Abstract Regular Polytopes of Finite Irreducible Coxeter Groups

Malcolm Hoong Wai Chen   Peter Rowley

Submitted: Jan 13, 2025; Accepted: Aug 27, 2025; Published: Oct 17, 2025

Abstract

Here, for W the Coxeter group D_n where $n > 4$, it is proved that the maximal rank of an abstract regular polytope for W is $n-1$ if n is even and n if n is odd. Further it is shown that W has abstract regular polytopes of rank r for all r such that $3 \leqslant r \leqslant n-1$, if n is even, and $3 \leqslant r \leqslant n$, if n is odd. The possible ranks of abstract regular polytopes for the exceptional finite irreducible Coxeter groups are also determined.

Mathematics Subject Classifications: 52B11, 20B25, 20F55

1 Introduction

Finite Coxeter groups appear in many guises in the mathematics literature – as Weyl groups of semisimple Lie algebras, reflection groups, and automorphism groups of regular polytopes to mention a few. It is the last area that is of interest here. We recall that $\mathrm{Sym}(n+1)$, the symmetric group of degree $n+1$, also the Coxeter group of type A_n is the automorphism group of a regular n-simplex, while the Coxeter group of type B_n is the automorphism group of the n-cube and its dual, the n-cross polytope.

In this paper we examine abstract regular polytopes of the finite irreducible Coxeter groups W. As is documented in McMullen and Schulte [27], this is equivalent to investigating the C-strings of W, and we follow that approach. A *C-string* of a group G is a set of involutions $S = \{s_1, \dots, s_r\}$ of G which generates the group. Additionally, setting $I = \{1, \dots, r\}$ and, for $J \subseteq I, W_J = \langle s_j \mid j \in J \rangle$, they must satisfy

(i) for all $J, K \subseteq I$, $W_J \cap W_K = W_{J \cap K}$ (intersection property); and

(ii) $s_i s_j = s_j s_i$ for all $i, j \in I$ with $|i - j| \geqslant 2$ (string property).

Department of Mathematics, University of Manchester, Manchester M13 9PL, United Kingdom (malcolmhoongwai.chen@postgrad.manchester.ac.uk, peter.j.rowley@manchester.ac.uk).

We refer to r as the *rank* of the C-string, equivalently, the rank of the associated abstract regular polytope, and also call (G, S) a *string C-group.* The maximal rank for a C-string of W will be denoted by $r_{\max}(W)$. If W is either of type A_n or B_n, then their defining presentation in terms of fundamental (or simple) reflections yields a C-string of rank n (see Section 5.5 of [18] for the intersection property). Moreover $r_{\max}(W) = n$ when $W \cong \mathrm{A}_n$, a consequence of a recent deep result by Whiston [30] which uses the classification of finite simple groups. For $W \cong \mathrm{B}_n$, $r_{\max}(W) = n$ when n is even and $r_{\max}(W) = n+1$ when n is odd. In the latter case the corresponding C-string is degenerate, meaning the string is disconnected. See Theorem 8(ii) for more details. If we insist on only considering non-degenerate strings, then we have $r_{\max}(W) = n$. On the other hand, $\mathrm{I}_2(m)$, identified as $\mathrm{Dih}(2m)$ are also well known to only exhibit rank two polytopes given by the m-gon. Now in the remaining infinite family of finite irreducible Coxeter groups we have D_n whose Dynkin diagrams are not strings. Our first result concerns the maximal rank of C-strings for this family. Noting that D_4 has no C-strings, we have the following theorem.

Theorem 1. *Suppose that W is the Coxeter group D_n with $n \geqslant 5$. If n is even, then $r_{\max}(W) = n - 1$ and if n is odd, then $r_{\max}(W) = n$.*

The question as to the existence of abstract regular polytopes for intermediate rank values is the subject of the next theorem.

Theorem 2. *Suppose that W is the Coxeter group D_n with $n \geqslant 5$. Then W has a C-string of rank r for all r with $3 \leqslant r \leqslant r_{\max}(W)$.*

To complete the picture we consider the exceptional finite irreducible Coxeter groups.

Theorem 3. *Suppose that W is an exceptional finite irreducible Coxeter group.*

(i) If W is one of $\mathrm{I}_2(m), \mathrm{H}_3, \mathrm{H}_4, \mathrm{F}_4$, then $r_{\max}(W)$ is the Coxeter rank of W.

(ii) If W is one of $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$, then $r_{\max}(W)$ is 5, 6, and 7, respectively.

The quest to classify string C-groups in a given family of groups started from early experimental results by Hartley [16] and also the joint work of Leemans and Vauthier [26] which resulted in atlases of abstract regular polytopes for small groups. More efficient computer algorithms have also been developed which successfully enumerated the C-strings of some sporadic simple groups [17, 21, 22]. This experimental data has led to some interesting conjectures which were later proven theoretically. In particular, the possible ranks of C-strings has been investigated for some infinite families of almost simple groups, namely the Suzuki groups $\mathrm{Sz}(q)$ [19], Ree groups ${}^2\mathrm{G}_2(q)$ [25], groups with socle $\mathrm{PSL}(2, q)$ [11, 23, 24], groups $\mathrm{PSL}(3, q)$ and $\mathrm{PGL}(3, q)$ [6], groups $\mathrm{PSL}(4, q)$ [4], symmetric groups [13], alternating groups [10, 14], orthogonal and symplectic groups [3, 2]. We refer the interested reader to a recent survey article by Leemans [20] for more details of these investigations.

Only a few family of groups are known to give rise to string C-groups of arbitrarily large rank. As already noted, the highest possible rank of a string C-group representation for $\mathrm{Sym}(n)$ is $n-1$, and for $n \geqslant 12$, the highest rank for $\mathrm{Alt}(n)$ is $\lfloor \frac{n-1}{2} \rfloor$ [10]. Furthermore, $\mathrm{Sym}(n)$ has C-strings of rank r for every $3 \leqslant r \leqslant n-1$ [13], and for $n \geqslant 12$, $\mathrm{Alt}(n)$ also has C-strings of rank r for every $3 \leqslant r \leqslant \lfloor \frac{n-1}{2} \rfloor$ [14]. Likewise for all integers $k, m \geqslant 2$, the orthogonal groups $\mathrm{O}^{\pm}(2m, 2^k)$ and the symplectic groups $\mathrm{Sp}(2m, 2^k)$ have string C-group representations of rank $2m$ and $2m+1$ [3], respectively. Interestingly in [5], Brooksbank proved a 'rank reduction theorem' which can be applied on known string C-groups to obtain C-strings of the same group with smaller ranks, and consequently it was shown that $\mathrm{O}^{\pm}(2m, 2^k)$ and $\mathrm{Sp}(2m, 2^k)$ have string C-group representations of rank r for every $3 \leqslant r \leqslant 2m$ and $3 \leqslant r \leqslant 2m+1$, respectively. For recent work on polytopes of rank $n/2$ for a transitive group of degree n see [15]. Finally, we mention a further study in which two interesting families of C-strings for B_n are constructed [28].

This paper is structured as follows. In Section 2, we state some preliminary results on Coxeter groups and string C-groups. Lemma 4 plays an important role as our investigations into $W \cong \mathrm{D}_n$ are conducted in $\mathrm{Sym}(2n)$. Section 3 begins with Theorem 12 showing that $r_{\max}(W) < n$ when n is even. The remainder of this section focuses on constructing various C-strings for W and then Lemmas 9 and 10 together with Theorem 11 are called upon. Our final section gives a census of abstract regular polytopes for the exceptional finite irreducible Coxeter groups.

2 Preliminaries

We recall that a *Coxeter group* is a group W with presentation $\langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} \rangle$ where $m_{ii} = 1$ and $m_{ij} = m_{ji} \geqslant 2$ is a positive integer or infinity for every $1 \leqslant i < j \leqslant n$ (infinity being read as no relation). The *Coxeter rank* of W is n. Associated with this presentation is the *Coxeter diagram* where the nodes correspond to the s_i with a bond between s_i and s_j if $i \neq j$ and $m_{ij} > 2$. If this diagram is connected, then W is said to be an *irreducible* Coxeter group. Coxeter [12] (see also [18]) classified the finite irreducible Coxeter groups. There are four infinite families denoted A_n, B_n, D_n, $\mathrm{I}_2(m)$, and six exceptional groups H_3, H_4, F_4, E_6, E_7, E_8 (we shall blur the distinction between the group and its root system).

As already mentioned in Section 1, we shall focus on the Coxeter groups D_n and the exceptional groups. Suppose W is isomorphic to D_n, where $n > 4$. Then $W = SN$ where $S \cong \mathrm{Sym}(n)$ and N is a normal subgroup of W with N an elementary abelian subgroup of order 2^{n-1}. The subgroup N will sometimes be called the subgroup of (even) sign changes. In this paper we frequently view W as a subgroup of $\mathrm{Sym}(2n)$, as specified in our first lemma.

Lemma 4. *[18, (2.10)] Let $\beta_0 = (1, n+1)(2, n+2)$ and $\beta_i = (i, i+1)(n+i, n+i+1)$ for every $1 \leqslant i \leqslant n-1$ be permutations in* $\mathrm{Sym}(2n)$*. Then $\langle \beta_0, \beta_1, \ldots, \beta_{n-1} \rangle$ is isomorphic to* D_n.

We mention that the permutation representation of D_n in Lemma 4 may also be viewed as its action on the $2n$ facets of the n-cube (or equivalently, the $2n$ vertices of the n-crosspolytope).

In a similar fashion, we also have the well-known characterization of $\mathrm{Sym}(n)$.

Lemma 5. *[18, (6.4)] Let G be a group with presentation $\langle s_1, \ldots, s_{n-1} \mid (s_i s_j)^{m_{ij}} \rangle$ where $m_{ii} = 1$, $m_{ij} = 2$ if $|i-j| \geqslant 2$ and $m_{ij} = 3$ if $|i-j| = 1$ for every $1 \leqslant i, j \leqslant n-1$. Then G is isomorphic to* $\mathrm{Sym}(n)$.

Observe that in Lemma 4, $W \cong \mathrm{D}_n$ is a transitive subgroup of $\mathrm{Sym}(2n)$. Also note, using the notation from Lemma 4 that $\langle \beta_1, \ldots, \beta_{n-1} \rangle \cong \mathrm{Sym}(n)$, while products of an even number of transpositions of the form $(i, n+i)$, for some $1 \leqslant i \leqslant n$ yields the normal elementary abelian subgroup of sign changes. The next lemma will be used frequently in Section 3.

Lemma 6. *Suppose that $W \cong \mathrm{D}_n$ with N being the subgroup of sign changes. If $H \leqslant W$ is such that $W = HN$ and $H \cap N \nleqslant Z(W)$, then $W = H$.*

Proof. Since N is abelian and $W = HN$, $H \cap N \trianglelefteq W$. The only non-trivial normal subgroup of W contained in N is $Z(W)$ (if n is even) and N. Thus, as $H \cap N \nleqslant Z(W)$, $H \cap N = N$ which implies $H = HN = W$. $\square$

Now let G be a group generated by involutions $s_1, \ldots, s_r$ with the ordering such that $s_i s_j = s_j s_i$ (or equivalently, $(s_i s_j)^2 = 1$) if $|i-j| \geqslant 2$ for every $i, j \in I = \{1, \ldots, r\}$. Then $(G, \{s_1, \ldots, s_r\})$ is said to be a *string group generated by involutions*. Set $G_J = \langle s_j \mid j \in J \rangle$ for every $J \subseteq I$, and sometimes we may write $G_{i_1,\ldots,i_k}$ in place of $G_{\{i_1,\ldots,i_k\}}$. Its *Schläfli type* is the sequence $\{p_1, \ldots, p_{r-1}\}$ where p_i is the order of $s_i s_{i+1}$ for every $1 \leqslant i \leqslant r-1$. If a symbol p_i appears k times in adjacent places in the sequence we may write p_i^k in place of writing p_i k times. So a string group generated by involutions will be a string C-group if the intersection condition also holds.

A set R of elements of a group G is *independent* if for every $g \in R$, $g \notin \langle R \setminus \{g\} \rangle$. Let $\mu(G)$ denote the maximum size of an independent subset of G.

Theorem 7. *Suppose that $G \cong \mathrm{Sym}(m)$. Then*

(i) $\mu(G) \leqslant m-1$.

(ii) Assume that $m \geqslant 5$. If (G, S) is a string group generated by involutions and S is an independent set of size $m-1$, then S is the set of Coxeter generators for G.

Proof. (i) This is a theorem of Whiston [30]. (ii) This follows from Cameron and Cara [8] which classifies independent sets of size $m-1$ in $\mathrm{Sym}(m)$ when $m \geqslant 7$, while $m = 5, 6$ can be checked using Magma [1]. $\square$

Theorem 7 is deployed in the proof of Theorem 12, as is our next theorem which gives an upper bound for the rank of a C-string of transitive permutation groups. This is due to Cameron, Fernandes, Leemans and Mixer and is as follows.

Theorem 8. *[9, Theorem 1.2] Let G be a string C-group of rank r which is isomorphic to a transitive subgroup of* $\mathrm{Sym}(n)$ *other than* $\mathrm{Sym}(n)$ *or* $\mathrm{Alt}(n)$. *Then one of the following holds.*

(i) $r \leqslant n/2$.

(ii) $n \equiv 2 \pmod 4$ *and* $G \cong \mathrm{B}_n \cong C_2 \wr \mathrm{Sym}(n/2)$ *with Schläfli type* $\{2, 3^{n/2-2}, 4\}$.

(iii) G *is imprimitive and is one of the following.*

(a) $G = 6\mathrm{T}9$ *with Schläfli type* $\{3, 2, 3\}$.

(b) $G = 6\mathrm{T}11$ *with Schläfli type* $\{2, 3, 3\}$.

(c) $G = 6\mathrm{T}11$ *with Schläfli type* $\{2, 3, 4\}$.

(d) $G = 8\mathrm{T}45$ *with Schläfli type* $\{3, 4, 4, 3\}$.

Here, nTj *is the* j*-th conjugacy class among transitive subgroups of* $\mathrm{Sym}(n)$ *according to Butler and McKay [7].*

(iv) G *is primitive. In this case,* $n = 6$ *and* G *is obtained from the degree six permutation representation of* $\mathrm{Sym}(5) \cong \mathrm{PGL}_2(5)$ *and is the group of the 4-simplex of Schläfli type* $\{3, 3, 3\}$.

We will also make use of the following lemmas to verify that the generating set of involutions we construct indeed give us string C-groups.

Lemma 9. *[27, 2E16(a) and 11A10] Let* $(G, \{s_1, \ldots, s_r\})$ *be a string group generated by involutions where* $G_{1,\ldots,r-1}$ *and* $G_{2,\ldots,r}$ *are string C-groups. Then* G *is a string C-group if either of the following conditions are satisfied.*

(i) $G_{1,\ldots,r-1} \cap G_{2,\ldots,r} = G_{2,\ldots,r-1}$.

(ii) $s_r \notin G_{1,\ldots,r-1}$ *and* $G_{2,\ldots,r-1}$ *is maximal in* $G_{2,\ldots,r}$.

Lemma 10. *[27, 2E16(b)] Let* $(G, \{s_1, \ldots, s_r\})$ *be a string group generated by involutions such that* $G_{1,\ldots,r-1}$ *is a string C-group and* $G_{1,\ldots,r-1} \cap G_{k,\ldots,r} = G_{k,\ldots,r-1}$ *for every* $2 \leqslant k \leqslant r-1$*. Then* G *is also a string C-group.*

Once we have constructed a C-string of maximal rank, we will use the following rank reduction theorem by Brooksbank and Leemans to obtain C-strings of smaller ranks.

Theorem 11. *[5, Theorem 1.1 and Corollary 1.3] Let* $(G, \{s_1, \ldots, s_n\})$ *be a non-degenerate string C-group of rank* $n \geqslant 4$ *with Schläfli type* $\{p_1, \ldots, p_{n-1}\}$*. If* $s_1 \in \langle s_1 s_3, s_4 \rangle$*, then* $(G, \{s_2, s_1 s_3, s_4, \ldots, s_n\})$ *is a string C-group of rank* $n-1$*. In particular,* G *has a C-string of rank* $n-i$ *for every* $0 \leqslant i \leqslant t$*, where*

$$t = \max\{j \in \{0, \ldots, n-3\} \mid \textit{for all } i \in \{0, \ldots, j\},\ p_{3+i} \textit{ is odd}\}.$$

3 C-Strings for D_n

Theorem 12. *Let n be even with $n \geqslant 6$. If $W \cong \mathrm{D}_n$, then $r_{\max}(W) < n$.*

Proof. By Lemma 4 we may regard W as a subgroup of $\mathrm{Sym}(2n)$ and hence, appealing to Theorem 8, $r_{\max}(W) \leqslant n$. So we must show that W has no C-strings of rank n. Assuming that W has a C-string S with $|S| = n$, we seek a contradiction. Let $S = \{s_1, \ldots, s_n\}$ with $s_i s_j = s_j s_i$ for all $i, j \in I = \{1, \ldots, n\}$ with $|i - j| \geqslant 2$. Recalling the bar convention, put $\overline{W} = W/N$ where N is the subgroup of W consisting of the even sign changes. So $\overline{W} \cong \mathrm{Sym}(n)$.

(12.1) For $i \in I, s_i \notin Z(W)$.

Suppose that $s_i \in Z(W)$ and set $H = \langle S \setminus \{s_i\}\rangle$. Then $W/Z(W) = HZ(W)/Z(W)$ and therefore $HZ(W) = W = HN$ with $H \cap N \nleqslant Z(W)$. Thus $W = H$ by Lemma 6, against S satisfying the intersection condition. This proves (12.1).

Let $T \subseteq S$ be such that $\overline{T}$ is an independent generating set for $\overline{W}$. Hence $|\overline{T}| \leqslant n - 1$ by Theorem 7(i). Further $\overline{T}$ is a connected string in $\overline{W}$. For if not then $\overline{T} = \overline{T}_1 \cup \overline{T}_2$, $T_i \subseteq S$ with $[\overline{T}_1, \overline{T}_2] = 1$ and $\overline{T}_1 \neq \varnothing \neq \overline{T}_2$. But then $\langle \overline{T}_1\rangle\langle\overline{T}_2\rangle = \overline{W}$ with $[\langle\overline{T}_1\rangle, \langle\overline{T}_2\rangle] = 1$, contrary to $\overline{W} \cong \mathrm{Sym}(n)$.

(12.2) $|\overline{T}| = n - 1$ or $n - 2$. Moreover, if $|\overline{T}| = n - 2$, then $\overline{T} = \{\overline{s}_2, \ldots, \overline{s}_{n-1}\}$.

Suppose that $|\overline{T}| \leqslant n-3$. Then there exists $s \in S$ such that $[T, s] = 1$. From $\langle\overline{T}\rangle = \overline{W}$, we get $\overline{s} \in Z(\overline{W})$ and then, as $\overline{W} \cong \mathrm{Sym}(n)$, we have $s \in N$. Hence, using (12.1) and Lemma 6, $W = \langle T \cup \{s\}\rangle$ which is impossible as S satisfies the intersection property. Thus $|\overline{T}| = n - 1$ or $n - 2$. A similar argument applies to show $\overline{T} = \{\overline{s}_2, \ldots \overline{s}_{n-1}\}$ when $|\overline{T}| = n - 2$.

(12.3) $|\overline{T}| \neq n - 1$.

Suppose that $|\overline{T}| = n-1$. We may assume, as $\overline{T}$ is connected, that $\overline{T} = \{\overline{s}_1, \ldots, \overline{s}_{n-1}\}$. Put $\overline{X} = \langle\overline{s}_1, \ldots, \overline{s}_{n-2}\rangle$. By Theorem 7(ii) $\overline{T}$ consists of the Coxeter generators for $\overline{W}$. Thus $\overline{X} \cong \mathrm{Sym}(n-1)$. Because $[s_n, s_i] = 1$ for $i = 1, \ldots, n-2$, $\overline{s}_n \in C_{\overline{W}}(\overline{X})$. Since $C_{\overline{W}}(\overline{X}) = 1$, $s_n \in N$. Now $n - 1$ being odd implies $C_N(\overline{X}) = Z(W)$ giving $s_n \in Z(W)$, against (12.1). This rules out $|\overline{T}| = n - 1$.

(12.4) $|\overline{T}| \neq n - 2$.

Assume that $|\overline{T}| = n - 2$. By (12.2) we have $\overline{T} = \{\overline{s}_2, \ldots, \overline{s}_{n-1}\}$. Put $X = \langle s_2, \ldots, s_{n-1}\rangle$ and $X_1 = \langle s_1, s_2, \ldots, s_{n-1}\rangle$. Then $W = XN = X_1N$ and, as $X \leqslant X_1, X_1 = X(X_1 \cap N)$. If $X_1 \cap N = 1$, then $X_1 \cong \mathrm{Sym}(n)$ and so $X = X_1$. But this contradicts the intersection condition. Thus $X_1 \cap N \neq 1$. If $X_1 \cap N \neq Z(W)$, then Lemma 6 forces

$X_1 = W$, also a contradiction. Hence $X_1 \cap N = Z(W)$ and consequently $X_1 = XZ(W)$. A similar argument applies for $X_n = \langle s_2, \ldots, s_{n-1}, s_n \rangle$ to also yield $X_n = XZ(W)$. But then

$$W = \langle s_1, \ldots, s_n \rangle \leqslant XZ(W) \neq W,$$

so proving (12.4).

Combining (12.2), (12.3) and (12.4) yields the desired contradiction, so establishing Theorem 12. $\square$

In the remainder of this section we construct examples of C-strings for D_n.

For every integer $n \geqslant 5$, we define $t_1, \ldots, t_n$ in $\mathrm{Sym}(2n)$ as follows.

$$t_1 = \prod_{j=2}^{n} (j, n+j),$$
$$t_i = (i-1, i)(n+i-1, n+i) \text{ for every } 2 \leqslant i \leqslant n.$$

Lemma 13. *For every $1 \leqslant i, j \leqslant n$, we have*

$$\textit{order of } t_i t_j = \begin{cases} 1, & \textit{if } \ i = j, \\ 2, & \textit{if } \ |i-j| \geqslant 2, \\ 3, & \textit{if } \ |i-j| = 1 \textit{ and } \{i,j\} \neq \{1,2\}, \\ 4, & \textit{if } \ \{i,j\} = \{1,2\}. \end{cases}$$

Proof. Each of the elements $t_1, \ldots, t_n$ are involutions as they are defined as products of pairwise disjoint transpositions. Note that

$$t_1 t_2 = (1, 2, n+1, n+2) \prod_{j=3}^{n} (j, n+j),$$
$$t_i t_{i+1} = (i-1, i+1, i)(n+i-1, n+i+1, n+i) \text{ for every } 2 \leqslant i \leqslant n.$$

For every $2 \leqslant i, j \leqslant n$ with $|i-j| \geqslant 2$, t_i and t_j are disjoint, whereas for every $3 \leqslant k \leqslant n$, note that

$$t_1 t_k = (k-1, n+k)(k, n+k-1) \prod_{\substack{2 \leqslant j \leqslant n \\ j \notin \{k-1, k\}}} (j, n+j).$$

As such, $t_i t_j$ is a product of pairwise disjoint transpositions for every $1 \leqslant i, j \leqslant n$ with $|i-j| \geqslant 2$. The lemma follows from examining each of the products $t_i t_j$ in disjoint cycle notation. $\square$

Lemma 14. *$\langle t_1, \ldots, t_n \rangle$ is a string C-group.*

Proof. We prove by induction on k that $\langle t_1, \ldots, t_k \rangle$ is a string C-group for every $3 \leqslant k \leqslant n$. For the base case $k = 3$, it is easy to verify that $\langle t_1, t_2 \rangle \cap \langle t_2, t_3 \rangle = \langle t_2 \rangle$. So now suppose that $k > 3$ and that $\langle t_1, \ldots, t_{k-1}\rangle$ is a string C-group. Now by Lemmas 5 and 13, $\langle t_2, \ldots, t_k\rangle \cong \mathrm{Sym}(k)$ is a string C-group and $\langle t_2, \ldots, t_{k-1}\rangle \cong \mathrm{Sym}(k-1)$ which is maximal in $\langle t_2, \ldots, t_k\rangle$. We also have $t_k \notin \langle t_1, \ldots, t_{k-1}\rangle$ because t_k moves $k-1$ to k, which are in different $\langle t_1, \ldots, t_k\rangle$-orbits. It follows from Lemma 9 that $\langle t_1, \ldots, t_k\rangle$ is a string C-group. $\square$

Lemma 15. *Let n be odd and $n \geqslant 5$. Then $\langle t_1, \ldots, t_n\rangle$ is isomorphic to D_n.*

Proof. Let $G = \langle t_1, \ldots, t_n\rangle$, $H = \langle \beta_0, \beta_1, \ldots, \beta_{n-1}\rangle$ where $\beta_0 = (1, n+1)(2, n+2)$ and $\beta_i = (i, i+1)(n+i, n+i+1)$ for every $1 \leqslant i \leqslant n-1$. Then $H \cong \mathrm{D}_n$ by Lemma 4 and we want to show that $G = H$. Since $t_i = \beta_{i-1}$ for every $2 \leqslant i \leqslant n$, it suffices to show that $\beta_0 \in G$ and $t_1 \in H$.

We note that

$$t_1t_2 = (1, 2, n+1, n+2)\prod_{j=3}^{n}(j, n+j),$$

whence $\beta_0 = (1, n+1)(2, n+2) = (t_1t_2)^2 \in G$. Now let $\alpha_{i,j} = (i, n+i)(j, n+j)$ and $\gamma_{i,j} = (i, j)(n+i, n+j)$ for every $2 \leqslant i, j \leqslant n$. Note that $\alpha_{2,3} = \beta_0^{\beta_2\beta_1} \in H$. Now for $4 \leqslant k \leqslant n$, we have

$$\gamma_{2,k} = {\beta_2}^{\prod_{j=4}^{k}\beta_{j-1}} \text{ and } \gamma_{3,k} = {\beta_2}^{\gamma_{2,k}}.$$

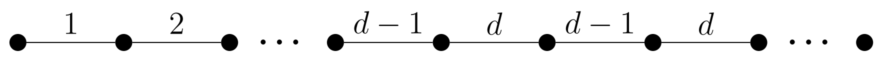
It follows that for every $4 \leqslant i, j \leqslant n$, we have $\alpha_{i,j} = {\alpha_{2,3}}^{\gamma_{2,i}\gamma_{3,j}} \in H$ and hence, $t_1 = \alpha_{2,3}\alpha_{4,5}\cdots\alpha_{n-1,n} \in H$. $\square$

Theorem 16. *Let n be odd and $n \geqslant 5$. Then $\{t_1, \ldots, t_n\}$ is a rank n C-string of D_n with Schläfli type $\{4, 3^{n-2}\}$. Therefore, $r_{\max}(\mathrm{D}_n) = n$ and there is a rank r C-string of D_n for every $3 \leqslant r \leqslant n$.*

Proof. The first assertion follows from Lemmas 13, 14 and 15. The second and third assertion follows from Theorem 8 and Theorem 11, respectively. $\square$

To describe our C-strings, we will use an interesting graph called the string C-group permutation representation (CPR) graph, first introduced in [29]. Let $(G, \{s_1, \ldots, s_r\})$ be a string C-group identified as a permutation group of degree m. Then the *CPR graph* of G is an r-edge-labelled multigraph with vertex set $\{1, \ldots, m\}$ such that $\{i, j\}$ is an edge of label k if s_k moves i to j. Recall the notation for $W \cong \mathrm{D}_n$ when $W \leqslant \mathrm{Sym}(2n)$ as described in Section 2. That is, $W = SN$ where $S \cong \mathrm{Sym}(n)$ and N is the subgroup of even sign changes.

Theorem 17. *[13, Lemma 21] Let $n \geqslant 5$ and $3 \leqslant d \leqslant n-2$. Then $\mathrm{Sym}(n)$ has a C-string of rank d with Schläfli type $\{3^{d-3}, 6, n-d+2\}$ whose CPR graph can be obtained by appending $n-d$ vertices and edges with labels $d-1$ and d alternatively to the CPR graph of the d-simplex, as follows.*



Lemma 18. *Let $G = \langle s_1, \ldots, s_r\rangle \leqslant W \leqslant \mathrm{Sym}(2n)$ with $W \cong \mathrm{D}_n$ be a string group generated by involutions. If $G_{1,\ldots,r-1} = S \cong \mathrm{Sym}(n)$ is a string C-group and $s_r \in N\backslash Z(G)$, then $G = W$ and is also a string C-group.*

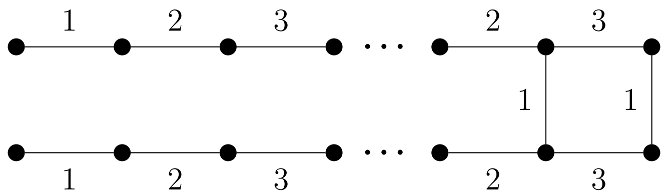
Proof. By Lemma 6 we have $G = W$. Let $2 \leqslant k \leqslant r-1$. Since $s_r \in N$, $G_{k,\ldots,r} \leqslant G_{k,\ldots,r-1}N$, and so $G_{k,\ldots,r} = G_{k,\ldots,r-1}(G_{k,\ldots,r}\cap N)$ by the Dedekind law. Then by Dedekind law again,

$$\begin{aligned} G_{1,\ldots,r-1}\cap G_{k,\ldots,r} &= G_{1,\ldots,r-1}\cap (G_{k,\ldots,r-1}(G_{k,\ldots,r}\cap N)) \\ &= G_{k,\ldots,r-1}(G_{1,\ldots,r-1}\cap G_{k,\ldots,r}\cap N) \\ &= G_{k,\ldots,r-1}, \end{aligned}$$

as $G_{1,\ldots,r-1}\cap N = S\cap N = 1$. Thus the lemma follows from Lemma 10. □

Let $n \geqslant 6$ be an even integer for the remainder of this section. We first construct rank three C-strings of D_n, defining t_1, t_2, t_3 in $\mathrm{Sym}(2n)$ as follows.

$$\begin{aligned} t_1 &= (1,2)(n+1,n+2)(n-1,2n-1)(n,2n), \\ t_2 &= (2,3)(4,5)\cdots(n-2,n-1)(n+2,n+3)(n+4,n+5)\cdots(2n-2,2n-1), \\ t_3 &= (3,4)(5,6)\cdots(n-1,n)(n+3,n+4)(n+5,n+6)\cdots(2n-1,2n). \end{aligned}$$



Lemma 19. *$\{t_1, t_2, t_3\}$ is a rank 3 C-string of D_n with Schläfli type $\{12, n-1\}$.*

Proof. Each of the elements t_1, t_2, t_3 are involutions as they are defined as products of pairwise disjoint transpositions. Note that

$$\begin{aligned} t_1t_2 &= (1,3,2)(n-1,2n-2,2n-1,n-2)(n,2n) \\ &\quad\ (4,5)\cdots(n-4,n-3)(n+4,n+5)\cdots(2n-4,2n-3), \\ t_2t_3 &= (2,4,\ldots,n,n-1,n-3,\ldots,3)(n+2,n+4,\ldots,2n,2n-1,2n-3,\ldots,n+3), \\ t_1t_3 &= (1,2)\cdots(n-3,n-2)(n-1,2n)(n,2n-1). \end{aligned}$$

Since $\langle t_1, t_2\rangle \cong \mathrm{Dih}(24)$, its elements are of the form ${t_1}^{\varepsilon}(t_1t_2)^k$ for some integers $\varepsilon \in \{0,1\}$ and $0 \leqslant k \leqslant 11$. Also since $\langle t_2, t_3\rangle \leqslant S\cap \mathrm{Stab}_{\mathrm{Sym}(2n)}(1)$, we have

$$\begin{aligned} \langle t_1,t_2\rangle \cap \langle t_2,t_3\rangle &\leqslant \langle t_1,t_2\rangle \cap \big(S\cap \mathrm{Stab}_{\mathrm{Sym}(2n)}(1)\big) \\ &= (\langle t_1,t_2\rangle \cap S)\cap \big(\langle t_1,t_2\rangle \cap \mathrm{Stab}_{\mathrm{Sym}(2n)}(1)\big) \\ &= \langle t_1,(t_1t_2)^4\rangle \cap \langle t_1,(t_1t_2)^3\rangle = \langle t_2\rangle. \end{aligned}$$

It follows that $\langle t_1, t_2\rangle \cap \langle t_2, t_3\rangle = \langle t_2\rangle$ and so $\langle t_1, t_2, t_3\rangle$ is a string C-group with Schläfli type $\{12, n-1\}$. We now show that $H = \langle t_1, t_2, t_3\rangle$ is isomorphic to D_n. Note that $\overline{H} = S$ by Theorem 17 and so $HN = \mathrm{D}_n$. Since

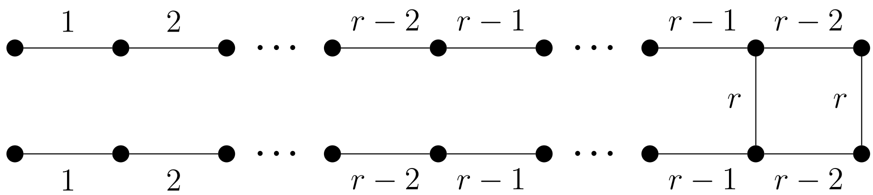
$$(n-2, 2n-2)(n-1, 2n-1) = (t_1t_2)^6 \in N$$

and $(n-2, 2n-2)(n-1, 2n-1) \notin Z(\mathrm{D}_n)$. Then calling upon Lemma 6 yields $H = \mathrm{D}_n$. $\square$

We now construct rank r C-strings of D_n for every $4 \leqslant r \leqslant n-1$ as follows. We use Theorem 17 to construct a rank $r-1$ C-string for the subgroup $S \cong \mathrm{Sym}(n)$ of D_n, and then append an element in N. We note that the construction depends on the parity of r.

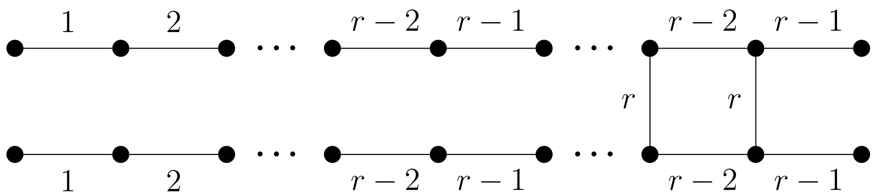
If r is odd, we define $t_1, \ldots, t_r$ in $\mathrm{Sym}(2n)$ as follows.

$$\begin{aligned}
t_i &= (i, i+1)(n+i, n+i+1) \text{ for every } 1 \leqslant i \leqslant r-3,\\
t_{r-2} &= (r-2, r-1)(r, r+1)\cdots(n-1, n)\\
&\qquad (n+r-2, n+r-1)(n+r, n+r+1)\cdots(2n-1, 2n),\\
t_{r-1} &= (r-1, r)(r+1, r+2)\cdots(n-2, n-1)\\
&\qquad (n+r-1, n+r)(n+r+1, n+r+2)\cdots(2n-2, 2n-1),\\
t_r &= (n-1, 2n-1)(n, 2n).
\end{aligned}$$



If r is even, we define $t_1, \ldots, t_r$ in $\mathrm{Sym}(2n)$ as follows.

$$\begin{aligned}
t_i &= (i, i+1)(n+i, n+i+1) \text{ for every } 1 \leqslant i \leqslant r-3,\\
t_{r-2} &= (r-2, r-1)(r, r+1)\cdots(n-2, n-1)(n+r-2, n+r-1)\\
&\qquad (n+r, n+r+1)\cdots(2n-2, 2n-1),\\
t_{r-1} &= (r-1, r)(r+1, r+2)\cdots(n-1, n)(n+r-1, n+r)\cdots(2n-1, 2n),\\
t_r &= (n-2, 2n-2)(n-1, 2n-1).
\end{aligned}$$



Lemma 20. *Let n be even and $n \geqslant 6$. Then for every $4 \leqslant r \leqslant n-1$, $\{t_1, \ldots, t_r\}$ is a rank r C-string of D_n with Schläfli type $\{3^{r-4}, 6, n-r+3, 4\}$.*

Proof. Each of the elements $t_1, \ldots, t_r$ are involutions as they are defined as products of pairwise disjoint transpositions. Note that for every $1 \leqslant k \leqslant r-3$, we have

$$t_k t_r = \begin{cases} (k,k+1)(n+k,n+k+1)(n-1,2n-1)(n,2n) & \text{, if } r \text{ is odd} \\ (k,k+1)(n+k,n+k+1)(n-2,2n-2)(n-1,2n-1) & \text{, if } r \text{ is even} \end{cases},$$

which are products of pairwise disjoint transpositions because

$$k+1 \leqslant (r-3)+1 \leqslant ((n-1)-3)+1 = n-3 < n-2.$$

Likewise, if r is odd we have

$$\begin{aligned} t_{r-2}t_r &= (n-1,2n)(n,2n-1)(r-2,r-1)(r,r+1)\cdots(n-3,n-2) \\ &\quad (n+r-2,n+r-1)(n+r,n+r+1)\cdots(2n-3,2n-2), \end{aligned}$$

whereas if r is even we have

$$\begin{aligned} t_{r-2}t_r &= (n-2,2n-1)(n-1,2n-2)(r-2,r-1)(r,r+1)\cdots(n-4,n-3) \\ &\quad (n+r-2,n+r-1)(n+r,n+r+1)\cdots(2n-4,2n-3), \end{aligned}$$

which are also products of pairwise disjoint transpositions. Finally, we have

$$\begin{aligned} t_{r-1}t_r &= (n-1,n-2,2n-1,2n-2)(n,2n) \\ &\quad (r-1,r)(r+1,r+2)\cdots(n-4,n-3) \\ &\quad (n+r-1,n+r)(n+r+1,n+r+2)\cdots(2n-4,2n-3) \end{aligned}$$

for r odd and we also have

$$\begin{aligned} t_{r-1}t_r &= (n-2,n-3,2n-2,2n-3)(n-1,n,2n-1,2n) \\ &\quad (r-1,r)(r+1,r+2)\cdots(n-5,n-4) \\ &\quad (n+r-1,n+r)(n+r+1,n+r+2)\cdots(2n-5,2n-4), \end{aligned}$$

for r even. It follows from Lemma 18 that $\langle t_1, \ldots, t_r\rangle = \mathrm{D}_n$ is a string C-group with Schläfli type $\{3^{r-4}, 6, n-r+3, 4\}$. ☐

Theorem 21. *Let n be even and $n \geqslant 6$. Then $r_{\max}(\mathrm{D}_n) = n-1$ and there is a rank r C-string of D_n for every $3 \leqslant r \leqslant n-1$.*

Proof. The theorem follows from Lemmas 19, 20 and Theorem 12. ☐

Together Theorems 12, 16 and 21 prove Theorems 1 and 2.

4 Exceptional Finite Irreducible Coxeter Groups

The following table lists the number of abstract regular polytopes (up to isomorphism and duality with the number of self-dual polytopes in brackets) for each exceptional Coxeter groups computed using Magma [1].

Table 1: Number of polytopes for each exceptional Coxeter group.

Group	Total	Rank 3	Rank 4	Rank 5	Rank 6	Rank 7	Rank $\geqslant 8$
H_3	8(1)	8(1)	0	0	0	0	0
H_4	59(6)	45(2)	14(4)	0	0	0	0
F_4	5(1)	3(0)	2(1)	0	0	0	0
E_6	147(18)	87(12)	50(4)	10(2)	0	0	0
E_7	3662(10)	1577(10)	1525(0)	465(0)	95(0)	0	0
E_8	11689(142)	6746(117)	3584(22)	986(2)	310(0)	63(1)	0

References

[1] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3–4):235–265, 1997.

[2] P. A. Brooksbank. On the ranks of string C-group representations for symplectic and orthogonal groups. *Polytopes and Discrete Geometry. Contemp. Math.*, 764:31–41, 2021.

[3] P. A. Brooksbank, J. T. Ferrara, and D. Leemans. Orthogonal groups in characteristic 2 acting on polytopes of high rank. *Discrete Comput. Geom.*, 63(3):656–669, 2020.

[4] P. A. Brooksbank and D. Leemans. Polytopes of large rank for $\mathrm{PSL}(4, \mathbb{F}_q)$. *J. Algebra*, 452:390–400, 2016.

[5] P. A. Brooksbank and D. Leemans. Rank reduction of string C-group representations. *Proc. Amer. Math. Soc.*, 147(12):5421–5426, 2019.

[6] P. A. Brooksbank and D. A. Vicinsky. Three-dimensional classical groups acting on polytopes. *Discrete Comput. Geom.*, 44(3):654–659, 2010.

[7] G. Butler and J. McKay. The transitive groups of degree up to eleven. *Comm. Algebra*, 11(8):863–911, 1983.

[8] P. J. Cameron and P. Cara. Independent generating sets and geometries for symmetric groups. *J. Algebra*, 258(2):641–650, 2002.

[9] P. J. Cameron, M. E. Fernandes, D. Leemans, and M. Mixer. String C-groups as transitive subgroups of S_n. *J. Algebra*, 447:468–478, 2016.

[10] P. J. Cameron, M. E. Fernandes, D. Leemans, and M. Mixer. Highest rank of a polytope for A_n. *Proc. Lond. Math. Soc. (3)*, 115(1):135–176, 2017.

[11] T. Connor, J. De Saedeleer, and D. Leemans. Almost simple groups with socle $\mathrm{PSL}(2, q)$ acting on abstract regular polytopes. *J. Algebra*, 423:550–558, 2015.

[12] H. S. M. Coxeter. The complete enumeration of finite groups of the form $R_i^2 = (R_iR_j)^{k_{ij}} = 1$. *J. Lond. Math. Soc.*, 10(1):21–25, 1935.

[13] M. E. Fernandes and D. Leemans. Polytopes of high rank for the symmetric groups. *Adv. Math.*, 228(6):3207–3222, 2011.

[14] M. E. Fernandes and D. Leemans. String C-group representations of alternating groups. *Ars Math. Contemp.*, 17(1):291–310, 2019.

[15] M. E. Fernandes and C. A. Piedade. Regular polytopes of rank $n/2$ for transitive groups of degree n. 2024. Preprint, arXiv:2407.16003.

[16] M. I. Hartley. An atlas of small regular abstract polytopes. *Period. Math. Hungar.*, 53(1–2):149–156, 2006.

[17] M. I. Hartley and A. Hulpke. Polytopes derived from sporadic simple groups. *Contrib. Discrete Math.*, 5(2):106–118, 2010.

[18] J. E. Humphreys. *Reflection Groups and Coxeter Groups.* Cambridge University Press, 1990.

[19] D. Leemans. Almost simple groups of Suzuki type acting on polytopes. *Proc. Amer. Math. Soc.*, 134(12):3649–3651, 2006.

[20] D. Leemans. String C-group representations of almost simple groups: a survey. *Polytopes and Discrete Geometry*. *Contemp. Math.*, 764:157–178, 2021.

[21] D. Leemans and M. Mixer. Algorithms for classifying regular polytopes with a fixed automorphism group. *Contrib. Discrete Math.*, 7(2):105–118, 2012.

[22] D. Leemans and J. Mulpas. The string C-group representations of the Suzuki, Rudvalis and O'Nan sporadic groups. *Art. Discrete Appl. Math.*, 5(3):Article #P3.09, 12 pp., 2022.

[23] D. Leemans and E. Schulte. Groups of type $L_2(q)$ acting on polytopes. *Adv. Geom.*, 7(4):529–539, 2007.

[24] D. Leemans and E. Schulte. Polytopes with groups of type $\mathrm{PGL}_2(q)$. *Ars Math. Contemp.*, 2(2):163–171, 2009.

[25] D. Leemans, E. Schulte, and H. Van Maldeghem. Groups of Ree type in characteristic 3 acting on polytopes. *Ars Math. Contemp.*, 14(2):209–226, 2018.

[26] D. Leemans and L. Vauthier. An atlas of abstract regular polytopes for small groups. *Aequationes Math.*, 72(3):313–320, 2006.

[27] P. McMullen and E. Schulte. *Abstract Regular Polytopes.* Cambridge University Press, 2002.

[28] R. Nicolaides and P. Rowley. Two families of unravelled abstract regular polytopes in B_n. *J. Group Theory*, 25(6):1133–1148, 2022.

[29] D. Pellicer. CPR graphs and regular polytopes. *European J. Combin.*, 29(1):59–71, 2008.

[30] J. Whiston. Maximal independent generating sets of the symmetric group. *J. Algebra*, 232(1):255–268, 2000.