

Transversal Panconnectedness in Graph Collections

Wanting Sun^a Guanghui Wang^{b,c} Lan Wei^b

Submitted: Jan 19, 2025; Accepted: Jul 15, 2025; Published: Oct 17, 2025

© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

Given a collection $\mathcal{G} = \{G_1, G_2, \dots, G_m\}$ of graphs on the common vertex set V of size n , a graph H with vertices in V is *rainbow* in $\mathcal{G}$ if there exists an injection $\varphi : E(H) \rightarrow [m]$ such that $e \in E(G_{\varphi(e)})$ for all $e \in E(H)$. We say H is a *transversal* in $\mathcal{G}$ if $|E(H)| = m$. Denote $\delta(\mathcal{G}) := \min \{\delta(G_i) : i \in [m]\}$. For vertices $x, y \in V$, let $d_{\mathcal{G}}(x, y)$ be the length of the shortest rainbow path (if exists) connecting x and y in $\mathcal{G}$. We say $\mathcal{G}$ is *transversal panconnected* if for any two vertices $x, y \in V$, there exists a rainbow path on k vertices inside $\mathcal{G}$ joining x and y for every integer $k \in [d_{\mathcal{G}}(x, y) + 1, n]$. In this paper, we provide a tight bound on $\delta(\mathcal{G})$ to ensure that $\mathcal{G}$ is transversal panconnected. It generalizes the result of [Williamson, Period. Math. Hungar., 1977] to the transversal version and improves the result of [Li, Li and Li, J. Graph Theory, 2023].

Mathematics Subject Classifications: 05C38

1 Introduction

1.1 Transversal in graph collections

The problem of determining whether an n -vertex graph contains a given subgraph is a central topic in extremal graph theory. In this paper, we consider the generalization of this problem to graph collections.

Let $\mathcal{G} = \{G_1, G_2, \dots, G_m\}$ be a collection of not necessarily distinct graphs with common vertex set V . We often think of each G_i having the color i . Let H be a graph on the vertex set V . We say H is *rainbow* in $\mathcal{G}$ if there exists an injection $\varphi : E(H) \rightarrow [m]$ such that $e \in E(G_{\varphi(e)})$ for all $e \in E(H)$. Furthermore, if $|E(H)| = m$, then φ is a

^aData Science Institute, Shandong University, Jinan, China (wtsun@sdu.edu.cn).

^bSchool of Mathematics, Shandong University, Jinan, China (ghwang@sdu.edu.cn, lanwei@mail.sdu.edu.cn).

^cState Key Laboratory of Cryptography and Digital Economy Security, Shandong University, Jinan, China.

bijection, and we refer to H as a *transversal* in $\mathcal{G}$. Transversal often appears in infinitary combinatorics under several similar definitions (see, [1, 18]).

The following general question about transversals was first proposed by Joos and Kim in [25].

Question 1 ([25]). Let H be a graph with m edges, and $\mathcal{G} = \{G_1, G_2, \dots, G_m\}$ be a collection of graphs with common vertex set V . What properties imposed on $\mathcal{G}$ guarantee a transversal copy of H ?

By taking $G_1 = G_2 = \dots = G_m$, we need to study properties for $\mathcal{G}$ such that each graph in $\mathcal{G}$ contains H as a subgraph. However, this alone is not always sufficient. For example, Aharoni, DeVos, de la Maza, Montejano and Šámal [2] proved that if $\mathcal{G} = \{G_1, G_2, G_3\}$ is a collection of graphs on a common vertex set of size n and $\min_{i \in [3]} |E(G_i)| > (\frac{26-2\sqrt{7}}{81})n^2$, then $\mathcal{G}$ contains a transversal triangle. Moreover, the constant $\frac{26-2\sqrt{7}}{81} > \frac{1}{4}$ is optimal. However, Mantel's theorem states that every n -vertex graph with more than $\lfloor \frac{n^2}{4} \rfloor$ edges must contain a triangle.

Hamiltonicity of graphs is one of the fundamental problems in extremal graph theory and structural graph theory. In 1952, Dirac [17] proved that every n -vertex graph ($n \geq 3$) with minimum degree at least $\frac{n}{2}$ contains a Hamilton cycle. Aharoni [2] conjectured that Dirac's theorem can be extended to the transversal version. It was asymptotically solved by Cheng, Wang and Zhao [15], and completely confirmed by Joos and Kim [25]. Let $\delta(G)$ be the minimum degree of a graph G and define $\delta(\mathcal{G}) := \min\{\delta(G_i) : i \in [m]\}$ to be the minimum degree of $\mathcal{G}$.

Theorem 2 ([25]). Let $n \in \mathbb{N}$ and $n \geq 3$. Suppose $\mathcal{G} = \{G_1, \dots, G_n\}$ is a collection of graphs on a common vertex set V of size n . If $\delta(\mathcal{G}) \geq \frac{n}{2}$, then $\mathcal{G}$ contains a transversal Hamilton cycle.

Bradshaw [6] considered the bipartite analogue of Theorem 2, extending the result of Moon and Moser [33] to the transversal version. Bowtell, Morris, Pehova and Staden [5] established the minimum degree condition that guarantees a Hamilton cycle for any given color pattern. Cheng and Staden [13] established a stability result for transversal Hamilton cycles by using a transversal blow-up lemma [12]. Cheng, Sun, Wang and Wei [14] characterized all graph collections consisting of n graphs on a common vertex set V of size n with minimum degree at least $\frac{n}{2} - 1$ and without transversal Hamilton cycles. Recently, Chakraborti, Kim, Lee and Seo [8, 9] studied the Hamiltonicity in a collection of tournaments.

In addition to research on Hamiltonicity, many other classical results in extremal graph theory have been generalized. Cheng, Han, Wang and Wang [11] investigated the minimum degree condition to guarantee the existence of transversal K_t -factors in (hyper)graph collections. This gives an asymptotic version of the rainbow Hajnal-Szemerédi theorem [22]. Montgomery, Műyesser and Pehova [32] proved the asymptotically tight transversal versions of Hajnal-Szemerédi theorem [22] and Kühn-Osthus theorem [27] on factors, and a transversal generalization of the theorem of Komlós, Sárközy and Szemerédi [26] on

spanning trees. Chakraborti, Im, Kim and Liu [7] provided a transversal generalization of the celebrated bandwidth theorem by Böttcher, Schacht and Taraz [4]. Gupta, Hamann, Müyesser, Parczyk and Sgueglia [21] provided a general approach to extend several classical Dirac-type results to the transversal version. For more results on transversal settings, we refer readers to [3, 19, 20, 23, 30, 31, 36] and our survey [35].

1.2 Our results

Denote by P_n , C_n , K_n and $K_{t,n-t}$ the path, cycle, complete graph and complete bipartite graph with partite sets of size t and $n - t$ on n vertices, respectively. A cycle containing all the vertices of a graph G is said to be a *Hamilton cycle* of the graph. A *Hamilton path* of G is a path containing all the vertices of G . A graph G is *Hamiltonian connected* if there exists a Hamilton path between any two distinct vertices of G . A graph G is *panconnected* if for any two vertices x, y , there exists a path P_k joining x and y inside G for every integer $k \in [d(x, y) + 1, n]$, where $d(x, y)$ is the length of the shortest path connecting x and y in G .

Some derivative problems related to Hamiltonicity have also received a lot of attention, including Hamiltonian connected, panconnected and so on (see, [10, 24, 37, 38, 39]). Chen, Li and Zhao [10] and Williamson [39] established the Dirac-type conditions for a graph to be Hamiltonian connected and panconnected, respectively.

Theorem 3 ([10, 39]). *Let G be a graph with order n .*

- (i) *If $n \geq 3$ and $\delta(G) \geq \frac{n+1}{2}$, then G is Hamiltonian connected.*
- (ii) *If $n \geq 4$ and $\delta(G) \geq \frac{n}{2} + 1$, then G is panconnected.*

In this paper, we are interested in generalizing the degree condition for Hamiltonian connected and panconnected to the transversal version.

Let $\mathcal{G} = \{G_1, \dots, G_{n-1}\}$ be a collection of graphs on a common vertex set V of size n . We say $\mathcal{G}$ is *transversal Hamiltonian connected* if for any two distinct vertices $x, y \in V$, there is a transversal Hamilton path inside $\mathcal{G}$ joining x and y ; $\mathcal{G}$ is *transversal panconnected* if for every two distinct vertices x and y , there exists a rainbow path P_k joining x and y inside $\mathcal{G}$ for every integer $k \in [d_{\mathcal{G}}(x, y) + 1, n]$.

Our first result gives the Dirac-type condition to guarantee a graph collection is transversal Hamiltonian connected. This generalizes Theorem 3 (i) to the transversal version.

Theorem 4. *Let $n \in \mathbb{N}$ and $n \geq 3$. Suppose $\mathcal{G} = \{G_1, \dots, G_{n-1}\}$ is a collection of graphs on a common vertex set V of size n . If $\delta(\mathcal{G}) \geq \frac{n+1}{2}$, then $\mathcal{G}$ is transversal Hamiltonian connected.*

Notice that the degree condition in Theorem 4 is best possible. Indeed, let $\mathcal{G} = \{G_1, \dots, G_{n-1}\}$ be a collection of $n - 1$ bipartite graphs on a common vertex partition $A \cup B$ with $|A| = \lfloor \frac{n}{2} \rfloor$ and $|B| = \lceil \frac{n}{2} \rceil$, and $G_i = G_i[A, B] = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. It is routine to check

that $\delta(\mathcal{G}) \geq \lfloor \frac{n}{2} \rfloor$ and $\mathcal{G}$ does not contain transversal Hamilton paths with two endpoints in A .

The proof of Theorem 4 draws inspiration from the work of Joos and Kim [25], who established the transversal version of Dirac's theorem by constructing auxiliary digraphs.

For two vertex disjoint graphs G and H , the disjoint union of G and H is denoted by $G \cup H$. Let $G \vee H$ be the graph formed by adding an edge between every vertex in G and every vertex in H to $G \cup H$. Let n be an odd integer and H_1 be an empty graph with $\frac{n-1}{2}$ vertices, and let H_2 be a graph on $\frac{n+1}{2}$ vertices with $\delta(H_2) \geq 1$ such that one component of H_2 is a single edge. Define $\mathcal{F}_m$ as a collection of m copies of $H_1 \vee H_2$.

Li, Li and Li [28] studied the rainbow panconnectedness of a graph collection consisting of n graphs.

Theorem 5 ([28]). *Suppose $\mathcal{G} = \{G_1, \dots, G_n\}$ is a collection of graphs on a common vertex set V of size n . If $\delta(\mathcal{G}) \geq \frac{n+1}{2}$, then $\mathcal{G}$ is rainbow panconnected unless $\mathcal{G} = \mathcal{F}_n$.*

Observe that the edges of a rainbow path joining two distinct vertices only come from at most $n - 1$ graphs in the collection $\mathcal{G} = \{G_1, \dots, G_n\}$. That is to say, Theorem 5 uses one more color than we need. Motivated by this, our next result determines the Dirac-type condition for a graph collection consisting of $n - 1$ graphs to be transversal panconnected, extending Theorem 3 (ii) to the transversal version.

Theorem 6. *Let $n \geq 4$ be an even integer or a sufficiently large odd integer. Assume that $\mathcal{G} = \{G_1, \dots, G_{n-1}\}$ is a collection of graphs on a common vertex set V of size n . If $\delta(\mathcal{G}) \geq \frac{n+1}{2}$, then $\mathcal{G}$ is transversal panconnected unless $\mathcal{G} = \mathcal{F}_{n-1}$.*

By applying Theorem 4 and Theorem 6 with $G_1 = G_2 = \dots = G_{n-1} = G$, we can deduce Theorem 3 (i) and Theorem 3 (ii) (n is sufficiently large if it is odd), respectively.

1.3 Notation and organization

First of all, we state that all graph theoretic terminology and notation not defined in this paper is the same as that used in the textbook [16].

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G , denote by $N_G(v)$ the neighborhood of v . We usually omit the subscripts when the graph G is clear from the context.

For any vertex subsets $X, Y \subseteq V$, $G[X]$ denotes the graph on X whose edges are precisely the edges of G with both ends in X . We write $G - X$ for $G[V \setminus X]$, i.e., $G - X$ is obtained from G by deleting all the vertices in X and their incident edges. If $X = \{v\}$ is a singleton, we write $G - v$ rather than $G - \{v\}$. Let $G[X, Y]$ be the subgraph of G with vertex set $X \cup Y$ and edge set $\{xy : x \in X, y \in Y \text{ and } xy \in E(G)\}$. If $E(G) = \emptyset$, then we simply write $G = \emptyset$. For any vertex set V of size n , we say $A \cup B$ is an *equitable partition* of V if $|A| + |B| = n$ and $||A| - |B|| \leq 1$.

For a digraph D , we denote its vertex set by $V(D)$ and its arc set by $A(D)$. For vertices u and v in D , the arc from u to v is denoted by $\vec{uv}$. Given $v \in V(D)$, let $N_D^+(v)$ and $N_D^-(v)$ be the out-neighborhood and in-neighborhood of v in D , respectively. Denote by $d_D^+(v)$ and $d_D^-(v)$ the out-degree and in-degree of v , respectively.

Let $\mathcal{G} = \{G_i : i \in [m]\}$ be a graph collection on a common vertex set V and H be a rainbow subgraph inside $\mathcal{G}$. For a graph G_i and a vertex $v \in V$, let $N_i(v) := N_{G_i}(v)$ for each $i \in [m]$. Denote by $\text{col}(H)$ the set of colors appearing in H .

For a positive integer n , we write $[n] := \{1, 2, \dots, n\}$ and $[a, b] := \{a, a+1, \dots, b\}$ for two positive integers $a < b$. Our calculations are based on the sense of modulo arithmetic.

The rest of the paper is organized as follows. In Section 2 and Section 3, we prove Theorem 4 and Theorem 6, respectively. Some concluding remarks are presented in Section 4.

2 Proof of Theorem 4

In this section, we prove Theorem 4. The following lemma is obtained by Li, Li and Li [28], which can be used to find a long rainbow path inside a graph collection.

Lemma 7 ([28]). *Let $n \geq 3$ be an integer. Suppose $\mathcal{G} = \{G_1, \dots, G_n\}$ is a collection of graphs with common vertex set V of size n , and $\delta(\mathcal{G}) \geq \frac{n-1}{2}$. Then one of the following holds:*

- $\mathcal{G}$ has a rainbow cycle of length at least $n-1$;
- n is odd and $\mathcal{G}$ consists of n copies of $K_1 \vee (2K_{\frac{n-1}{2}})$.

Proof of Theorem 4. It is routine to check that the result holds for $n \leq 5$. In what follows, we assume that $n \geq 6$. We prove it by contradiction. Suppose that $\mathcal{G}$ is not transversal Hamiltonian connected. That is, there are two vertices $x, y \in V$ such that $\mathcal{G}$ contains no transversal Hamilton path with endpoints x and y . Let $H_i = G_i - \{x, y\}$ for all $i \in [n-1]$ and $\mathcal{H} = \{H_1, \dots, H_{n-1}\}$. Note that $\delta(\mathcal{H}) \geq \frac{n+1}{2} - 2 = \frac{n-3}{2}$.

Claim 8. *If n is even, then $\mathcal{H}$ has a rainbow cycle of length $n-2$; if n is odd, then $\mathcal{H}$ has a rainbow cycle of length at least $n-3$.*

Proof of the Claim 8. If n is even, then $\delta(\mathcal{H}) \geq \frac{n}{2} - 1$. Together with Theorem 2, $\mathcal{H}$ contains a rainbow cycle of length $n-2$. In what follows, it suffices to consider n is odd.

Assume that C_l is a longest rainbow cycle inside $\mathcal{H}$. Suppose that $l < n-3$. In view of Lemma 7, we know that every $n-2$ graphs of $\mathcal{H}$ consists of $n-2$ copies of $K_1 \vee (2K_{\frac{n-3}{2}})$. It follows that $\mathcal{H}$ consists of $n-1$ copies of $K_1 \vee (2K_{\frac{n-3}{2}})$. Hence, there are two subsets U, W of $V \setminus \{x, y\}$ and a vertex $z \in V \setminus \{x, y\}$ such that $U \cap W = \{z\}$ and $H_i[U] = H_i[W] = K_{\frac{n-1}{2}}$ for all $i \in [n-1]$. For each vertex $u \in V \setminus \{x, y, z\}$, since $\delta(\mathcal{G}) \geq \frac{n+1}{2}$, we must have $ux, uy \in E(G_i)$ for all $i \in [n-1]$. Therefore, $(H_i - z) \vee (2K_1)$ is a subgraph of G_i for each $i \in [n-1]$. Now, it is easy to verify that $\mathcal{G}$ contains a transversal Hamilton path with endpoints x and y , a contradiction. Thus, $\mathcal{H}$ contains a rainbow cycle of length at least $n-3$. $\square$

We proceed by considering the following two cases.

Case 1. $\mathcal{H}$ has a rainbow cycle of length $n-2$.

Let $C_{n-2} = v_1 \dots v_{n-2} v_1$ be a rainbow cycle of length $n-2$ in $\mathcal{H}$, where $v_i v_{i+1} \in E(G_i)$ for each $i \in [n-2]$. Recall that $\{x, y\} = V \setminus V(C_{n-2})$. Let

$$I_1 = \{i \in [n-2] : xv_i \in E(G_i)\}, \quad I_2 = \{i \in [n-2] : yv_{i+1} \in E(G_{n-1})\}.$$

Clearly, $|I_2| \geq \lceil \frac{n-1}{2} \rceil$ and $I_1 \cup I_2 \subseteq [n-2]$. Suppose that $I_1 \cap I_2 \neq \emptyset$, choose $i_0 \in I_1 \cap I_2$. Hence, $xv_{i_0} \dots v_1 v_{n-2} \dots v_{i_0+1} y$ is a transversal Hamilton path inside $\mathcal{G}$ with endpoints x and y , a contradiction. Thus, $I_1 \cap I_2 = \emptyset$. Consequently, $|I_1| \leq \lfloor \frac{n-3}{2} \rfloor$.

Now, we utilize the following auxiliary digraph D with vertex set V and arc set

$$A(D) = \bigcup_{i \in [n-2]} \{\overrightarrow{v_i u} : v_i u \in E(G_i) \text{ and } u \neq v_{i+1}\}.$$

Clearly, $d_D^+(x) = d_D^+(y) = 0$ and $d_D^+(v_i) \geq \lceil \frac{n+1}{2} \rceil - 1 = \lceil \frac{n-1}{2} \rceil$ for each $i \in [n-2]$. Then, $|A(D)| \geq (n-2) \cdot \lceil \frac{n-1}{2} \rceil$. It follows from $|I_1| \leq \lfloor \frac{n-3}{2} \rfloor$ that $d_D^-(x) \leq \lfloor \frac{n-3}{2} \rfloor$. By symmetry, $d_D^-(y) \leq \lfloor \frac{n-3}{2} \rfloor$. Therefore,

$$|A(D - x - y)| \geq (n-2) \cdot \lceil \frac{n-1}{2} \rceil - (n-3) > (n-2) \cdot \lceil \frac{n-3}{2} \rceil.$$

Without loss of generality, assume that $d_D^-(v_1) = \max\{d_D^-(v_i) : i \in [n-2]\}$. Thus, $d_D^-(v_1) \geq \lceil \frac{n-1}{2} \rceil$. Let

$$I_3 = \{i \in [n-2] : xv_i \in E(G_1)\}.$$

Obviously, $|I_3| \geq \lceil \frac{n-1}{2} \rceil$ and $I_2 \cup I_3 \subseteq [n-2]$. Therefore, $I_2 \cap I_3 \neq \emptyset$ and choose $i_1 \in I_2 \cap I_3$. If $i_1 = 1$, then $xv_1 v_{n-2} \dots v_2 y$ is a transversal Hamilton path from x to y with colors “ $1, n-2, \dots, 2, n-1$ ”, a contradiction. Hence $i_1 \neq 1$. If $v_1 v_2 \in E(G_{i_1})$, then $xv_{i_1} \dots v_1 v_{n-2} \dots v_{i_1+1} y$ is a transversal Hamilton path from x to y with colors “ $1, i_1 - 1, \dots, 2, i_1, n-2, \dots, i_1 + 1, n-1$ ”, a contradiction. Therefore, $v_1 v_2 \notin E(G_{i_1})$.

Delete $v_1 v_2$ and $v_{i_1} v_{i_1+1}$ from C_{n-2} , then we get two disjoint rainbow paths $xv_{i_1} \dots v_2$ and $yv_{i_1+1} \dots v_1$, see Figure 1 (I). Let

$$K_1 = \{k \in [2, n-3] \setminus \{i_1\} : v_2 v_{k+1} \in E(G_{i_1})\}, \quad K_2 = \{k \in [2, n-3] \setminus \{i_1\} : v_1 v_k \in E(G_k)\}.$$

Notice that $|K_1| \geq \lceil \frac{n+1}{2} \rceil - 3 = \lceil \frac{n-5}{2} \rceil$, $|K_2| \geq d_D^-(v_1) - 1 = \lceil \frac{n-3}{2} \rceil$ and $|K_1 \cup K_2| \leq n-5$. Therefore, $K_1 \cap K_2 \neq \emptyset$ and choose $k_0 \in K_1 \cap K_2$. Notice that $k_0 \neq i_1$. If $k_0 > i_1$ (resp. $k_0 < i_1$), then $xv_{i_1} \dots v_2 v_{k_0+1} \dots v_{n-2} v_1 v_{k_0} \dots v_{i_1+1} y$ (resp. $xv_{i_1} \dots v_{k_0+1} v_2 \dots v_{k_0} v_1 v_{n-2} \dots v_{i_1+1} y$) is a transversal Hamilton path inside $\mathcal{G}$ (see Figure 1 (II)), whose colors are “ $1, i_1 - 1, \dots, 2, i_1, k_0 + 1, \dots, n-2, k_0, \dots, i_1 + 1, n-1$ ” (resp. “ $1, i_1 - 1, \dots, k_0 + 1, i_1, 2, \dots, k_0 - 1, k_0, n-2, \dots, i_1 + 1, n-1$ ”), a contradiction.

Case 2. $\mathcal{H}$ has no rainbow cycle of length $n-2$.

In view of Claim 8, we know that n is odd and $\mathcal{H}$ has a rainbow cycle of length $n-3$, say $C_{n-3} = v_1 \dots v_{n-3} v_1$ with $v_i v_{i+1} \in E(G_i)$ for $i \in [n-3]$. Denote $\{z\} = V \setminus \{v_1, \dots, v_{n-3}, x, y\}$. Let

$$I_1 = \{i \in [n-3] : zv_i \in E(G_{n-2})\}, \quad I_2 = \{i \in [n-3] : zv_{i+1} \in E(G_{n-1})\}.$$

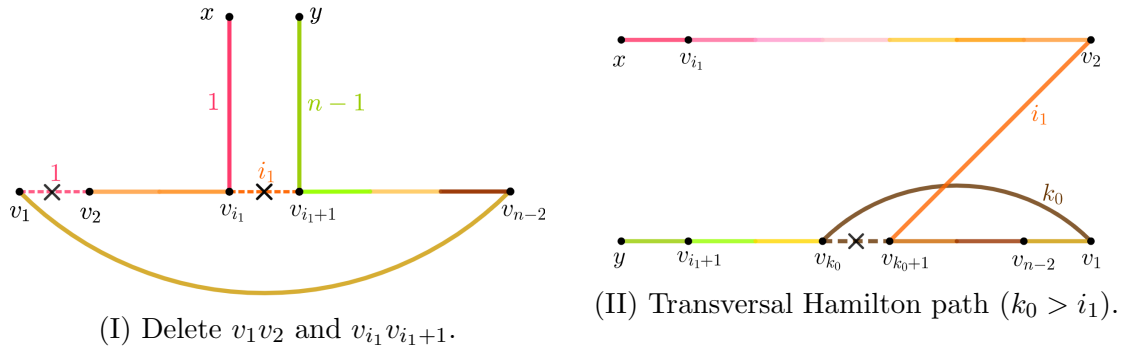


Figure 1

Hence $|I_1| \geq \frac{n-3}{2}$, $|I_2| \geq \frac{n-3}{2}$ and $I_1 \cup I_2 \subseteq [n-3]$. Since $\mathcal{H}$ contains no rainbow cycle of length $n-2$, one has $I_1 \cap I_2 = \emptyset$ and therefore $|I_1| = |I_2| = \frac{n-3}{2}$. Together with $\delta(\mathcal{G}) \geq \frac{n+1}{2}$, we conclude that $N_{n-2}(z) = N_{n-1}(z) = \{x, y, v_1, v_3, \dots, v_{n-4}\}$ or $\{x, y, v_2, v_4, \dots, v_{n-3}\}$.

Similar to Case 1, we utilize the following auxiliary digraph D with vertex set V and arc set

$$A(D) = \bigcup_{i \in [n-3]} \{\overrightarrow{v_i u} : v_i u \in E(G_i) \text{ and } u \neq v_{i+1}\}.$$

Clearly, $d_D^+(x) = d_D^+(y) = d_D^+(z) = 0$ and $d_D^+(v_i) \geq \frac{n-1}{2}$ for each $i \in [n-3]$. Let

$$A_1 = \{a \in [n-3] : zv_a \in E(G_a)\}, \quad A_2 = \{a \in [n-3] : zv_{a+1} \in E(G_{n-2})\}.$$

If $A_1 \cap A_2 \neq \emptyset$, then choose $a_0 \in A_1 \cap A_2$. Hence $zv_{a_0} \dots v_1 v_{n-3} \dots v_{a_0+1} z$ is a rainbow cycle of length $n-2$ in $\mathcal{H}$, a contradiction. Thus, $A_1 \cap A_2 = \emptyset$. It follows from $|I_1| = \frac{n-3}{2}$ that $|A_2| = \frac{n-3}{2}$. Since $A_1 \cup A_2 \subseteq [n-3]$, one has $d_D^-(z) = |A_1| \leq \frac{n-3}{2}$.

Let

$$A_3 = \{a \in [n-3] : xv_a \in E(G_a)\}.$$

If $A_2 \cap A_3 \neq \emptyset$, then choose $a_1 \in A_2 \cap A_3$. Recall that $yz \in E(G_{n-1})$. Hence, $xv_{a_1} \dots v_1 v_{n-3} \dots v_{a_1+1} zy$ is a transversal Hamilton path with endpoints x and y , a contradiction. Therefore, $A_2 \cap A_3 = \emptyset$. Note that $|A_2| = \frac{n-3}{2}$ and $A_2 \cup A_3 \subseteq [n-3]$. Hence $d_D^-(x) = |A_3| \leq \frac{n-3}{2}$. Similarly, $d_D^-(y) \leq \frac{n-3}{2}$. Therefore,

$$|A(D-x-y-z)| \geq |A(D)| - d_D^-(x) - d_D^-(y) - d_D^-(z) \geq (n-3) \cdot \frac{n-1}{2} - 3 \cdot \frac{n-3}{2} = (n-3) \cdot \frac{n-4}{2}.$$

Without loss of generality, assume that $d_D^-(v_1) = \max\{d_D^-(v_i) : i \in [n-3]\}$. Thus, $d_D^-(v_1) \geq \frac{n-3}{2}$ (since n is odd). Let

$$A_4 = \{a \in [n-3] : xv_a \in E(G_1)\}.$$

It is easy to see that $|A_4| \geq \frac{n-3}{2}$ and $A_2 \cup A_4 \subseteq [n-3]$. Next, we proceed by considering the following subcases.

Subcase 2.1. $N_{n-2}(z) = N_{n-1}(z) = \{x, y, v_2, v_4, \dots, v_{n-3}\}$.

In this subcase, $A_2 = \{1, 3, \dots, n-4\}$. If $A_2 \cap A_4 = \emptyset$, then $A_4 = \{2, 4, \dots, n-3\}$. Recall that $d_D^-(v_1) \geq \frac{n-3}{2}$ and $v_2 \notin N_D^-(v_1)$. Therefore, there exists an odd integer $m \in [3, n-3]$ such that $v_1 v_m \in A(D)$, i.e., $v_1 v_m \in E(G_m)$. Since $z v_{m+1} \in E(G_{n-2})$, we know that $x v_2 \dots v_m v_1 v_{n-3} \dots v_{m+1} z y$ is a transversal Hamilton path inside $\mathcal{G}$ with endpoints x, y and colors “ $1, \dots, m, n-3, \dots, m+1, n-2, n-1$ ” (see Figure 2 (I)), a contradiction. Hence $A_2 \cap A_4 \neq \emptyset$.

Choose $a_2 \in A_2 \cap A_4$. Delete $v_1 v_2$ and $v_{a_2} v_{a_2+1}$ from the rainbow cycle C_{n-3} , we get two disjoint rainbow paths $x v_{a_2} \dots v_2$ and $z v_{a_2+1} \dots v_1$. By a similar argument as Case 1, one has $a_2 \neq 1$ and $v_1 v_2 \notin E(G_{a_2})$. Let

$$K_1^1 = \{k \in [2, n-4] : v_2 v_{k+1} \in E(G_{a_2})\}, \quad K_2^1 = \{k \in [2, n-4] : v_1 v_k \in E(G_k)\}.$$

It is routine to check that $|K_1^1| \geq \frac{n-5}{2}$, $|K_2^1| \geq \frac{n-3}{2}$ and $K_1^1 \cup K_2^1 \subseteq [2, n-4]$. Thus, $K_1^1 \cap K_2^1 \neq \emptyset$ and there exists $k_1 \in K_1^1 \cap K_2^1$. Suppose that $k_1 > a_2$ (resp. $k_1 < a_2$). Then $x v_{a_2} \dots v_2 v_{k_1+1} \dots v_{n-3} v_1 v_{k_1} \dots v_{a_2+1} z y$ (resp. $x v_{a_2} \dots v_{k_1+1} v_2 \dots v_{k_1} v_1 v_{n-3} \dots v_{a_2+1} z y$) is a transversal Hamilton path inside $\mathcal{G}$ with endpoints x and y , whose colors are “ $1, a_2-1, \dots, 2, a_2, k_1+1, \dots, n-3, k_1, \dots, a_2+1, n-2, n-1$ ” (resp. “ $1, a_2-1, \dots, k_1+1, a_2, 2, \dots, k_1-1, k_1, n-3, \dots, a_2+1, n-2, n-1$ ”), a contradiction. Therefore, $k_1 = a_2$. Recall that $N_{n-1}(z) = \{x, y, v_2, v_4, \dots, v_{n-3}\}$. Thus, $v_2 z \in E(G_{n-1})$ and $v_{a_2} \dots v_2 z v_{a_2+1} \dots v_{n-3} v_1 v_{a_2}$ is a rainbow cycle of length $n-2$ inside $\mathcal{H}$ with colors “ $a_2-1, \dots, 2, n-1, n-2, a_2+1, \dots, n-3$ ”, which is also a contradiction (see Figure 2 (II)).

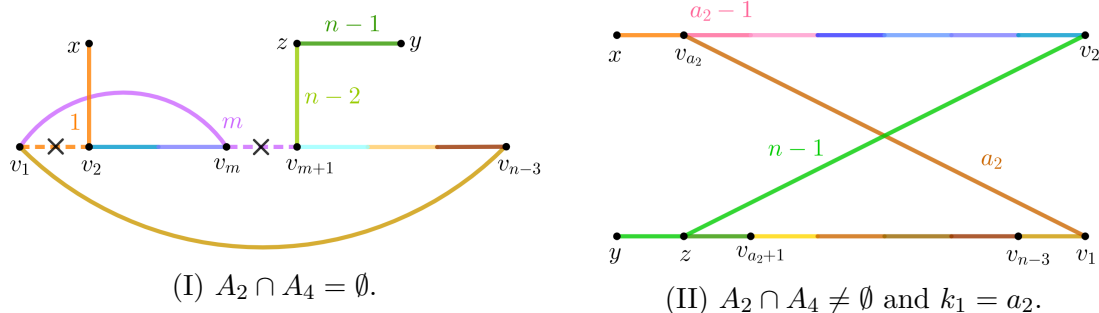


Figure 2

Subcase 2.2. $N_{n-2}(z) = N_{n-1}(z) = \{x, y, v_1, v_3, \dots, v_{n-4}\}$.

In this subcase, $A_2 = \{2, 4, \dots, n-3\}$. If $A_2 \cap A_4 = \emptyset$, then $A_4 = \{1, 3, \dots, n-4\}$. Notice that $z v_1 \in E(G_{n-1})$ and $z v_{n-4} \in E(G_{n-2})$. Therefore, $z v_1 \dots v_{n-4} z$ is another rainbow cycle of length $n-3$ inside $\mathcal{H}$, denoted by C'_{n-3} . Clearly, $V \setminus V(C'_{n-3}) = \{x, y, v_{n-3}\}$ and $[n] \setminus \text{col}(C'_{n-3}) = \{n-3, n-4\}$. By the symmetry of z and v_{n-3} , we obtain $N_{n-3}(v_{n-3}) = N_{n-4}(v_{n-3}) = \{x, y, v_2, v_4, \dots, v_{n-5}, z\}$ or $\{x, y, v_1, v_3, \dots, v_{n-4}\}$. This implies that $z v_1 \dots v_{n-3} x$ is a transversal Hamilton path inside $\mathcal{G}$ with endpoints x and y , which is a contradiction. Thus, $A_2 \cap A_4 \neq \emptyset$.

Let $a_3 \in A_2 \cap A_4$ and

$$K_1^2 = \{k \in [2, n-4] : v_2 v_{k+1} \in E(G_{a_3})\}, \quad K_2^2 = \{k \in [2, n-4] : v_1 v_k \in E(G_k)\}.$$

Clearly, $K_1^2 \cap K_2^2 \neq \emptyset$, i.e., there exists $k_2 \in K_1^2 \cap K_2^2$. By a similar discussion as Subcase 2.1, we obtain $k_2 = a_3$. Notice that $v_1 z \in E(G_{n-2})$, then $xv_{a_3} \dots v_2 v_{a_3+1} \dots v_{n-3} v_1 z y$ is a transversal Hamilton path with endpoint x and y with colors “ $1, a_3-1, \dots, 2, a_3, \dots, n-1$ ”, which is also a contradiction.

This completes the proof of Theorem 4. $\square$

3 Proof of Theorem 6

In this section, we prove Theorem 6. We first give three preliminaries. The first one considers the rainbow pancyclicity in a collection of graphs under the Dirac-type condition, which was obtained by [29].

Let $\mathcal{G} = \{G_1, \dots, G_n\}$ be a collection of graphs on a common vertex set of size n . We say that $\mathcal{G}$ is *transversal pancyclic* if there is a rainbow cycle of length ℓ in $\mathcal{G}$ for each integer $\ell \in [3, n]$.

Theorem 9 ([29]). *Let $n \geq 3$ be an integer and $\mathcal{G} = \{G_1, \dots, G_n\}$ be a collection of graphs on a common vertex set of size n . If $\delta(\mathcal{G}) \geq \frac{n}{2}$, then either $\mathcal{G}$ is transversal pancyclic or $\mathcal{G}$ consists of n copies of $K_{\frac{n}{2}, \frac{n}{2}}$.*

The subsequent two lemmas were obtained by [14], which investigate the existence of transversal Hamilton paths and cycles in graph collections.

Theorem 10 ([14]). *Let $\mathcal{G} = \{G_1, \dots, G_{n-1}\}$ be a collection of graphs on a common vertex set of size n . If $\delta(\mathcal{G}) \geq \frac{n-1}{2}$, then $\mathcal{G}$ has a transversal Hamilton path.*

Given integers $s, t \geq 0$, let $\mathcal{R}_s^t$ be the graph collection on a common vertex set of even size n obtained by taking s copies of $K_{\frac{n}{2}} \cup K_{\frac{n}{2}}$ and t copies of $K_{\frac{n}{2}, \frac{n}{2}}$, where they are defined on the same equitable partition.

Theorem 11 ([14]). *For sufficiently large even integer n , let $\mathcal{C}$ be a set of n colors, and let $\mathcal{G} = \{G_i : i \in \mathcal{C}\}$ be a collection of graphs on a common vertex set V of size n and $\delta(\mathcal{G}) \geq \frac{n}{2} - 1$. Assume that $\mathcal{G}$ contains no transversal Hamilton cycles. Then one of the following holds:*

- (i) $\mathcal{G}$ is a spanning collection of $\mathcal{R}_{n-t}^t$ for some odd integer $t \in [3, n-1]$;
- (ii) there exists a partition $A \cup B$ of V with $|A| = \frac{n}{2} + 1$ such that either $E(G_i[A]) \subseteq \{uv\}$ for fixed $u, v \in A$ and all $i \in \mathcal{C}$, or $G_i[A] = \emptyset$ for all but at most one $i \in \mathcal{C}$;
- (iii) there exists an equitable partition $A \cup B$ of V such that $\mathcal{G}[A, B]$ contains no rainbow 2-matching, that is, one of the following holds:
 - (a) $G_i = G_i[A] \cup G_i[B] = 2K_{\frac{n}{2}}$ for all but at most one $i \in \mathcal{C}$;
 - (b) there are vertices $u, u' \in A$ and $w, w' \in B$ such that $E(G_1[A, B]) = \{uw, u'w'\}$, $E(G_2[A, B]) = \{uw', u'w\}$ and $G_j = G_j[A] \cup G_j[B] = 2K_{\frac{n}{2}}$ for all $j \in \mathcal{C} \setminus [2]$;
 - (c) there is a vertex $u \in A$ such that $E(G_i[A, B]) = E(G_i[\{u\}, B])$ for all $i \in \mathcal{C}$;

- (d) there are vertices $u \in A$ and $v \in B$ such that $E(G_1[A, B]) \subseteq \{uw : w \in B\} \cup \{wv : w \in A\}$ and $E(G_i[A, B]) \subseteq \{uv\}$ for all $i \in \mathcal{C} \setminus \{1\}$.

Now, we are ready to give the proof of Theorem 6.

Proof of Theorem 6. Suppose that $\mathcal{G} \neq \mathcal{F}_{n-1}$. Let $x, y \in V$ be two arbitrary distinct vertices and $k \in [d_{\mathcal{G}}(x, y) + 2, n - 1]$ be an arbitrary integer. In view of Theorem 4, we know that $\mathcal{G}$ is transversal Hamiltonian connected. Together with the definition of $d_{\mathcal{G}}(x, y)$, it suffices to prove that $\mathcal{G}$ contains a rainbow path P_k with endpoints x and y . Let $H_i = G_i - \{x, y\}$ for each $i \in [n - 1]$ and $\mathcal{H} = \{H_i : i \in [n - 1]\}$. We proceed by considering the following two cases.

Case 1. n is even.

Since n is even, one has $\delta(\mathcal{G}) \geq \frac{n}{2} + 1$ and $\delta(\mathcal{H}) \geq \frac{n}{2} - 1$. In view of Theorem 9, we know that either $\mathcal{H}$ has a rainbow cycle of length $n - 3$ or every $n - 2$ graphs of $\mathcal{H}$ consists of $n - 2$ copies of $K_{\frac{n-2}{2}, \frac{n-2}{2}}$.

We first assume that $\mathcal{H}$ has a rainbow cycle of length $n - 3$. Let $C = v_1 \dots v_{n-3} v_1$ be such a rainbow cycle with $v_i v_{i+1} \in E(G_i)$ for each $i \in [n - 3]$. Suppose that $\{z\} = V \setminus (V(C) \cup \{x, y\})$ and $[n - 1] \setminus \text{col}(C) = \{n - 2, n - 1\}$. Let

$$I_{n-1} = \{i \in [n - 3] : xv_{i+k-3} \in E(G_{n-1})\}, \quad I_{n-2} = \{i \in [n - 3] : yv_i \in E(G_{n-2})\}.$$

Clearly, $|I_{n-1}| \geq \frac{n}{2} - 1$, $|I_{n-2}| \geq \frac{n}{2} - 1$ and $I_{n-1} \cup I_{n-2} \subseteq [n - 3]$. Therefore, $I_{n-1} \cap I_{n-2} \neq \emptyset$ and choose $i_0 \in I_{n-1} \cap I_{n-2}$. It is routine to check that $\mathcal{G}$ contains a rainbow path $P_k = yv_{i_0} v_{i_0+1} \dots v_{i_0+k-3} x$ with endpoints x and y , as desired.

Now, we consider that every $n - 2$ graphs of $\mathcal{H}$ consists of $n - 2$ copies of $K_{\frac{n-2}{2}, \frac{n-2}{2}}$. It follows that $\mathcal{H}$ consists of $n - 1$ copies of $K_{\frac{n-2}{2}, \frac{n-2}{2}}$. For each vertex $u \in V \setminus \{x, y\}$, since $\delta(\mathcal{G}) \geq \frac{n}{2} + 1$, we must have $ux, uy \in E(G_i)$ for all $i \in [n - 1]$. Therefore, $K_{\frac{n-2}{2}, \frac{n-2}{2}} \vee (2K_1)$ is a subgraph of G_i for each $i \in [n - 1]$. It is easy to verify that $\mathcal{G}$ is transversal panconnected, as desired.

Case 2. n is odd.

When n is odd, we have $\delta(\mathcal{H}) \geq \frac{n+1}{2} - 2 = \frac{n-3}{2}$. In view of Theorem 10, we know that $\mathcal{H}$ contains a rainbow Hamilton path $\tilde{P}$. If all vertices in $V(\tilde{P})$ are adjacent to both x and y in G_i for each $i \in [n - 1]$, then it is obvious that $\mathcal{G}$ contains a rainbow path P_k with endpoints x and y , as desired. Hence, it suffices to consider that there exists a vertex $z \in V \setminus \{x, y\}$ and a color $c \in [n - 1]$ such that at least one of $\{x, y\}$ is not adjacent to z in G_c . Without loss of generality, assume that $c = n - 1$ and $xz \notin E(G_{n-1})$.

Let $H'_i = G_i - \{x, y, z\}$ for each $i \in [n - 2]$ and $\mathcal{H}' = \{H'_i : i \in [n - 2]\}$. Obviously, $\delta(\mathcal{H}') \geq \frac{n+1}{2} - 3 = \frac{n-5}{2}$. We distinguish our proof into the following two subcases.

Subcase 2.1. $\mathcal{H}'$ contains a rainbow Hamilton cycle.

In this subcase, assume that $C = v_1 \dots v_{n-3} v_1$ is the rainbow Hamilton cycle of $\mathcal{H}'$, where $v_i v_{i+1} \in E(G_i)$ for each $i \in [n - 3]$. Notice that $\{x, y, z\} = V \setminus V(C)$. Let

$$I_{n-1} = \{i \in [n - 3] : xv_{i+k-3} \in E(G_{n-1})\}, \quad I_{n-2} = \{i \in [n - 3] : yv_i \in E(G_{n-2})\}.$$

Since $xz \notin E(G_{n-1})$, one has $|I_{n-1}| \geq \frac{n-1}{2}$. Furthermore, $|I_{n-2}| \geq \frac{n-3}{2}$ and $I_{n-1} \cup I_{n-2} \subseteq [n-3]$. Hence $I_{n-1} \cap I_{n-2} \neq \emptyset$. Thus, $\mathcal{G}$ contains a rainbow path P_k with endpoints x and y , as desired.

Subcase 2.2. $\mathcal{H}'$ does not contain rainbow Hamilton cycles.

Based on Theorem 11, we know that every $n-3$ graphs in $\mathcal{H}'$ must be the same graph collection described in Theorem 11. Therefore, one may obtain the characterization of all graphs in $\mathcal{H}'$, leading to the following three cases.

(i) It is routine to check that $\mathcal{H}'$ consists of either t copies of $K_{\frac{n-3}{2}, \frac{n-3}{2}}$ and $n-2-t$ copies of $K_{\frac{n-3}{2}} \cup K_{\frac{n-3}{2}}$, or $t+1$ copies of $K_{\frac{n-3}{2}, \frac{n-3}{2}}$ and $n-3-t$ copies of $K_{\frac{n-3}{2}} \cup K_{\frac{n-3}{2}}$, where $t \in [3, n-4]$ is odd. Applying Theorem 11 again yields that $\mathcal{H}'$ contains a rainbow Hamilton cycle in each case, a contradiction.

(ii) There exists a partition $A \cup B$ of $V \setminus \{x, y, z\}$ with $|A| = \frac{n-1}{2}$ such that either $E(H'_i[A]) \subseteq \{uv\}$ for fixed $u, v \in A$ and all $i \in [n-2]$, or $H'_i[A] = \emptyset$ for all but at most one $i \in [n-2]$. Here, we only discuss the former case, the latter one can be proved by a similar discussion, so we omit the proof here.

Since $\delta(\mathcal{H}') \geq \frac{n-5}{2}$ and $\delta(\mathcal{G}) \geq \frac{n+1}{2}$, for each $i \in [n-2]$ we have

- $G_i[A \setminus \{u, v\}, B \cup \{x, y, z\}] = K_{\frac{n-5}{2}, \frac{n+1}{2}}$;
- u, v are adjacent to all but at most one vertex of $B \cup \{x, y, z\}$ in G_i ;
- $N_i(z) \cap (B \cup \{x, y\}) \neq \emptyset$.

Assume that $A = \{u_1, \dots, u_{\frac{n-1}{2}}\}$ and $B = \{w_1, \dots, w_{\frac{n-5}{2}}\}$. Without loss of generality, assume that $u = u_{\frac{n-3}{2}}$ and $v = u_{\frac{n-1}{2}}$.

We first consider that k is odd. Then $3 \leq k \leq n-2$. Note that $u_{\frac{n-3}{2}}$ is adjacent to all but at most one vertex in $B \cup \{x, y\}$ in each G_i with $i \in [n-2]$. Hence, by suitable relabeling, we always guarantee that $\mathcal{G}$ contains a rainbow path, $P_k := xu_1w_1 \dots u_{\frac{k-3}{2}}w_{\frac{k-3}{2}}u_{\frac{k-1}{2}}y$, along with $col(P_k) \subseteq [n-2]$, as desired.

Next, we assume that k is even. Then $4 \leq k \leq n-1$. Recall that $u_{\frac{n-3}{2}}, u_{\frac{n-1}{2}}$ are adjacent to all but at most one vertex of $B \cup \{x, y, z\}$ in G_i with $i \in [n-2]$. If $u_{\frac{n-3}{2}}u_{\frac{n-1}{2}} \in E(G_{i_0}[A])$ for some $i_0 \in [n-1]$, then by suitable relabeling, $\mathcal{G}$ contains a rainbow path $P_k := xu_1w_1 \dots u_{\frac{k-4}{2}}w_{\frac{k-4}{2}}u_{\frac{n-3}{2}}u_{\frac{n-1}{2}}y$, along with $u_{\frac{n-3}{2}}u_{\frac{n-1}{2}} \in E(G_{i_0})$ and $col(P_k - u_{\frac{n-3}{2}}u_{\frac{n-1}{2}}) \subseteq [n-2] \setminus \{i_0\}$ (see Figure 3 (I)), as desired. Similarly, we can show that such a rainbow path P_k exists if $E(G_{n-1}[A]) \neq \emptyset$ or $N_i(z) \cap \{x, y\} \neq \emptyset$ for some $i \in [n-1]$. Hence, in what follows, we assume that $E(G_i[A]) = \emptyset$ and $N_i(z) \cap \{x, y\} = \emptyset$ for all $i \in [n-1]$. Therefore, $G_i[A, B \cup \{x, y, z\}] = K_{\frac{n-1}{2}, \frac{n+1}{2}}$ and $N_i(z) \cap B \neq \emptyset$ for all $i \in [n-1]$.

If $k \neq 4$, then by relabeling, we obtain that $P_k := xu_1zw_1u_2w_2 \dots u_{\frac{k-4}{2}}w_{\frac{k-4}{2}}u_{\frac{k-2}{2}}y$ is a rainbow path on k vertices inside $\mathcal{G}$, along with $col(P_k) \subseteq [n-2]$, see Figure 3 (II). Hence it suffices to consider $k = 4$.

If $N_i(x) \cap B \neq \emptyset$ or $N_i(y) \cap B \neq \emptyset$ for some $i \in [n-1]$, then it is easy to see that $\mathcal{G}$ contains a rainbow path P_4 with endpoints x and y , as desired. Hence, if $\mathcal{G}$ contains no

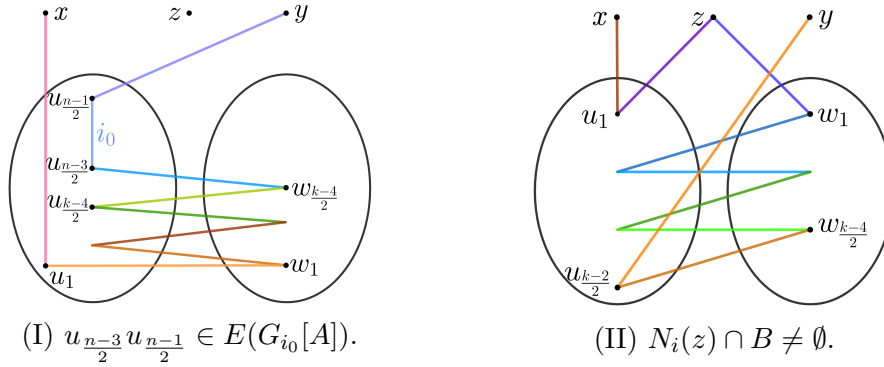


Figure 3: Subcase 2.2 (ii) and k is even.

rainbow P_4 with endpoints x and y , we must have $G_i[A] = \emptyset$ and $N_i(x) \cap (B \cup \{z\}) = N_i(y) \cap (B \cup \{z\}) = \emptyset$ for all $i \in [n-1]$. That is to say, $\mathcal{G} = \mathcal{F}_{n-1}$, a contradiction.

(iii) There exists an equitable partition $A \cup B$ of $V \setminus \{x, y, z\}$ such that $\mathcal{H}'[A, B]$ contains no rainbow 2-matching. It follows that $H'_i[A]$ and $H'_i[B]$ are almost complete graphs for each $i \in [n-2]$. Thus, for $3 \leq k \leq \frac{n+1}{2}$, the rainbow path P_k can be easily found by using vertices in A and colors in $[n-2]$. In the following, we consider $\frac{n+3}{2} \leq k \leq n-1$. In fact, such a rainbow path P_k can be found by utilizing the vertex z or edges in $E(G_i[A, B])$ to connect vertices in A and B . Here, we only illustrate the first two cases in Theorem 11 (iii), and the rest two cases can be discussed similarly.

(a) There exists an equitable partition $A \cup B$ of $V \setminus \{x, y, z\}$ and a color $i_1 \in [n-2]$ such that $H'_i = H'_i[A] \cup H'_i[B] = 2K_{\frac{n-3}{2}}$ for all $i \in [n-2] \setminus \{i_1\}$. Assume that $A = \{u_1, \dots, u_{\frac{n-3}{2}}\}$ and $B = \{w_1, \dots, w_{\frac{n-3}{2}}\}$. It follows from $\delta(\mathcal{G}) \geq \frac{n+1}{2}$ that $G_{i_1}[\{x, y, z\}, A \cup B] = K_{3, n-3}$ for all $i \in [n-2] \setminus \{i_1\}$.

If $E(G_{i_1}[A, B]) = \emptyset$, then $G_i[A] \cup G_i[B] = 2K_{\frac{n-3}{2}}$ and $G_i[\{x, y, z\}, A \cup B] = K_{3, n-3}$ for all $i \in [n-2]$. Thus, $\mathcal{G}$ contains a rainbow path $P_k := xu_1u_2 \dots u_{\frac{n-3}{2}}zw_1 \dots w_{k-\frac{n+3}{2}}y$ with $\text{col}(P_k) \subseteq [n-2]$ (see Figure 4 (I)), as desired. If $E(G_{i_1}[A, B]) \neq \emptyset$, then without loss of generality, assume that $u_{\frac{n-3}{2}}w_1 \in E(G_{i_1})$. Thus, $\mathcal{G}$ contains a rainbow path $P_k := xu_1u_2 \dots u_{\frac{n-3}{2}}w_1 \dots w_{k-\frac{n+1}{2}}y$, along with $u_{\frac{n-3}{2}}w_1 \in E(G_{i_1})$ and $\text{col}(P_k - u_{\frac{n-3}{2}}w_1) \subseteq [n-2] \setminus \{i_1\}$ (see Figure 4 (II)), as desired.

(b) There are vertices $u, u' \in A$ and $w, w' \in B$ such that $E(H'_1[A, B]) = \{uw, u'w'\}$, $E(H'_2[A, B]) = \{uw', u'w\}$ and $H'_j = H'_j[A] \cup H'_j[B] = 2K_{\frac{n-3}{2}}$ for all $j \in [3, n-2]$. Assume that $A = \{u_1, \dots, u_{\frac{n-3}{2}}\}$ and $B = \{w_1, \dots, w_{\frac{n-3}{2}}\}$. Let $u = u_{\frac{n-3}{2}}$ and $w = w_1$. Since $\delta(\mathcal{G}) \geq \frac{n+1}{2}$, we have

- $G_i[A] \cup G_i[B] = 2K_{\frac{n-3}{2}}$ for each $i \in [n-2]$;
- all vertices in $A \cup B$ are adjacent to all vertices of $\{x, y, z\}$ in G_i for each $i \in [3, n-2]$.

It is easy to see that $\mathcal{G}$ contains a rainbow path $P_k := xu_1u_2 \dots u_{\frac{n-3}{2}}w_1 \dots w_{k-\frac{n+1}{2}}y$, along with $u_{\frac{n-3}{2}}w_1 \in E(G_1)$ and $\text{col}(P_k - u_{\frac{n-3}{2}}w_1) \subseteq [2, n-2]$, as desired.

This completes the proof of Theorem 6. $\square$

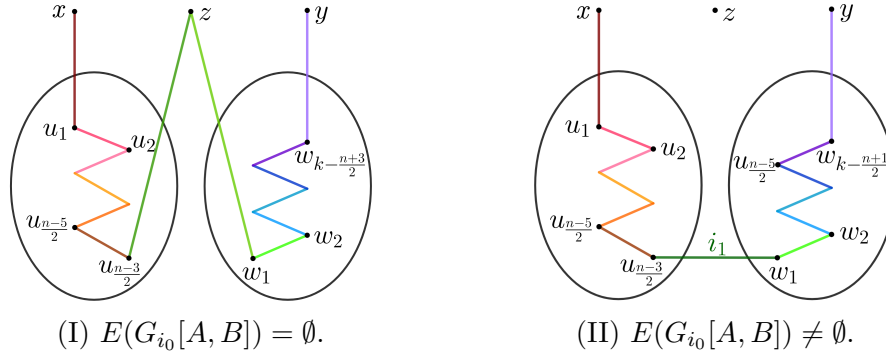


Figure 4: Subcase 2.2 (iii) (a).

4 Concluding remarks

In this paper, we determine the Dirac-type condition for transversal panconnectedness in graph collections. However, in Theorem 6, we require the order n to be sufficiently large when it is odd. This assumption stems from our reliance on Theorem 11, where the same condition is imposed. We speculate that Theorem 6 holds for all n . In fact, it suffices to prove the following: Let $n \geq 4$ be an even integer and $\mathcal{G} = \{G_1, \dots, G_{n+1}\}$ be a collection of graphs on a common vertex set of size n . If $\delta(\mathcal{G}) \geq \frac{n}{2} - 1$, then either $\mathcal{G}$ contains rainbow Hamilton cycles or $\mathcal{G}$ is some exceptional graph collection.

Define $\sigma_2(G) := \min\{d(u) + d(v) : u, v \in V(G), uv \notin E(G)\}$. Ore [34] proved that every n -vertex 2-connected graph G with $\sigma_2(G) \geq n + 1$ is Hamiltonian connected. It is natural to study the Ore-type condition for transversal Hamiltonian connectivity in graph collections.

Acknowledgements

We would like to express our gratitude to the anonymous reviewers for their valuable comments that greatly improved the presentation of this paper. The work is supported by the Natural Science Foundation of Shandong Province (ZR2024QA023) and the China Postdoctoral Science Foundation (2023M742092), the National Key Research and Development Program (2023YFA1009603) and the Natural Science Foundation of China (12231018).

References

- [1] R. Aharoni, C. St. J. A. Nash-Williams, and S. Shelah, “A general criterion for the existence of transversals,” *Proc. London Math. Soc.* (3), vol. 47, no. 1, pp. 43–68, 1983.
- [2] R. Aharoni, M. DeVos, S. González Hermosillo de la Maza, A. Montejano, and R. Šámal, “A rainbow version of Mantel’s theorem,” *Adv. Comb.*, pp. Paper No. 2, 12, 2020.

- [3] M. Anastos and D. Chakraborti, “Robust Hamiltonicity in families of Dirac graphs,” [arXiv:2309.12607](#), 2023.
- [4] J. Böttcher, M. Schacht, and A. Taraz, “Proof of the bandwidth conjecture of Bollobás and Komlós,” *Math. Ann.*, vol. 343, no. 1, pp. 175–205, 2009.
- [5] C. Bowtell, P. Morris, Y. Pehova, and K. Staden, “Universality for transversal Hamilton cycles,” [arXiv:2310.04138](#), 2023.
- [6] P. Bradshaw, “Transversals and bipancyclicity in bipartite graph families,” *Electron. J. Combin.*, vol. 28, no. 4, #P4.25, 2021.
- [7] D. Chakraborti, S. Im, J. Kim, and H. Liu, “A bandwidth theorem for graph transversals,” [arXiv:2302.09637](#), 2023.
- [8] D. Chakraborti, J. Kim, H. Lee, and J. Seo, “Hamilton transversals in tournaments,” *Combinatorica*, vol. 44, no. 6, pp. 1381–1400, 2024.
- [9] D. Chakraborti, J. Kim, H. Lee, and J. Seo, “Transversal cycles and paths in tournaments,” [arXiv:2407.14300](#), 2024.
- [10] D. Chen, Z. Li, and K. Zhao, “Dirac type condition and Hamiltonian-connected graphs,” *Quaest. Math.*, vol. 34, no. 4, pp. 521–525, 2011.
- [11] Y. Cheng, J. Han, B. Wang, and G. Wang, “Rainbow spanning structures in graph and hypergraph systems,” *Forum Math. Sigma*, vol. 11, pp. Paper No. e95, 20, 2023.
- [12] Y. Cheng and K. Staden, “Transversals via regularity,” [arXiv:2306.03595](#), 2023.
- [13] Y. Cheng and K. Staden, “Stability of transversal Hamilton cycles and paths,” [arXiv:2403.09913](#), 2024.
- [14] Y. Cheng, W. Sun, G. Wang, and L. Wei, “Transversal Hamilton paths and cycles,” [arXiv:2406.13998](#), 2024.
- [15] Y. Cheng, G. Wang, and Y. Zhao, “Rainbow pancyclicity in graph systems,” *Electron. J. Combin.*, vol. 28, no. 3, #P3.24, 2021.
- [16] R. Diestel, *Graph theory*, vol. 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth ed., 2017.
- [17] G. A. Dirac, “Some theorems on abstract graphs,” *Proc. London Math. Soc. (3)*, vol. 2, pp. 69–81, 1952.
- [18] P. Erdős, A. Hajnal, and E. C. Milner, “On sets of almost disjoint subsets of a set,” *Acta Math. Acad. Sci. Hungar.*, vol. 19, pp. 209–218, 1968.
- [19] A. Ferber, J. Han, and D. Mao, “Dirac-type problem of rainbow matchings and Hamilton cycles in random graphs,” [arXiv:2211.05477](#), 2022.
- [20] J. Gao, H. Lu, J. Ma, and X. Yu, “On the rainbow matching conjecture for 3-uniform hypergraphs,” *Sci. China Math.*, vol. 65, no. 11, pp. 2423–2440, 2022.
- [21] P. Gupta, F. Hamann, A. Müyesser, O. Parczyk, and A. Sgueglia, “A general approach to transversal versions of Dirac-type theorems,” *Bull. Lond. Math. Soc.*, vol. 55, no. 6, pp. 2817–2839, 2023.

- [22] A. Hajnal and E. Szemerédi, “Proof of a conjecture of P. Erdős,” in *Combinatorial theory and its applications, I-III (Proc. Colloq., Balatonfüred, 1969)*, vol. 4 of *Colloq. Math. Soc. János Bolyai*, pp. 601–623, North-Holland, Amsterdam-London, 1970.
- [23] J. Han, J. Hu, and D. Yang, “A robust version of the multipartite Hajnal–Szemerédi theorem,” [arXiv:2311.00950](https://arxiv.org/abs/2311.00950), 2023.
- [24] G. R. T. Hendry, “Extending cycles in graphs,” *Discrete Math.*, vol. 85, no. 1, pp. 59–72, 1990.
- [25] F. Joos and J. Kim, “On a rainbow version of Dirac’s theorem,” *Bull. Lond. Math. Soc.*, vol. 52, no. 3, pp. 498–504, 2020.
- [26] J. Komlós, G. N. Sárközy, and E. Szemerédi, “Spanning trees in dense graphs,” *Combin. Probab. Comput.*, vol. 10, no. 5, pp. 397–416, 2001.
- [27] D. Kühn and D. Osthus, “The minimum degree threshold for perfect graph packings,” *Combinatorica*, vol. 29, no. 1, pp. 65–107, 2009.
- [28] L. Li, P. Li, and X. Li, “Rainbow structures in a collection of graphs with degree conditions,” *J. Graph Theory*, vol. 104, no. 2, pp. 341–359, 2023.
- [29] L. Li, P. Li, and X. Li, “Rainbow pancyclicity in a collection of graphs under the Dirac-type condition,” *Acta Math. Appl. Sin. Engl. Ser.*, vol. 40, no. 2, pp. 269–274, 2024.
- [30] H. Lu and Y. Wang, “Rainbow perfect matchings in 3-partite 3-uniform hypergraphs,” [arXiv:2408.08523](https://arxiv.org/abs/2408.08523), 2024.
- [31] H. Lu, Y. Wang, and X. Yu, “A better bound on the size of rainbow matchings,” *J. Combin. Theory Ser. A*, vol. 195, pp. Paper No. 105700, 19, 2023.
- [32] R. Montgomery, A. Müyesser, and Y. Pehova, “Transversal factors and spanning trees,” *Adv. Comb.*, pp. Paper No. 3, 25, 2022.
- [33] J. Moon and L. Moser, “On Hamiltonian bipartite graphs,” *Israel J. Math.*, vol. 1, pp. 163–165, 1963.
- [34] O. Ore, “Hamilton connected graphs,” *J. Math. Pures Appl. (9)*, vol. 42, pp. 21–27, 1963.
- [35] W. Sun, G. Wang, and L. Wei, “Transversal structures in graph systems: a survey,” [arXiv:2412.01121](https://arxiv.org/abs/2412.01121), 2024.
- [36] Y. Tang, B. Wang, G. Wang, and G. Yan, “Rainbow Hamilton cycle in hypergraph system,” [arXiv:2302.00080](https://arxiv.org/abs/2302.00080), 2023.
- [37] I. Tomescu, “On Hamiltonian-connected regular graphs,” *J. Graph Theory*, vol. 7, no. 4, pp. 429–436, 1983.
- [38] B. Wei, “Hamiltonian paths and Hamiltonian connectivity in graphs,” *Discrete Math.*, vol. 121, no. 1-3, pp. 223–228, 1993.
- [39] J. E. Williamson, “Panconnected graphs. II,” *Period. Math. Hungar.*, vol. 8, no. 2, pp. 105–116, 1977.