# Inequalities Associated with the Root Sequences of *P*-recursive Sequences

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**Abstract**

The Turán inequalities and the Laguerre inequalities are closely related to the Laguerre-Pólya class and the Riemann hypothesis. These inequalities have been extensively studied in the literature. In this paper, we propose a method to determine a positive integer $N$ such that the sequences $\{\sqrt[n]{a_n}/n!\}_{n\geqslant N}$ and $\{\sqrt[n+1]{a_{n+1}}/(\sqrt[n]{a_n}n!)\}_{n\geqslant N}$ satisfy the higher order Turán inequality and the Laguerre inequality of order two for a P-recursive sequence $\{a_n\}_{n\geqslant 1}$.

**Mathematics Subject Classifications:** 05A20, 41A60

## 1 Introduction

A sequence $\{a_n\}_{n\geqslant 1}$ of real numbers is said to satisfy the *Turán inequality* or to be *log-concave* if for all $n \geqslant 2$,

$$a_n^2 \geqslant a_{n-1}a_{n+1}. \tag{1}$$

The sequence $\{a_n\}_{n\geqslant 1}$ is said to satisfy the *higher order Turán inequality* or the *cubic Newton inequality* if for all $n \geqslant 2$,

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_n a_{n+2}) - (a_n a_{n+1} - a_{n-1}a_{n+2})^2 \geqslant 0. \tag{2}$$

The Turán inequalities and the higher order Turán inequalities are related to the Laguerre-Pólya class of real entire functions. A real entire function

$$\psi(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \tag{3}$$

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is said to be in the Laguerre-Pólya class, denoted $\psi(x) \in \mathcal{LP}$, if

$$\psi(x) = cx^m e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty}(1 + x/x_k)e^{-x/x_k},$$

where $c,\ \beta,\ x_k$ are real numbers, $\alpha \geqslant 0$, $m$ is a nonnegative integer and $\sum x_k^{-2} < \infty$. Pólya and Schur [13] proved that if a real entire function $\psi(x) \in \mathcal{LP}$, its Maclaurin coefficients satisfy the Turán inequality (1). And Dimitrov [4] established that the Maclaurin coefficients of a real function $\psi(x) \in \mathcal{LP}$ satisfy the higher order Turán inequality (2).

In recent years, many combinatorial sequences have been demonstrated to satisfy the inequality (2). Chen, Jia and Wang [3] proved that the partition function $p(n)$ satisfies the higher order Turán inequality when $n \geqslant 95$, which was conjectured by Chen [2]. Griffin, Ono, Rolen and Zagier [6] showed that the partition function $p(n)$ satisfies the order $d \geqslant 1$ Turán inequality for sufficiently large $n$. Liu and Zhang [12] proved that the overpartition function $\bar{p}(n)$ satisfies the higher order Turán inequality for $n \geqslant 16$. Wang [15] presented a unified approach to studying the higher order Turán inequality for the sequence $\{a_n/n!\}_{n\geqslant 0}$ when $a_n$ satisfies a second-order linear recurrence. Moreover, Hou and Li [7] proposed a method to determine a specific $N$ such that the higher order Turán inequality holds for a P-recursive sequence $\{a_n\}_{n\geqslant N}$.

A sequence $\{a_n\}_{n\geqslant 1}$ satisfies the Laguerre inequality of order $m$ if for $n \geqslant 1$,

$$L_m(a_n) := \frac{1}{2}\sum_{k=0}^{2m}(-1)^{k+m}\binom{2m}{k}a_{n+k}a_{2m-k+n} \geqslant 0. \tag{4}$$

A polynomial $f(x)$ satisfies the Laguerre inequality if

$$f'(x)^2 - f(x)f''(x) \geqslant 0.$$

Laguerre [10] stated that the Laguerre inequality holds for $f(x)$ if $f(x)$ is a polynomial with only real zeros. Jensen [9] introduced the $n$-th generalization of the Laguerre inequality as

$$L_n(f(x)) := \frac{1}{2}\sum_{k=0}^{2n}(-1)^{n+k}\binom{2n}{k}f^{(k)}(x)f^{(2n-k)}(x) \geqslant 0, \tag{5}$$

where $f^{(k)}(x)$ denotes the $k$th derivative of $f(x)$. By choosing the function $f(x)$ to have Taylor coefficients $a_n$ through the specialization $x = 0$, the above inequality (5) transforms into the inequality (4).

Wang and Yang [16] established that the partition function $p(n)$, the overpartition function $\bar{p}(n)$ and several other combinatorial sequences satisfy the Laguerre inequality of order two. Dou and Wang [5] found $N(m)$ for $3 \leqslant m \leqslant 10$, such that the Laguerre inequality of order $m$ holds for the partition function $p(n)$ when $n \geqslant N(m)$. Wagner [14] showed that the partition function $p(n)$ satisfies the Laguerre inequality of any order when $n$ is sufficiently large. Furthermore, Li [11] gave a method to find the explicit integer $N$

such that the Laguerre inequality of order two holds for a P-recursive sequence $\{a_n\}_{n\geqslant N}$.

In this paper, we provide the sufficient conditions for the sequences related to the root sequences to satisfy the higher order Turán inequality and the Laguerre inequality of order two. We introduce the concept of a *root sequence* denoted by $\{\sqrt[n]{a_n}\}_{n\geqslant 1}$, which corresponds to a nonnegative real suquence $\{a_n\}_{n\geqslant 1}$. Also, we study the sequence $\{a_n\}_{n\geqslant 1}$ which is a *P-recursive sequence.* Recall that a sequence $\{a_n\}_{n\geqslant 1}$ is called a P-recursive sequence of order $d$ if it satisfies a recurrence relation of the form

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0,$$

where $p_i(n)$ are polynomials in $n$.

We can solve the given problem by establishing upper and lower bounds on $u_n = \sqrt[n-1]{a_{n-1}}\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n}^2$. To achieve this, we use the asymptotic expression of $u_n$. Hou and Li [8] provided the asymptotic expansion of $\log u_n$, then we can derive the asymptotic expansion of $u_n$,

$$\begin{aligned}1 - \frac{\mu_0}{n^2} + \sum_{j=1}^{\rho-1} \frac{\mu_j(j/\rho-1)(j/\rho-2)}{n^{3-j/\rho}} + \frac{r(2\log n - 3)}{n^3}\\ + \sum_{s=0}^{M} \frac{\tilde{b}_s(1+s/\rho)(2+s/\rho)}{n^{3+s/\rho}} + \cdots + o\left(\frac{1}{n^{1+M/\rho}}\right),\end{aligned}$$

where $\tilde{b}_0 = \log b_0$ and $\tilde{b}_i$ is a polynomial in $b_1/b_0, \ldots, b_i/b_0$.

Based on the asymptotic expression of $a_n$, we can establish both upper and lower bounds for $a_n$ as well as for the ratio $a_{n+1}/a_n$. Similarly, according to the asymptotic form of $u_n$, we can derive upper and lower bounds for $u_n$. Consequently, we introduce a method to demonstrate that the sequence $\{\sqrt[n]{a_n}/n!\}_{n\geqslant N}$ and $\{\sqrt[n+1]{a_{n+1}}/(\sqrt[n]{a_n}n!)\}_{n\geqslant N}$ satisfy the higher order Turán inequality and the Laguerre inequality of order two. Finally, we provide examples to illustrate the proof method.

This paper is organized as follows. In Section 2, we begin by utilizing the bounds of $a_n$ and $a_{n+1}/a_n$ provided by [8] to establish the criteria for calculating the bounds of $u_n$. Subsequently, we give a method to compute $N$, such that the sequences $\{\sqrt[n]{a_n}/n!\}_{n\geqslant N}$ and $\{\sqrt[n+1]{a_{n+1}}/(\sqrt[n]{a_n}n!)\}_{n\geqslant N}$ satisfy the higher order Turán inequality. In Section 3, we use the upper and lower bounds derived in Section 2 to establish sufficient conditions for proving that the sequences $\{\sqrt[n]{a_n}/n!\}_{n\geqslant N}$ and $\{\sqrt[n+1]{a_{n+1}}/(\sqrt[n]{a_n}n!)\}_{n\geqslant N}$ satisfy the Laguerre inequality of order two.

## 2 The higher order Turán inequality

In this section, we will first present a systematic approach to prove that the sequence $\left\{\sqrt[n]{a_n}/n!\right\}_{n\geqslant 1}$ satisfies the higher order Turán inequality when $\{a_n\}_{n\geqslant 1}$ is a P-recursive sequence.

Hou and Li [7] provided a method to prove that the higher order Turán inequality hold for the P-recursive sequence $\{a_n\}_{n\geqslant 1}$.

**Lemma 1. ([7, Theorem 5.2])** Let

$$t(x, y) = 4(1-x)(1-y) - (1-xy)^2. \tag{6}$$

If there exists an integer $N$ and two rational functions $f_n$ and $g_n$ about $n$, such that for all $n \geqslant N$,

$$f_n < a_{n-1}a_{n+1}/a_n^2 < g_n$$

and

$$t(f_n, f_{n+1}) > 0,\ t(f_n, g_{n+1}) > 0,\ t(g_n, f_{n+1}) > 0,\ t(g_n, g_{n+1}) > 0.$$

Then $\{a_n\}_{n\geqslant N}$ satisfies the higher order Turán inequality.

We will use the lemma above to prove the sequence $\{\sqrt[n]{a_n}/n!\}_{n\geqslant 1}$ satisfies the higher order Turán inequality. To accomplish this, we need to determine the lower and upper bounds of $u_n = \sqrt[n-1]{a_{n-1}}\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n}^2$. Additionally, we will provide a criterion for establishing these bounds.

**Theorem 2.** *Let $\{a_n\}_{n\geqslant 1}$ be a positive sequence. Suppose we can find a lower bound $s_n$ and an upper bound $S_n$ of $a_n$, a lower bound $f_n$ and an upper bound $g_n$ of $r_n = \frac{a_{n+1}}{a_n}$, and two rational functions $\widetilde{f_n}$ and $\widetilde{g_n}$, such that for $n \geqslant N$,*

$$(n^2-n)\log f_n - (n^2+n-2)\log g_{n-1} + 2\log s_{n-1} > (n-1)n(n+1)\log \widetilde{f_n},$$

*and*

$$(n^2-n)\log g_n - (n^2+n-2)\log f_{n-1} + 2\log S_{n-1} < (n-1)n(n+1)\log \widetilde{g_n}.$$

*Then we have $\widetilde{f_n} < u_n < \widetilde{g_n}$ for $n \geqslant N$.*

*Proof.* In order to prove $\widetilde{f_n} < u_n < \widetilde{g_n}$, it suffices to prove

$$\widetilde{f_n}^{(n-1)n(n+1)} < \frac{a_{n-1}^{n(n+1)}a_{n+1}^{n(n-1)}}{a_n^{2(n-1)(n+1)}} < \widetilde{g_n}^{(n-1)n(n+1)}.$$

Equivalently, we need to verify the following inequalities

$$(n-1)n(n+1)\log \widetilde{f_n} < n(n+1)\log a_{n-1} + n(n-1)\log a_{n+1} - 2(n-1)(n+1)\log a_n$$

and

$$n(n+1)\log a_{n-1} + n(n-1)\log a_{n+1} - 2(n-1)(n+1)\log a_n < (n-1)n(n+1)\log \widetilde{g_n}.$$

By substituting $a_n = a_{n-1}r_{n-1}$ and $a_{n+1} = a_n r_n$ into the above two inequalities, we obtain

$$(n-1)n(n+1)\log \widetilde{f_n} < (n^2-n)\log r_n - (n^2+n-2)\log r_{n-1} + 2\log a_{n-1}$$

and

$$(n^2-n)\log r_n - (n^2+n-2)\log r_{n-1} + 2\log a_{n-1} < (n-1)n(n+1)\log \widetilde{g_n}.$$

Consequently, the theorem is proven immediately. $\square$

Using the above lower and upper bounds and Lemma 1, we can establish the criteria that enable the sequence $\{\sqrt[n]{a_n}/n!\}_{n\geqslant 1}$ to satisfy the higher order Turán inequality.

**Theorem 3.** *Let $\{a_n\}_{n\geqslant 1}$ be a positive sequence,*

$$p_n = \frac{n}{n+1}\widetilde{f_n},\ q_n = \frac{n}{n+1}\widetilde{g_n}.$$

*If there exists an integer $N$ such that for $n \geqslant N$,*

$$t(p_n, p_{n+1}) > 0,\ t(p_n, q_{n+1}) > 0,\ t(q_n, p_{n+1}) > 0,\ t(q_n, q_{n+1}) > 0.$$

*Then the sequence $\{\sqrt[n]{a_n}/n!\}_{n\geqslant N}$ satisfies the higher order Turán inequality.*

Similarly, we can give the sufficient conditions for the sequence $\left\{\sqrt[n+1]{a_{n+1}}/(\sqrt[n]{a_n}n!)\right\}_{n\geqslant 1}$ to satisfy the higher order Turán inequality.

**Theorem 4.** *Let $\{a_n\}_{n\geqslant 1}$ be a positive sequence,*

$$\widetilde{p_n} = \frac{n}{n+1}\frac{\widetilde{f_{n+1}}}{\widetilde{g_n}},\ \widetilde{q_n} = \frac{n}{n+1}\frac{\widetilde{g_{n+1}}}{\widetilde{f_n}}.$$

*If there exists an integer $N$ such that for $n \geqslant N$,*

$$t(\widetilde{p_n}, \widetilde{p_{n+1}}) > 0,\ t(\widetilde{p_n}, \widetilde{q_{n+1}}) > 0,\ t(\widetilde{q_n}, \widetilde{p_{n+1}}) > 0,\ t(\widetilde{q_n}, \widetilde{q_{n+1}}) > 0.$$

*Then the sequence $\left\{\sqrt[n+1]{a_{n+1}}/(\sqrt[n]{a_n}n!)\right\}_{n\geqslant N}$ satisfies the higher order Turán inequality.*

Based on the Lemma 1, the proof of Theorems 3 and 4 is obvious. So we omit the proof process. Next, we will provide an example to illustrate the application of the above theorems.

**Theorem 5. ([17, Conjecture 4.7])** Let

$$B_n = \sum_{k=1}^{n} \frac{2}{n(n+1)^2}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$$

be the Baxter number. Then the sequences $\{\sqrt[n]{B_n}/n!\}_{n\geqslant 2}$ and $\left\{\sqrt[n+1]{B_{n+1}}/(\sqrt[n]{B_n}n!)\right\}_{n\geqslant 2}$ satisfy the higher order Turán inequality.

*Proof.* By Zeilberger's algorithm, we get that $B_n$ satisfies the following recurrence relation,

$$(n+3)(n+4)B_{n+1} = (7n^2+21n+12)B_n + 8n(n-1)B_{n-1}$$

with initial values

$$B_0 = 1,\ B_1 = 1,\ B_2 = 2,\ B_3 = 6.$$

By utilizing algorithm `Asy` from the `Mathematica` package `P-rec.m`, we can obtain the asymptotic expansion of $B_n$,

$$C \cdot 8^n n^{-4}\left(1-\frac{22}{3n}+\frac{955}{27n^2}+o\left(\frac{1}{n^2}\right)\right),$$

where $C$ is a constant.

Then employing the algorithm `RootLog`, we derive a lower bound $f_n$ and an upper bound $g_n$ for $r_n = B_{n+1}/B_n$ when $n \geqslant 753$,

$$f_n = 8-\frac{32}{n}+\frac{413}{3n^2},\ g_n = 8-\frac{32}{n}+\frac{419}{3n^2}.$$

Next, we will demonstrate by mathematical induction that $s_n = 8^n n^{-5}$ is a lower bound and $S_n = 8^n n^{-3}$ is an upper bound for $B_n$ when $n \geqslant 3$. Assuming that $s_n < B_n < S_n$, then we will prove $s_{n+1} < B_{n+1} < S_{n+1}$. On the one hand, we have

$$B_{n+1} = r_n B_n > f_n s_n$$

and

$$\frac{f_n s_n - s_{n+1}}{s_n} = \frac{413+1969n+3674n^2+3290n^3+1345n^4+173n^5+24n^6}{3n^2(1+n)^5},$$

which is positive for $n \geqslant 1$. On the other hand, we have

$$B_{n+1} = r_n B_n < g_n S_n$$

and

$$\frac{S_{n+1} - g_n S_n}{S_n} = \frac{-419-1161n-993n^2-203n^3+24n^4}{3n^2(1+n)^3},$$

which is positive for $n \geqslant 13$. Therefore, we can conclude that $s_n < B_n < S_n$ for $n \geqslant 753$. Checking the first 752 items, we finally get that

$$s_n < B_n < S_n, \quad \forall\, n \geqslant 3.$$

By utilizing the asymptotic expansion of $B_n$, we obtain the asymptotic expression of $u_n = \sqrt[n-1]{B_{n-1}}\sqrt[n+1]{B_{n+1}}/\sqrt[n]{B_n}^2$ as follows,

$$1+\left(\frac{12}{n^3}-\frac{8}{n^3}\log n\right)+o\left(\frac{1}{n^3}\right).$$

Consequently, we will prove using Theorem 2 that $\widetilde{f_n} = 1-\frac{1}{n^2}$ serves as a lower bound and $\widetilde{g_n} = 1-\frac{8}{n^3}$ acts as an upper bound for $u_n$ when $n \geqslant 14$. Let

$$D_1(n) = (n^2-n)\log f_n - (n^2+n-2)\log g_{n-1} + 2\log s_{n-1} - (n-1)n(n+1)\log \widetilde{f_n}$$

and

$$D_2(n) = (n-1)n(n+1)\log \widetilde{g_n} - (n^2-n)\log g_n + (n^2+n-2)\log f_{n-1} - 2\log S_{n-1}.$$

Hence, we need to prove that $D_1(n) > 0$ and $D_2(n) > 0$ when $n \geqslant 14$. We know that $D_1^{(4)}(n)$ is a rational function of $n$ and based on the largest real root of the numerator of $D_1^{(4)}(n)$, we get that $D_1^{(4)}(n) > 0$ for $n \geqslant 32$. Then we obtain the following formulae by `Mathematica`,

$$\lim_{n\to+\infty} D_1(n) = \infty, \quad \lim_{n\to+\infty} D_1^{'}(n) = 1, \quad \lim_{n\to+\infty} D_1^{''}(n) = \lim_{n\to+\infty} D_1^{'''}(n) = 0.$$

Thus, we deduce that

$$D_1^{'''}(n) < 0, \quad D_1^{''}(n) > 0, \quad D_1^{'}(n) > 0, \quad \forall\, n \geqslant 32.$$

Since $D_1(32) > 0$, we conclude that $D_1(n) > 0$ for $n \geqslant 32$. By a similiar proof process, we can show that $D_2(n) > 0$ for $n \geqslant 44$. Consequently, we have $\widetilde{f_n} < u_n < \widetilde{g_n}$ for $n \geqslant 44$. By verifying the initial values, we derive that

$$\widetilde{f_n} < u_n < \widetilde{g_n}, \quad \forall\, n \geqslant 14.$$

Then when $n \geqslant 14$, let

$$\begin{aligned} p_n &= \frac{n}{n+1}\widetilde{f_n} = \frac{n}{n+1}\left(1 - \frac{1}{n^2}\right), \\ q_n &= \frac{n}{n+1}\widetilde{g_n} = \frac{n}{n+1}\left(1 - \frac{8}{n^3}\right). \end{aligned}$$

By direct calculation, we find that

$$t(p_n, p_{n+1}) = \frac{4}{n(1+n)^2},$$

which is positive for $n \geqslant 1$.

$$t(p_n, q_{n+1}) = \frac{-49 + 240n + 122n^2 + 100n^3 + 15n^4 + 4n^5}{n^2(1+n)^4(2+n)^2},$$

which is positive for $n \geqslant 1$.

$$t(q_n, p_{n+1}) = \frac{-32 + 48n + 3n^2 + 4n^3}{n^2(1+n)^4},$$

which is positive for $n \geqslant 1$.

$$\begin{aligned} t(q_n, q_{n+1}) = \frac{4}{n^4(1+n)^6(2+n)^2}&(-784 + 672n + 728n^2 + 840n^3 + 529n^4 + 325n^5 + 98n^6 \\ &+ 34n^7 + 5n^8 + n^9), \end{aligned}$$

which is positive for $n \geqslant 1$.

Therefore, we can conclude that the sequence $\{\sqrt[n]{B_n}/n!\}_{n\geqslant 14}$ satisfies the higher order Turán inequality according to Theorem 3. By examining the initial terms, we can further deduce that sequence $\{\sqrt[n]{B_n}/n!\}_{n\geqslant 2}$ satisfies the higher order Turán inequality.

Next when $n \geqslant 14$, let

$$\begin{aligned}\widetilde{p_n} &= \frac{n}{n+1}\frac{\widetilde{f_{n+1}}}{\widetilde{g_n}} = \frac{n^5(2+n)}{(1+n)^3\left(-8+n^3\right)},\\ \widetilde{q_n} &= \frac{n}{n+1}\frac{\widetilde{g_{n+1}}}{\widetilde{f_n}} = \frac{n^3(7+4n+n^2)}{(1+n)^5}.\end{aligned}$$

So we can get that

$$\begin{aligned}& t(\widetilde{p_n},\widetilde{p_{n+1}})\\ =& \frac{4}{(-2+n)^2(-1+n)^2(1+n)^3(2+n)^4(4+2n+n^2)^2(7+4n+n^2)^2}(40320+178496n\\ &+290816n^2+145312n^3-177824n^4-322592n^5-192458n^6-18235n^7+39578n^8\\ &+20340n^9+35n^{10}-3304n^{11}-1083n^{12}+71n^{14}+15n^{15}+n^{16})>0, \quad \forall n\geqslant 4.\end{aligned}$$

$$\begin{aligned}& t(\widetilde{p_n},\widetilde{q_{n+1}})\\ =& \frac{4}{(-2+n)^2(1+n)^3(2+n)^8(4+2n+n^2)^2}(6144+36864n+94720n^2+132864n^3\\ &+102144n^4+26944n^5-25056n^6-29952n^7-14232n^8-3036n^9+168n^{10}+283n^{11}\\ &+81n^{12}+12n^{13}+n^{14})>0, \quad \forall n\geqslant 4.\end{aligned}$$

$$\begin{aligned}& t(\widetilde{q_n},\widetilde{p_{n+1}})\\ =& \frac{4}{(-1+n)^2(1+n)^5(2+n)^6(7+4n+n^2)}(360+2468n+7086n^2+7107n^3-1451n^4\\ &-8433n^5-6858n^6-2217n^7+38n^8+261n^9+88n^{10}+14n^{11}+n^{12})>0, \quad \forall n\geqslant 3.\end{aligned}$$

$$\begin{aligned}& t(\widetilde{q_n},\widetilde{q_{n+1}})\\ =& \frac{4}{(1+n)^5(2+n)^{10}}(384+3456n+14080n^2+31088n^3+41512n^4+36784n^5+23084n^6\\ &+10531n^7+3536n^8+868n^9+151n^{10}+17n^{11}+n^{12})>0, \quad \forall n\geqslant 1.\end{aligned}$$

Consequently, we know that the sequence $\{\sqrt[n+1]{B_{n+1}}/(\sqrt[n]{B_n}n!)\}_{n\geqslant 14}$ satisfies the higher Turán inequality by Theorem 4. After verifying the first 13 items, we ultimately conclude that the sequence $\{\sqrt[n+1]{B_{n+1}}/(\sqrt[n]{B_n}n!)\}_{n\geqslant 2}$ satisfies the higher Turán inequality. $\square$

By using the `Mathematica` package `P-rec.m` (which is available at [1]), we can show that the higher order Turán inequality holds for the two sequences associated with the root sequence of many combinatorial sequences, such as Fine numbers $\{f_n\}_{n\geqslant 4}$, Motzkin numbers $\{M_n\}_{n\geqslant 2}$, Cohen numbers $\{C_n\}_{n\geqslant 2}$, large Schröder numbers $\{S_n\}_{n\geqslant 2}$, the numbers of the set of all tree-like polyhexes with $n+1$ hexagons $\{h_n\}_{n\geqslant 2}$, the numbers of walks on cubic lattice with $n$ steps, starting and finishing on the $xy$ plane and never going below it $\{w_n\}_{n\geqslant 2}$, the numbers of $n\times n$ matrices with nonnegative entries and every row and column sum 2 $\{t_n\}_{n\geqslant 2}$, Domb numbers $\{D_n\}_{n\geqslant 2}$ and so on. Then we list the lower and upper bounds of $u_n = \sqrt[n-1]{a_{n-1}}\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n}^2$ for $n \geqslant N$.

Table 1: The lower and upper bounds

| | the lower bound | the upper bound | $N$ |
|---|---|---|---|
| $f_n$ | $1-\frac{1}{n^2}$ | $1-\frac{3}{n^3}$ | 6 |
| $M_n$ | $1-\frac{1}{n^2}$ | $1-\frac{3}{n^3}$ | 11 |
| $C_n$ | $1-\frac{1}{n^2}$ | $1-\frac{5}{n^3}$ | 9 |
| $S_n$ | $1-\frac{1}{n^2}$ | $1-\frac{3}{n^3}$ | 9 |
| $h_n$ | $1-\frac{1}{n^2}$ | $1-\frac{3}{n^3}$ | 6 |
| $w_n$ | $1-\frac{1}{n^2}$ | $1-\frac{3}{n^3}$ | 26 |
| $t_n$ | $1-\frac{2}{n^2}+\frac{1}{n^3}$ | $1-\frac{1}{n^2}$ | 5 |
| $D_n$ | $1-\frac{1}{n^2}$ | $1-\frac{3}{n^3}$ | 6 |

## 3 The Laguerre inequality of order 2

In this section, we will give sufficient conditions for the sequences $\{\sqrt[n]{a_n}/n!\}_{n\geqslant 1}$ and $\{\sqrt[n+1]{a_{n+1}}/(\sqrt[n]{a_n}n!)\}_{n\geqslant 1}$ to satisfy the Laguerre inequality of order two, where the sequence $\{a_n\}_{n\geqslant 1}$ is a P-recursive sequence. Li [11] presented a method to determine the explicit value of $N$ such that Laguerre inequality of order two holds for $\{a_n\}_{n\geqslant N}$.

**Lemma 6. ([11, Theorem 5.1])** If there exists an integer $N$, two rational functions $f_n$ and $g_n$, such that for all $n \geqslant N$,

$$f_n < \frac{a_{n-1}a_{n+1}}{a_n^2} < g_n,$$

and

$$f_{n-1}f_n^2 f_{n+1} - 4g_n + 3 > 0.$$

Then $\{a_n\}_{n\geqslant N}$ satisfies Laguerre inequality of order two.

We will utilize the lemma mentioned above to provide sufficient conditions for the sequences $\{\sqrt[n]{a_n}/n!\}_{n\geqslant 1}$ and $\{\sqrt[n+1]{a_{n+1}}/(\sqrt[n]{a_n}n!)\}_{n\geqslant 1}$ to satisfy the Laguerre inequality of order two. Similarly to the previous section, let $\widetilde{f_n}$ and $\widetilde{g_n}$ represent the lower and upper bounds of $u_n = \sqrt[n-1]{a_{n-1}}\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n}^2$, respectively.

**Theorem 7.** *Let $\{a_n\}_{n\geqslant 1}$ be a positive sequence and*

$$p_n = \frac{n}{n+1}\widetilde{f_n},\ q_n = \frac{n}{n+1}\widetilde{g_n}.$$

*If there exists an integer $N$ such that for $n \geqslant N$,*

$$p_{n-1}p_n^2p_{n+1} - 4q_n + 3 > 0,$$

*then the sequence $\{\sqrt[n]{a_n}/n!\}_{n\geqslant N}$ satisfies the Laguerre inequality of order two.*

**Theorem 8.** *Let $\{a_n\}_{n\geqslant 1}$ be a positive sequence and*

$$\widetilde{p_n} = \frac{n}{n+1}\frac{\widetilde{f_{n+1}}}{\widetilde{g_n}},\ \widetilde{q_n} = \frac{n}{n+1}\frac{\widetilde{g_{n+1}}}{\widetilde{f_n}}.$$

*If there exists an integer $N$ such that for $n \geqslant N$,*

$$\widetilde{p_{n-1}}\widetilde{p_n}\widetilde{q_n}\widetilde{p_{n+1}} - 4\widetilde{q_n} + 3 > 0,$$

*then the sequence $\{\sqrt[n+1]{a_{n+1}}/(\sqrt[n]{a_n}n!)\}_{n\geqslant N}$ satisfies the Laguerre inequality of order two.*

By applying Lemma 6, we can directly obtain the two theorems above, thus we omit the proof process. Subsequently, we will illustrate the application of the theorems with an example.

**Theorem 9.** *Let $H_n$ denote the number of $n \times n$ $(0,1)$-matrices with row and column sum 2. The sequences $\{\sqrt[n]{H_n}/n!\}_{n\geqslant 1}$ and $\{\sqrt[n+1]{H_{n+1}}/(\sqrt[n]{H_n}n!)\}_{n\geqslant 2}$ satisfy the Laguerre inequality of order two.*

*Proof.* According to Zeilberger's algorithm, we have the following recursion relation for $H_n$,

$$2H_n - 2n(n-1)H_{n-1} - n(n-1)^2H_{n-2} = 0,$$

with initial values

$$H_0 = 1,\ H_1 = 0,\ H_2 = 1,\ H_3 = 6.$$

Using the `Mathematica` package `P-rec.m`, we obtain the asymptotic expression of $H_n$,

$$C \cdot e^{-2n}n^{\frac{1}{2}+2n}\left(1 - \frac{5}{24n} - \frac{47}{1152n^2} + o\left(\frac{1}{n^2}\right)\right),$$

where $C$ is a constant.

Additionally, for $n \geqslant 5$, we can establish a lower bound $f_n$ and an upper bound $g_n$ for $r_n = H_{n+1}/H_n$ as follows,

$$f_n = n^2 + \frac{3n}{2} + \frac{3}{4} + \frac{1}{4n} - \frac{19}{16n^2},\ g_n = n^2 + \frac{3n}{2} + \frac{3}{4} + \frac{1}{4n} + \frac{13}{16n^2}.$$

Then we aim to prove that $s_n = e^{-2n} n^{\frac{1}{4}+2n}$ serves as a lower bound and $S_n = e^{-2n} n^{1+2n}$ serves as an upper bound of $H_n$ when $n \geqslant 5$ by induction. Suppose that $s_n < H_n < S_n$, we need to prove $s_{n+1} < H_{n+1} < S_{n+1}$. Since that

$$f_n s_n < H_{n+1} = r_n H_n < g_n S_n.$$

Thus, it is sufficient to prove

$$s_{n+1} < f_n s_n,\ g_n S_n < S_{n+1}.$$

For $n \geqslant 5$, we define

$$\Delta_1(n) = \log f_n + \log s_n - \log s_{n+1},\ \Delta_2(n) = \log S_{n+1} - \log g_n - \log S_n.$$

Consequently, we know that $\Delta_1^{''}(n)$ and $\Delta_2^{''}(n)$ are rational functions about $n$. By analyzing the largest roots of their numerators, we conclude that $\Delta_1^{''}(n) > 0$ and $\Delta_2^{''}(n) > 0$ for $n \geqslant 3$. By calculation, we also establish that

$$\lim_{n\to+\infty} \Delta_1^{'}(n) = \lim_{n\to+\infty} \Delta_2^{'}(n) = 0.$$

Given that $\lim_{n\to\infty} \Delta_1(n) = \lim_{n\to\infty} \Delta_2(n) = 0,\ \Delta_1(3) > 0$ and $\Delta_2(3) > 0$, we derive that $\Delta_1(n) > 0$ and $\Delta_2(n) > 0$ when $n \geqslant 5$. Then examining the initial values, we observe that for $n \geqslant 5$,

$$s_n < H_n < S_n.$$

From the asymptotic expansion of $H_n$, we can derive the asymptotic expansion of $u_n = \sqrt[n-1]{H_{n-1}}\sqrt[n+1]{H_{n+1}}/\sqrt[n]{H_n}^2$,

$$1 - \frac{2}{n^2} + \left(-\frac{3}{2n^3} + \frac{\log n}{n^3}\right) + o\left(\frac{1}{n^3}\right).$$

Following a similar proof process as above, we can establish that $\widetilde{f_n} = 1 - \frac{2}{n^2} + \frac{1}{n^3}$ is a lower bound and $\widetilde{g_n} = 1 - \frac{1}{n^2}$ is an upper bound for $u_n$ when $n \geqslant 5$.

Therefore, when $n \geqslant 5$, let

$$\begin{aligned} p_n &= \frac{n}{n+1}\widetilde{f_n} = \frac{n}{n+1}\left(1 - \frac{2}{n^2} + \frac{1}{n^3}\right), \\ q_n &= \frac{n}{n+1}\widetilde{g_n} = \frac{n}{n+1}\left(1 - \frac{1}{n^2}\right). \end{aligned}$$

Through calculation, we can obtain that

$$\begin{aligned} &p_{n-1}p_n^2 p_{n+1} - 4q_n + 3 \\ = &\frac{2 + 3n - 14n^2 - 6n^3 + 62n^4 + 74n^5 + 26n^6 + 2n^7}{n^4(1+n)^4(2+n)} > 0, \quad \forall\, n \geqslant 1. \end{aligned}$$

Thus, based on the Theorem 7, it can be concluded that the sequence $\{\sqrt[n]{H_n}/n!\}_{n\geqslant 5}$ satisfies the Laguerre inequality of order two. After checking the first 4 items, we finally deduce that the sequence $\{\sqrt[n]{H_n}/n!\}_{n\geqslant 1}$ satisfies the Laguerre inequality of order two.

Next, when $n \geqslant 5$, we define

$$\widetilde{p_n} = \frac{n}{n+1}\frac{\widetilde{f_{n+1}}}{\widetilde{g_n}} = \frac{n^4(1+3n+n^2)}{(-1+n)(1+n)^5},$$
$$\widetilde{q_n} = \frac{n}{n+1}\frac{\widetilde{g_{n+1}}}{\widetilde{f_n}} = \frac{n^5(2+n)}{(1+n)^3(1-2n+n^3)}.$$

Consequently, we can get that

$$\begin{aligned}&\widetilde{p_{n-1}}\widetilde{p_n}\widetilde{q_n}\widetilde{p_{n+1}} - 4\widetilde{q_n} + 3\\ =&\frac{1}{(-2+n)(-1+n)(1+n)^4(2+n)^4(-1+n+n^2)}(-96-336n-144n^2+893n^3\\ &+1458n^4+597n^5-227n^6-192n^7-15n^8+6n^9),\end{aligned}$$

which is positive for $n \geqslant 8$. According to Theorem 8, we know that the sequence $\{\sqrt[n+1]{H_{n+1}}/(\sqrt[n]{H_n}n!)\}_{n\geqslant 8}$ satisfies the Laguerre inequality of order two. By verifying the initial values, we can further conclude that the sequence $\{\sqrt[n+1]{H_{n+1}}/(\sqrt[n]{H_n}n!)\}_{n\geqslant 2}$ satisfies the Laguerre inequality of order two. $\square$

The method described above allows us to prove that two sequences associated with the root sequence of Fine numbers $\{f_n\}_{n\geqslant 3}$ and two sequences related to the root sequence of other sequences in Section 2 satisfy the Laguerre inequality of order two for $n \geqslant 1$.

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## References

[1] https://github.com/hou-tju/root-sequences.

[2] W.Y.C. Chen. The spt-function of Andrews. In *Surveys in Combinatorics*, volume 440 of *London Math. Soc. Lecture Note Ser.*, pages 141–203. Cambridge Univ. Press, Cambridge, 2017.

[3] W.Y.C. Chen, D.X.Q. Jia and L.X.W. Wang. Higher order Turán inequalities for the partition function. *Trans. Amer. Math. Soc.*, 372(3): 2143–2165, 2019.

[4] D.K. Dimitrov. Higher order Turán inequalities. *Proc. Amer. Math. Soc.*, 126(7): 2033–2037, 1998.

[5] L.-M. Dou and L.X.W. Wang. Higher order Laguerre inequalities for the partition function. *Discrete Math.*, 346(6): 113366, 2023.

[6] M. Griffin, K. Ono, L. Rolen and D. Zagier. Jensen polynomials for the Riemann zeta function and other sequences. *Proc. Natl. Acad. Sci. USA*, 116(23): 11103–11110, 2019.

[7] Q.-H. Hou and G. Li. Log-concavity of $P$-recursive sequences. *J. Symb. Comput.*, 107: 251–268, 2021.

[8] Q.-H. Hou and Z. Li. Log-behavior of the root sequences of P-recursive sequences. arXiv:2310.19234vi, 2023.

[9] J.L.W.V. Jensen. Recherches sur la théorie des équations. *Acta Math.*, 36: 181–195, 1913.

[10] E. Laguerre. Oeuvres, volume 1. Gaauthier-Villars, Paris, 1989.

[11] G.-J. Li. $\ell$-Log-momotonic and Laguerre inequality of P-recursive sequences. arXiv:2206.13922vi, 2022.

[12] E.Y.S. Liu and H.W.J. Zhang. Inequalities for the overpartition function. *Ramanujan J.*, 54: 485–509, 2021.

[13] G. Pólya and J. Schur. Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen. *J. Reine Angew. Math.*, 144: 89–113, 1914.

[14] I. Wagner. On a new class of Laguree-Pólya type functions with applications in number theory. *Pacific J. Math.*, 320(1): 177–192, 2022.

[15] L.X.W. Wang. Higher order Turán inequalities for combinatorial sequences. *Adv. Appl. Math.*, 110: 180–196, 2019.

[16] L.X.W. Wang and E.Y.Y. Yang. Laguerre inequalities for discrete sequences. *Adv. Appl. Math.*, 139: 102357, 2022.

[17] J.J.Y. Zhao. Inequalities associated with the Baxter number. *Glas. Mat.*, 58(1): 1–16, 2023.