Forbidden subgraphs and complete partitions

John Byrne[a]   Michael Tait[b]   Craig Timmons[c]

Abstract

A graph is called an (r, k)-graph if its vertex set can be partitioned into r parts, each having at most k vertices and there is at least one edge between any two parts. Let $f(r, H)$ be the minimum k for which there exists an H-free (r, k)-graph. In this paper we build on the work of Axenovich and Martin, obtaining improved bounds on this function when H is a complete bipartite graph or an even cycle. Some of these bounds are best possible up to a constant factor and confirm a conjecture of Axenovich and Martin in several cases.

Mathematics Subject Classifications: 05C35, 05C25

1 Introduction

Let $r \geqslant 2$ and k be positive integers and write $[r]$ for $\{1, 2, \ldots, r\}$. A graph G is an (r, k)-*graph* if there is a partition of $V(G)$ into sets $V_1, V_2, \ldots, V_r$ such that $|V_i| \leqslant k$ for $1 \leqslant i \leqslant r$, and for all $1 \leqslant i < j \leqslant r$, there is an $e \in E(G)$ with one endpoint in V_i and the other in V_j. Given an (r, k)-graph G, if each V_i induces a connected subgraph, then contracting each V_i to a single vertex shows that K_r is a minor of G. Consequently, an (r, k)-graph with partition $V_1, \ldots, V_r$ is sometimes called a k-*split of* K_r. A more general notion of k-splits exists and the study of splits goes back to Heawood [22]. Originally graph splits were mostly studied with respect to topological properties (see for example [16]). More recently, the study of splits has been considered from an extremal perspective. There are several interesting questions one can ask, such as the largest complete minor in an H-free graph [7, 26] or in a graph without a sparse cut [25].

In another direction, if each V_i is an independent set, then the parts $V_1, \ldots, V_r$ give a proper vertex coloring of G. This leads to the achromatic number of a graph which we define now. A *complete r-coloring* of a graph G is a proper vertex coloring $c : V(G) \to [r]$

[a]Department of Mathematics, University of Delaware, Newark, Delaware, USA (jpbyrne@udel.edu).

[b]Department of Mathematics & Statistics, Villanova University, Villanova, Pennsylvania, USA (michael.tait@villanova.edu). Research, partially supported by NSF grants DMS-2011553 and DMS-2245556.

[c]Department of Mathematics and Statistics, California State University, Sacramento, California, USA (craig.timmons@csuc.edu). Research partially supported by Sac State (P3) Program.

such that between any two distinct color classes there is at least one edge. The *achromatic number* of G, denoted $\chi_a(G)$, is the maximum r for which G has a complete r-coloring. If there is no restriction on the V_i's, then we have a *pseudocomplete r-coloring* of G. Pseudocomplete colorings drop the restriction that the coloring must be proper, and we write $\psi(G)$ for the *pseudoachromatic number* of G. Complete and pseudocomplete colorings of graphs and their associated parameters have a large body of work devoted to them. Researchers have studied them for specific graph families [11, 13, 14, 29, 31] and hypergraphs [15, 21]. The surveys [12, 23] contain additional references.

Kostochka [24] and Thomason [33] used pseudocomplete colorings in their influential work on the minimum degree forcing a complete minor, so there are useful relationships between these two notions (see Section 3 of [33] or Bollobás, Reed, Thomason [6] for brief discussions). The focus of this paper is a function, considered by Barbanera and Ueckerdt [4] and studied further by Axenovich and Martin [3], that falls into the branch of extremal partitioning problems with (possibly) some condition on the parts. For a graph H, define

$$f(r, H) = \min\{k : \text{exists an } H\text{-free } (r, k)\text{-graph}\}.$$

Assuming that H has minimum degree at least 3, Axenovich and Martin give a simple construction that takes an H-free (r, k)-graph and produces an H-free graph with a K_r minor by adding at most $k + 1$ vertices and at most $2k$ edges to each V_i so that the new color classes are connected. Regarding coloring, if G is an (r, k)-graph, then $r \leqslant \psi(G)$. Furthermore, any complete or pseudocomplete r-coloring defines an (r, k)-partition for some k. Since we will consider only H-free graphs in this paper and removing edges cannot create a copy of H, any graph with partition $V_1, \ldots, V_r$ that we consider may be assumed to have each V_i an independent set. So bounds for $f(r, H)$ can imply bounds on achromatic numbers and pseudoachromatic numbers and vice versa. However, in this paper we want to minimize the largest size of a color class in a pseudocomplete r-coloring rather than maximizing the number of color classes. Because of these differing objectives, we phrase all of our work in the language of [3] instead of using the terminology from these graph coloring and minor problems. It would be interesting to explore these connections in more depth, but that is not the focus of this paper.

For a set $\mathcal{H}$ of graphs, write $\text{ex}(n, \mathcal{H})$ for the Turán number of $\mathcal{H}$. This is the maximum number of edges in an n-vertex graph that does not contain any graph in $\mathcal{H}$ as a subgraph. When $\mathcal{H} = \{H\}$, we write $\text{ex}(n, H)$ instead of $\text{ex}(n, \mathcal{H})$. A simple observation is that if G is an H-free (r, k)-graph, then

$$\binom{r}{2} \leqslant e(G) \leqslant \text{ex}(rk, H).$$

The first inequality is true since there is at least one edge between any two of the r parts. The second holds because G is H-free with at most rk vertices. For fixed r, this inequality gives a lower bound on k which implies a lower bound on $f(r, H)$. Any lower bound on $f(r, H)$ obtained using

$$\binom{r}{2} \leqslant \text{ex}(rk, H) \tag{1}$$

will be called the "trivial (lower) bound". The inequality $\binom{r}{2} \leqslant e(G)$ also holds if r is replaced by $\chi_a(G)$, $\psi(G)$, or the Hadwiger number of G which is the largest r such that K_r is a minor of G (see Proposition 1 of [26] for example).

If H is not bipartite, a simple argument given in [3] shows that $f(r, H) \leqslant 2$ for all $r \geqslant 1$. For this reason we will assume that H is bipartite. Supposing for a moment that $\text{ex}(N, H) = \Theta(N^\eta)$ for some $0 < \eta < 2$, the trivial lower bound implies $f(r, H) = \Omega(r^{2/\eta-1})$. Axenovich and Martin proved an upper bound that differs from this lower bound by a logarithmic factor under an assumption on the Turán number of H.

Theorem 1 (Axenovich, Martin [3]). *If H is a bipartite graph that is not a forest with $\text{ex}(N, H) = \Theta(N^\eta)$, then there are positive constants c and C such that for all r,*

$$cr^{2/\eta-1} \leqslant f(r, H) \leqslant Cr^{2/\eta-1} \log^{1/\eta} r.$$

They conjectured that the trivial lower bound (1) gives the correct order of magnitude. If true, it shows that the logarithmic factor in Theorem 1 can be removed.

Conjecture 2 (Axenovich, Martin [3]). *Let H be a bipartite graph that is not a forest. There is a positive constant $c = c(H)$ such that for any positive integers r and k for which $\text{ex}(rk, H) > c\binom{r}{2}$, there is an H-free (r, k)-graph, i.e., $f(r, H) < k$.*

Using the incidence graph of an affine plane, they proved that Conjecture 2 is true for C_4.

Theorem 3 (Axenovich, Martin [3]). *The function $f(r, C_4)$ satisfies*

$$r^{1/3} - o(r^{1/3}) \leqslant f(r, C_4) \leqslant 2r^{1/3} + o(r^{1/3}).$$

While this paper was under review, Taranchuk and the third author [32] removed the factor of 2 in the upper bound and so $f(r, C_4) = (1 + o(1))r^{1/3}$. Now for any $d \geqslant 1$, C_4 is a subgraph of $K_{2,d+1}$ so that $f(r, K_{2,d+1}) \leqslant f(r, C_4) = (1 + o(1))r^{1/3}$. Combining this with the asymptotic formula $\text{ex}(N, K_{2,d+1}) = \frac{\sqrt{d}}{2}N^{3/2} + o(N^{3/2})$, proved by Füredi [18], and the trivial bound (1) gives

$$\frac{1}{d^{1/3}}r^{1/3} - o(r^{1/3}) \leqslant f(r, K_{2,d+1}) \leqslant r^{1/3} + o(r^{1/3}).$$

For fixed $d \geqslant 2$, Theorem 3 proves Conjecture 2 for $H = K_{2,d+1}$. However, this result does not capture the dependence of $f(r, K_{2,d+1})$ on d. Using the projective norm graphs, we prove an upper bound on $f(r, K_{2,d+1})$ to one that gives the correct dependence on d. Additionally, we confirm Conjecture 2 for $K_{t,s}$ whenever $s > (t - 1)!$.

Theorem 4. *(a) For $d \geqslant 3$,*

$$\frac{1}{d^{1/3}}r^{1/3} - o(r^{1/3}) \leqslant f(r, K_{2,d+1}) \leqslant \frac{2}{d^{1/3}} \cdot \frac{1}{(1 - 1.5d^{-1/2})^{5/3}}r^{1/3} + o(r^{1/3}).$$

(b) For $t \geqslant 3$,

$$\frac{1}{((t-1)! - t)^{1/(2t-1)}}r^{1/(2t-1)} - o(r^{1/(2t-1)}) \leqslant f(r, K_{t,(t-1)!+1}) \leqslant 2r^{1/(2t-1)} + o(r^{1/(2t-1)}).$$

Unless $d > 34$, the upper bound in part (a) of Theorem 4 does not improve upon the bound $f(r, K_{2,d+1}) \leqslant (1 + o(1))r^{1/3}$.

Next, we turn our attention towards even cycles by confirming Conjecture 2 for C_6 and C_{10}.

Theorem 5. *For the even cycles C_6 and C_{10},*

$$f(r, C_6) \leqslant 2r^{1/2} + o(r^{1/2}) \quad and \quad f(r, C_{10}) \leqslant 2r^{2/3} + o(r^{2/3}).$$

As in the case of C_4 and inspired by our approach, the upper bounds were improved, while this paper was being reviewed, to $f(r, C_6) \leqslant r^{1/2} + o(r^{1/2})$ and $f(r, C_{10}) \leqslant r^{2/3} + o(r^{2/3})$ in [32].

The graphs used to prove Theorem 5 were constructed by Wenger [35]. This family of graphs is not $C_{2\ell}$-free when $\ell \neq 2, 3, 5$. One of the more notable unsolved problems in extremal graph theory is to determine the order of magnitude of $\text{ex}(n, C_{2\ell})$ for $\ell \notin \{2, 3, 5\}$. With the order of magnitude of $\text{ex}(N, C_{2\ell})$ not known in these cases, one cannot apply Theorem 1 directly. The authors of [3] also proved the following result which relaxes the assumption $\text{ex}(N, H) = \Theta(N^\eta)$.

Theorem 6 (Axenovich, Martin [3])**.** *Let H be a bipartite graph that is not a forest and such that for any sufficiently large N, $CN^a \leqslant \text{ex}(N, H) \leqslant C'N^b$ for positive constants C, C' and exponents a, b with $b - a < \frac{(2-b)(b-1)}{5-b}$.*

- *If $\text{ex}(nk, H) \geqslant 12n^2 \log n$, then $f(n, H) \leqslant k + o(k)$.*

- *If $\text{ex}(nk, H) < \binom{n}{2}$, then $f(n, H) \geqslant k$.*

The densest constructions of $C_{2\ell}$-free graphs are due to Lazebnik, Ustimenko, and Woldar [27]. They show that

$$\text{ex}(N, C_{2\ell}) = \Omega(N^{1 + \frac{2}{3\ell - 3 + \varepsilon}})$$

where $\varepsilon = 0$ if k is odd and $\varepsilon = 1$ if k is even. Here the exponent in the best known lower bound is too far from that of the best known upper bound $\text{ex}(N, C_{2\ell}) = O(N^{1 + \frac{1}{\ell}})$ to satisfy the assumption $b - a < \frac{(2-b)(b-1)}{5-b}$. However, in the proof of Theorem 6 the only use of this assumption is to obtain an H-free graph whose maximum degree is not too large compared to its average degree. Since the graphs in [27] are regular, the proof of Theorem 6 applies to them giving the following.

Proposition 7. *For $\ell \geqslant 2$, we have $f(r, C_{2\ell}) \leqslant Cr^{\frac{3\ell-2}{3\ell+2}}(\log r)^{\frac{3\ell}{3\ell+2}}$.*

We were unable to find an explicit partition of the $C_{2\ell}$-free graphs which would improve this upper bound above.

For integers $K, \ell \geqslant 2$, the *theta graph* $\theta_{K,\ell}$ is the graph consisting of $K \geqslant 2$ internally disjoint paths of length ℓ all having the same two vertices as endpoints. Observe that $\theta_{2,\ell} = C_{2\ell}$. The magnitude of $\text{ex}(n, C_8)$ is unknown, yet Verstraëte and Williford [34] discovered an algebraic construction that gives the lower bound in the asymptotic formula $\text{ex}(n, \theta_{3,4}) = \Theta(n^{5/4})$. Using their construction, we confirm Conjecture 2 for $\theta_{3,4}$.

Theorem 8. *There are positive constants c and C such that for all r,*

$$cr^{3/5} < f(r, \theta_{3,4}) < Cr^{3/5}.$$

Finally we turn to bounds for arbitrary forbidden subgraphs. The proof of Theorem 1 uses a random partition of the vertex set of an H-free graph. Thus, it remains to find deterministic algorithms for constructing H-free (r, k)-graphs. We describe in Section 3.3 such an algorithm which can be used when there exists a dense pseudorandom H-free graph.

Theorem 9. *Let H be a bipartite graph with no vertices of degree 1. Suppose there exists an H-free d-regular graph G on n vertices, where $d = n^a$ and $a < \frac{1}{3}$. Let $\rho = \max\{\rho_2, -\rho_n\}$ if G is not bipartite and $\rho = \rho_2$ if G is bipartite and assume that $\rho = O(d^{\frac{1}{2}})$. Then Algorithm 15 terminates with a complete partition of a graph G', with $m = \frac{1}{4}n^{\frac{1+a}{2}}/(\log n)^{\frac{1}{2}}$ parts and maximum part size at most $Cm^{\frac{2}{1+a}-1}(\log m)^{\frac{1}{1+a}}$, where C is an absolute constant.*

If G is optimal in the sense that $\mathrm{ex}(N, H) = \Theta(N^{1+a})$, then this result provides the same upper bound as Theorem 1. Thus, Theorem 9 can be viewed as a deterministic version of Theorem 1. We believe that the pseudorandomness condition imposed on G is not unreasonably strong, since many constructions of H-free graphs satisfy the condition. For example, this applies to the incidence graph of a generalized quadrangle and hexagon for the cases of $H = C_6, C_{10}$ and to the $C_{2\ell}$-free graphs constructed by Lubotzky, Phillips, and Sarnak [28].

In Section 2, we give some elementary upper and lower bounds. In Section 3, we prove our results. Section 4 contains some concluding remarks and open problems.

2 Elementary Bounds

The first result of this section is an easy consequence of (1).

Proposition 10. *Let $\mathcal{H}$ be a family of graphs with $\mathrm{ex}(N, \mathcal{H}) \leqslant CN^e$ for some positive constant C. If there exists an $\mathcal{H}$-free (r, k)-graph, then*

$$\frac{(r-1)^2}{2} \leqslant C(rk)^e.$$

Proof. Suppose that there exists an $\mathcal{H}$-free (r, k)-graph G. The number of edges of G is at least $\binom{r}{2}$ and is at most $\mathrm{ex}(rk, \mathcal{H})$. Therefore, $\binom{r}{2} \leqslant e(G) \leqslant \mathrm{ex}(rk, \mathcal{H})$ which implies $\frac{(r-1)^2}{2} \leqslant C(rk)^e$. $\qquad\square$

As r tends to infinity, Proposition 10 gives the asymptotic lower bound

$$f(r, H) \geqslant (1 - o_r(1))\frac{r^{2/e-1}}{(2C)^{1/e}}. \tag{2}$$

It is noted in [3] that

$$f(r, H) = 2 \text{ for all non-bipartite } H \text{ with } r \geqslant |V(H)|. \tag{3}$$

We give provide a proof for completeness. Given a non-bipartite graph H and an integer $r \geqslant 2$, let $V_1, \ldots, V_r$ be disjoint sets, each containing two vertices. In each part V_i, color one vertex red and the other blue. Let G be any graph obtained by connected all parts using edges that join pairs of vertices of different colors. The two color classes form independent sets in G. Since G is bipartite and H is not, the graph G is H-free. This shows $f(r, H) \leqslant 2$ for all $r \geqslant 2$. If $r \geqslant |V(H)|$, then there is no H-free $(r, 1)$-graph since an $(r, 1)$-graph must be K_r, and H is a subgraph of K_r. We conclude that $f(r, H) = 2$.

Because of (3), one assumes that $\mathcal{H}$ contains at least one bipartite graph when investigating $f(r, \mathcal{H})$.

3 Proofs of Theorems 4, 5, 8, and 9.

3.1 Complete bipartite graphs

In this subsection we begin by proving the following result which will be used to prove Theorem 4.

Theorem 11. *Suppose that q is an even power of an odd prime and that d is a positive integer for which $d | q - 1$. For any $t \geqslant 2$ and positive integers $a \leqslant h$ for which $ha = \frac{q-1}{d}$,*

$$f(q^{t-1}a, K_{t,(t-1)!d^{t-1}+1}) \leqslant a + h + O(h^{2/5}). \tag{4}$$

Proof. Let q be an even power of an odd prime and d be a positive divisor of $q - 1$. Let $t \geqslant 2$ be an integer and $N : \mathbb{F}_{q^{t-1}} \to \mathbb{F}_q$ be the $\mathbb{F}_q$-norm on $\mathbb{F}_{q^{t-1}}$ defined by $N(x) = x^{1+q+q^2+\cdots+q^{t-2}}$. We write $\mathbb{F}_q^*$ for the group of non-zero elements of $\mathbb{F}_q$ under multiplication and assume that θ is a generator. Let $\mathcal{K}_d = \langle \theta^{(q-1)/d} \rangle$ so that $\mathcal{K}_d$ is the unique subgroup of $\mathbb{F}_q^*$ of order d. Write $\mathbb{F}_q^*/\mathcal{K}_d$ for the quotient group of $\mathbb{F}_q^*$ modulo $\mathcal{K}_d$. This is also a cyclic group whose order is $\frac{q-1}{d}$. We define $\mathcal{B}_d(q, t)$ to be the bipartite graph with parts $\mathcal{P} = \mathbb{F}_{q^{t-1}} \times (\mathbb{F}_q^*/\mathcal{K}_d)$ and $\mathcal{L} = \mathbb{F}_{q^{t-1}} \times (\mathbb{F}_q^*/\mathcal{K}_d)$ where subscripts are used to distinguish vertices in the two parts. The vertex $(x, b\mathcal{K}_d)_{\mathcal{P}}$ in $\mathcal{P}$ is adjacent to the vertex $(y, c\mathcal{K}_d)_{\mathcal{L}}$ in $\mathcal{L}$ if and only if

$$N(x + y) \in bc\mathcal{K}_d.$$

The graph $\mathcal{B}_d(q, t)$ comes from the projective norm graph by taking quotients, in a similar spirit to Füredi's construction of $K_{2,t+1}$ free graphs [18].

Claim 12. *The graph $\mathcal{B}_d(q, t)$ is $K_{t,(t-1)!d^{t-1}+1}$−free.*

Proof of Claim 12. Following the argument in [2], it is straightforward to prove that $\mathcal{B}_1(q, t)$ is $K_{t,(t-1)!+1}$-free. We briefly outline the argument for completeness. In this case $d = 1$ so $\mathcal{K}_1$ is just the multiplicative identity in $\mathbb{F}_q^*$ and the quotient $\mathbb{F}_q^*/\mathcal{K}_1$ is the

same as $\mathbb{F}_q^*$. Fix t distinct vertices $(x_1, b_1)_{\mathcal{P}}, \ldots, (x_t, b_t)_{\mathcal{P}}$ in $\mathcal{P}$. A common neighbor $(y, c)_{\mathcal{L}}$ must be a solution to the system

$$N(x_1 + Y) = b_1 C$$
$$\vdots$$
$$N(x_t + Y) = b_t C.$$

Using Lemma 4 and the proof of Theorem 5 in [2], there are at most $(t-1)!$ solutions to this system. Hence, there is no $K_{t,(t-1)!+1}$ in $\mathcal{B}_1(q,t)$ with t vertices in $\mathcal{P}$. By symmetry, there is no $K_{t,(t-1)!+1}$ with t vertices in $\mathcal{L}$. We conclude $\mathcal{B}_1(q,t)$ is $K_{t,(t-1)!+1}$-free.

Next we prove $\mathcal{B}_d(q,t)$ is $K_{t,(t-1)!d^{t-1}+1}$-free. Again, we follow [2] (specifically the argument presented on pages 287 and 288). As seen in the previous paragraph, there is a symmetry between $\mathcal{P}$ and $\mathcal{L}$ and so we handle both cases at once by omitting the subscripts. Fix t distinct elements $(x_1, b_1\mathcal{K}_d), \ldots, (x_t, b_t\mathcal{K}_d) \in \mathbb{F}_{q^{t-1}} \times (\mathbb{F}_q^*/\mathcal{K}_d)$. Let $(y, c\mathcal{K}_d)$ be a common neighbor of these vertices in $\mathcal{B}_d(q,t)$. Thus, $N(x_i + y) \in b_i c \mathcal{K}_d$ for $1 \leqslant i \leqslant t$. As in [2], this system can be reduced to a system of $t-1$ equations of the form $N(X_i + Y) \in b_i \mathcal{K}_d$ where $1 \leqslant i \leqslant t-1$. For any choice of d^{t-1} elements from $b_1\mathcal{K}_d, \ldots, b_{t-1}\mathcal{K}_d$, there are at most $(t-1)!$ solutions. Hence, the total number of solutions is at most $(t-1)!d^{t-1}$. This implies $(x_1, b_1\mathcal{K}_d), \ldots, (x_t, b_t\mathcal{K}_d)$ have at most $(t-1)!d^{t-1}$ common neighbors in the graph $\mathcal{B}_d(q,t)$. This completes the proof of the Claim 12. $\square$

With Claim 12 established, we may now return to the proof of Theorem 11. Let a and h be positive integers with $a \leqslant h$ and $ha = \frac{q-1}{d}$. Since $\mathbb{F}_q^*/\mathcal{K}_d$ is a cyclic group with ha elements, we can choose a subgroup H with $|H| = h$. Let A be a set of coset representatives of H in $\mathbb{F}_q^*/\mathcal{K}_d$. Thus, $|H| = h$, $|A| = a$, and

$$\mathbb{F}_q^*/\mathcal{K}_d = \bigcup_{b\mathcal{K}_d \in A} b\mathcal{K}_d H. \tag{5}$$

Partition $\mathcal{P}$ into the sets

$$\mathcal{P}_{x,h_1\mathcal{K}_d} = \{(x, a_1\mathcal{K}_d h_1\mathcal{K}_d)_{\mathcal{P}} : a_1\mathcal{K}_d \in A\} \quad \text{where} \quad x \in \mathbb{F}_{q^{t-1}} \text{ and } h_1\mathcal{K}_d \in H.$$

Similarly, partition $\mathcal{L}$ into the sets

$$\mathcal{L}_{y,a_2\mathcal{K}_d} = \{(y, a_2\mathcal{K}_d h_2\mathcal{K}_d)_{\mathcal{L}} : h_2\mathcal{K}_d \in H\} \quad \text{where} \quad y \in \mathbb{F}_{q^{t-1}} \text{ and } a_2\mathcal{K}_d \in A.$$

Fix $x, y \in \mathbb{F}_{q^{t-1}}$ and $h_1\mathcal{K}_d \in H$, $a_2\mathcal{K}_d \in A$. There is an edge between $\mathcal{P}_{x,h_1\mathcal{K}_d}$ and $\mathcal{L}_{y,a_2\mathcal{K}_d}$ if and only if

$$N(x + y) \in a_1 h_1 a_2 h_2 \mathcal{K}_d$$

for some $a_1\mathcal{K}_d \in A$ and $h_2\mathcal{K}_d \in H$.

First suppose $N(x+y) \neq 0$. Then $N(x+y)$ is some element of $\mathbb{F}_q^*$, say α. By (5) there is a $b\mathcal{K}_d \in A$ and $h\mathcal{K}_d \in H$ such that $\alpha\mathcal{K}_d = (b\mathcal{K}_d)(h\mathcal{K}_d) = bh\mathcal{K}_d$. Choose $a_1\mathcal{K}_d \in A$ such that $a_1\mathcal{K}_d = ba_2^{-1}\mathcal{K}_d$. Choose $h_2\mathcal{K}_d \in H$ such that $h_2\mathcal{K}_d = h_1^{-1}h\mathcal{K}_d$. We then have that

$$\alpha\mathcal{K}_d = bh\mathcal{K}_d = a_1 a_2 h_1 h_2 \mathcal{K}_d = a_1 h_1 a_2 h_2 \mathcal{K}_d.$$

Therefore, $N(x + y) \in \alpha\mathcal{K}_d = a_1 h_1 a_2 h_2 \mathcal{K}_d$ and so there is an edge with one endpoint in $\mathcal{P}_{x, h_1 \mathcal{K}_d}$ and the other in $\mathcal{L}_{y, a_2 \mathcal{K}_d}$.

Now suppose that $N(x + y) = 0$. This holds if and only if $y = -x$. In this case, $\mathcal{B}_d(q, t)$ does not have an edge between the parts $\mathcal{P}_{x, h_1 \mathcal{K}_d}$ and $\mathcal{L}_{y, a_2 \mathcal{K}_d}$. For $x \in \mathbb{F}_{q^{t-1}}$, let $\mathcal{D}_x$ be the subgraph induced by

$$\left(\bigcup_{h_1 \mathcal{K}_d \in H} \mathcal{P}_{x, h_1 \mathcal{K}_d} \right) \cup \left(\bigcup_{a_2 \mathcal{K}_d \in A} \mathcal{L}_{-x, a_2 \mathcal{K}_d} \right).$$

The sets in this union are pairwise disjoint and $\mathcal{D}_x$ is an R-partite graph (inducing no edges) with $R = |H| + |A|$. We will add a small number of new vertices and edges to $\mathcal{D}_x$. Add at most $f(R, K_{t, (t-1)! d^{t-1}+1})$ vertices to each of the R parts such that the subgraph induced by these new vertices is $K_{t, (t-1)! d^{t-1}+1}$-free, and there is an edge between any two of the R parts. Write $\mathcal{P}'_{x, h_1 \mathcal{K}_d}$ and $\mathcal{L}'_{-x, a_2 \mathcal{K}_d}$ for these new sets, which contain $\mathcal{P}_{x, h_1 \mathcal{K}_d}$ and $\mathcal{L}_{-x, a_2 \mathcal{K}_d}$, respectively. Observe that $|\mathcal{P}'_{x, h_1 \mathcal{K}_d}| \leqslant |\mathcal{P}_{x, h_1 \mathcal{K}_d}| + f(R, K_{t, (t-1)! d^{t-1}+1})$ and similarly for $\mathcal{L}'_{-x, a_2 \mathcal{K}_d}$.

Let $\psi : \left\{ \mathcal{L}'_{y, b \mathcal{K}_d} : y \in \mathbb{F}_{q^{t-1}}, b \mathcal{K}_d \in A \right\} \to \left\{ \mathcal{P}'_{x, h \mathcal{K}_d} : x \in \mathbb{F}_{q^{t-1}}, h \mathcal{K}_d \in H \right\}$ be an arbitrary injection. Since $|A| \leqslant |H|$, such an injection exists. For each $y \in \mathbb{F}_{q^{t-1}}$ and $b \mathcal{K}_d \in A$, let

$$V_{y, b \mathcal{K}_d} = \psi \left(\mathcal{L}'_{y, b \mathcal{K}_d} \right) \cup \mathcal{L}'_{y, b \mathcal{K}_d}.$$

We obtain a graph $G = G(q, t, d, H)$ whose vertex set is the union of the $V_{y, b \mathcal{K}_d}$'s, which is a $K_{t, (t-1)! d^{t-1}+1}$-free (r, k)-graph with

$$r = q^{t-1} |A| = \frac{q^{t-1}(q-1)}{d|H|} \qquad \text{and} \qquad k \leqslant |A| + |H| + 2 f(|A| + |H|, K_{t, (t-1)! d^{t-1}+1}).$$

We then have

$$f(q^{t-1} |A|, K_{t, (t-1)! d^{t-1}+1}) \leqslant |A| + |H| + 2 f(|A| + |H|, K_{t, (t-1)! d^{t-1}+1}). \tag{6}$$

Using Theorem 1 (and the known upper bound on $\mathrm{ex}(n, K_{t, (t-1)! d^{t-1}+1})$),

$$2 f(R, K_{t, (t-1)! d^{t-1}+1}) \leqslant 2 C R^{\frac{1}{2t-1}} \log^{\frac{t}{2t-1}} R \leqslant C' |H|^{2/5}$$

which gives an upper bound on the last term in (6). Here we have used $|A| \leqslant |H|$ and the inequality $\frac{1}{2t-1} \leqslant \frac{1}{3} < \frac{2}{5}$. With this estimate, we have completed the proof since $a = |A|$ and $h = |H|$. $\qquad\square$

Proof of Theorem 4 part (a). Let $d \geqslant 3$ be an integer and D be the unique positive integer for which $D(D+1) < d \leqslant (D+1)(D+2)$. Observe that

$$(D + 1/2)^2 = D(D+1) + \frac{1}{4} < d \leqslant (D+1)(D+2) < (D+3/2)^2$$

where the first inequality holds because $D(D+1) < d$ and d, D are integers. Thus,

$$\sqrt{d} - \frac{3}{2} < D < \sqrt{d} - \frac{1}{2}. \tag{7}$$

Next, since D and $D+1$ are relatively prime, the system $x + 1 \equiv 0 \pmod{D}$ and $x - 1 \equiv 0 \pmod{D+1}$ has a unique solution, say $x_0 \pmod{D(D+1)}$. Since $x_0 \equiv D - 1 \pmod{D}$ and $x_0 \equiv 1 \pmod{D+1}$, the least residue x_0 is relatively prime with both D and $D+1$. By Dirichlet's Theorem on Arithmetic Progressions, there are infinitely many primes p with $p \equiv x_0 \pmod{D(D+1)}$. Let $\mathcal{S}$ be the set of all such primes so that for each $p \in \mathcal{S}$, $p + 1 \equiv 0 \pmod{D}$ and $p - 1 \equiv 0 \pmod{D+1}$.

By Theorem 11 with $t = 2$, $d = D(D+1)$, $a = \frac{p-1}{D+1}$, $h = \frac{p+1}{D}$, and $q = p^2$ where $p \in \mathcal{S}$, we have

$$f\left(p^2 \cdot \frac{p-1}{D+1}, K_{2,D(D+1)+1}\right) \leqslant \left(\frac{1}{D} + \frac{1}{D+1}\right)p + O(p^{2/5}) = \frac{2D+1}{D(D+1)}p + O(p^{2/5}).$$

If $R_p := \dfrac{p^2(p-1)}{D+1}$, then using monotonicity and (7),

$$\begin{aligned}
f(R_p, K_{2,d+1}) \leqslant f(R_p, K_{2,D(D+1)+1}) &\leqslant \frac{2D+1}{D(D+1)}((D+1)R_p)^{1/3} + O(R_p^{2/15}) \\
&= \frac{2D+1}{D(D+1)^{2/3}}R_p^{1/3} + O(R_p^{2/15}) \\
&< \frac{2D+1}{D^{5/3}}R_p^{1/3} + O(R_p^{2/15}) \\
&< \frac{2\sqrt{d}}{(\sqrt{d} - 3/2)^{5/3}}R_p^{1/3} + O(R_p^{2/15}) \\
&= \frac{2}{d^{1/3}} \cdot \frac{1}{(1 - 1.5d^{-1/2})^{5/3}}R_p^{1/3} + O(R_p^{2/15}).
\end{aligned}$$

The last step will be to use The Prime Number Theorem in Arithmetic Progressions to go from $f(R_p, K_{2,d+1}) < \frac{2}{d^{1/3}} \cdot \frac{1}{(1-1.5d^{-1/2})^{5/3}}R_p^{1/3} + O(R_p^{2/15})$ for $p \in \mathcal{S}$ to

$$f(r, K_{2,d+1}) < \frac{2}{d^{1/3}} \cdot \frac{1}{(1 - 1.5d^{-1/2})^{5/3}}r^{1/3} + o(r^{1/3}).$$

For $x > 0$, let $\pi_{D(D+1),x_0}(x)$ be the number of primes p such that $p \equiv x_0 \pmod{D(D+1)}$ and $p \leqslant x$. Let $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$ and ϕ denote the Euler phi function. By Theorem 1.3 in [5], there are positive constants x_D and c_D depending only on D such that

$$\left| \pi_{D(D+1),x_0}(x) - \frac{\mathrm{Li}(x)}{\phi(D(D+1))} \right| < \frac{c_D x}{(\log x)^2} \tag{8}$$

for all $x > x_D$.

Let $0 < \epsilon < 1$. Let r be a positive integer that we will take sufficiently large in terms of ϵ and D. Define $f_{r,D}(z) = z^3 - z^2 - r(D+1)$. The function $f_{r,D}$ is strictly increasing on $(1, \infty)$, continuous, and has one real root $x_1 = (1 + o_r(1))(r(D+1))^{1/3}$ which tends to infinity with r. Let r be any integer chosen large enough so that $x_D < x_1 < 2(r(D+1))^{1/3}$ and $\frac{(\log x_1)^2}{\log((1+\epsilon)x_1)} > \frac{3c_D D^2}{\epsilon}$. Then, using (8) and this last inequality,

$$
\begin{aligned}
\pi_{D(D+1), x_0}((1+\epsilon)x_1) - \pi_{D(D+1), x_0}(x_1) &\geqslant \frac{1}{\phi(D(D+1))} \left(\int_{x_1}^{(1+\epsilon)x_1} \frac{dt}{\log t} \right) - \frac{3c_D x_1}{(\log(x_1))^2} \\
&> \frac{\epsilon x_1}{D^2 \log((1+\epsilon)x_1)} - \frac{3c_D x_1}{(\log x_1)^2} > 0.
\end{aligned}
$$

Let p be a prime with $x_1 \leqslant p \leqslant (1+\epsilon)x_1$ and $p \equiv x_0 \pmod{D(D+1)}$. Write $R_p = \frac{p^2(p-1)}{D+1}$ as before. Since x_1 is a root of $f_{r,D}(z)$, we have $0 = f_{r,D}(x_1) \leqslant f_{r,D}(p) \leqslant f_{r,D}(x_1 + \epsilon x_1)$ which implies

$$
0 \leqslant p^3 - p^2 - r(D+1) \leqslant (3\epsilon + 3\epsilon^2 + \epsilon^3)x_1^3 - (2\epsilon + \epsilon^2)x_1^2.
$$

Thus,

$$
r \leqslant \frac{p^2(p-1)}{D+1} \leqslant r + \frac{(3\epsilon + 3\epsilon^2 + \epsilon^3)x_1^3 - (2\epsilon + \epsilon^2)x_1^2}{D+1} < r + \frac{7\epsilon x_1^3}{D+1} < r(1 + 56\epsilon)
$$

where we have used $x_1 < 2(r(D+1))^{1/3}$ for the last inequality. We now have that p is a prime in $\mathcal{S}$ with $r \leqslant R_p < r(1 + 56\epsilon)$. Hence,

$$
\begin{aligned}
f(r, K_{2,d+1}) &\leqslant f(R_p, K_{2,d+1}) < \frac{2}{d^{1/3}} \cdot \frac{1}{(1 - 1.5d^{-1/2})^{5/3}} R_p^{1/3} + O(R_p^{2/15}) \\
&< \frac{2}{d^{1/3}} \cdot \frac{1}{(1 - 1.5d^{-1/2})^{5/3}} r^{1/3}(1 + 56\epsilon)^{1/3} + O(r^{2/15}).
\end{aligned}
$$

As ϵ can be made arbitrarily small by taking r large, we have

$$
f(r, K_{2,d+1}) < \frac{2}{d^{1/3}} \cdot \frac{1}{(1 - 1.5d^{-1/2})^{5/3}} r^{1/3} + o(r^{1/3}). \qquad \square
$$

While the proof of Theorem 4 is valid for $d \geqslant 3$, it does not necessarily produce the best constant. First, for all d we have the upper bound $f(r, K_{2,d+1}) \leqslant f(r, C_4) = (1 + o(1))r^{1/3}$. For small d, one can use the inequality $f\left(p^2 \cdot \frac{p-1}{D+1}, K_{2,D(D+1)+1}\right) \leqslant \left(\frac{1}{D} + \frac{1}{D+1}\right)p + O(p^{2/5})$ to obtain some improvements. However, these improvements do not have a matching lower bound and require $D(D+1) < d \leqslant (D+1)(D+2)$. The smallest d for which this argument beats $f(r, K_{2,d+1}) \leqslant (1 + o(1))r^{1/3}$ is $d = 13$ (so $D = 3$) giving $f(r, K_{2,14}) \leqslant (0.9261 + o(1))r^{1/3}$.

Proof of Theorem 4 part (b). Let $d = 1$. The condition $d|q - 1$ is satisfied for any q. Taking $a = \sqrt{q} - 1$ and $h = \sqrt{q} + 1$ gives

$$f(q^{t-1}(\sqrt{q} - 1), K_{t,(t-1)!+1}) \leqslant 2\sqrt{q} + O_t(q^{1/5})$$

for all $t \geqslant 2$. As in [3], this upper bound and a density of primes argument proves $f(r, K_{t,(t-1)!+1}) \leqslant 2r^{1/(2t-1)} + o(r^{1/(2t-1)})$.

The lower bound follows from Proposition 10 and the best-known upper bound on $\mathrm{ex}(N, K_{s,t})$ [19, 30]. □

3.2   Cycles

Using graphs constructed by Wenger we prove Theorem 5. Instead of using Wenger's definiton of these graphs, we will use a different set of equations to define adjacency. The resulting graphs are isomorphic and an explicit isomorphism is given in [8].

Let q be a power of a prime, $M \geqslant 1$ be an integer, $\mathcal{P} = \{(p_1, p_2, \ldots, p_{M+1})_{\mathcal{P}} : p_i \in \mathbb{F}_q\}$, and $\mathcal{L} = \{(\ell_1, \ell_2, \ldots, \ell_{M+1})_{\mathcal{L}} : \ell_i \in \mathbb{F}_q\}$. Let $W_M(q)$ be the bipartite graph with parts $\mathcal{P}$ and $\mathcal{L}$ where $(p_1, p_2, \ldots, p_{M+1})_{\mathcal{P}}$ is adjacent to $(\ell_1, \ell_2, \ldots, \ell_{M+1})_{\mathcal{L}}$ if and only if $\ell_{j+1} + p_{j+1} = l_j p_1$ for $1 \leqslant j \leqslant M$. The graph $W_M(q)$ has $2q^{M+1}$ vertices and is q-regular. For $M \in \{1, 2, 4\}$, $W_M(q)$ is C_{2M+2}-free. These graphs were constructed by Wenger [35] and have been studied extensively.

Proof of Theorem 5. Let $M = 2$ and consider the bipartite graph $W_2(q)$ which is C_6-free. Partition $\mathcal{P}$ and $\mathcal{L}$ into q^2 sets by letting

$$\mathcal{P}_{p_1,p_3} = \{(p_1, p_2, p_3)_{\mathcal{P}} : p_2 \in \mathbb{F}_q\}, \quad p_1, p_3 \in \mathbb{F}_q,$$

and

$$\mathcal{L}_{\ell_1,\ell_2} = \{(\ell_1, \ell_2, \ell_3)_{\mathcal{L}} : \ell_3 \in \mathbb{F}_q\}, \quad \ell_1, \ell_2 \in \mathbb{F}_q.$$

Fix $p_1, p_3, \ell_1, \ell_2 \in \mathbb{F}_q$. There is exactly one edge between $\mathcal{P}_{p_1,p_3}$ and $\mathcal{L}_{\ell_1,\ell_2}$ because the two equations $\ell_2 + p_2 = \ell_1 p_1$ and $\ell_3 + p_3 = \ell_2 p_1$ uniquely determine p_2 and ℓ_3. Arbitrarily pair up the $\mathcal{P}_{p_1,p_3}$'s with the $\mathcal{L}_{\ell_1,\ell_2}$'s in a 1-to-1 fashion giving q^2 parts, each containing $2q$ vertices. This partition shows that $W_2(q)$ is a $(q^2, 2q)$-graph. Using a density of primes argument as in the proof of Theorem 4 gives $f(r, C_6) \leqslant 2r^{1/2} + o(r^{1/2})$.

Next we prove the upper bound on $f(r, C_{10})$. Let $M = 4$ and consider $W_4(q)$. Partition $\mathcal{P}$ and $\mathcal{L}$ into q^3 sets by letting

$$\mathcal{P}_{p_1,p_3,p_5} = \{(p_1, p_2, p_3, p_4, p_5)_{\mathcal{P}} : p_2, p_4 \in \mathbb{F}_q\}, \quad p_1, p_3, p_5 \in \mathbb{F}_q,$$

and

$$\mathcal{L}_{\ell_1,\ell_2,\ell_4} = \{(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5)_{\mathcal{L}} : \ell_3, \ell_5 \in \mathbb{F}_q\}, \quad \ell_1, \ell_2, \ell_4 \in \mathbb{F}_q.$$

Fix $p_1, p_3, p_5, \ell_1, \ell_2, \ell_4$. The system of equations

$$
\begin{aligned}
\ell_2 + p_2 &= \ell_1 p_1 \\
\ell_3 + p_3 &= \ell_2 p_1 \\
\ell_4 + p_4 &= \ell_3 p_1 \\
\ell_5 + p_5 &= \ell_4 p_1.
\end{aligned}
$$

has a unique solution for p_2, ℓ_3, p_4, and ℓ_5. This implies that there is exactly one edge between $\mathcal{P}_{p_1,p_3,p_5}$ and $\mathcal{L}_{\ell_1,\ell_2,\ell_4}$. The last part of the argument is almost the same as the C_6 case and leads to the upper bound $f(r, C_{10}) \leqslant 2r^{2/3} + o(r^{2/3})$. $\qquad\square$

Proof of Theorem 8. Let q be an even power of an odd prime and μ be a root of an irreducible quadratic over $\mathbb{F}_{\sqrt{q}}$. An element $x_j \in \mathbb{F}_q$ can be written uniquely as $x_j = x_{1,j} + x_{2,j}\mu$ where and $x_{1,j}, x_{2,j} \in \mathbb{F}_{\sqrt{q}}$. Let G be the bipartite graph with parts

$$\mathcal{P} = \{(v_1, v_2, v_3, v_4)_{\mathcal{P}} : v_j \in \mathbb{F}_q\} \text{ and } \mathcal{L} = \{(w_1, w_2, w_3, w_4)_{\mathcal{L}} : w_j \in \mathbb{F}_q\}.$$

Vertices $(v_1, v_2, v_3, v_4)_{\mathcal{P}}$ and $(w_1, w_2, w_3, w_4)_{\mathcal{L}}$ are adjacent if and only if

$$
\begin{aligned}
v_2 + w_2 &= v_1 w_1 \\
v_4 + w_3 &= v_1^2 w_1 \\
v_3 + w_4 &= v_1 w_1^2.
\end{aligned}
$$

This graph is $\theta_{3,4}$-free [34]. Write each coordinate v_j as $v_j = v_{1,j} + v_{2,j}\mu$ and $w_j = w_{1,j} + w_{2,j}\mu$ where $v_{1,j}, v_{2,j}, w_{1,j}, w_{2,j} \in \mathbb{F}_{\sqrt{q}}$. Partition $\mathcal{P}$ into the $q^{5/2}$ sets

$$\mathcal{P}_{v_1, v_{3,2}, v_4} = \{(v_1, v_{2,1} + v_{2,2}\mu, v_{3,1} + v_{3,2}\mu, v_{4,1} + v_{4,2}\mu)_{\mathcal{P}} : v_{2,1}, v_{2,2}, v_{3,1} \in \mathbb{F}_{\sqrt{q}}\}$$

where $v_1, v_4 \in \mathbb{F}_q$, and $v_{3,2} \in \mathbb{F}_{\sqrt{q}}$. Each of these parts contains $q^{3/2}$ vertices. Partition $\mathcal{L}$ into the sets

$$\mathcal{L}_{w_1, w_2, w_{4,1}} = \{(w_1, w_2, w_{3,1} + w_{3,2}\mu, w_{4,1} + w_{4,2}\mu)_{\mathcal{L}} : w_{3,1}, w_{3,2}, w_{4,2} \in \mathbb{F}_{\sqrt{q}}\}$$

where $w_1, w_2 \in \mathbb{F}_q$ and $w_{4,1} \in \mathbb{F}_{\sqrt{q}}$.

Fix two parts $\mathcal{P}_{v_1, v_{3,2}, v_4}$ and $\mathcal{L}_{w_1, w_2, w_{4,1}}$. There will be an edge between these two parts provided that there exists $v_{2,1}, v_{2,2}, v_{3,1}, w_{3,1}, w_{3,2}, w_{4,2} \in \mathbb{F}_{\sqrt{q}}$ such that

$$
\begin{aligned}
v_{2,1} + v_{2,2}\mu + w_{2,1} + w_{2,2}\mu &= v_1 w_1 \\
v_{4,1} + v_{4,2}\mu + w_{3,1} + w_{3,2}\mu &= v_1^2 w_1 \\
v_{3,1} + v_{3,2}\mu + w_{4,1} + w_{4,2}\mu &= v_1 w_1^2.
\end{aligned}
$$

The first two equations are solved by choosing $v_{2,1} + v_{2,2}\mu$ to equal $v_1 w_1 - w_{2,1} - w_{2,2}\mu$, and $w_{3,1} + w_{3,2}\mu$ to equal $v_1^2 w_1 - v_{4,1} - v_{4,2}\mu$. The last equation can be solved by choosing $v_{3,1}$ and $w_{4,2}$ so that $v_{3,1} + w_{4,2}\mu = v_1 w_1^2 - v_{3,2}\mu - w_{4,1}$. This gives a unique solution to this system of equations and so we have an edge between these parts. Take an arbitrary pairing

of the $\mathcal{P}_{v_1,v_{3,2},v_4}$'s and $\mathcal{L}_{w_1,w_2,w_{4,1}}$'s to obtain a $(p^5, 2p^3)$-graph. Hence, $f(q^{5/2}, \theta_{3,4}) \leqslant 2q^{3/2}$ for q an even power of an odd prime. By a density of primes argument,

$$f(r, \theta_{3,4}) \leqslant 2r^{3/5} + o(r^{3/5}).$$

The trivial bound together with $\mathrm{ex}(N, \theta_{3,4}) = O(N^{5/4})$ proved by Faudree and Simonovits [17] implies that $f(r, \theta_{3,4}) = \Omega(r^{3/5})$. $\qquad\square$

3.3 Partitioning pseudorandom graphs

Let G be a graph on n vertices with adjacency matrix A. Then A has n real eigenvalues $\rho_1 \geqslant \cdots \geqslant \rho_n$. We will make use of the expander-mixing lemma which relates the second largest eigenvalue to the distribution of the edges. This result can be traced back to at least as early as a theorem about designs in Haemers' Ph.D Thesis [20]. For the non-bipartite and bipartite versions stated below in graph-theoretic terms, we refer to [1] and [10] respectively. If $U, W \subseteq V(G)$ then we define $e(U, W) = |\{(u, w) : u \in U, w \in W, u \text{ is adjacent to } w\}|$.

Theorem 13 (Expander-Mixing Lemma)**.** *Suppose that G is a d-regular graph with $\rho = \max\{\rho_2, -\rho_n\}$. Then for any $U, W \subseteq V(G)$, we have*

$$\left| e(U, W) - \frac{d}{n}|U||W| \right| \leqslant \rho\sqrt{|U||W|}.$$

Theorem 14 (Bipartite Expander-Mixing Lemma)**.** *If G is a d-regular bipartite graph with parts X and Y where $\rho_2 = \rho$, then for any $U \subseteq X, W \subseteq Y$, we have*

$$\left| e(U, W) - \frac{2d}{n}|U||W| \right| \leqslant \rho\sqrt{|U||W|}.$$

Before stating our algorithm we fix some notation. Let $V = V(G)$, and if G is bipartite let (X, Y) be a balanced bipartition. For convenience let A, B both denote V if G is not bipartite, and let $A = X$, $B = Y$ if G is bipartite. For $U, W \subseteq V$, let $N(U)$ be the set of vertices not in U which are adjacent to a vertex in U. Unless otherwise indicated, all vertices and adjacency relations are within the graph G.

Algorithm 15. Let H and G be graphs satisfying the assumptions of Theorem 9.

1. Initialize $V_1 = \cdots = V_m = \emptyset$.

2. For each i, add $n^{\frac{1-a}{2}}(\log n)^{\frac{1}{2}}$ vertices in G to V_i such that each vertex added is not already contained in any other V_j and is also at distance at least 3 from every other vertex in V_i. If G is bipartite then all added vertices should also belong to X.

3. For $1 \leqslant i \leqslant m$, define $\mathcal{S}_i = \{j : j \neq i, e(V_i, V_j) = 0\}$ and $s_i = |\mathcal{S}_i|$. If for every i we have $s_i < n^{\frac{1-a}{2}}(\log n)^{\frac{1}{2}}$, proceed to step 4. Otherwise, for each $1 \leqslant i \leqslant m$, choose the vertex in $B - (N(V_i) \cup N(N(V_i))) - (V_1 \cup \cdots \cup V_m)$ which is adjacent to the maximum number of vertices in $\bigcup_{j \in \mathcal{S}_i} V_j$ and add it to V_i. Return to the beginning of step 3.

4. For each i, for each $j \in \mathcal{S}_i$, create a new vertex in V_i and make it adjacent to some vertex in V_j which was added to V_j in one of the previous steps.

Proof of Theorem 9. Is it possible to perform step 2, because at any point during step 2 when we want to add a vertex to V_i, we have

$$|A - (N(V_i) \cup N(N(V_i))) - (V_1 \cup \cdots \cup V_m)| \geqslant \frac{n}{2} - d^2|V_i| - m \cdot |V_1|$$
$$\geqslant \frac{n}{2} - n^{2a + \frac{1-a}{2}}(\log n)^{\frac{1}{2}} - \frac{1}{4}n^{\frac{1+a}{2} + \frac{1-a}{2}}.$$

The choice of a ensures that both exponents in the right-hand side are at most 1 and the right-hand side is positive for n sufficiently large.

To show it is possible to perform step 3, we track how the number s_i changes each time a vertex is added. We assume that i is a value for which the inequality $s_i \geqslant n^{\frac{1-a}{2}}(\log n)^{1/2}$ is true. Let $s_i(t)$ be the value of s_i after t iterations of step 3 have been performed. Now during any of the first $2an^{\frac{1-a}{2}}(\log n)^{\frac{1}{2}}$ iterations of step 3 we have

$$|B - (N(V_i) \cup N(N(V_i))) - (V_1 \cup \cdots \cup V_m)| \geqslant \frac{n}{2} - n^{2a}(n^{\frac{1-a}{2}}(\log n)^{\frac{1}{2}} + 2an^{\frac{1-a}{2}}(\log n)^{\frac{1}{2}})$$
$$- \frac{1}{4}\frac{n^{\frac{1+a}{2}}}{(\log n)^{\frac{1}{2}}}(n^{\frac{1-a}{2}}(\log n)^{\frac{1}{2}} + 2an^{\frac{1-a}{2}}(\log n)^{\frac{1}{2}}).$$

The choice of a ensures that the right-hand side is $\Theta(n)$. Set $W = B - (N(V_i) \cup N(N(V_i))) - (V_1 \cup \cdots \cup V_m)$. To apply the expander mixing lemma we would like

$$\frac{d\left|\bigcup_{j \in \mathcal{S}_i(t)} V_j\right||W|}{n} \gg \rho\sqrt{\left|\bigcup_{j \in \mathcal{S}_i(t)} V_j\right||W|},$$

in other words $\Theta(n) = |W| \gg \frac{\rho^2 n^2}{d^2\left|\bigcup_{j \in \mathcal{S}_i(t)} V_j\right|}$. Noting that $\rho = O(d^{\frac{1}{2}})$ and $\left|\bigcup_{j \in \mathcal{S}_i(t)} V_j\right| \geqslant s_i(t)n^{\frac{1-a}{2}}(\log n)^{\frac{1}{2}} \geqslant n^{1-a}(\log n)$ we see that this condition is met. Therefore,

$$e\left(W, \bigcup_{j \in \mathcal{S}_i(t)} V_j\right) \geqslant \frac{1}{2}\frac{ds_i(t)n^{\frac{1-a}{2}}(\log n)^{\frac{1}{2}}|W|}{n} = \frac{1}{2n^{\frac{1-a}{2}}}|W|s_i(t)(\log n)^{\frac{1}{2}}.$$

Thus, there exists a vertex in W with at least $\frac{1}{2n^{\frac{1-a}{2}}}s_i(t)(\log n)^{\frac{1}{2}}$ neighbors in $\left|\bigcup_{j \in \mathcal{S}_i(t)} V_j\right|$, and so in iteration t of step 3, a vertex is added to V_i which has at least $\frac{1}{2n^{\frac{1-a}{2}}}s_i(t)(\log n)^{\frac{1}{2}}$ neighbors in $\left|\bigcup_{j \in \mathcal{S}_i(t)} V_j\right|$. Now since each of the sets V_j has all of its vertices at distance at least 3 from each other, each neighbor of the vertex added belongs to a different set V_j (where $j \in \mathcal{S}_i(t)$). It follows that

$$s_i(t+1) \leqslant s_i(t) - \frac{1}{2n^{\frac{1-a}{2}}}(\log n)^{\frac{1}{2}}s_i(t).$$

Using $s_i(0) \leqslant m \leqslant n^{\frac{1+a}{2}}/(\log n)^{\frac{1}{2}}$, we then have that as long as $|W|$ is large enough as above,

$$s_i(t) \leqslant \frac{n^{\frac{1+a}{2}}}{(\log n)^{\frac{1}{2}}} \left(1 - \frac{1}{2n^{\frac{1-a}{2}}} (\log n)^{\frac{1}{2}} \right)^t.$$

Using the estimate $\log(1 - x) \leqslant -x$, we therefore have

$$\log s_i(t) \leqslant \left(\frac{1+a}{2} \log n - \frac{1}{2} \log \log n \right) - t \frac{(\log n)^{\frac{1}{2}}}{2n^{\frac{1-a}{2}}}.$$

Step 3 terminates when for all i, $s_i(t) < n^{\frac{1-a}{2}} (\log n)^{\frac{1}{2}}$. Solving for t in the inequality

$$\left(\frac{1+a}{2} \log n - \frac{1}{2} \log \log n \right) - t \frac{(\log n)^{\frac{1}{2}}}{2n^{\frac{1-a}{2}}} \leqslant \log \left(n^{\frac{1-a}{2}} (\log n)^{\frac{1}{2}} \right)$$

we see that this is guaranteed to happen by iteration $t = 2a(\log n)^{\frac{1}{2}} n^{\frac{1-a}{2}}$.

In step 4, none of the vertices created creates a copy of H, since H has no vertex of degree 1.

We calculate the maximum size of a part in the resulting (m, k)-graph. For each V_i, $n^{\frac{1-a}{2}} (\log n)^{\frac{1}{2}}$ vertices were added in step 2, at most $2an^{\frac{1-a}{2}} (\log n)^{\frac{1}{2}}$ were added in step 3, and at most $n^{\frac{1-a}{2}} (\log n)^{\frac{1}{2}}$ were added in step 4. Therefore we have $f(m, H) \leqslant 4n^{\frac{1-a}{2}} (\log n)^{\frac{1}{2}}$. Choose C such that $C \left(\frac{1}{4} \right)^{\frac{2}{1+a} - 1} \left(\frac{1}{2} \right)^{\frac{1}{1+a}} \geqslant 4$. Using the fact that

$$\log m = \frac{1+a}{2} \log n - \log 4 - \frac{1}{2} \log \log n \geqslant \frac{1}{2} \log n$$

for n large enough, we find that

$$Cm^{\frac{2}{1+a} - 1} (\log m)^{\frac{1}{1+a}} \geqslant 4n^{\frac{1-a}{2}} (\log n)^{\frac{1}{2}} \geqslant f(m, H).$$

$\square$

4 Concluding Remarks

In light of Theorem 3, it would be interesting to determine an asymptotic formula for $f(r, K_{2,d+1})$. It seems possible that $f(r, K_{2,d+1}) = (r/d)^{1/3} + o(r^{1/3})$ for all $d \geqslant 1$. This is true for $d = 1$ by the result of [32].

We proved that $f(r, \theta_{3,4}) = \Theta(r^{3/5})$. It was proved by Conlon [9] that for every $\ell \geqslant 2$, there is a K such that $\mathrm{ex}(N, \theta_{K,\ell}) = \Omega_\ell(N^{1+1/\ell})$, and an upper bound that matches in order of magnitude is in [17]. It would be interesting to determine the order of magnitude for $f(r, \theta_{t,\ell})$ for $t \geqslant K$ and in such cases where it can be done it would further be interesting to determine the dependence on t.

References

[1] N. Alon and F.R.K. Chung. Explicit construction of linear sized tolerant networks. *Discrete Math.*, 72(1):15–19, 1988.

[2] N. Alon, L. Rónyai, and T. Szabó. Norm-graphs: Variations and applications. *J. Combin. Theory Ser. B*, 76:280–290, 1999.

[3] M. Axenovich and R. Martin. Splits with forbidden subgraphs. *Discrete Math.*, 345(2), 2022.

[4] F. Barbanera. Covering numbers in Ramsey problems. Master's thesis, Karlsruhe Institute of Technology, 2019.

[5] M. Bennett, G. Martin, K. O'Bryant, and A. Rechnitzer. Explicit bounds for primes in arithmetic progressions. *Illinois J. Math.*, 62(1–4):427–532, 2018.

[6] B. Bollobás, B. Reed, and A. Thomason. An extremal function for the achromatic number. *Graph structure theory* in Contemp. Math., 147(1–4):161–165, 1993.

[7] M. Bucić, J. Fox, and B. Sudakov. Clique minors in graphs with a forbidden subgraph. *Random Structures Algorithms.*, 60(3):327–338, 2022.

[8] S. Cioabă, F. Lazebnik, and W. Li. On the spectrum of wenger graphs. *J. Combin. Theory Ser. B*, 107:132–139, 2014.

[9] D. Conlon. Graphs with few paths of prescribed length between any two vertices. *Bull. Lond. Math. Soc.*, 51(6):1015–1021, 2019.

[10] S. De Winter, J. Schillewaert, and J. Verstraete. Large incidence-free sets in geometries. *Electron. J. of Combin.*, 19(4):#P24, 2012.

[11] M. Dębski, Z. Lonc, and P. Rzążewski. Achromatic and harmonious colorings of circulant graphs. *J. Graph Theory*, 87(1):18–34, 2018.

[12] K. Edwards. The harmonious chromatic number and the achromatic number. In *Surveys in combinatorics, 1997*, pages 13–48. Cambridge University Press, 1997.

[13] K. Edwards. Achromatic number of fragmentable graphs. *J. Graph Theory*, 65(2):94–114, 2010.

[14] K. Edwards. Achromatic number of collections of paths and cycles. *Discrete Math.*, 313(19):1856–1860, 2013.

[15] K. Edwards and P. Rzążewski. Complete colourings of hypergraphs. *Discrete Math.*, 343(2):111673, 2020.

[16] D. Eppstein, P. Kindermann, S. Kobourov, G. Liotta, A. Lubiw, A. Maignan, D. Mondal, H. Vosoughpour, S. Whitesides, and S. Wismath. On the planar split thickness of graphs. *Algorithmica*, 80:977–994, 2018.

[17] R. Faudree and M. Simonovits. On a class of degenerate extremal graph problems. *Combinatorica*, 3:83–93, 1983.

[18] Z. Füredi. New asymptotics for bipartite Turán numbers. *J. Combin. Theory Ser. A*, 75:141–144, 1996.

[19] Z. Füredi. An upper bound on Zarankiewicz problem. *Combin. Prob. Comput.*, 5(1):29–33, 1996.

[20] W.H. Haemers. *Eigenvalue techniques in design and graph theory.* PhD thesis, Mathematics and Computer Science, 1979.

[21] N. Haghparast, M. Hasanvand, and Y. Ohno. The existence of uniform hypergraphs for which the interpolation property of complete coloring fails. *Discrete Math.*, 345(3):112722, 2022.

[22] P. J. Heawood. Map-colour theorem. *Proc. London. Math. Soc.*, 2(1):161–175, 1949.

[23] F. Hughes and G. MacGillivray. The achromatic number of graphs: a survey and some new results. *Bull. Inst. Combin. Appl*, 19:27–56, 1997.

[24] A. V. Kostochka. The minimum hadwiger number for graphs with a given mean degree. (russian). *Metody Diskret. Analiz.*, 38:37–58, 1982.

[25] M. Krivelevich and R. Nenadov. Complete minors in graphs without sparse cuts. *Int. Math. Res. Not. IMRN*, 12:8996–9015, 2021.

[26] M. Krivelevich and B. Sudakov. Minors in expanding graphs. *Geom. Fun. Anal.*, 19:294–331, 2009.

[27] F. Lazebnik, V. A. Ustimenko, and A. J. Woldar. A new series of dense graphs of high girth. *Bull. Amer. Math. Soc.*, 32:73–79, 1995.

[28] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8:261–277, 1988.

[29] G. MacGillivray and A. Rodriguez. The achromatic number of the union of paths. *Discrete Math.*, 231(1-3):331–335, 2001.

[30] V. Nikiforov. A contribution to the Zarankiewicz problem. *Linear Algebra Appl.*, 432(6):1405–1411, 2010.

[31] Y. Roichman. On the achromatic number of hypercubes. *J. Combin. Theory Series B*, 79(2):177–182, 2000.

[32] V. Taranchuk and C. Timmons. Achromatic colorings of polarity graphs. *Finite Fields Appl.*, 99:102497, 2024.

[33] A. Thomason. *Extremal Functions for Graph Minors*, pages 359–380. Springer Berlin Heidelberg, 2006.

[34] J. Verstraëte and J. Williford. Graphs without theta subgraphs. *J. Combin. Theory Ser. B*, 134:76–87, 2019.

[35] R. Wenger. Extremal graphs with no C^4's, C^6's, or C^{10}'s. *J. Combin. Theory Ser. B*, 52:113–116, 1991.