u_i \in F_j$ for all $i, j \in [\ell]$ with $i < j$.

The *order* of a semi-ladder is the length ℓ of the sequences forming a semi-ladder, and the *semi-ladder index* of $\mathcal{S}$ is the largest order of a semi-ladder present in $\mathcal{S}$. Same for comatchings and ladders.
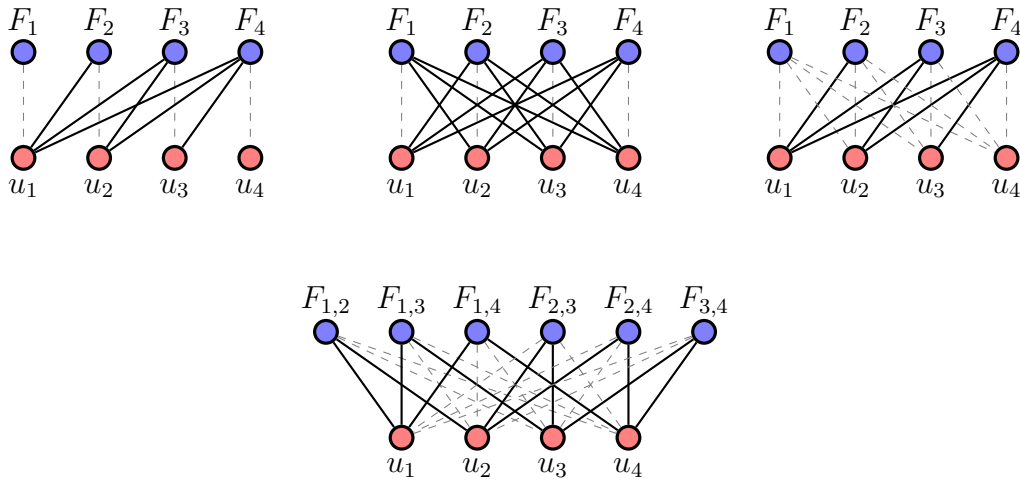
Figure 1: A semi-ladder of order 4, a comatching of order 4, a ladder of order 4, and a 2-shattered set of size 4. Solid edges represent membership, dashed gray edges represent non-membership, lack of an edge represents no specification.

Note that every co-matching is a semi-ladder and also every ladder is a semi-ladder. As observed in [5], Ramsey's Theorem implies that every semi-ladder of a large order contains a co-matching of a large order or a ladder of a large order. In particular, we have the following: a class of set systems has bounded semi-ladder index if and only if it has bounded co-matching index and bounded ladder index.

We proceed to the next parameter.

Definition 5. Let $\mathcal{S} = (U, \mathcal{F})$ be a set system. We say that a set $X \subseteq U$ is *2-shattered* in $\mathcal{S}$ if for every pair of distinct elements $x, y \in X$ there exists $F \in \mathcal{F}$ such that $F \cap X = \{x, y\}$. The *2VC dimension* of $\mathcal{S}$ is the largest cardinality of a subset of U that is 2-shattered in $\mathcal{S}$.

The key fact about set systems of bounded 2VC dimension that we will exploit is the result of Ding, Seymour, and Winkler [4] about the connection between the packing number and the hitting number in set systems whose duals have bounded 2VC dimension. The following statement is the reformulation of this result through duality of set systems.

Theorem 6 ([4, (1.1)], reformulated through duality). *There exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the following holds. Suppose $\mathcal{S} = (U, \mathcal{F})$ is a set system of 2VC dimension at most h such that $\bigcup \mathcal{F} = U$ and $\mathsf{Gaif}(\mathcal{S})$ does not contain any independent set of size larger than k. Then there exists a subfamily $\mathcal{Q} \subseteq \mathcal{F}$ such that $\bigcup \mathcal{Q} = U$ and $|\mathcal{Q}| \leqslant f(h, k)$.*

We note that all the introduced parameters — the semi-ladder/co-matching/ladder indices and the 2VC dimension — are closed under taking induced subsystems: if $\mathcal{S} = (U, \mathcal{F})$ is a set system and $W \subseteq U$ and $\mathcal{Q} \subseteq \mathcal{F}$, then the value of any of these parameters for $\mathcal{S}[W, \mathcal{Q}]$ is not greater than for $\mathcal{S}$.

Set systems of balls. For a graph G, vertex u, and $d \in \mathbb{N}$, the *radius-d ball* around u is defined as

$$\mathsf{Ball}_d^G(u) := \{v \in V(G) \mid \mathrm{dist}_G(u,v) \leqslant d\},$$

where $\mathrm{dist}_G(\cdot, \cdot)$ is the distance metric in G. We define the set system of radius-d balls in G as follows:

$$\mathsf{Balls}_d(G) := (V(G), \{\mathsf{Ball}_d^G(u) \colon u \in V(G)\}).$$

Naturally, for a graph class $\mathscr{C}$, we may define the corresponding classes of set systems:

$$\mathsf{Balls}_d(\mathscr{C}) := \{\mathsf{Balls}_d(G) \colon G \in \mathscr{C}\}.$$

It turns out that provided d is fixed and $\mathscr{C}$ has bounded expansion, the class of set systems $\mathsf{Balls}_d(\mathscr{C})$ has bounded semi-ladder index and bounded 2VC dimension. This was proved even in a larger generality of nowhere dense classes, which subsume classes of bounded expansion. The result for the semi-ladder index is due to Fabiański et al. [5], and for the 2VC dimension is due to Pilipczuk and Siebertz [17]

Lemma 7. *For every $d \in \mathbb{N}$ and graph class of bounded expansion $\mathscr{C}$, the class of set systems $\mathsf{Balls}_d(\mathscr{C})$ has bounded semi-ladder index.*

Proof. As proved in [5, Lemma 14], the class $\mathsf{Balls}_d(\mathscr{C})$ has bounded semi-ladder index, provided $\mathscr{C}$ is *uniformly quasi-wide*. And it is known that classes that are uniformly quasi-wide exactly coincide with nowhere dense classes [12], while every class of bounded expansion is nowhere dense. $\qquad\square$

Lemma 8. *For every $d \in \mathbb{N}$ and graph class of bounded expansion $\mathscr{C}$, the class of set systems $\mathsf{Balls}_d(\mathscr{C})$ has bounded 2VC-dimension.*

Proof. As proved in [17, Corollary 11], if a graph G does not contain K_t as a depth-d minor, then the 2VC dimension of the set system $\mathsf{Balls}_d(G)$ is bounded by $t-1$. Since $\mathscr{C}$ has bounded expansion, it is in particular nowhere dense, hence there exists a $t \in \mathbb{N}$ such that no member of $\mathscr{C}$ contains K_t as a depth-d minor. So the 2VC dimension of $\mathsf{Balls}_d(\mathscr{C})$ is bounded by $t-1$. $\qquad\square$

3 Subchromatic number

In this section we discuss another parameter of graphs and set systems: the subchromatic number. Contrary to the parameters discussed in Section 2, the definition is based on the existence of certain colorings.

Definition 9. A *cluster graph* is a graph whose every connected component is a clique. For a graph G, a *subcoloring* of G is a function $\lambda \colon V(G) \to C$, for some palette of colors C, satisfying the following property: for every color $i \in C$, the induced subgraph $G[\lambda^{-1}(i)]$ is a cluster graph. The *subchromatic number* of a graph G is the minimum number of colors needed for a subcoloring of G. The *subchromatic number* of a set system $\mathcal{S}$ is the subchromatic number of its Gaifman graph $\mathsf{Gaif}(\mathcal{S})$.

For a graph G and $r \in \mathbb{N}$, the *rth power* of G, denoted G^r, is the graph with the same vertex set as G and adjacency defined as follows: any two vertices u, v are adjacent in G^r if and only if $\operatorname{dist}(u, v) \leqslant r$. Naturally, for a graph class $\mathscr{C}$, we denote $\mathscr{C}^r := \{G^r : G \in \mathscr{C}\}$. As proved by Nešetřil et al. [14], powers of bounded expansion classes have bounded subchromatic number, in the following sense.

Theorem 10 ([14, Corollary 11])**.** *For every $r \in \mathbb{N}$ and graph class of bounded expansion $\mathscr{C}$, the graph class $\mathscr{C}^r$ has bounded subchromatic number.*

It can be easily seen that for any graph G and $d \in \mathbb{N}$, the graph $\mathsf{Gaif}(\mathsf{Balls}_d(G))$ is equal to G^{2d}. Therefore, Theorem 10 implies that whenever $\mathscr{C}$ is a graph class of bounded expansion and $d \in \mathbb{N}$ is fixed, the class of set systems $\mathsf{Balls}_d(\mathscr{C})$ has bounded subchromatic number. Unfortunately, for the proof of Theorem 2 we will need a stronger property — that this statement also hold for the hereditary closure of $\mathsf{Balls}_d(\mathscr{C})$. (Note here that the subchromatic number of a set system is *not* closed under taking induced subsystems.) More precisely, we will need the following.

Theorem 11. *For every $d \in \mathbb{N}$ and graph class of bounded expansion $\mathscr{C}$, the hereditary closure of the class of set systems $\mathsf{Balls}_d(\mathscr{C})$ has bounded subchromatic number.*

We were not able to derive Theorem 11 directly from Theorem 10, so we resorted to proposing a completely new proof, based on a different proof method and relying on much stronger tools. In particular, we will use techniques related to logic and first-order transductions.

The remainder of this section is devoted to the proof of Theorem 11.

Transductions. We assume reader's familiarity with the basic terminology of first-order logic on graphs and first-order transductions; see [16] for a broad introduction. Let us only recall that a (first-order) transduction T consists of a finite set of unary predicates C and a first-order formula $\varphi(x, y)$ working over C-colored graphs, where a *C-colored graph* is a graph G expanded with an interpretation of each predicate from C as a subset of vertices of G. We assume $\varphi(x, y)$ to be symmetric: $G \models \varphi(u, v) \Leftrightarrow \varphi(v, u)$ for every C-colored graph G and vertices u, v. Now, we say that H is *transducible* from G using T if H can be obtained from G by means of the following procedure:

- Arbitrarily choose the interpretation of the predicates from C as subsets of vertices of G, thus obtaining a C-colored graph G^+.

- Obtain a graph H^+ from G^+ by interpreting the new adjacency relation using φ: $V(H^+) = V(G^+)$, and two vertices u, v are adjacent in H^+ if and only if $\varphi(u, v)$ holds in G^+.

- Drop all the predicates from C and output an arbitrary induced subgraph of H^+ as H.

For a graph G, by $\mathsf{T}(G)$ we denote the set of all graphs H that are transducible from G using T. And for a graph class $\mathscr{C}$, we define $\mathsf{T}(\mathscr{C}) \coloneqq \bigcup_{G \in \mathscr{C}} \mathsf{T}(G)$. For two graph classes $\mathscr{C}, \mathscr{D}$, we say that $\mathscr{D}$ is *transducible* from $\mathscr{C}$ if $\mathscr{D} \subseteq \mathsf{T}(\mathscr{C})$ for some transduction T. Since transductions are compositional — the composition of two transductions is again a transduction — the relation of transducibility is transitive: If $\mathscr{D}$ is transducible from $\mathscr{C}$, and $\mathscr{E}$ is transducible from $\mathscr{D}$, then $\mathscr{E}$ is transducible from $\mathscr{C}$.

We call a graph class $\mathscr{C}$ *monadically dependent* if the class of all graphs is not transducible from $\mathscr{C}$. Note that by transitivity, if $\mathscr{D}$ is transducible from $\mathscr{C}$ and $\mathscr{D}$ is not monadically dependent, then $\mathscr{C}$ is not monadically dependent either. As proved by Adler and Adler [1], every nowhere dense class of graphs is monadically dependent. This in particular applies to every class of bounded expansion.

Proof plan. We say that a graph class $\mathscr{D}$ has *structurally bounded expansion* if $\mathscr{D}$ is transducible from some class $\mathscr{C}$ of bounded expansion. The main technical hurdle will be in proving the following statement.

Theorem 12. *Let $t \in \mathbb{N}$ and $\mathscr{D}$ be a graph class of structurally bounded expansion such that no member of $\mathscr{D}$ contains $K_{t,t}$ as an induced subgraph. Then $\mathscr{D}$ has bounded subchromatic number.*

Before we discuss the approach to Theorem 12, let us derive Theorem 11 from it.

Proof of Theorem 11 using Theorem 12. Let $\mathscr{E}$ be the hereditary closure of $\mathsf{Balls}_d(\mathscr{C})$ and $\mathscr{D} \coloneqq \mathsf{Gaif}(\mathscr{E})$, that is, $\mathscr{D}$ is the class of Gaifman graphs of the set systems from $\mathscr{E}$. Our goal is to prove that $\mathscr{D}$ has bounded subchromatic number.

Unraveling the definitions, we have the following: for every graph $H \in \mathscr{D}$, there exists a graph $G \in \mathscr{C}$ and two subsets of vertices $A, B \subseteq V(G)$ such that $V(H) = A$ and any two distinct vertices $u, v \in A$ are adjacent in H if and only if there exists $w \in B$ such that $\mathrm{dist}_G(u, w) \leqslant d$ and $\mathrm{dist}_G(v, w) \leqslant d$.

This characterization is clearly expressible as a first-order transduction, which marks the set B using a unary predicate, interprets the new edge relation using the formula $\varphi(x, y) = \exists z\ B(z) \wedge \delta_d(x, z) \wedge \delta_d(y, z)$, where $\delta_d(x, y)$ is the first-order formula verifying that the distance between x and y is at most d, and restricts the vertex set to A. Thus, we have the following.

Claim 13. *$\mathscr{D}$ is transducible from $\mathscr{C}$.*

Further, we observe that graphs from $\mathscr{D}$ exclude some complete bipartite as an induced subgraph.

Claim 14. *There exists $t \in \mathbb{N}$ such that no graph from $\mathscr{D}$ contains $K_{t,t}$ as an induced subgraph.*

Proof of the claim. Suppose, for contradiction, that for every $t \in \mathbb{N}$ there exists a graph in $\mathscr{D}$ that contains $K_{t,t}$ as an induced subgraph. By the characterization of the graphs from $\mathscr{D}$ explained above, this means that for every $t \in \mathbb{N}$, there is a graph $G \in \mathscr{C}$,

vertex subsets $A, B \subseteq V(G)$, and distinct vertices $u_1, \ldots, u_t, v_1, \ldots, v_t \in A$, such that the following two conditions are satisfied:

- For every pair $(i, j) \in [t] \times [t]$, there exists a vertex $w_{i,j} \in B$ such that $\mathrm{dist}_G(u_i, w_{i,j}) \leqslant d$ and $\mathrm{dist}_G(v_j, w_{i,j}) \leqslant d$.

- For every pair $(i, j) \in [t] \times [t]$ with $i \neq j$, there is no vertex $w \in B$ that would satisfy the conditions $\mathrm{dist}_G(u_i, w) \leqslant d$ and $\mathrm{dist}_G(u_j, w) \leqslant d$, and there is no vertex $w \in B$ that would satisfy the conditions $\mathrm{dist}_G(v_i, w) \leqslant d$ and $\mathrm{dist}_G(v_j, w) \leqslant d$.

Note that the second point above implies that for each $(i, j) \in [t] \times [t]$, the only two vertices among $\{u_1, \ldots, u_t, v_1, \ldots, v_t\}$ that are at distance at most d from $w_{i,j}$ are u_i and v_j. In particular, the vertices $w_{i,j}$ for $(i, j) \in [t] \times [t]$ are pairwise different.

Consider now a transduction that marks the set $\{u_i, v_i \colon i \in [t]\}$ using a unary predicate P, interprets a new edge relation using the formula $\varphi(x, y) = \exists z\, P(z) \wedge \delta_d(x, z) \wedge \delta_d(y, z)$, and outputs the subgraph induced by the vertices $\{w_{i,j} \colon (i, j) \in [t] \times [t]\}$. Note that in the output graph, vertices $w_{i,j}$ and $w_{i',j'}$ are adjacent if and only if $i = i'$ or $j = j'$, hence the output graph is the so-called $t \times t$ *rook graph*. Thus, applying the transduction described above to $\mathscr{C}$ allows us to conclude that the class of rook graphs is transducible from $\mathscr{C}$. However, it is known that the class of rook graphs is not monadically dependent, see e.g. [16, Lemma 3]. This implies that $\mathscr{C}$ is not monadically dependent either, a contradiction with the assumption that $\mathscr{C}$ has bounded expansion. ⌟

Since $\mathscr{C}$ has bounded expansion, from Claim 13 we infer that $\mathscr{D}$ has structurally bounded expansion. Together with Claim 14, this ensures that the assumptions of Theorem 12 are satisfied. So we conclude that $\mathscr{D}$ has bounded subchromatic number. $\square$

We are left with proving Theorem 11. The main idea is to use the characterization of graph classes of structurally bounded expansion through *low shrubdepth colorings*, due to Gajarský et al. [6]. This allows us to split the proof of Theorem 11 into a sequence of three steps, each achieving a greater generality. First we treat cographs of bounded depth, then classes of bounded shrubdepth, and finally classes of structurally bounded expansion.

Cographs of bounded depth. Recall that a *cotree* of a graph G is a rooted tree T satisfying the following properties:

- The leaf set of T coincides with the vertex set of G.

- The non-leaf nodes of T are divided into two types: the *join nodes* and the *union nodes*. We require that T encodes the adjacency relation of G in the following way: For any two distinct vertices u, v of G, u and v are adjacent in G if and only if the lowest common ancestor in T is a join node.

In other words, if x is a join node in T, then leaves from different subtrees rooted at the children of x are adjacent; and if x is a union node, then they are non-adjacent. The *depth* of a cotree T is the maximum number of nodes on a root-to-leaf path in T. We

say that G is a *cograph* if G admits a cotree, and G is a cograph of *depth* at most d if G admits a cotree of depth at most d.

Clearly, cographs are closed under complementation (replacing all edges with non-edges and vice versa), because this corresponds to swapping join and union nodes in a cotree. Also, it is known that cographs are *perfect*: the chromatic number of a cograph is equal to the maximum size of a clique contained in it. From these two facts we can trivially derive the following.

Observation 15. *Let G be a cograph that does not contain an independent set of size k. Then the vertex set of G can be partitioned into at most $k-1$ cliques in G.*

Our first goal is to prove the conclusion of Theorem 12 for cographs of bounded depth.

Lemma 16. *Suppose G is a cograph of depth at most d such that G does not contain $K_{t,t}$ as an induced subgraph, for some positive $t \in \mathbb{N}$. Then the subchromatic number of G is at most $1 + (d-1)(t-1)$.*

Proof. We proceed by induction on the depth d. For $d = 1$ the claim is trivial, for G consists of one vertex. Suppose then that $d \geqslant 2$. Let then T be a cotree of G of depth at most d. For a node x of T, by V_x we denote the set of leaves of T that are descendants of x (where every node is considered a descendant of itself), and we write $G_x := G[V_x]$. Further, let r be the root of T. We distinguish two cases, depending on the type of r.

If r is a union node, then for every child x of r, we may find by induction a subcoloring λ_x of G_x with $1 + (d-2)(t-1)$ colors. Noting that G is the disjoint union of graphs G_x over the children x of r, a subcoloring of G can be constructed by taking the union of colorings λ_x, using for each of them the same palette of $1 + (d-2)(t-1) \leqslant 1 + (d-1)(t-1)$ colors.

Suppose now that r is a join node. We note the following:

Claim 17. *There is at most one child x of r in T such that G_x contains an independent set of size t.*

Proof of the claim. If there are two distinct children x, x' such that G_x contains an independent set I of size t and $G_{x'}$ contains an independent set I' of size t, then the set $I \cup I'$ induces a $K_{t,t}$ in G; a contradiction. ⌟

Let then x be the unique child of r such that G_x contains an independent set of size t; or we set x to be any child of r, if there is no child satisfying this property. By induction, we may fix a subcoloring λ_x of G_x using at most $1 + (d-2)(t-1)$ colors. Next, by Observation 15, for each child y different from x we may partition V_y into cliques $K_y^1, \ldots, K_y^{t-1}$ (some possibly empty). For $i \in [t-1]$, define

$$K^i := \bigcup \{K_y^i : y \text{ is a child of } r \text{ different from } x\}.$$

Note that each set K^i is a clique in G, for r is a join node. Therefore, the mapping $\lambda' : (V(G) - V_x) \to [t-1]$ defined by assigning color i to every vertex of K^i is a subcoloring of $G - V_x$. We may now obtain a subcoloring of G with at most $1 + (d-2)(t-1) + (t-1) = 1 + (d-1)(t-1)$ colors by taking the union of subcolorings λ_x and λ', using disjoint palettes for them. □

Classes of bounded shrubdepth. Next, we treat classes of bounded shrubdepth. Shrubdepth is a graph parameter introduced by Ganian et al. [7]. The underlying decomposition notion is called *connection model*, and intuitively, it generalizes the concept of a cograph of bounded depth by allowing a more elaborate mechanism for encoding the adjacency relation in a bounded-depth tree.

Definition 18. Let G be a graph. A *connection model* for G consists of a finite set of labels Λ, a labelling $\lambda \colon V(G) \to \Lambda$, a rooted tree T whose leaf set coincides with the vertex set of G, and, for every non-leaf node x of T, a symmetric relation $M_x \subseteq \Lambda \times \Lambda$. We require that for any two distinct vertices u, v, we have

$$u, v \text{ are adjacent in } G \qquad \text{if and only if} \qquad (\lambda(u), \lambda(v)) \in M_x,$$

where x is the lowest common ancestor of u and v in T. We say that a class of graphs $\mathscr{D}$ has *bounded shrubdepth* if there exists $d, m \in \mathbb{N}$ such that every member of $\mathscr{D}$ admits a connection model of depth at most d and using a label set of size m. As with cographs, the *depth* of a connection model is the maximum number of nodes on a root-to-leaf path in T.

Note the following.

Observation 19. *Let $(\Lambda, \lambda, T, \{M_x\}_{x \in V(T) - V(G)})$ be a connection model of depth at most d of a graph G. Then for every $i \in \Lambda$, the graph $G[\lambda^{-1}(i)]$ is a cograph of depth at most d.*

Proof. A cotree of depth at d of $G[\lambda^{-1}(i)]$ can be constructed by taking T, removing all the nodes which do not have a descendant leaf with label i, and making every remaining non-leaf node x a join node if $(i, i) \in M_x$, and a union node if $(i, i) \notin M_x$. $\qquad \square$

This allows us to easily lift the conclusion of Lemma 16 to the setting of classes of bounded shrubdepth.

Lemma 20. *Let $t \in \mathbb{N}$ and $\mathscr{D}$ be a graph class of bounded shrubdepth such that no member of $\mathscr{D}$ contains $K_{t,t}$ as an induced subgraph. Then $\mathscr{D}$ has bounded subchromatic number.*

Proof. By definition, there are $d, m \in \mathscr{D}$ such that every member of $\mathscr{D}$ admits a connection model of depth at most d and using m labels. By Observation 19, we conclude that for every $G \in \mathscr{D}$, the vertex set of G can be partitioned into at most m subsets, each inducing a cograph of depth at most d. We may now apply Lemma 16 to obtain a subcoloring of each of these cographs. Assuming the obtained subcolorings use disjoint palettes, we may construct a subcoloring of G by taking their union. This subcoloring uses at most $m \cdot (1 + (d-1)(t-1))$ colors, which is a constant depending only on $\mathscr{D}$ and t. $\qquad \square$

Classes of structurally bounded expansion. We will use the following result of Gajarský et al. [6] that classes of structurally bounded expansion admit *low shrubdepth colorings*.

Theorem 21 ([6]). *Let $\mathscr{D}$ be a graph class of structurally bounded expansion. Then for every $p \in \mathbb{N}$, there is a finite palette of colors C_p and a graph class $\mathscr{E}_p$ of bounded shrubdepth such that the following holds: For every graph $G \in \mathscr{C}$, there is a coloring $\alpha \colon V(G) \to C_p$ such that for every subset of colors $I \subseteq C_p$ with $|I| \leqslant p$, we have $G[\alpha^{-1}(I)] \in \mathscr{E}_p$.*

Now, Theorem 21 allows us to easily lift the conclusion of Lemma 20 to the setting of classes of structurally bounded expansion. Thus we finish the proof of Theorem 12, restated below for convenience.

Theorem 12. *Let $t \in \mathbb{N}$ and $\mathscr{D}$ be a graph class of structurally bounded expansion such that no member of $\mathscr{D}$ contains $K_{t,t}$ as an induced subgraph. Then $\mathscr{D}$ has bounded subchromatic number.*

Proof. Consider any $G \in \mathscr{D}$. By Theorem 21 applied to $\mathscr{D}$ and $p = 1$, we find a coloring $\alpha\colon V(G) \to C$ such that each color in α induces a graph belonging to $\mathscr{E}$, where $C := C_1$ and $\mathscr{E} := \mathscr{E}_1$ is the finite palette and the class of bounded shrubdepth provided by Theorem 21. Further, by Lemma 20, for each color $i \in C$ we may find a subcoloring λ_i of $G[\alpha^{-1}(i)]$ using at most q colors, where q is a constant depending only on $\mathscr{E}$ and t. Assuming that each coloring λ_i uses a disjoint palette of q colors, we find that $\bigcup_{i \in C} \lambda_i$ is a subcoloring of G using at most $|C| \cdot q$ colors. This proves that the subchromatic number of every graph in $\mathscr{D}$ is bounded by $|C| \cdot q$, which is a constant depending only on $\mathscr{D}$ and t. $\qquad\square$

4 Main proof

In this section, we prove Theorem 3 and then derive Theorem 2 from it. The idea for the proof of Theorem 3 is to apply induction on the semi-ladder index. The main push is encapsulated in the following technical lemma.

Lemma 22. *Suppose $\mathcal{S} = (U, \mathcal{F})$ is a set system satisfying the following conditions:*

- *The 2VC dimension of $\mathcal{S}$ is bounded by a constant $h \in \mathbb{N}$.*

- *The subchromatic number of $\mathcal{S}$ is bounded by a constant $s \in \mathbb{N}$.*

- *We have $U \notin \mathcal{F}$; that is, $\mathcal{F}$ does not contain the set consisting of all the elements of the universe.*

Then there is a constant $k \in \mathbb{N}$, depending only on h and s, and a subcoloring λ of $\mathsf{Gaif}(\mathcal{S})$ with at most k colors so that the following condition is satisfied: for every color i and connected component X of $\mathsf{Gaif}(\mathcal{S})[\lambda^{-1}(i)]$, the semi-ladder index of the set system $\mathcal{S}[X, \mathcal{F}]$ is strictly smaller than that of $\mathcal{S}$.

Proof. Denote $G := \mathsf{Gaif}(\mathcal{S})$. Since $\mathcal{S}$ has subchromatic number at most s, we may find a subcoloring $\kappa\colon U \to [s]$ of G. Recall that for each $i \in [s]$, the graph $G[\kappa^{-1}(i)]$ is a cluster graph, that is, a disjoint union of cliques. We will call those cliques *clusters* of colors i; and a *cluster* is a cluster of color i, for some $i \in [s]$. Thus, the vertex set of G is partitioned into clusters, each being a clique in G, and distinct clusters of the same color are not adjacent. We let $\mathcal{Y}$ be the set of clusters.

Consider any cluster $Y \in \mathcal{Y}$. Note that $\mathsf{Gaif}(\mathcal{S}[Y, \mathcal{F}])$ is the complete graph on vertex set Y, hence it does not contain any independent set of size larger than 1. Further, as $\mathcal{S}[Y, \mathcal{F}]$ is an induced subsystem of $\mathcal{S}$, the 2VC dimension of $\mathcal{S}[Y, \mathcal{F}]$ is at most h.

Therefore, from Theorem 6 we infer that there exists $\mathcal{Q}_Y \subseteq \mathcal{F}$ such that $\bigcup \mathcal{Q}_Y \supseteq Y$ and $|\mathcal{Q}_Y| \leqslant f(h,1)$, where f is the function provided by Theorem 6. Therefore, denoting $m := f(h,1)$, we may partition Y into subsets $Y^1, \ldots, Y^{m_Y}$ for some $m_Y \leqslant m$ so that each subset Y^j, for $j \in [m_Y]$, is contained in some $F \in \mathcal{F}$.

The following claim is the key step of the whole proof.

Claim 23. *For each $Y \in \mathcal{Y}$ and $j \in [m_Y]$, the semi-ladder index of the set system $\mathcal{S}[Y^j, \mathcal{F}]$ is strictly smaller than that of $\mathcal{S}$.*

Proof of the claim. As we argued, there exists $F \in \mathcal{F}$ such that $Y^j \subseteq F$. Since $U \notin \mathcal{F}$, we have $F \neq U$ and therefore there exists $u \in U$ such that $u \notin F$. Observe now that every semi-ladder in $\mathcal{S}[Y^j, \mathcal{F}]$, say of order ℓ, can be extended by the pair (u, F) to a semi-ladder in $\mathcal{S}$ of order $\ell + 1$. It follows that the semi-ladder index of $\mathcal{S}[Y^j, \mathcal{F}]$ is strictly smaller than the semi-ladder index of $\mathcal{S}$. ⌟

Define now a mapping $\lambda : U \to [s] \times [m]$ by setting

$$\lambda(u) := (\kappa(u), j),$$

where $j \in [m]$ is such that $u \in Y^j$ for $Y \in \mathcal{Y}$ that contains u. By construction, the connected components of the subgraphs induced by the colors of λ are exactly the sets Y^j, for $Y \in \mathcal{Y}$ and $j \leqslant m_Y$. Clearly, each of them is a clique in G, hence λ is a subcoloring of G with $k := sm$ colors. And Claim 23 implies that each such component induces a subsystem of strictly smaller semi-ladder index, as required. $\square$

We are ready to complete the proof of Theorem 3. We recall it for convenience.

Theorem 3. *There is a function $f : \mathbb{N}^3 \to \mathbb{N}$ such that the following holds. Suppose $\mathcal{S}$ is a set system with semi-ladder index at most ℓ, 2VC dimension at most h, and whose every induced subsystem has subchromatic number at most s, for some $\ell, h, s \in \mathbb{N}$. Then the strong odd chromatic number of $\mathcal{S}$ is at most $f(\ell, h, s)$.*

Proof. We fix h and s for the remainder of the proof and proceed by induction on the semi-ladder index ℓ. Let $\mathcal{S} = (U, \mathcal{F})$. Our goal is to prove a slightly stronger statement: we will construct a strong odd coloring $\sigma : U \to [f(\ell, h, s)]$ such that also in the whole universe U, every color appears zero times or an odd number of times. A coloring having this property will be called a *totally strong odd coloring*. The number of colors $f(\ell, h, s)$ will be governed by the following recursion:

$$\begin{aligned} f(0, h, s) &:= 2, \\ f(\ell, h, s) &:= k \cdot f(\ell - 1, h, s) \cdot 2^{f(\ell-1,h,s)+1} \qquad \text{for } \ell \geqslant 1, \end{aligned}$$

where k is the constant provided by Lemma 22 for h and s. For convenience, denote $p_\ell := f(\ell, h, s)$.

Consider first the base case $\ell = 0$. That $\mathcal{S}$ has semi-ladder index 0 means that $\mathcal{F} = \emptyset$ or $\mathcal{F} = \{U\}$. In any case, a totally strong odd coloring of $\mathcal{S}$ with two colors can be

obtained by either assigning all the elements of U the same color, provided $|U|$ is odd, or arbitrarily splitting U into two colors of odd cardinality, provided $|U|$ is even.

We proceed to the induction step. First, we may assume that $U \notin \mathcal{F}$. Indeed, since we anyway aim at constructing a totally strong odd coloring, constructing it for the subsystem obtained by removing U from $\mathcal{F}$ will automatically satisfy also the requirement imposed by U. Note here that removing U from $\mathcal{F}$ means passing to an induced subsystem, in which all the relevant parameters — the semi-ladder index, the 2VC dimension, and the maximum subchromatic number among induced subsystems — are still bounded by ℓ, h, s, respectively.

Having assumed that $U \notin \mathcal{F}$, we may apply Lemma 22 to construct a subcoloring $\lambda \colon U \to [k]$ of $G \coloneqq \mathsf{Gaif}(\mathcal{S})$ with the following property: for every color $i \in [k]$ and a connected component X of $G[\lambda^{-1}(i)]$, the induced subsystem $\mathcal{S}[X, \mathcal{F}]$ has semi-ladder index at most $\ell - 1$. Similarly to the proof of Lemma 22, such connected components will be called *clusters*, and a cluster of color i is one whose elements are colored i in λ. The set of all clusters will be denoted $\mathcal{X}$; this is a partition of U.

By induction, for every cluster $X \in \mathcal{X}$ we may find a totally strong odd coloring $\sigma_X \colon X \to [p_{\ell-1}]$ of $\mathcal{S}[X, \mathcal{F}]$. We now classify clusters X according to their color in λ and the set of colors present within them in σ_X. That is, for $i \in [k]$ and $T \subseteq [p_{\ell-1}]$, we let $\mathcal{X}^{i,T}$ be the set of all clusters $X \in \mathcal{X}$ such that

- X is of color i; and

- $\sigma_X(X) = T$.

Consider any $X \in \mathcal{X}^{i,T}$ and note that since σ_X is totally strong odd, for each color $j \in [p_{\ell-1}]$ the number of elements of X colored j under σ_X is zero if $j \notin T$ and odd if $j \in T$.

Now, for each $i \in [k]$ and $T \subseteq [p_{\ell-1}]$, define a partition $\{\mathcal{X}_1^{i,T}, \mathcal{X}_2^{i,T}\}$ of $\mathcal{X}^{i,T}$ as follows:

- If $|\mathcal{X}^{i,T}|$ is odd, then $\mathcal{X}_1^{i,T} \coloneqq \mathcal{X}^{i,T}$ and $\mathcal{X}_2^{i,T} \coloneqq \emptyset$.

- Otherwise, we let $\{\mathcal{X}_1^{i,T}, \mathcal{X}_2^{i,T}\}$ be an arbitrary partition of $\mathcal{X}^{i,T}$ into two subsets of odd size.

Denote $P \coloneqq [k] \times 2^{[p_{\ell-1}]} \times [2]$ for convenience.

With all the necessary objects defined, we may construct the sought coloring $\sigma \colon U \to [p_{\ell-1}] \times P$ as follows: for every $u \in U$, say belonging to a cluster $X \in \mathcal{X}_a^{i,T}$ for some $(i, T, a) \in P$, we set
$$\sigma(u) \coloneqq (\sigma_X(u), i, T, a).$$

Clearly, σ uses at most $k \cdot p_{\ell-1} \cdot 2^{p_{\ell-1}+1} = p_\ell$ colors. So it remains to verify that σ is a totally strong odd coloring of $\mathcal{S}$.

Consider any $F \in \mathcal{F}$. Since the elements of the sets $\bigcup \mathcal{X}_a^{i,X}$ for different tuples $(i, T, a) \in P$ are colored with disjoint palettes, it suffices to show that for each tuple $(i, T, a) \in P$, every color appears in $F \cap \bigcup \mathcal{X}_a^{i,X}$ either zero times or an odd number of times. As $\bigcup \mathcal{X}_a^{i,X} \subseteq \lambda^{-1}(i)$, we have that $G[\bigcup \mathcal{X}_a^{i,X}]$ is an induced subgraph of $G[\lambda^{-1}(i)]$,

and hence it is a cluster graph. Note that $F \cap \bigcup \mathcal{X}_a^{i,X}$ is a clique in $G[\bigcup \mathcal{X}_a^{i,X}]$, by the definition of the Gaifman graph. It follows that $F \cap \bigcup \mathcal{X}_a^{i,X}$ is entirely contained in a single cluster of $\mathcal{X}_a^{i,X}$, or in other words, there exists $X \in \mathcal{X}_a^{i,X}$ such that $F \cap \bigcup \mathcal{X}_a^{i,X} = F \cap X$. It now follows from σ_X being a strong odd coloring of $\mathcal{S}[X, \mathcal{F}]$ that every color appears in $F \cap X$ either zero times or an odd number of times.

We are left with checking that in total in the whole U, every color $(j, i, T, a) \in [p_{\ell-1}] \times P$ appears either zero times or an odd number of times. By construction, this color does not appear at all if $j \notin T$, and otherwise it appears exactly in the clusters belonging to $\mathcal{X}_a^{i,T}$, and in each of them it is featured an odd number of times. Since $|\mathcal{X}_a^{i,T}|$ is either zero or odd by construction, it follows that (j, i, T, a) appears either zero times or an odd number of times in the whole U. $\qquad \square$

We conclude by deriving Theorem 2 from all the tools we have established.

Theorem 2. *Let $d \in \mathbb{N}$ and $\mathscr{C}$ be a graph class of bounded expansion. Then there exists $c \in \mathbb{N}$ such that for every graph $G \in \mathscr{C}$ there exists a proper coloring of G with c colors satisfying the following condition: for every vertex u of G and color i, the number of vertices of color i that are at distance at most d from u is either zero or odd.*

Proof. Let $\mathscr{D}$ be the hereditary closure of the class of set systems $\mathsf{Balls}_d(\mathscr{C})$. By Lemmas 7 and 8 and Theorem 11, there are $\ell, h, s \in \mathbb{N}$ depending only on d and $\mathscr{C}$ such that $\mathscr{D}$ has semi-ladder index at most ℓ, 2VC dimension at most h, and subchromatic number at most s. Further, since $\mathscr{C}$ has bounded expansion, it also has bounded chromatic number, say by $b \in \mathbb{N}$.

Consider any graph $G \in \mathscr{C}$ and let $\mathcal{S} = (V(G), \mathcal{F}) := \mathsf{Balls}_d(G)$. Let $\alpha \colon V(G) \to [b]$ be a proper coloring of G with b colors. For each color $i \in [b]$, let $\mathcal{S}_i := \mathcal{S}[\alpha^{-1}(i), \mathcal{F}]$. Noting that $\mathcal{S}_i \in \mathscr{D}$, we may apply to Theorem 3 to obtain a strong odd coloring $\sigma_i \colon \alpha^{-1}(i) \to [k]$ of $\mathcal{S}_i$, where $k := f(\ell, h, s)$ and f is the function provided by Theorem 3; thus k is a constant depending only on $\mathscr{C}$ and d. We now define a coloring $\phi \colon V(G) \to [b] \times [k]$ with $c := bk$ colors by setting

$$\phi(u) := (\alpha(u), \sigma_{\alpha(u)}(u)).$$

Clearly ϕ is proper, because it is a refinement of the proper coloring α. Also, that each σ_i is a strong odd coloring of $\mathcal{S}_i$ implies that ϕ is a strong odd coloring of $\mathcal{S}$. So ϕ satisfies all the requested properties. $\qquad \square$

5   Conclusions

In this paper we argued that every graph class of bounded expansion has bounded strong odd chromatic number. In fact, we proved a more general statement, where the zero-or-odd condition on the number of times a color is featured is applied to balls of a fixed radius d, instead of distance-1 neighborhoods. The zero-or-odd condition appeared essentially only in the proof of Theorem 3. It is not hard to see that the same proof method goes through with minimal modifications also if this condition is replaced by the following

more general one: in every ball of radius d, every color is featured either zero times, or 1 mod m times, where $m \in \mathbb{N}$ is a fixed modulus. We leave the details to the reader.

The key ingredient in our proof is Theorem 3, which states that bounding the three parameters — the semi-ladder index, the 2VC dimension, and the maximum subchromatic number among induced subsystems — suffices to derive a bound on the strong odd chromatic number of a set system. This result may serve as a motivation to seek a characterization of hereditary classes of set systems that have bounded strong odd chromatic number. It is easy to see that the boundedness of the ladder index, of the co-matching index, and of the 2VC dimension, are all necessary conditions for such a characterization, because the families of induced obstructions for those parameters have unbounded strong odd chromatic number. However, it is not clear whether bounding the subchromatic number is necessary as well, and to what extent.

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