A Spectral Version of the Theorem of Zykov and Erdős

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Submitted: Apr 30, 2025; Accepted: Sep 22, 2025; Published: Oct 17, 2025

Abstract

Zykov and Erdős showed independently that for $2 \leqslant s \leqslant r$, the maximum number of copies of K_s among all K_{r+1}-free n-vertex graphs is achieved uniquely on the complete balanced r-partite n-vertex graph (Turán graph $T_{n,r}$). When $s = 2$, it is the classical theorem of Turán. Nikiforov proved a spectral version of Turán's Theorem. In this paper, we give a spectral version of the theorem by Zykov and Erdős. Our result is a generalization of Nikiforov's Theorem and a theorem of Liu and Bu.

Mathematics Subject Classifications: 05C50, 05C35

1 Introduction

For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. The number of edges and the spectral radius of G is denoted by $e(G)$ and $\rho(G)$ respectively. Let K_s be a complete graph with s vertices. A K_s contained in a graph G is called an s-clique in G. For an integer $s \geqslant 1$, let $C_s(G)$ be the set of all s-subsets of $V(G)$ which form s-cliques in G and let $c_s(G) = |C_s(G)|$. The Turán graph, denoted by $T_{n,r}$, is the balanced complete r-partite graph with n vertices. For a given graph F, we say that G is F-free if G contains no subgraph isomorphic to F. The Turán number of F, denoted by $ex(n, F)$, is the maximum number of edges an F-free graph on n vertices can have. A classical theorem of Turán [40] determines $ex(n, K_{r+1})$.

Theorem 1 (Turán [40], 1941)**.** *Let G be a K_{r+1}-free graph on n vertices, then $e(G) \leqslant e(T_{n,r})$, and equality holds if and only if $G \cong T_{n,r}$.*

In 2007, Nikiforov [27] proved the spectral version of Turán's Theorem.

Theorem 2 (Nikiforov [27], 2007)**.** *Let G be a K_{r+1}-free graph on n vertices, then $\rho(G) \leqslant \rho(T_{n,r})$, and equality holds if and only if $G \cong T_{n,r}$.*

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In fact, it is easy to see that Theorem 2 implies Theorem 1 by using the facts $\rho(G) \geqslant \frac{2e(G)}{n}$ and $e(T_{n,r}) = \lfloor \frac{n}{2}\rho(T_{n,r}) \rfloor$.

Zykov [52] and Erdős [8] generalized the theorem of Turán by replacing the maximum number of edges in a K_{r+1}-free graph by the maximum number of copies of K_s in a K_{r+1}-free graph for all $2 \leqslant s \leqslant r$.

Theorem 3 (Zykov [52], 1949, Erdős [8], 1962)**.** *Let G be a K_{r+1}-free graph with n vertices. Then for any $2 \leqslant s \leqslant r$, $c_s(G) \leqslant c_s(T_{n,r})$, and equality holds if and only if $G \cong T_{n,r}$.*

The spectral Turán problems have also attracted considerable attentions in the last two decades, see, e.g., [2, 23, 28] for cliques, [11] for matchings, [7, 16, 17, 29] for cycles, [30] for color-critical graphs, [1, 33] for complete bipartite graphs, [25, 38] for planar and outerplanar graphs, [9, 42, 50] for more extremal problems of planar graphs, [3, 13, 39, 48, 51] for graphs without minors, [31] for a spectral Erdős–Stone–Bollobás theorem, [32] for the spectral stability theorem, [6] for a spectral Erdős–Sós theorem, [46] for a spectral Erdős–Pósa theorem, [19] for a spectral Erdős—Rademacher theorem, [20] for a spectral Erdős–Faudree–Rousseau theorem, [22] for a spectral Lovász–Simonovits theorem. [4, 47] for friendship graphs, [18, 21] for bowties, [34, 49] for book graphs, [5] for odd wheels and [10, 41] for a characterization of spectral extremal graphs for a class of graphs.

The purpose of this paper is to establish a spectral version of Theorem 3, which can be viewed as a spectral counterpart of results on generalized Turán numbers of cliques. We do this via the concept of s-clique tensor introduced by Liu and Bu [26].

For a matrix A, let A^{T} denote the transpose of A. An m-order n-dimensional real tensor over $\mathbb{R}$ is a multi-dimensional array with n^m entries, and every entry is a real number. The concepts of eigenvalue and eigenvector of tensors is independently introduced by Qi [35] and Lim [24] as follows.

Definition 4 (Qi [35], 2005. Lim [24], 2005)**.** Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an m-order n-dimensional real tensor with $m \geqslant 2$. If there exist a number $\lambda \in \mathbb{C}$ and a nonzero vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ satisfying

$$\sum_{i_2,\cdots,i_m=1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_i^{m-1}, \text{ for any } 1 \leqslant i \leqslant n,$$

then λ is called an eigenvalue of $\mathcal{A}$, and $\mathbf{x}$ is an eigenvector of $\mathcal{A}$ corresponding to λ.

The adjacency spectral radius of $\mathcal{A}$, denoted by $\rho(\mathcal{A})$, is the maximum modulus of the eigenvalues of $\mathcal{A}$. For an n-dimensional vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and an m-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$, let

$$\mathbf{x}^{\mathrm{T}}\mathcal{A}\mathbf{x} := \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

Let $\mathbb{R}^n_{+}$ and $\mathbb{R}^n_{++}$ be the set of nonnegative real vectors and positive real vectors with dimension n, respectively. A symmetric tensor is a tensor whose entry is invariant under any permutation of its indices.

Lemma 5 (Qi [36], 2013). *Let $\mathcal{A}$ be an m-order n-dimensional nonnegative symmetric tensor with $m \geqslant 2$. Then $\rho(\mathcal{A})$, the spectral radius of $\mathcal{A}$, satisfies*

$$\rho(\mathcal{A}) = \max\left\{\mathbf{x}^{\mathrm{T}}\mathcal{A}\mathbf{x} : \mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i^m = 1\right\}.$$

As a generalization of adjacency matrix, Liu and Bu [26] introduced the definition of s-clique tensor.

Definition 6 (Liu–Bu [26], 2023). Let G be a graph with n vertices. The s-clique tensor of G is an s-order n-dimensional tensor $\mathcal{A}_s(G) = (a_{i_1 \cdots i_s})$, where the entry $a_{i_1 \cdots i_s}$ satisfies

$$a_{i_1 \cdots i_s} = \begin{cases} \frac{1}{(s-1)!}, & \{i_1, \ldots, i_s\} \in C_s(G), \\ 0, & \text{otherwise.} \end{cases}$$

The s-clique spectral radius of G is defined as the spectral radius of the s-clique tensor $\mathcal{A}_s(G)$, denoted by $\rho_s(G)$. Obviously, when $s = 2$, $\mathcal{A}_2(G)$ is the adjacency matrix, and $\rho_2(G)$ is the spectral radius of G. From Definition 6 and Lemma 5, we get that

$$\rho_s(G) = \max\left\{s \sum_{\{i_1, \cdots, i_s\} \in C_s(G)} x_{i_1} x_{i_2} \cdots x_{i_s} : \mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}_+^n, \sum_{i=1}^{n} x_i^s = 1\right\}. \quad (1)$$

Besides, Liu and Bu [26] proved an analogous result of Turán's Theorem for s-clique tensor. They also got the extension of the inequality of $\rho(G) \geqslant \frac{2e(G)}{n}$.

Theorem 7 (Liu–Bu [26], 2023). *Let G be a K_{r+1}-free graph with n vertices. Then $\rho_r(G) \leqslant \rho_r(T_{n,r})$, and equality holds if and only if $G \cong T_{n,r}$.*

Theorem 8 (Liu–Bu [26], 2023). *Let G be a graph with n vertices. Then $c_r(G) \leqslant \frac{n}{r}\rho_r(G)$. Furthermore, if the number of r-cliques containing i is equal for all $i \in V(G)$, then the equality holds.*

Let M_t denote a matching of size t. Recently, we established a spectral version of the generalized Erdős-Gallai's Theorem, that is, determining the maximum s-clique spectral radius among all M_t-free n-vertex graphs. Let n, t be positive integers, where $t \geqslant 2$ and $n \geqslant 2t$. Define $F_{t-1}(n) = ([n], E)$, where $E = \{\{a, b\} : \{a, b\} \cap \{1, \ldots, t-1\} \neq \emptyset.\}$.

Theorem 9 (Yu–Peng [45], 2026). *Let G be an M_t-free graph with n vertices. Then for any $s \geqslant 2$ and sufficiently large n, $\rho_s(G) \leqslant \max\{\rho_s(K_{2t-1}), \rho_s(F_{t-1}(n))\}$.*

A spectral generalized Alon-Frankl theorem is given in [43]. In this paper, we extend Theorem 7 from $s = r$ to $2 \leqslant s \leqslant r$. Our result is also a generalization of Theorem 2.

Theorem 10. *Let G be a K_{r+1}-free graph with n vertices. Then for any $2 \leqslant s \leqslant r$, $\rho_s(G) \leqslant \rho_s(T_{n,r})$, and equality holds if and only if $G \cong T_{n,r}$.*

Remark 11. For any $2 \leqslant s \leqslant r$, the number of s-cliques containing i is equal for all $i \in V(T_{n,r})$ when $r|n$. In this case, by Theorem 8, $c_s(T_{n,r}) = \frac{n}{s}\rho_s(T_{n,r})$. Therefore, combining Theorems 8 and 10, we have that for a K_{r+1}-free graph G of order n, $c_s(G) \leqslant \frac{n}{s}\rho_s(G) \leqslant \frac{n}{s}\rho_s(T_{n,r}) = c_s(T_{n,r})$.

The main idea of the proof is to apply Zykov symmetrization method as in [23] and [26] (we need to overcome some obstacles not appearing in [26]). The motivation to write this note is that Theorem 10 can be viewed as a spectral version of the theorem of Zykov [52] and Erdős [8] (Theorem 3) on generalized Turán numbers.

2 Preliminaries

In this section, we will introduce some necessary lemmas for the proof.

Definition 12 (Yang–Yang [44], 2011)**.** Let $\mathcal{A}$ be an m-order n-dimensional tensor. If there exists some nonempty proper subset $I \subseteq [n]$ such that $a_{i_1 i_2 \cdots i_m} = 0$ holds for any $i_1 \in I$ and $\{i_2, \ldots, i_m\} \nsubseteq I$, then $\mathcal{A}$ is called weakly reducible. If $\mathcal{A}$ is not weakly reducible, then $\mathcal{A}$ is weakly irreducible.

Lemma 13 (Friedland–Gaubert–Han [12], 2013)**.** *Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor. If $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is the unique eigenvalue of $\mathcal{A}$ which corresponds to an eigenvector $\mathbf{x} \in \mathbb{R}^n_{++}$, and the eigenvector is unique up to a positive scaling coefficient.*

For two m-order n-dimensional real tensors $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A} - \mathcal{B}$ is a nonnegative but nonzero tensor, then we write $\mathcal{B} < \mathcal{A}$.

Lemma 14 (Khan-Fan [15], 2015)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two m-order n-dimensional nonnegative tensors. If $\mathcal{A} < \mathcal{B}$ and $\mathcal{B}$ is weakly irreducible, then $\rho(\mathcal{A}) < \rho(\mathcal{B})$.*

Let $\mathcal{A}$ and $\mathcal{B}$ be two m-order n-dimensional tensors. If there exists a permutation matrix P of order n such that $\mathcal{B} = P\mathcal{A}P^{\mathrm{T}}$, then we say that $\mathcal{A}$ and $\mathcal{B}$ are permutational similar. In [14], Hu et al. defined the (triangular) block tensors. Subsequently, Shao et al. [37] generalized the concept.

Definition 15 (Hu–Huang–Ling–Qi [14], 2013)**.** Let $\mathcal{A}$ be a tensor with dimension n and order m. If there exists some integer k with $1 \leqslant k \leqslant n-1$ such that $a_{i_1 i_2 \ldots i_m} = 0$ if $i_1 \leqslant k$ and at least one of $\{i_2, \ldots, i_m\}$ is greater than k. Then $\mathcal{A}$ is called a k-lower triangular block tensor, or simply lower triangular block tensor.

Definition 16 (Shao–Shan–Zhang [37], 2013)**.** Let $\mathcal{A}$ be a tensor with dimension n and order m. Let $n_1, \ldots, n_r$ be positive integers with $n_1 + \cdots + n_r = n$, where $r \geqslant 1$. For $1 \leqslant i \leqslant r$, let

$$I_i = \left\{ (\sum_{j=1}^{i-1} n_j) + 1, \ldots, \sum_{j=1}^{i} n_j \right\} \subseteq [n]$$

and write $\mathcal{A}[I_i] = \mathcal{A}_i$. Suppose that for each $1 \leqslant i \leqslant r-1$, the subtensor $\mathcal{A}[I_i \cup \cdots \cup I_r]$ is an n_i-lower triangular block tensor, then $\mathcal{A}$ is called an $(n_1, \ldots, n_r)$-lower triangular block tensor with the diagonal blocks $\mathcal{A}_1, \ldots, \mathcal{A}_r$.

Lemma 17 (Shao–Shan–Zhang [37], 2013)**.** *Let $\mathcal{A}$ be an m-order n-dimensional symmetric tensor with $m \geqslant 2$. Then there exists a positive integer $r \geqslant 1$ such that $\mathcal{A}$ is permutational similar to some $(n_1, n_2, \ldots, n_r)$-lower triangular block tensor, where all the diagonal blocks $\mathcal{A}_1, \ldots, \mathcal{A}_r$ are weakly irreducible, and all the entries of $\mathcal{A}$ not in these diagonal blocks are zero. Furthermore, we have $\rho(\mathcal{A}) = max\{\rho(\mathcal{A}_1), \ldots, \rho(\mathcal{A}_r)\}$.*

In [26], Liu and Bu also introduced the concept of s-clique walk and s-clique connectedness. An s-clique walk of a graph G is a sequence of vertices,where each pair of consecutive vertices are contained in some s-clique of G. G is s-clique connected if for any pair of vertices $u, v \in V(G)$, there exists an s-clique walk connecting them. The following lemma indicates that there is a relation between s-clique connectedness of G and weakly irreducibility of $\mathcal{A}_s(G)$.

Lemma 18 (Liu–Bu [26], 2023)**.** *For an n-vertex graph G and its s-clique tensor $\mathcal{A}_s(G)$, $\mathcal{A}_s(G)$ is weakly irreducible if and only if G is s-clique connected.*

3 Proof of Theorem 10

Let G^* be an n-vertex K_{r+1}-free graph attaining the maximum s-clique spectral radius, our goal is to show that $G^* = T_{n,r}$. We can assume that each edge of G^* is contained in at least one s-clique. Indeed, if there are some edges in G^* that are not contained in any s-clique, then we can obtain a new K_{r+1}-free graph G' from G^* by deleting all of them. Since an edge not contained in any s-clique contributes 0 to the value of $\mathcal{A}_s(G^*)$, by Definition 6, $\mathcal{A}_s(G^*) = \mathcal{A}_s(G')$. Moreover, $\rho_s(G^*) = \rho_s(G')$. Suppose that we have shown the conclusion of Theorem 10 under the assumption that edges of G^* not contained in an s-clique has been deleted, i.e., suppose that we have shown that $G' = T_{n,r}$. Since G^* and G' have the same vertex set, $E(G') \subseteq E(G^*)$ and $G' = T_{n,r}$ is a maximal K_{r+1}-free graph, we have G^* must be $T_{n,r}$. From now on, we assume that G^* is an n-vertex K_{r+1}-free graph attaining the maximum s-clique spectral radius and each edge of G^* is contained in at least one s-clique.

For any $u \in V(G^*)$, the set of all vertices adjacent to u is denoted by $N_{G^*}(u)$. Sometimes, we will eliminate the subscript and write $N(u)$ if it does not cause confusion. If u and v are adjacent in G^*, we will abbreviate the edge $\{u, v\} \in E(G^*)$ as uv. For an edge subset $E \subseteq E(G^*)$, $G^* - E$ denotes the graph obtained by deleting all edges of E. Similarly, $G^* + E$ denotes the graph obtained by adding all edges of E.

Lemma 19. *G^* is s-clique connected, and $\mathcal{A}_s(G^*)$ is weakly irreducible.*

Proof. We will prove it by contradiction. If G^* is not s-clique connected, then there must exist an s-clique connected component G_1 with $\rho_s(G_1) = \rho_s(G^*)$ by Lemmas 17 and 18. Since G^* is not s-clique connected, there are two vertices u and v which cannot be connected by an s-clique walk. Obviously, $u \in V(G_1)$ and $v \in V(G_1)$ cannot hold simultaneously as G_1 is s-clique connected. Without loss of generality, assume that $v \notin V(G_1)$. Since $v \notin V(G_1)$, $|N_{G^*}(v) \cap V(K_s)| \leqslant s-2$ holds for any s-clique K_s of G_1. Let K_s^* be an s-clique which contains the maximum number of vertices of $N_{G_1}(v)$ and $V(K_s^*) = \{w_1, w_2, \ldots, w_s\}$. Without loss of generality, let $N_{G_1}(v) \cap V(K_s^*) = \{w_1, w_2, \ldots, w_t\}$. Clearly, $t \leqslant s-2$. Let $G' = G^*[V(G_1) \cup \{v\}] + \{vw_{t+1}, \ldots, vw_{s-1}\}$. We claim that G' is K_{r+1}-free. Otherwise, there is a subgraph H isomorphic to K_{r+1} in G'. This K_{r+1} contains v and possibly contains some or all vertices in $\{w_{t+1}, w_{t+2}, \ldots, w_{s-1}\}$. Since $r+1 > s$, remove v from this K_{r+1}, G_1 has an s-clique containing at least $s-(s-1-t) = t+1$ vertices of $N_{G_1}(v)$, a contradiction to the choice of K_s^*. Let G'' be the graph obtained from G_1 by adding a new isolated vertex. Obviously $\rho_s(G'') = \rho_s(G_1)$ and $\mathcal{A}_s(G'') < \mathcal{A}_s(G')$. Hence, by Lemma 14, $\rho_s(G') > \rho_s(G'') = \rho_s(G_1) = \rho_s(G^*)$, a contradiction to the choice of G^*. Therefore, G^* is s-clique connected. Then by Lemma 18, $\mathcal{A}_s(G^*)$ is weakly irreducible. □

By Lemmas 19 and 13, we know that up to a positive scaling coefficient, there exists a unique positive eigenvector corresponding to the s-clique spectral radius $\rho_s(G^*)$, denoted by $\mathbf{x}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$. For any vertex $i \in [n]$, we have

$$\rho_s(G^*) x_i^{s-1} = \sum_{\{i, i_2, \ldots, i_s\} \in C_s(G^*)} x_{i_2} \cdots x_{i_s}. \tag{2}$$

For simplicity, for any $i \in [n]$, define

$$W_{G^*}^s(i, \mathbf{x}) = \sum_{\{i, i_2, \ldots, i_s\} \in C_s(G^*)} x_{i_2} \cdots x_{i_s}. \tag{3}$$

Lemma 20. *G^* is a complete r-partite graph.*

Proof. First of all, we can assume that $n \geqslant r+1$. Otherwise, it is a trivial case and Lemma 20 holds obviously.

Claim. For any two non-adjacent vertices u and v, $N(u) = N(v)$.

Suppose on the contrary, there are two non-adjacent vertices u and v with $N(u) \neq N(v)$. Without loss of generality, let $w \in N(u) \setminus N(v)$.

Case 1. $W_{G^*}^s(v, \mathbf{x}) < W_{G^*}^s(u, \mathbf{x})$ or $W_{G^*}^s(v, \mathbf{x}) < W_{G^*}^s(w, \mathbf{x})$. If $W_{G^*}^s(v, \mathbf{x}) < W_{G^*}^s(u, \mathbf{x})$, let $G' = G^* - \{vi | i \in N_{G^*}(v)\} + \{vi | i \in N_{G^*}(u)\}$. Obviously, G' is K_{r+1}-free, and

$$\begin{aligned}
\rho_s(G') &\geqslant \mathbf{x}^{\mathrm{T}} \mathcal{A}_s(G') \mathbf{x} \\
&= \mathbf{x}^{\mathrm{T}} \mathcal{A}_s(G^*) \mathbf{x} - s x_v W_{G^*}^s(v, \mathbf{x}) + s x_v W_{G^*}^s(u, \mathbf{x}) \\
&> \mathbf{x}^{\mathrm{T}} \mathcal{A}_s(G^*) \mathbf{x} = \rho_s(G^*).
\end{aligned}$$

If $W_{G^*}^s(v, \mathbf{x}) < W_{G^*}^s(w, \mathbf{x})$, let $G' = G^* - \{vi | i \in N_{G^*}(v)\} + \{vi | i \in N_{G^*}(w)\}$. Obviously, G' is K_{r+1}-free, and

$$\rho_s(G') \geqslant \mathbf{x}^{\mathrm{T}} \mathcal{A}_s(G') \mathbf{x}$$

$$
\begin{aligned}
&= \mathbf{x}^{\mathrm{T}} \mathcal{A}_s(G^*)\mathbf{x} - s x_v W^s_{G^*}(v,\mathbf{x}) + s x_v W^s_{G^*}(w,\mathbf{x}) \\
&> \mathbf{x}^{\mathrm{T}} \mathcal{A}_s(G^*)\mathbf{x} = \rho_s(G^*).
\end{aligned}
$$

Case 2. $W^s_{G^*}(v,\mathbf{x}) \geqslant W^s_{G^*}(u,\mathbf{x})$ and $W^s_{G^*}(v,\mathbf{x}) \geqslant W^s_{G^*}(w,\mathbf{x})$.

Let $G'' = G^* - \{ui | i \in N_{G^*}(u)\} - \{wi | i \in N_{G^*}(w)\} + \{ui | i \in N_{G^*}(v)\} + \{wi | i \in N_{G^*}(v)\}$. Obviously, G'' is K_{r+1}-free, and

$$
\begin{aligned}
\rho_s(G'') &\geqslant \mathbf{x}^{\mathrm{T}} \mathcal{A}_s(G'')\mathbf{x} \\
&= \mathbf{x}^{\mathrm{T}} \mathcal{A}_s(G^*)\mathbf{x} - s x_u W^s_{G^*}(u,\mathbf{x}) + s x_u W^s_{G^*}(v,\mathbf{x}) \\
&\quad - s x_w W^s_{G^*}(w,\mathbf{x}) + s x_w W^s_{G^*}(v,\mathbf{x}) + \sum_{\{u,w,i_3,\ldots,i_s\} \in C_s(G^*)} x_u x_w x_{i_3} \cdots x_{i_s} \\
&> \mathbf{x}^{\mathrm{T}} \mathcal{A}_s(G^*)\mathbf{x} = \rho_s(G^*).
\end{aligned}
$$

The last inequality derives from the fact that $\mathbf{x}$ is positive and each edge of G is contained in at least one s-clique.

So, in both Cases 1 and 2, we can always get a K_{r+1}-free graph with larger s-clique spectral radius, a contradiction. The claim holds. Hence there is a vertex-disjoint partition of $V(G^*)$ such that any two non-adjacent vertices are in the same set. Therefore, G is a complete multipartite graph. Let G^* be a complete t-partite graph with partition sets $V_1, V_2, \ldots, V_t$ and $t \leqslant r$. We claim that $t = r$. If $t < r$, then we can assume that $t \geqslant s$. Otherwise, there is no s-clique in G^* and $\rho_s(G^*) = 0$. Clearly this is a contradiction. Since $t < r$, we know that at least one partition set contains at least two vertices. Without loss of generality, let V_1 be the partition set with at least two vertices in it and $u, v \in V_1$. We can construct a complete $(t+1)$-partite graph G' with partition $u, V_1 \setminus \{u\}, V_2, \ldots, V_t$. It is clear that $\mathcal{A}_s(G^*) < \mathcal{A}_s(G')$, so $\rho_s(G^*) < \rho_s(G')$ by Lemma 14. And since $t < r$, G' is also K_{r+1}-free, This is a contradiction. So it can be concluded that G^* is a complete r-partite graph. □

Lemma 21. $G^* \cong T_{n,r}$.

Proof. From Lemma 20, G^* is a complete r-partite graph. Let $V(G^*) = V_1 \cup \cdots \cup V_r$ be the vertex partition of G^* and the size of each part be $n_1, n_2, \ldots, n_r$, respectively. If $G^* \ncong T_{n,r}$, then there exists a pair of parts with the difference in sizes at least 2. Without loss of generality, let $n_1 - n_2 \geqslant 2$, $u \in V_1$ and $v \in V_2$. Since all vertices in the same part have the same neighborhood, by the property of an eigenvector, all vertices in the same part have the component with the same value in $\mathbf{x}$. Consequently,

$$
\rho_s(G^*) x_u^{s-1} = n_2 x_v A + B \tag{4}
$$

$$
\rho_s(G^*) x_v^{s-1} = n_1 x_u A + B, \tag{5}
$$

where

$$
A = \sum_{\{u,v,i_3,\ldots,i_s\} \in C_s(G^*)} x_{i_3} \cdots x_{i_s},
$$

and

$$B=\sum_{\substack{\{u,i_2,\ldots,i_s\}\in C_s(G^*),\\ \{i_2,\ldots,i_s\}\cap V_2=\varnothing}} x_{i_2}\cdots x_{i_s}=\sum_{\substack{\{v,i_2,\ldots,i_s\}\in C_s(G^*),\\ \{i_2,\ldots,i_s\}\cap V_1=\varnothing}} x_{i_2}\cdots x_{i_s}.$$

Subtracting (5) from (4), we get

$$\rho_s(G^*)x_u^{s-1}+n_1x_uA=\rho_s(G^*)x_v^{s-1}+n_2x_vA. \tag{6}$$

Thus, we get $x_u<x_v$ as $n_1-n_2\geqslant 2$. And from (4) and $x_u<x_v$, we obtain

$$\rho_s(G^*)x_u^{s-2}=n_2A\frac{x_v}{x_u}+\frac{B}{x_u}>n_2A. \tag{7}$$

Let G' be the graph obtained from G^* by deleting all edges incident with u, and adding edges connecting u with all vertices in $N_{G^*}(v)\setminus\{u\}$. Since $n_1-n_2\geqslant 2$, we have $G'\ncong G^*$. Obviously, G' is K_{r+1}-free. Furthermore,

$$\begin{aligned}\rho_s(G')-\rho_s(G^*)&\geqslant \mathbf{x}^{\mathrm{T}}\mathcal{A}_s(G')\mathbf{x}-\mathbf{x}^{\mathrm{T}}\mathcal{A}_s(G^*)\mathbf{x}\\ &=(n_1-1)x_u^2A-n_2x_ux_vA\\ &=((n_1-1)x_u-n_2x_v)x_uA.\end{aligned}$$

However, by (6),

$$\begin{aligned}&((n_1-1)x_u-n_2x_v)(\rho_s(G^*)x_v^{s-1}+n_2x_vA)\\ &=(n_1-1)x_u(\rho_s(G^*)x_v^{s-1}+n_2x_vA)-n_2x_v(\rho_s(G^*)x_u^{s-1}+n_1x_uA)\\ &=x_ux_v((n_1-1)(\rho_s(G^*)x_v^{s-2}+n_2A)-n_2(\rho_s(G^*)x_u^{s-2}+n_1A))\\ &=x_ux_v(\rho_s(G^*)x_v^{s-2}(n_1-1)-n_2(\rho_s(G^*)x_u^{s-2}+A))\\ &\geqslant x_ux_v(\rho_s(G^*)x_v^{s-2}(n_2+1)-n_2(\rho_s(G^*)x_u^{s-2}+A))\\ &=x_ux_v(n_2\rho_s(G^*)(x_v^{s-2}-x_u^{s-2})+\rho_s(G^*)x_v^{s-2}-n_2A)>0,\end{aligned}$$

where the first inequality comes from $n_1-n_2\geqslant 2$ and the second inequality is derived from the fact $x_v>x_u$ and (7). So $(n_1-1)x_u-n_2x_v>0$, which implies that $\rho_s(G')-\rho_s(G^*)>0$. This leads to a contradiction. So $G^*\cong T_{n,r}$ and the proof is complete. □

Remarks. The concept $\rho_s(G)$ can be viewed as the spectral counterpart of the number of copies of K_s in G. Thus, Theorem 10 can be viewed as the spectral counterpart of the theorem of Erdős and Zykov on $ex(n,K_s,K_{r+1})$ (the maximum number of copies of K_s in an n-vertex K_{r+1}-free graphs), where $s\leqslant r$. It would be interesting to obtain spectral counterparts of generalized Turán numbers $ex(n,H,F)$ for general structures F and H rather than complete graph. In the case $r|n$, the conclusion in this paper can imply the theorem of Zykov and Erdős, we do not see any reason why this is not true for the case that r does not divide n. With more technical estimation, it is possible to obtain the implication, but we will not go through tedious and technical calculations.

Acknowledgements

We are grateful to both reviewers for reading the manuscript very carefully, spending a lot of time in checking the details, and giving us valuable comments to help improve the presentation. This research is supported by National natural science foundation of China(Nos. 12571363 and 12371327) and National Natural Science Foundation of Hunan Province (Grant No. 2025JJ30003).

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