

The First Roe Homology Group of Locally Finite Graphs

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Abstract

We give a decomposition of the first group of so-called “Roe” homology of locally finite, connected graphs. We show that this group can be decomposed as a direct sum of two terms: the first counts the number of ends of the graph, while the second measures the existence of cycles that are not decomposable into smaller cycles (in some suitably coarse sense).

Mathematics Subject Classifications: 18G85, 51F30, 05C63

1 Introduction

We study the coarse structure of the cycle space of locally finite, connected graphs. In similar settings the coarse structure of the cycle space has an influence on the value of several coarse invariants, such as combinatorial cost of graph sequences and their β -invariants [Ele07] or cost of orbit equivalence relations [Gab00]. In [Win24] for example, an analysis of the cycles leads to insights about geometric property (T). Roe homology¹ is a certain homology theory for coarse spaces, defined by Roe [Roe03] and which specialises to (locally finite, connected) graphs via metric spaces [Roe06, Mit01]. In the present paper, we describe the first Roe homology group of locally finite, connected graphs. As we will see, this group can be understood by analysing the cycle space of the graph at hand. It is a quotient of a cycle space introduced by Bonnington and Richter [BR03]. However, excluding effects of the space of ends of the graph, we will be able to reduce our analysis to that of the quotient of a smaller cycle space, introduced and studied by Diestel et al. [Die05, DK04a, DK04b, DK04c]. The main difference between their and our analysis lies in the fact that for Diestel all cycles are created equal, while for us their length matters and must be taken into account. Essential in these papers and also ours is the study of so-called thin sums of cycles (i.e. well-defined infinite sums) [BG11]. The results of this paper arose as a first step towards an understanding of the (first) uniformly finite

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¹This denomination does not appear to be commonplace.

homology group, introduced by Block and Weinberger [BW92]. The first Roe homology group can be more thoroughly described than that of uniformly finite homology, with fewer restrictions on the graph. Some of the preliminary results presented here will be useful in a follow-up paper. For this reason some of our constructions will be more explicit than strictly necessary.

The remainder of this section is dedicated to a cursory presentation and commentary of the main results of the paper. Fix a locally finite, connected graph X (all vertices have finite degree), and consider the increasing Rips complexes $R_r X$ ($r \in \mathbb{N}$) defined over X . On any one of these complexes, one can define a chain complex whose chains are functions from simplices to the coefficient ring A , or “infinite formal sums” of simplices. The inclusion of chains corresponding to a given Rips complex to a larger Rips complex are chain maps, and one can take the inductive limit of those chain complexes. The homology $H_\bullet^\infty(X, A)$ of this inductive limit is what we call *Roe homology*—it is a special case of Roe’s coarse homology, defined in greater generality (see e.g. [Roe03]). In particular, $H_\bullet^\infty(X, A)$ is invariant under quasi-isometries [Mos09, Step 3 in the proof of Theorem 12].

We focus only on the *first* group of Roe homology, $H_1^\infty(X, A)$, which enjoys a relatively intuitive combinatorial description. Recall that a (A -valued) flow on a locally finite graph X is an assignment, to each (preliminarily oriented) edge of X , of a value in A , in such a way that, for all vertices v of X , the sum of values of edges pointing to v is equal to the sum of values of edges pointing away from v (i.e., “in=out”).

Theorem 6. If X is a locally finite, connected graph and A a ring, then

$$H_1^\infty(X, A) = \frac{Z_1(X, A)}{\mathcal{C}_\infty(X, A)},$$

where $Z_1(X, A)$ is the space of all (A -valued) flows on X , and $\mathcal{C}_\infty(X, A)$ is the space of all flows which can be decomposed as (potentially infinite) sums of cycles of bounded length.

In a sense, this description tells us that $H_1^\infty(X, A)$ measures the existence of flows which do not arise as “local” phenomena.

In [Dia15], Diana shows that for a (locally finite) tree T with a finite number n of ends, $H_1^\infty(T, A)$ has dimension $n - 1$ ². If T has infinitely many ends, a similar description can be obtained:

Theorem 8. If T is a locally finite tree, and $\mathcal{B}$ a carefully chosen³ set of birays on T which will be seen to satisfy $|\mathcal{B}| + 1 = |\text{Ends } T|$, then

$$H_1^\infty(T, A) \cong A^{\mathcal{B}}.$$

Thus, as can be guessed from theorem 6, the group $H_1^\infty(T, A)$ is entirely characterised by the ends of T .

²Diana’s result uses uniformly finite homology. However for locally finite finite-ended trees, Roe and uniformly finite homology agree.

³Such a set always exists.

We then obtain a decomposition of $H_1^\infty(X, A)$ in the general case. Recall that if T is a subtree of the graph X , it is called *end-faithful* if the inclusion $T \subseteq X$ induces a bijection $\text{Ends } T \rightarrow \text{Ends } X$.

Theorem 12. Let X be a locally finite, connected graph and T an end-faithful subtree of X . Then

$$H_1^\infty(X, A) \cong H_1^\infty(T, A) \oplus \frac{\mathcal{C}(X, A)}{\mathcal{C}_\infty(X, A)},$$

where $\mathcal{C}(X, A)$ is the space of flows on X that can be written as (potentially infinite) sums of cycles.

Note that $\mathcal{C}_\infty(X, A)$ differs from $\mathcal{C}(X, A)$ in that the latter includes sums of cycles of arbitrarily large length, while in the former, each sum only consists of cycles of at most a given length. So the theorem states that the first homology of X is the direct sum of two terms: the first comes from the ends of X , while the second measures the extent to which “sums of cycles” differ from “sums of cycles of bounded length”.

From this decomposition, the following consequences are clear:

- If X has at least two ends, its first homology is non-trivial.
- If X has cycles of arbitrarily large length that cannot be decomposed into smaller cycles (think: chaining together increasingly long cycle graphs), its first homology is also non-trivial. This depends on the coefficient ring. It might vanish for one ring but not for another.

Finally, in the case where $A = \mathbb{Z}/2\mathbb{Z}$, we can further improve our understanding of $H_1^\infty(X, A)$. The proof of this theorem uses both compactness/finiteness of $A = \mathbb{Z}/2\mathbb{Z}$, and the fact that A is a field:

Theorem 18. If X is a locally finite, connected graph and $A = \mathbb{Z}/2\mathbb{Z}$, then $\frac{\mathcal{C}(X, A)}{\mathcal{C}_\infty(X, A)}$ is non-vanishing if and only if it is infinite-dimensional.

2 Flows, cycles, and H_1^∞

2.1 Conventions on graphs

For the purposes of this paper, a *graph* X is the data of a set VX of *vertices* and a set $EX \subseteq \mathcal{P}_2(VX)$ of *edges*, where $\mathcal{P}_2(VX)$ denotes the subsets of cardinality two. Two vertices, u, v are *neighbours*, written $u \sim v$, if $\{u, v\} \in EX$ and the *valency* or *degree* of a vertex v is the number of its neighbours. The graph X is said to be *locally finite* if all vertices of X have finite valency. An *orientation* of the edge $\{u, v\}$ is a choice of ordering of its vertices, that is either one of the (ordered) pairs (u, v) or (v, u) . An orientation of the graph is then a choice of orientation for all of its edges; an oriented graph is equivalently the data of a set V of vertices and a set E of ordered pairs representing edges, with the restriction that $(u, u) \notin E$ and $(u, v) \in E \Rightarrow (v, u) \notin E$. A *walk* in X is a sequence of vertices, each neighbouring its successor in the sequence. A walk in which no vertex

appears more than once is a *path*. The index set of the sequence can be $\{0, \dots, n\}$ for some positive integer n , $\mathbb{N}$, or $\mathbb{Z}$. In the latter two cases, and assuming the walk is a path, it is called a *ray* and a *biray*, respectively. The *length* of the walk is n in the first case, and infinite in the latter two. A walk is *closed* if it is finite and such that the first and last vertex are identical. A *circuit* is a closed walk where no edge is traversed more than once. A circuit in which no vertex appears more than once, except the first-and-last, appearing exactly twice, is also called a *cycle*⁴. The graph X is *connected* if any pair of its vertices appear in some common path. From now on, all graphs are assumed to be locally finite, oriented and connected. The (*induced*) *distance* on the (connected) graph X is the metric on VX obtained by setting

$$d(u, v) = \text{length of a shortest path containing both } u \text{ and } v.$$

From local finiteness, it follows that all metric balls $B_r(v) = \{u \in VX : d(u, v) \leq r\}$ are finite. Furthermore, using connectivity, X is the union of its metric balls centered at any of its vertices: $VX = \bigcup_{r \in \mathbb{N}} B_r(v) \ \forall v \in VX$. Endowing the (vertex set of the) graph X with the discrete topology, the compact sets are just the finite ones, and the above property allows (a simple form of) “exhaustion by compacts”.

Say that two rays in X are *close*, if for any given compact $K \subset VX$ they can be joined by a path that avoids K ; being close is an equivalence relation on the set of rays. An *end* of X is an equivalence class of rays under “being close”. Let us write $\text{Ends } X$ for the set of ends of X ; it is common knowledge that, under the assumptions of connectivity and local finiteness, $\text{Ends } X$ can also be described as the inverse limit of the inverse system indexed over compact sets $K \subseteq VX$, whose objects are the sets of connected components of $X - K$ ⁵, and whose morphisms map, for $K \subseteq K'$, a connected component C' of $X - K'$ to the unique connected component C of $X - K$ with $C' \subseteq C$ [DK03].

Finally, given two (connected) graphs X and Y , a map $f : VX \rightarrow VY$ is said to be a *quasi-isometry* if there exists a constant $K > 0$ such that:

- f is K -Lipschitz, i.e. $d(f(x), f(x')) \leq Kd(x, x')$ ⁶.
- There exists a *quasi-inverse* for f , that is a K -Lipschitz map $g : VY \rightarrow VX$ such that

$$d((g \circ f)(x), x) \leq K \quad \text{and} \quad d((f \circ g)(y), y) \leq K.$$

X and Y are said to be *quasi-isometric* if there exists a quasi-isometry from one to the other; this is an equivalence relation.

⁴This paper also deals with homological cycles. To avoid confusion we will never use this terminology but talk about flows instead.

⁵ $X - K$ is the graph obtained by removing from X all vertices in K and all edges adjacent to at least one vertex in K .

⁶For general metric spaces one needs (K, K) -Lipschitz and writes $\leq Kd + K$, but for (connected) graphs Kd suffices.

2.2 Simplicial complexes

Recall that a simplicial complex is the data of a set V of vertices, plus a subset X of the set of finite subsets $\mathcal{P}_f(V)$ of V , such that X

- contains all singletons $\{v\}$ for $v \in V$; and
- is closed under subsets: if $\tau \subseteq \sigma \in X$, then $\tau \in X$.

The elements of X are called simplices. Let us write X_n for the simplices of cardinality $n + 1$, called n -simplices. The n -skeleton of X is the simplicial complex consisting of all simplices of X of cardinality at most $n + 1$; the 1-skeleton of a simplicial complex is (equivalent to) a graph. X is said to be locally finite if its 1-skeleton is locally finite. The faces of an n -simplex σ are the $(n - 1)$ -simplices contained in σ ; we write $\tau < \sigma$ to say that τ is a face of σ . Finally, an *orientation* of X consists (for our purposes) in a choice of sign $[\tau : \sigma] = \pm$ for each simplex σ and face τ of σ , such that:

- if θ is a common face of $\tau \neq \tau'$, both (distinct) faces of σ , then $[\theta : \tau'] \cdot [\tau' : \sigma] = -[\theta : \tau] \cdot [\tau : \sigma]$;
- for all $v \in V$, we have $[\emptyset : \{v\}] = +$.

From now on, assume that all simplicial complexes are endowed with a choice of orientation (it does not matter which one). Any graph defines a simplicial complex; an orientation of the graph is equivalent to an orientation of the induced simplicial complex. Fix $r \in \mathbb{N}^{>0}$ and a graph X ; the r -th *Rips complex* $R_r(X)$ is the simplicial complex on vertex set VX , and with simplices:

$$\{\sigma \in \mathcal{P}_f(V) : d(u, v) \leq r \ \forall u, v \in \sigma\}.$$

In other words, the simplices are the finite subsets of vertices of diameter at most r . The r -th Rips complex $R_r(X)$ can alternatively be described as the simplicial complex whose n -simplices are cardinality- $(n + 1)$ cliques of the r -th graph power X^r of X ⁷. It is clear that the 1-skeleton of $R_1(X)$ is just X , and that if $r \leq s$, then $R_r(X) \subseteq R_s(X)$. If X is a locally finite graph, then $R_r(X)$ is also locally finite.

2.3 Thin sums

We will often view functions $f : S \rightarrow A$ from a set S to a coefficient ring A as infinite formal sums $\sum_{s \in S} f(s) \cdot s$. We might want to apply a function to each $s \in S$ and extend “linearly” to infinite sums, or sum an infinite number of such sums. For all of this to be well-defined, some care is needed; the point of this subsection is to give a general explanation of why all the constructions appearing later are well-defined and behave as expected; it is elementary and may safely be skipped. We follow terminology from [BG11].

⁷The r -th power of a graph has the same vertex set and an edge for pair of vertices at distances at most r .

Let $f \in A^S$, the *support* $\text{supp } f$ is the set of elements of S on which f is non-zero. A family $F \subseteq A^S$ is said to be *thin* (or *summable*) if any $s \in S$ lies in the support of only finitely many functions f in F . If F is thin, one can define the thin sum

$$\sum F := \sum_{f \in F} f : S \rightarrow A$$

of all elements of F . This sum can either be defined as the limit of its finite sub-sums under the product topology on A^S (A discrete), or, more elementarily, by setting:

$$\left(\sum F\right)(s) := \sum_{\substack{f \in F \\ s \in \text{supp } f}} f(s).$$

We also say that a family of subsets of S is *thin* if the corresponding family of indicator functions is itself thin. Although overly convoluted, one can rephrase the local finiteness of a graph X using the above terminology, by remarking that X is locally finite if and only if the family $\{\{u, v\} \mid u \sim v\}$ is thin.

Let $\phi : S \rightarrow A^T$ be a map such that $\phi[S] \subseteq A^T$ is thin. Given a function $f : S \rightarrow A$, one can define

$$\phi f : T \rightarrow A : t \mapsto \sum_{s \in S} f(s) \cdot \phi(s)(t),$$

which is well-defined by thinness of $\phi[S]$. Moreover, given any thin family $F \subseteq A^S$, the family $\{\phi f \mid f \in F\}$ will still be thin.

2.4 Roe Homology

Given a locally finite simplicial complex X and a choice of coefficient ring A , one can extend the usual definition of simplicial homology on X to a form of *homology with infinite support* (in a more general context also known as Borel-Moore, or closed, homology). This is the homology induced by the chain complex $(C_\bullet(X, A), d_\bullet(X, A))$ with:

- $C_n(X, A)$ the A -module of functions from X_n to A (generally interpreted as infinite formal sums of n -simplices);
- $d_n(X, A) : C_n(X, A) \rightarrow C_{n-1}(X, A)$ obtained by sending an n -simplex to the alternating sum of its faces, and extending “linearly” by thin sums:

$$f = \sum_{\sigma} f(\sigma) \cdot \sigma \quad \mapsto \quad \sum_{\sigma} \underbrace{\left(f(\sigma) \sum_{\tau < \sigma} [\tau : \sigma] \cdot \tau \right)}_{=: d_n(f(\sigma) \cdot \sigma)}.$$

A routine calculation (making essential use of the local finiteness condition) ensures that this is well-defined and yields a chain complex. With notation

$$Z_n(X, A) := \ker d_n(X, A), \quad B_n(X, A) := \text{im } d_{n+1}(X, A),$$

we get $H_n(X, A) = Z_n(X, A)/B_n(X, A)$.

For $r \leq s$, we have $C_\bullet(R_r(X), A) \subseteq C_\bullet(R_s(X), A)$ and this inclusion agrees with the boundary operators $d_\bullet$. We can therefore define the *Roe chain complex* $C_\bullet^\infty(X, A)$ by letting:

$$C_\bullet^\infty(X, A) := \bigcup_{r>0} C_\bullet(R_r(X), A),$$

and defining the boundary operator $d_\bullet^\infty(X, A)$ accordingly. Finally, the *Roe homology* (of X with coefficients in A) is simply the homology of this chain complex. With notation

$$Z_n^\infty(X, A) := \ker d_n^\infty(X, A), \quad B_n^\infty(X, A) := \operatorname{im} d_{n+1}^\infty(X, A),$$

we get $H_n^\infty(X, A) = Z_n^\infty(X, A)/B_n^\infty(X, A)$. Note that we also have

$$H_n^\infty(X, A) = \frac{\bigcup_r Z_n^\infty(R_r(X), A)}{\bigcup_r B_n^\infty(R_r(X), A)}.$$

If no confusion is possible, we will not index our chain complexes with the choice of base ring and/or simplicial complex. Furthermore, from now on, A will always denote the chosen coefficient ring; if the choice does not matter, we will usually drop it.

We state the following essential property of Roe homology without proof:

Theorem 1. *Roe homology is quasi-isometry invariant.*

Proof. Follows from [Mit01, Theorem 3.10]. □

2.5 Cycles, birays and flows

Fix a locally finite, connected graph X . Let us call any element of $Z_1(X)$ a *flow*. This terminology is motivated by the fact that, given an orientation of X , an element of $Z_1(X)$ can be described as an assignment to each edge e from u to v of some value $f(e) \in A$, interpreted as a quantity of “flow” running through e , from u to v ; the prescription $d_1 f = 0$ states the usual condition that the sum of incoming flow on a vertex is equal to the sum of outgoing flow. Let p be a path in X . The “flow” running through p is given by the element $\mathcal{D}p$, defined as the sum of edges appearing in p with sign, depending on whether the edge has the same or opposite orientation to that of p . In other words:

$$(\mathcal{D}p)(e) := \begin{cases} 0 & \text{if } e \notin \operatorname{supp} p, \\ 1 & \text{if } e \in \operatorname{supp} p \text{ and } e, p \text{ have equal orientation,} \\ -1 & \text{if } e \in \operatorname{supp} p \text{ and } e, p \text{ have opposite orientation.} \end{cases}$$

Of course this only gives us a flow (i.e. element of $Z_1(X)$) when the path is a biray. One can extend the definition of $\mathcal{D}$ to finite walks. Decompose the finite walk w into a concatenation of paths $\{p_i\}_{i=1}^n$, then define $\mathcal{D}w = \sum_{i=1}^n \mathcal{D}p_i$, which is independent of the chosen decomposition. This only defines a flow when the walk is closed.

We can single out two indecomposable (in an informal sense) units of flow on X ; those given by birays and cycles. Let $\mathfrak{S}$ denote the set of birays and cycles⁸. Any flow on X can be decomposed into a sum of birays and cycles:

Lemma 2. *Given any (A -valued) flow f on the locally finite, connected graph X , there exists a function $\lambda : \mathfrak{S} \rightarrow A$, such that $\{\lambda(s) \cdot \mathcal{D}s \mid s \in \mathfrak{S}\}$ is thin, and satisfying*

$$f = \sum_{s \in \mathfrak{S}} \lambda(s) \cdot \mathcal{D}s.$$

If λ is such that $\{\lambda(s) \cdot \mathcal{D}s \mid s \in \mathfrak{S}\}$ is thin, we call λ *locally finitely supported*. Given a locally finitely supported function $\lambda : \mathfrak{S} \rightarrow A$, let us write $\mathcal{D}\lambda$ for the sum $\sum_{s \in \mathfrak{S}} \lambda(s) \cdot \mathcal{D}s$.

Proof. Fix an enumeration $\{e_i\}_{i=1}^\infty$ of the edges of X . We will construct λ iteratively, starting with the zero function, iterating over edges and adding first cycles, then birays to λ in a locally finite manner.

Start with $\lambda = 0$, and iterate over each edge e_i in order:

- If e_i lies in some cycle of $\text{supp}(f - \mathcal{D}\lambda)$, add this cycle with correct weighting to λ , in such a way that now $e_i \notin \text{supp}(f - \mathcal{D}\lambda)$.
- Otherwise, do nothing.

Note that after each step, λ stays locally finite, and that after step i , $(\mathcal{D}\lambda)(e_i)$ stabilises. In the end (or rather, in the limit) $\text{supp}(f - \mathcal{D}\lambda)$ is a forest. Indeed, assume it contains a cycle and let e_i be the first edge in this cycle; the very construction of λ at step i yields a contradiction.

Since $f - \mathcal{D}\lambda$ is a flow, this forest has no vertex of degree 1, and any path can be extended into a biray. Now, iterate again over each edge e_i in order:

- If e_i lies in $\text{supp}(f - \mathcal{D}\lambda)$, add any biray containing e_i with correct weighing to λ , in such a way that now $e_i \notin \text{supp}(f - \mathcal{D}\lambda)$.
- Otherwise, do nothing.

Note that after each step, λ remains locally finite and that $\text{supp}(f - \mathcal{D}\lambda)$ stays a forest. At the limit all edges have been removed, and thus $\mathcal{D}\lambda = f$, as needed. $\square$

Remark 3. By inspecting the proof of the lemma, we can make the following observations. Given $s \in \mathfrak{S}$ a biray or cycle in the decomposition of a flow f (i.e. $\lambda(s) \neq 0$), it necessarily holds that $\text{supp}(\mathcal{D}s) \subseteq \text{supp}(f)$. This implies that if f is supported on a finite set of edges, the decomposition λ of f consists of a finite number of cycles, all supported on a subset of the support of f . In particular, if w is a closed walk, then the associated flow $\mathcal{D}w$ is simply a finite weighted sum of flows on cycles, all supported on a subset of the support of w .

⁸Since cycles are sequences of vertices, one cycle appears multiple times in this set by allowing each vertex to serve as starting point. It has a built-in orientation that can be reversed by reversing the indexing. Note that one could alternatively decide to have $\mathfrak{S}$ only contain one representative for all these elements.

Cycle space. Fix a locally finite, connected graph X . Let $\mathfrak{C} = \mathfrak{C}(X)$ denote the set of cycles of X and $\mathfrak{C}_r = \mathfrak{C}_r(X)$ the subset of $\mathfrak{C}(X)$ consisting of cycles of length at most r . As seen above, to any cycle $c \in \mathfrak{C}$, one can associate a function $\mathcal{D}c : EX \rightarrow A$ sending an edge to 0 if it does not lie in the cycle, and to ± 1 if it does, according to orientation; note that $\mathcal{D}c$ lies in $Z_1(X)$. Let $A^{(\mathfrak{C})}$ denote the set of functions from $\mathfrak{C}$ to A that are locally finitely supported. Formally,

$$A^{(\mathfrak{C})} := \{f : \mathfrak{C} \rightarrow A : \{\mathcal{D}c : c \in \text{supp } f\} \text{ is thin}\}.$$

It is easy to see that $A^{(\mathfrak{C})}$ is an A -module. This association can be extended to a linear map:

$$\mathcal{D} : A^{(\mathfrak{C}(X))} \rightarrow Z_1(X),$$

with essential use of the local finiteness condition. For any $r > 0$, we similarly get a module of functions $A^{(\mathfrak{C}_r(X))}$ and a restricted map

$$\mathcal{D}_r : A^{(\mathfrak{C}_r(X))} \rightarrow Z_1(X).$$

Actually, local finiteness of an element of $A^{(\mathfrak{C}_r(X))}$ comes for free since the local finiteness of the graph implies that only a finite number of cycles of a given length can contain a given edge. Thus, $A^{(\mathfrak{C}_r(X))}$ is simply the set of functions $\mathfrak{C}_r(X) \rightarrow A$.

Finally, let us write

$$\mathcal{C}(X, A) := \text{im } \mathcal{D}, \quad \mathcal{C}_r(X, A) := \text{im } \mathcal{D}_r, \quad \mathcal{C}_\infty(X, A) := \bigcup_r \text{im } \mathcal{D}_r.$$

Under the interpretation of $Z_1(X)$ as the space of all flows on X , the space $\mathcal{C}(X)$ [resp. $\mathcal{C}_r(X)$, $\mathcal{C}_\infty(X)$] corresponds to flows that can be written as any locally finite sum of cycles [resp. cycles of length at most r , cycles of uniformly bounded length]. Some remarks:

- Each $\mathcal{C}_\bullet(X, A)$ and $\mathcal{C}(X, A)$ is an A -module.
- $\mathcal{C}_1(X) \subseteq \mathcal{C}_2(X) \subseteq \dots \subseteq \mathcal{C}_\infty(X) \subseteq \mathcal{C}(X)$.
- If X is a tree, then all $\mathcal{C}_\bullet(X)$ and $\mathcal{C}(X)$ are trivial.

Remark 4. Note that we have chosen to generate $\mathcal{C}(X)$ through $\mathcal{D}$ as well-defined sums of cycles. However we could as well have defined $\mathfrak{C}(X)$ as the set of closed walks or the set of circuits, and the resulting space would have been the same. This follows from remark 3. Indeed, a collection of closed walks [resp. circuits] is thin if and only if the associated collection of cycles is thin, which guarantees that replacing cycles by closed walks [resp. circuits] in our definition of the cycle space does not change the result. Furthermore, since the cycles in the decomposition of a closed walk are necessarily at most as long as the original closed walk, the spaces $\mathcal{C}_r(X)$ are also unchanged.

In the sequel, when no confusion is possible, we will usually not distinguish between a cycle $c \in \mathfrak{C}$ and the element $\mathcal{D}c \in \mathcal{C}$ it defines (similarly for birays and sums of such).

Let us say that X has *A-large cycles* if the sequence $\mathcal{C}_1(X, A) \subseteq \mathcal{C}_2(X, A) \subseteq \dots$ does not stabilise (i.e., if $\mathcal{C}_\infty(X, A)$ is different from $\mathcal{C}_r(X, A)$ for all r).

Proposition 5. *Both having A -large cycles and the vanishing of $\mathcal{C}(X, A)/\mathcal{C}_\infty(X, A)$ are invariant under quasi-isometries.*

Proof. Let X, Y be two locally finite, connected graphs, and $\phi : VX \rightarrow VY$ a quasi-isometry with quasi-inverse $\psi : VY \rightarrow VX$ of constant K . If u, v are two vertices both in X or both in Y , fix a geodesic path $p_{u,v}$ joining u to v . Extend ϕ to a function $\phi : \mathcal{C}(X, A) \rightarrow \mathcal{C}(Y, A)$ (and similarly ψ) by

$$\phi(f) = \sum_{\{u,v\}=e \in EX} [v : e] f(e) \mathcal{D}p_{\phi u, \phi v}.$$

It is a quick check that ϕ behaves linearly with respect to thin sums. For a finite walk p in X of length l , denote by ϕp the finite walk in Y obtained by concatenating the segments $p_{\phi u_i, \phi u_{i+1}}$, for each pair of subsequent vertices (u_i, u_{i+1}) in p ; define similarly ψp if p is a finite walk in Y . The length of this walk is at most lK . For an edge $\{u, v\} = e \in EX$, oriented as $[v : e] = +$, define the closed walk c_e as the concatenation of e , then $p_{v, \psi \phi v}$, then the reverse of $\psi \phi e$ and then $p_{\psi \phi u, u}$. The distance of a vertex v and its image $\psi \phi v$ is at most K . So the length of c_e is at most $L = K^2 + 2K + 1$. It is then easily verified that

$$f - \psi \phi f = \sum_{e \in EX} f(e) \mathcal{D}c_e.$$

By using remark 3 one can further decompose $\mathcal{D}c_e$ into a sum of cycles and conclude that $f - \psi \phi f \in \mathcal{C}_L(X, A)$. Assume now that $\mathcal{C}_r(Y, A) = \mathcal{C}_\infty(Y, A)$, and let $f \in \mathcal{C}_s(X, A)$ for some s . So there is a $\lambda \in A^{(\mathfrak{C}_s(X))}$, such that $f = \mathcal{D}\lambda$, and we obtain

$$\phi f = \sum_{c \in \mathfrak{C}_s(X)} \lambda(c) \cdot \phi \mathcal{D}c.$$

So ϕf is a flow in $\mathcal{C}_{Ks}(Y, A)$, and thus by assumption in $\mathcal{C}_r(Y, A)$. By an analogous argument we find that $\psi \phi f \in \mathcal{C}_{Kr}(Y, A)$. Since $f = \psi \phi f + (f - \psi \phi f)$, we conclude that $f \in \mathcal{C}_{\max(Kr, L)}(X, A)$. Thus, $\mathcal{C}_\infty(X, A) = \mathcal{C}_{\max(Kr, L)}(X, A)$, as was to be shown.

To show that the vanishing of $\mathcal{C}(X, A)/\mathcal{C}_\infty(X, A)$ is invariant under quasi-isometries, one simply follows the same approach. $\square$

2.6 H_1^∞ in terms of flows.

In this subsection, we provide an equivalent definition of H_1^∞ , which can be summarised as “all flows” modulo “all sums of cycles of uniformly bounded lengths”.

Fix a locally finite, connected graph X , and choose, for each pair of vertices $u, v \in VX$, a geodesic path $p_{u,v}$ connecting u to v .

If $c = (c_0, \dots, c_n) \in \mathfrak{C}_r$ is a cycle in X of length $\leq r$, set $\sigma_i = \{c_0, c_i\}$ and $\tau_i = \{c_0, c_i, c_{i+1}\}$, resp. edges and triangles of $R_r(X)$. Defining $\lambda_i = [c_i : \sigma_i][\sigma_i : \tau_i]$ to take into account the chosen orientation, we associate to c its triangulation:

$$\Delta c = \sum_{i=1}^{n-1} \lambda_i \tau_i \in A^{(R_r(X))_2}.$$

This assignment is thin, and therefore extends to a linear map:

$$\Delta : A^{\mathfrak{C}_r} \rightarrow A^{(R_r(X))_2} = C_2(R_r(X)),$$

which, by construction, satisfies $d_2 \circ \Delta = \mathcal{D}$. In particular, we see that:

$$\mathcal{C}_r(X, A) = \mathcal{D}(A^{\mathfrak{C}_r}) = d_2 \circ \Delta[A^{\mathfrak{C}_r}] \subseteq d_2[C_2(R_r(X), A)] = B_1(R_r(X), A)$$

so that, since $\mathcal{C}_r(X, A) \subseteq Z_1(X, A)$:

$$\mathcal{C}_r(X, A) \subseteq B_1(R_r(X), A) \cap Z_1(X, A). \quad (1)$$

Conversely, given a triangle $\tau = \{u, v, w\} \in (R_r(X))_2$ and supposing $[\{u, v\} : \tau] = [v : \{u, v\}]$, consider the closed walk $w := p_{u,v}p_{v,w}p_{w,u} \in \mathfrak{C}_{3r}$ obtained by concatenating the geodesics. Then, decompose w into cycles according to remark 3 in such a way that $\mathcal{D}w = \mathcal{D}\lambda$. Finally, set $c\tau := \lambda$. Again, this assignment is thin, and extends to a linear map:

$$c : C_2(R_r(X)) = A^{(R_r(X))_2} \rightarrow A^{\mathfrak{C}_{3r}}.$$

Assume now that $f \in B_1(R_r(X), A) \cap Z_1(X, A)$, so that $f = d_2F$ with $F \in C_2(R_r(X), A)$. From the fact that $f \in Z_1(X, A)$ follows that $d_2F = \mathcal{D}cF$. This shows that:

$$B_1(R_r(X), A) \cap Z_1(X, A) \subseteq \mathcal{C}_{3r}(X, A). \quad (2)$$

By combining eq. (1), eq. (2) and the fact that $B_1^\infty(X, A) = \bigcup_r B_1^\infty(R_r(X), A)$, we conclude:

$$\mathcal{C}_\infty(X, A) = Z_1(X, A) \cap B_1^\infty(X, A). \quad (3)$$

Now fix some $r \in \mathbb{N}$ and consider the composite morphism:

$$\Phi_r : Z_1(X, A) \hookrightarrow Z_1(R_r(X), A) \twoheadrightarrow H_1(R_r(X), A).$$

We claim that Φ_r is surjective, and that its kernel satisfies:

$$\mathcal{C}_r \subseteq \ker \Phi_r \subseteq \mathcal{C}_{3r}.$$

Indeed, let $f = \sum_{e \in (R_r(X))_1} f(e) \cdot e$ be a representative of an element of $H_1(R_r(X), A)$, and define

$$\tilde{f} := \sum_{\{u,v\} =: e \in (R_r(X))_1} [v : e] f(e) \cdot \mathcal{D}p_{u,v}.$$

$\tilde{f}$ is obtained by replacing edges in $R_r(X)$ by geodesics in X . By construction, $\tilde{f}$ and f differ by an appropriate thin sum of triangles of $R_r(X)$, hence are equal in homology. This verifies surjectivity. The two inclusions involving the kernel follow from eqs. (1) and (2), since $\ker \Phi_r = Z_1(X, A) \cap B_1(R_r(X), A)$.

We now easily obtain the desired description of H_1^∞ :

Theorem 6. *If X is a locally finite, connected graph, then*

$$H_1^\infty(X, A) \cong \frac{Z_1(X, A)}{C_\infty(X, A)}.$$

Proof. We consider the composite morphism

$$\Phi_\infty : Z_1(X, A) \hookrightarrow Z_1^\infty(X, A) \rightarrow \frac{Z_1^\infty(X, A)}{B_1^\infty(X, A)} = \frac{\bigcup_r Z_1(R_r(X), A)}{\bigcup_r B_1(R_r(X), A)}.$$

Surjectivity of all Φ_r implies surjectivity of Φ . Furthermore, $\ker \Phi_\infty = C_\infty(X, A)$, since $\ker \Phi_\infty = \bigcup_r \ker \Phi_r$. $\square$

Lemma 7. *If X is a locally finite, connected graph with $C_r(X, A) = C_\infty(X, A)$, then, for any large enough s , the map Φ_s and the map Φ_∞ descend to isomorphisms:*

$$\tilde{\Phi}_s : \frac{Z_1(X, A)}{C_r(X, A)} \rightarrow H_1(R_s(X), A) \quad \text{and} \quad \tilde{\Phi}_\infty : \frac{Z_1(X, A)}{C_r(X, A)} \rightarrow H_1^\infty(X, A).$$

Furthermore, the following triangle of isomorphisms commutes:

$$\begin{array}{ccc} & \frac{Z_1(X, A)}{C_r(X, A)} & \\ \swarrow \tilde{\Phi}_s & & \searrow \tilde{\Phi}_\infty \\ H_1(R_s(X), A) & \xrightarrow{[\iota]} & H_1^\infty(X, A). \end{array}$$

Proof. That we have isomorphisms follows from the fact that $C_r(X, A) = C_s(X, A) = C_\infty(X, A)$ for large enough s , commutativity from the fact that all maps involved are appropriate quotients of inclusions. $\square$

3 The case of trees

The goal of this section is to understand the relation between H_1^∞ and (locally finite) trees. In the first part, we will compute $H_1^\infty(T)$, where T is a locally finite tree. The second part is devoted to understanding the relation between the homology of a (locally finite, connected) graph X and that of a tree that “represents” the ends of X .

3.1 $H_1^\infty(\text{tree})$

We will show (expanding on a result of [Dia15]) that given a locally finite tree T , there exists a set $\mathcal{B}$ of birays on T such that $H_1^\infty(T, A)$ can be identified with the set $A^\mathcal{B}$ of functions $\mathcal{B} \rightarrow A$. We will first briefly explain the properties we need $\mathcal{B}$ to satisfy and illustrate how to construct it. A formal description of its construction and verification of the properties it ought to satisfy is tedious, but straightforward. We choose to provide an illustration instead; see fig. 1.

Simplifying assumptions. Let us start with three easy steps. First, from the decomposition $H_1^\infty(T) = Z_1(T)/\mathcal{C}_\infty(T)$ and the fact that $\mathcal{C}_\infty$ is trivial for trees, it follows that $H_1^\infty(T) = Z_1(T)$. Second, all elements of $Z_1(T)$ are entirely supported on edges incident to vertices of degree at least 2, so that all vertices of degree 1 can be discarded from the tree without interfering with $Z_1(T)$. We can therefore safely assume that T has no vertices of degree 1. Finally, by local finiteness and a simple “unrolling” of high valency vertices, one can transform T into a tree T' with only vertices of valency 2 or 3, and such that at least one is of valency 2; it is then a simple matter to verify that $Z_1(T)$ and $Z_1(T')$ are isomorphic. From now on, let us therefore assume that T is a locally finite tree with all vertices of degree 2 or 3, and at least one of degree 2.

Comb decomposition. Let T be a tree (with only vertices of degrees 2 and 3 by assumption) and $v_0 \in VT$ be a vertex of degree 2. We will make use of a decomposition of T into a set of rays covering it, dubbed *comb decomposition*, and defined as an $\mathbb{N}$ -indexed sequence $(\mathcal{P}_n)_n$ of sets of rays on T , with the following properties:

1. Each edge of T is covered by exactly one ray in exactly one $\mathcal{P}_n$.
2. $\mathcal{P}_0$ contains exactly two rays, both starting at v_0 .
3. For any $n > k$, the only vertex that a ray in $\mathcal{P}_n$ shares with a ray in $\mathcal{P}_k$ is its starting point, and in that case k is $n - 1$.

Let T_n be the subtree of T covered by all rays in $\bigcup_{k < n} \mathcal{P}_k$.

Let us explain how to construct such a decomposition. First, $\mathcal{P}_0$ consists of any two edge-disjoint rays leaving v_0 . Then, assuming $\mathcal{P}_0, \dots, \mathcal{P}_n$ have been defined, let V_{n+1} consist of all vertices contained in some ray of $\mathcal{P}_n$, such that one edge incident to the vertex is not contained in any previously defined ray. Let $\mathcal{P}_{n+1}$ consist of one ray per vertex in V_{n+1} , starting at this vertex and leaving through the free edge in any direction, as long as it does not intersect T_n . This process can end after some n if the set V_{n+1} is empty, or continue indefinitely.

It is easy to see that this construction yields a family satisfying the required properties (see fig. 1).

Good set of birays. Given a comb decomposition $(\mathcal{P}_n)_n$ of T , we construct a set $\mathcal{B}$ of birays on T endowed with a well-ordering $\leq$ satisfying the following conditions:

1. Each edge is covered by at least one and at most three birays of $\mathcal{B}$.

For any edge $e \in ET$, let $\mathcal{B}(e)$ be the set of elements of $\mathcal{B}$ covering e . Let us say that $b \in \mathcal{B}$ is *last* on e if $b \in \mathcal{B}(e)$ and for all $b' \in \mathcal{B}(e)$, we have $b' \leq b$. For any $b \in \mathcal{B}$, define the set

$$M_b := \{e : b \text{ is last on } e\}.$$

2. Each set M_b is non-empty and connected, and $\bigcup_{b \in \mathcal{B}} M_b = ET$.
3. Whenever b is last on e , there exists some other edge e' such that $\mathcal{B}(e) = \mathcal{B}(e') \sqcup \{b\}$.

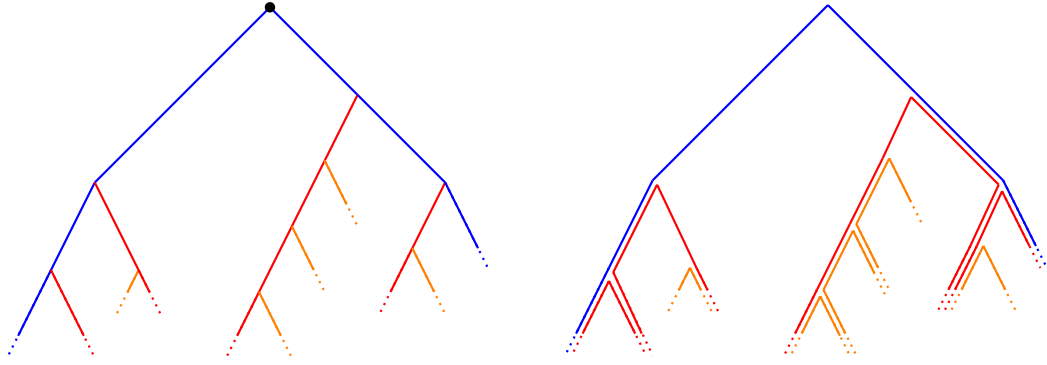


Figure 1: Comb decomposition of a tree and the corresponding set of birays $\mathcal{B}$. The black dot is v_0 .

The construction of the set $\mathcal{B}$ naturally follows from that of a comb decomposition. A preliminary (well-founded but not total) order structure comes naturally with it, and it suffices to extend it carefully to find our well-ordering on $\mathcal{B}$. Instead of explaining this construction precisely, let us refer to fig. 1: the construction can easily be deduced from this example.

We then have:

Theorem 8. *If $\mathcal{B}$ is a good set of birays for the tree T , then*

$$H_1^\infty(T, A) \cong A^{\mathcal{B}}.$$

Proof. Consider the morphism

$$\begin{aligned} \mathcal{D} : A^{\mathcal{B}} &\rightarrow Z_1(T, A) = H_1^\infty(T, A) \\ \phi &\mapsto \sum_{b \in \mathcal{B}} \phi(b) \cdot \mathcal{D}b. \end{aligned}$$

Since only a finite number of elements of $\mathcal{B}$ passes through any given edge, this map is well-defined, and A -linearity is clear.

It remains to verify that $\mathcal{D}$ is injective and surjective.

For injectivity, assume that $\phi \in A^{\mathcal{B}}$ is non-zero, and let b the first (w.r.t our order) biray with $\phi(b) \neq 0$. Take some $e \in M_b$ (item 2); by definition of M_b , the value of $\mathcal{D}(\phi)$ on e is equal to $[e : b]\phi(b)$ (where $[e : b]$ corrects the sign if the orientations of e and b disagree), which is non-zero by hypothesis.

For surjectivity, given an element $f \in Z_1(T, A)$, we construct a preimage ϕ of f by inductively defining its values on the rays $b \in \mathcal{B}$, using the well-ordering on $\mathcal{B}$. Fix some $b \in \mathcal{B}$ and assume that for all $b' < b$, $\phi(b')$ is defined and is such that whenever an edge $e \in ET$ is covered only by birays $b' < b$, the sum $\sum_{e \in b'} [e : b']\phi(b')$ is equal to $f(e)$ (this is an obviously necessary condition, since the value of $\mathcal{D}\phi$ on e is entirely defined by the coefficients associated to each $b' \in \mathcal{B}(e)$, and all elements therein have already been given

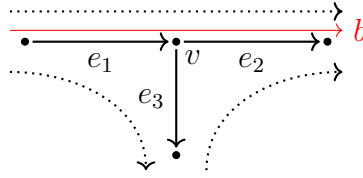


Figure 2: Local configuration in the inductive construction of ϕ .

a coefficient). Take any edge $e \in M_b$, and define

$$\phi(b) := f(e) - \sum_{e \in b' < b} [e : b'] \phi(b').$$

Let us verify that this definition does not depend on the choice of $e \in M_b$. By item 2, it suffices to compare the values obtained for two adjacent edges e_1, e_2 . Let v be the vertex shared by e_1, e_2 ; it is either of degree 2 or of degree 3, in which case it is incident to a third edge e_3 . We treat only the case of degree 3, the other one being similar but simpler. For definiteness, assume that the local configuration around v is as presented in fig. 2. In particular, the orientation of edges (thick arrows) and all birays involved (dotted thin arrows) are assumed to be as given by the figure. First, note that since $e_1, e_2 \in M_b$ all birays in $\mathcal{B}$ traversing e_1, e_2 come before b , which implies that this is also the case for e_3 : indeed, any biray traversing e_3 must then either traverse e_1 or e_2 . A few remarks (with index arithmetic mod 3):

- $\mathcal{B}(e_i) \cap \mathcal{B}(e_{i+1}) \cap \mathcal{B}(e_{i+2}) = \emptyset$ and $\mathcal{B}(e_i) \subseteq \mathcal{B}(e_{i+1}) \cup \mathcal{B}(e_{i+2})$.
- Any $b' \in \mathcal{B}(e_3)$ comes before b , so that $f(e_3) = \sum_{b' \in \mathcal{B}(e_3)} [e : b'] \phi(b')$ by the induction hypothesis.
- For $i = 1, 2$ and any $b' \in \mathcal{B}(e_i)$, $[e_i : b'] = +$. If $b' \in \mathcal{B}(e_3)$, then $[e_3, b']$ is $+$ if $b' \in \mathcal{B}(e_1)$ and $-$ otherwise. This is a consequence of our choice of orientations.

We then get (with $\mathcal{B}_i := \mathcal{B}(e_i)$ and $\mathcal{B}_{ij} := \mathcal{B}(e_i) \cap \mathcal{B}(e_j)$)

$$\begin{aligned} f(e_1) - f(e_2) &= f(e_3) && (f \text{ is a flow}) \\ &= \sum_{b' \in \mathcal{B}_3} [e_3 : b'] \phi(b') && (\text{second observation}) \\ &= \sum_{b' \in \mathcal{B}_{13}} \phi(b') - \sum_{b' \in \mathcal{B}_{23}} \phi(b') && (\text{decomposing \& third observation}) \\ &= \sum_{b' \in \mathcal{B}_{13}} \phi(b') + \sum_{b' \in \mathcal{B}_{12} - \{b\}} \phi(b') \\ &\quad - \sum_{b' \in \mathcal{B}_{12} - \{b\}} \phi(b') - \sum_{b' \in \mathcal{B}_{23}} \phi(b') && (\text{adding zero}) \\ &= \sum_{b' \in \mathcal{B}_1 - \{b\}} \phi(b') - \sum_{b' \in \mathcal{B}_2 - \{b\}} \phi(b') && (\text{grouping}) \end{aligned}$$

and a rebalancing of the sum yields the desired equality. By construction, the definition of ϕ on b satisfies the inductive hypothesis, and letting ϕ be the resulting map $\mathcal{B} \rightarrow A$, it is clear that $\mathcal{D}\phi = f$; surjectivity is verified. $\square$

3.2 Subtrees

Fix a locally finite, connected graph X , some vertex $v_0 \in VX$ and a tree T contained in X and containing v_0 . The set of ends of T is in natural bijection with the set of rays starting at v_0 ; recall that each such ray also defines an end of X . Consider the map

$$\text{Ends } T \rightarrow \text{Ends } X,$$

obtained by sending a ray starting at v_0 in T to the end of X it represents. T is called *end-respecting* if this map is injective, and *end-faithful* if it is bijective. As proved by Halin, end-faithful spanning trees always exist (for locally finite, connected graphs) [Die90].

Theorem 9. *Let X be a locally finite, connected graph and T an end-respecting subtree of X . Then*

$$Z_1(T) \cap \mathcal{C} = \{0\}.$$

In particular, the natural map $H_1^\infty(T) \rightarrow H_1^\infty(X)$ is injective.

To prove theorem 9, we will use a generalised version of Menger's theorem due to Aharoni [Aha87]:

Theorem 10 (Menger's theorem for countable graphs). *Given a connected countable graph X and any two disjoint sets $A, B \subseteq VX$, there exists a family P of vertex disjoint paths starting in A and ending in B , and a set $S \subseteq VX$ that is $A - B$ -separating (that is, there is no path in $X - S$ from A to B), such that S consists of the choice of precisely one vertex from each path of P .*

We can now proceed with proving the theorem.

Proof of theorem 9. The second part of the theorem follows from the first, using $H_1^\infty(T) = Z_1(T)$ (T being a tree) and the characterisation of $H_1^\infty(X)$ as $\frac{Z_1(X)}{\mathcal{C}_\infty}$.

To prove the first part, assume towards a contradiction that it does not hold. Then, there exists some non-zero element f of $Z_1(T)$ that also lies in $\mathcal{C}$: f can be written as some sum $\sum_c \lambda_c \mathcal{D}c$ of cycles of X . For notational convenience, let us discard the coefficients λ_c and let C be the set of cycles appearing in the sum, so that $f = \sum_{c \in C} \mathcal{D}c$. Fix some edge e_0 of T such that $f(e_0) \neq 0$; e_0 separates T into two connected components T_- and T_+ . Let $v_- = T_- \cap e_0$ and $v_+ = T_+ \cap e_0$. Let now P be the family of paths and S the set of vertices obtained by taking $A = VT_-$ and $B = VT_+$ in theorem 10. We claim that P is necessarily finite. Indeed, assume that P is infinite; we will construct two distinct rays in the tree T , hence defining different ends of T , but defining the same end of X , thus contradicting the assumption that T is end-respecting. For $p \in P$, let v_p and w_p be the start and end points of p , and p_T be the unique path in T connecting v_p to w_p . By assumption that P is infinite, there exists a biray b in T such that for any finite subpath b' of b , infinitely many of the paths p_T contain b' (this biray can be constructed by starting with $b_0 := e_0$ and inductively extending both sides by an edge in such a way that infinitely many $p \in P$ satisfy b_{i+1} being a subpath of p_T). Write $r_- := b \cap T_-$ and

$r_+ := b \cap T_+$, so that b is the concatenation of r_- taken in reverse, then e_0 , then r_+ ; since r_-, r_+ are contained in disjoint subtrees of T , they are distinct, and thus define different ends of T . For any $u \in VT_-$, define the subtree $T_-(u)$ of T_- with respect to u as the graph induced by the vertices w of T_- such that the unique path in T joining v_- to w passes through u ; define similarly $T_+(u)$ if $u \in T_+$. Now, let K be a compact in X . By finiteness of K , there is some vertex u_- far enough along r_- such that $T_-(u_-)$ does not intersect K ; there similarly exists a vertex u_+ far enough along r_+ . Consider now the subpath b' of b joining u_- and u_+ ; by construction b' is contained in infinitely many of the paths p_T ; which implies in particular that for infinitely many of the paths $p \in P$, we have $v_p \in T_-(u_-)$ and $w_p \in T_+(u_+)$. Again by finiteness of K , and the assumption that the paths in P are pairwise disjoint, there necessarily exists some $p \in P$ (among those with p_T containing b') that avoids K . Consider now the concatenation of paths $[u_-, v_p]p[w_p, u_+]$, where $[u_-, v_p]$ is the unique path in T_- (a fortiori in $T_-(u_-)$) connecting u_- to v_p , and similarly for $[w_p, u_+]$; by construction each of the components of this concatenation avoids K . Therefore, for any compact K , r_- and r_+ can be connected by a path in X avoiding K , thus implying that r_- and r_+ define the same end in X , and reaching the desired contradiction. We conclude that P , and hence S , is finite. Consider then $X - S$ and let L be the union of the connected components that contain some part of T_- . Then let $\tilde{E}$ be all edges not in L , incident to a vertex in S . For $e \in \tilde{E}$, let s_e be this vertex. Let $v_0 = e_0 \cap T_-$. We make two remarks:

- The value of f on $\tilde{E}$, defined as $\sum_{e \in \tilde{E}} [s_e : e] f(e)$, is equal to $[v_0 : e_0] f(e_0)$. Indeed, this can be seen by decomposing f into a sum of birays in T : each biray in this decomposition contributes to $f(e_0)$ in the same amount as it contributes to the value of f on $\tilde{E}$ (by construction of $\tilde{E}$).
- For any cycle c , the value $\sum_{e \in \tilde{E}} [s_e : e] \mathcal{D}c(e)$ of c on $\tilde{E}$ is zero. Indeed, any time an edge of $\tilde{E}$ is crossed by c , another one is crossed in the opposite direction.

Let C_S be the set of all circuits in C that contain some vertex of S . It is clear that $f - \sum C_S$ is zero on all of $\tilde{E}$. Since the set C_S is finite, our second remark implies that $f - \sum C_S$ has non-zero value on $\tilde{E}$, a contradiction. $\square$

Remark 11. In the proof above, verifying that the set P is finite is relatively tedious. All we actually require from the first part of the proof leading up to that is some finite set $S \subseteq VX$ that separates Ends T_- from Ends T_+ . As pointed out to us by a referee, a more direct proof is possible, given familiarity with the notion of Freudenthal compactification of graphs; for an introduction see e.g. [Die25]. Let us sketch the argument here. Let $|X|$ denote the Freudenthal compactification of the graph X . We will use a generalisation of Menger's theorem due to Bruhn et al. [BDS05] that applies to sets $A, B \subseteq |X|$, under the additional condition that $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. In the proof above, the sets of ends Ends T_- and Ends T_+ are disjoint compacts of $|T|$. By the continuity of the inclusion map $\iota : |T| \rightarrow |X|$, they are also compact in $|X|$, so that we may apply the generalised Menger's theorem, and get this time a set of birays P and a set of vertices S . To show that P (and hence S) is finite, assume it is not and argue by contradiction. Each biray p in P can be

split up as two rays, each defining an end, say e_p^+ and e_p^- . Let $(e_-, e_+) \in \mathcal{E}_- \times \mathcal{E}_+$ be a couple of ends defined as an accumulation point of the set $\{(e_p^-, e_p^+) \mid p \in P\} \subseteq \mathcal{E}_- \times \mathcal{E}_+$. These ends are distinct and can therefore be separated by some finite $F \in VX$, but there are also infinitely many (disjoint) birays in P that must intersect it: a contradiction. The remainder of the proof is the same.

4 Tree & cycle decomposition

By the result of the last section, we know that given a (locally finite, connected) graph X and an end-faithful tree T in X , the (first Roe) homology of T embeds in that of X . The following shows that this embedding is in fact a summand in a direct decomposition of $H_1^\infty(X)$ —the other summand being $\mathcal{C}/\mathcal{C}_\infty$.

Theorem 12. *Let X be a locally finite, connected graph and T an end-faithful subtree of X . Then*

$$H_1^\infty(X) \cong H_1^\infty(T) \oplus \frac{\mathcal{C}(X)}{\mathcal{C}_\infty(X)}.$$

Proof. We first show that $Z_1(T) + \mathcal{C}(X) = Z_1(X)$, i.e., that any $f \in Z_1(X)$ can be written as a sum $f = f_T + f_C$ with $f_T \in Z_1(T)$ and $f_C \in \mathcal{C}$.

Intuitively, we will use elements of $\mathcal{C}$ to “push” the flow f onto the tree T . First, decompose f into a sum of cycles and birays, using lemma 2. We can then write:

$$f = \sum_{e \in \text{Ends } X} \sum_{b \in \mathfrak{B}(e, e)} \lambda_b \mathcal{D}b + \sum_{e \neq e' \in \text{Ends } X} \sum_{b \in \mathfrak{B}(e, e')} \lambda_b \mathcal{D}b + \sum_{c \in \mathcal{C}} \lambda_c \mathcal{D}c,$$

where $\mathfrak{B}(e, e')$ is the set of birays coming from end e and going to end e' . Fix a ball $B(v_0, r)$ and consider all birays that appear in the decomposition of f and intersect the ball.

- If $b \in \mathfrak{B}(e, e)$ is a biray coming from the same end it is going to, there must exist a path outside of the ball connecting the two remaining strands of b . The concatenation of this path with the piece of b intersecting the ball yields a cycle, whose removal pushes b outside of the ball.
- If $b \in \mathfrak{B}(e, e')$ is a biray coming from a different end it is going to, there exists a biray b_T in the tree T with the same ends, and there exist two paths outside of the ball, connecting the matching strands of b and b_T . The concatenation of the two connecting paths with the appropriate pieces of b and b_T yields a closed walk, whose removal pushes b outside of $B(v_0, r) - T$. By remark 3 that closed walk can be replaced by a collection of cycles.

Thus, in each case, one can remove a collection of cycles from a biray to push it outside of $B(v_0, r) - T$. To guarantee this removal is thin when we repeat this procedure for a thin collection of birays some care is needed. To build the function f_C , proceed then as follows.

Start with some ball $B(v_0, r)$, and connect the birays intersecting the ball as explained above. Then, iteratively, choose a ball containing all connecting paths of the previous step, and find new connecting paths. This process will result in one infinite ladder for each biray b in some $\mathfrak{B}(e, e)$ (sides: two rays defined by b , rungs: the connecting paths), and one bi-infinite ladder for each biray b in some $\mathfrak{B}(e, e')$ (sides: b and the corresponding biray of T , rungs: the connecting paths). Each such ladder is decomposed into a sum of closed walks (and subsequently into cycles by remark 3). Let then

$$f_{\mathcal{C}} = \sum_b \lambda_b \mathcal{D}(\text{cycles in the ladder of } b) + \sum_{c \in \mathfrak{C}} \lambda_c \mathcal{D}c.$$

To see that this function is well-defined as a thin sum, note that:

- The part $\sum_{c \in \mathfrak{C}} \lambda_c \mathcal{D}c$ is thin to begin with, so that it suffices to ensure that the other one is thin.
- At each step of the process, only a finite number of cycles are used, since only a finite number of birays intersect any given finite ball (the original decomposition of f was thin to begin with). Furthermore, any edge can only be in the support of cycles in two consecutive steps of the construction (each new step adds cycles in a given “annulus”).

This shows that, indeed, $f_{\mathcal{C}}$ is thin and such that $f - f_{\mathcal{C}}$ is supported on T .

Combining the above with theorem 9, we conclude that

$$Z_1(T) \oplus \mathcal{C}(X) = Z_1(X),$$

and taking a quotient by $\mathcal{C}_{\infty}(X)$ yields, using theorem 6,

$$Z_1(T) \oplus \frac{\mathcal{C}(X)}{\mathcal{C}_{\infty}(X)} = \frac{Z_1(X)}{\mathcal{C}_{\infty}(X)} \cong H_1^{\infty}(X),$$

as was to be shown. □

5 More on cycles

As seen above, given a locally finite, connected graph X , and any end-faithful subtree T of X , we know that

$$H_1^{\infty}(X, A) \cong H_1^{\infty}(T, A) \oplus \mathcal{C}(X, A)/\mathcal{C}_{\infty}(X, A).$$

In section 3.1, we saw that the group $H_1^{\infty}(T, A)$ can be naturally described as the function space $A^{\mathcal{B}}$, for some well chosen set of birays $\mathcal{B}$ on T —we have found a basis of sorts for the left summand in the direct sum decomposition of $H_1^{\infty}(X, A)$. We will not reach such a simple description of the quotient $\mathcal{C}/\mathcal{C}_{\infty}$. Nevertheless, by an infinite Gaussian elimination process, we will find nested bases for the spaces $\mathcal{C}_r$ and $\mathcal{C}$ (but not $\mathcal{C}_{\infty}$), and see that as

soon as $\mathcal{C}_\infty \neq \mathcal{C}$, the quotient space $\mathcal{C}/\mathcal{C}_\infty$ is infinite dimensional (when A is a field, a necessary condition for Gaussian elimination to work in general). Thus, in the rest of this section, A will always denote a field.

From now on, given a fixed coefficient ring A , we call a subset $\mathcal{B}$ of $\mathcal{C}$ a *basis* if it is thin and is such that the map

$$A^{\mathcal{B}} \rightarrow \mathcal{C}; \quad \phi \mapsto \Sigma \phi := \sum_{b \in \mathcal{B}} \phi(b) \cdot b$$

is bijective. Note that this map is well-defined by virtue of the thinness of $\mathcal{B}$. Surjectivity of this map amounts to saying that the set $\mathcal{B}$ spans $\mathcal{C}$ under infinite sums; injectivity to the fact that no non-trivial infinite sum vanishes. We make the same definition for the subspaces $\mathcal{C}_r$ ($r < \infty$).

Theorem 13. *There exists a basis for $\mathcal{C}$.*

Proof. Simply apply Theorem 1 from [BG11] on $A^{(\mathcal{C})}$. □

5.1 Nested bases for the spaces $\mathcal{C}_r$

By theorem 13, there is a basis $\mathcal{B}$ of elements of $\mathcal{C}$. The goal here is to refine this basis into a nested sequence:

$$\mathcal{B}'_1 \subseteq \mathcal{B}'_2 \subseteq \dots \subseteq \mathcal{B}'_n \subseteq \mathcal{B}'_{n+1} \subseteq \dots \subseteq \mathcal{B}',$$

in such a way that each $\mathcal{B}'_i$ [resp. $\mathcal{B}'$] is a basis of the space $\mathcal{C}_i$ [resp. $\mathcal{C}$].

Choose first a well-ordering $\{g_\alpha | \alpha < \beta\}$ of $\mathcal{C}$ and an increasing sequence of ordinals $\beta_1 < \beta_2 < \dots < \beta_i < \beta_{i+1} < \dots < \beta$ ($i \in \mathbb{N}$) such that $\mathcal{C}_i = \{g_\alpha | \alpha < \beta_i\}$ for all $i \in \mathbb{N}$.

Let $\mathcal{B}$ be the basis of $\mathcal{C}$ provided by theorem 13. The set $\mathcal{B}$ is countable; let us choose an enumeration $\{b_i | i \in \mathbb{N}\}$ of $\mathcal{B}$. By theorem 13, there is an isomorphism $A^{\mathcal{B}} \leftrightarrow \mathcal{C}$. Any element g of $\mathcal{C}$ thus (uniquely) defines a function $\mathbb{N} \rightarrow A$ by means of our chosen enumeration of $\mathcal{B}$. From now on, we identify $\mathcal{C}$ and $(\mathbb{N} \rightarrow A) = A^{\mathbb{N}}$: if $g \in \mathcal{C}$, $g(i)$ is the coefficient associated to $b_i \in \mathcal{B}$ in the unique decomposition of g as a (infinite) sum of elements of $\mathcal{B}$.

Lemma 14. *Let X be a locally finite, connected graph. There exists a nested sequence of thin subsets of $\mathcal{C}$*

$$\mathcal{B}'_1 \subseteq \mathcal{B}'_2 \subseteq \dots \subseteq \mathcal{B}'_n \subseteq \mathcal{B}'_{n+1} \subseteq \dots \subseteq \mathcal{B}',$$

such that each $\mathcal{B}'_i$ is a basis of $\mathcal{C}_i$ and $\mathcal{B}'$ is a basis of $\mathcal{C}$.

Proof. Recall that we have chosen a well-ordering $\{g_\alpha | \alpha < \beta\}$ of the set $\mathcal{C}$, and ordinals $\beta_1 < \beta_2 < \dots$ such that $\mathcal{C}_i = \{g_\alpha | \alpha < \beta_i\}$ for all i , and that any element of $\mathcal{C}$ is identified with a map $\mathbb{N} \rightarrow A$ by means of theorem 13 and an enumeration $\{b_i\}_{i \in \mathbb{N}}$ of $\mathcal{B}$.

We will define a “Gaussian elimination” map

$$\nu : \beta \times \beta \times \mathbb{N} \rightarrow A$$

(whose application to the arguments τ, α, n we write $\nu_{\tau, \alpha}(n)$), inductively over its first parameter τ .

Define first the function $l : A^{\mathbb{N}} \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$l(g) := \min\{i : g(i) \neq 0\},$$

with by convention $l(0) = \infty$. Recall that by our identifications $\mathcal{B} \leftrightarrow \mathbb{N}$ and $\mathcal{C} \leftrightarrow A^{\mathbb{N}}$, both $l(g)$ and $g(n)$ are well-defined for $g \in \mathcal{C}$ and $n \in \mathbb{N}$; define also $g(\infty) = 0$.

Our inductive definition of ν is then the following.

$$\nu_{0, \alpha} = g_{\alpha},$$

$$\nu_{\tau, \alpha} = \begin{cases} \lim_{\sigma < \tau} \nu_{\sigma, \alpha} & \text{if } \tau \text{ is a limit ordinal,} \\ \nu_{\alpha, \alpha} & \text{otherwise and } \tau > \alpha, \\ \nu_{\tau-1, \alpha} - \frac{\nu_{\tau-1, \alpha}(l(\nu_{\tau-1, \tau-1}))}{\nu_{\tau-1, \tau-1}(l(\nu_{\tau-1, \tau-1}))} \cdot \nu_{\tau-1, \tau-1} & \text{otherwise,} \end{cases} \quad (4)$$

where the limit is a pointwise limit (meaning that at any coordinate $r \in \mathbb{N}$, the transfinite sequence $(\nu_{\sigma, \alpha}(r))_{\sigma < \tau}$, is eventually constant), and whenever $\nu_{\tau-1, \tau-1}$ is zero, the fraction in eq. (4) is set to zero, by convention.

From now on, we will write ν_{α} for the diagonal element $\nu_{\alpha, \alpha}$, and $\lambda_{\tau, \alpha}$ for the fraction $\frac{\nu_{\tau-1, \alpha}(l(\nu_{\tau-1, \tau-1}))}{\nu_{\tau-1, \tau-1}(l(\nu_{\tau-1, \tau-1}))}$ when $\tau \leq \alpha$ and τ is successor (again set to zero by convention when $\nu_{\tau-1, \tau-1} = 0$), or 0 for any other τ .

Let us briefly explain our interpretation of the map ν as an infinite Gaussian elimination. An element of $\mathcal{C}$ is uniquely determined by a map $\mathbb{N} \rightarrow A$, as explained above, which we view as an infinite row. The sequence $(g_{\alpha})_{\alpha < \beta}$ of elements of $\mathcal{C}$ is viewed as a list of rows: a matrix $(\nu_{0, \alpha}(n))_{\alpha < \beta, n \in \mathbb{N}}$. Finally, the matrix is modified by iterating over each row and subtracting (a suitable multiple of) it from the next rows; we thus get a “stack” of matrices: $(\nu_{\tau, \alpha}(n))_{\alpha, \tau < \beta, n \in \mathbb{N}}$.

We start with the matrix whose rows are the elements of $\mathcal{C}$, ordered along the ordinal β . At time $\tau = 0$, the matrix has not been modified. At time $\tau = \sigma + 1$, we pivot around the element $\nu_{\sigma, \sigma}$. Note that if the row α lies before or is the pivot (i.e., $\alpha \leq \sigma$), it is not modified. If row α lies after the pivot ($\alpha \geq \tau$), we must subtract the pivot from row α . In case τ is a limit ordinal, we just accumulate all changes that happened up until time τ .

We start by showing by induction on τ that:

(P1) For any $n \in \mathbb{N}$, the number of ordinals $\sigma \leq \tau$ such that $\nu_{\sigma}(n) \neq 0$ is finite.

(P2) The limit in eq. (4) is actually well-defined.

(P3) For any α and $\sigma < \min(\tau, \alpha)$, we have $\nu_{\tau, \alpha}(l(\nu_{\sigma})) = 0$.

(P4) The value of $\nu_{\tau, \alpha}$ is

$$\nu_{\tau, \alpha} = \nu_{0, \alpha} - \sum_{\sigma < \min\{\alpha, \tau\}} \lambda_{\sigma+1, \alpha} \cdot \nu_{\sigma}. \quad (5)$$

Indeed, assume the above holds for any $\theta < \tau$.

If $\tau = 0$: Then (P1), (P2), (P3) and (P4) hold trivially.

If τ is successor: (P1) is easy: by the induction hypothesis, $\{\sigma \leq \tau - 1 : \nu_\sigma(n) \neq 0\}$ is finite, and $\{\sigma \leq \tau : \nu_\sigma(n) \neq 0\}$ can contain at most one more element.

(P2) does not apply to successor ordinals.

For (P3), we see that if $\tau > \alpha$, then (by definition of ν)

$$\nu_{\tau,\alpha}(l(\nu_\sigma)) = \nu_\alpha(l(\nu_\sigma))$$

which is zero by the induction hypothesis (i.e. (P3) holds at $\tau = \alpha$). If $\tau \leq \alpha$, then

$$\nu_{\tau,\alpha} = \nu_{\tau-1,\alpha} - \lambda_{\tau,\alpha} \cdot \nu_{\tau-1},$$

which, evaluated at $l(\nu_{\tau-1})$, is zero by construction. If $\sigma < \tau - 1$, both $\nu_{\tau-1,\alpha}(l(\nu_\sigma))$ and $\nu_{\tau-1}(l(\nu_\sigma))$ are zero, hence so is $\nu_{\tau,\alpha}$.

Finally, (P4). If $\tau \leq \alpha$, then $\min\{\tau, \alpha\} = \tau$, and:

$$\begin{aligned} \nu_{\tau,\alpha} &= \nu_{\tau-1,\alpha} && - \lambda_{\tau,\alpha} \cdot \nu_{\tau-1} \\ &= \nu_{0,\alpha} - \sum_{\sigma < \tau-1} \lambda_{\sigma+1,\alpha} \cdot \nu_\sigma && - \lambda_{\tau,\alpha} \cdot \nu_{\tau-1} \\ &= \nu_{0,\alpha} - \sum_{\sigma < \min(\tau,\alpha)} \lambda_{\sigma+1,\alpha} \cdot \nu_\sigma \end{aligned}$$

by definition of $\nu_{\tau,\alpha}$. If $\tau > \alpha$, we get $\min\{\tau, \alpha\} = \alpha$, and

$$\begin{aligned} \nu_{\tau,\alpha} &= \nu_\alpha = \nu_{0,\alpha} + \sum_{\sigma < \alpha} \lambda_{\sigma+1,\alpha} \cdot \nu_\sigma \\ &= \nu_{0,\alpha} + \sum_{\sigma < \min\{\alpha,\tau\}} \lambda_{\sigma+1,\alpha} \cdot \nu_\sigma. \end{aligned}$$

If τ is limit: We first check (P1). It suffices to show that for any n , the number of ordinals $\sigma < \tau$ with $\nu_\sigma(n) \neq 0$ is finite. If $n = l(\nu_\sigma)$ for some $\sigma < \tau$, then the number of $\theta < \sigma$ for which $\nu_\theta(n) \neq 0$ is finite by the induction hypothesis (P1), and for any $\tau > \theta > \sigma$, $\nu_\theta(n) = 0$, also by the induction hypothesis (P3). Assume then that there exists some n which is not a leading coefficient (i.e., of the form $l(\nu_\sigma)$ for some $\sigma < \tau$) and in the support of infinitely many ν_σ (where $\sigma < \tau$). We can take n to be the least such integer, and for σ large enough, we will have:

$$\nu_\sigma(n) \neq 0, \quad \nu_\sigma(k) = 0, \quad \forall k < n,$$

so that n is actually the leading coefficient of ν_σ , a contradiction.

We now check (P2). Let α be arbitrary, then

$$\nu_{\tau,\alpha} = \lim_{\sigma < \tau} \left(\nu_{0,\alpha} - \sum_{\sigma' < \min\{\alpha,\sigma\}} \lambda_{\sigma'+1,\alpha} \cdot \nu_{\sigma'} \right)$$

which we need to verify to be well-defined. In other words, we need to check that for any n , the sequence

$$\sigma \mapsto \nu_{0,\alpha}(n) + \sum_{\sigma' < \min\{\alpha, \sigma\}} \lambda_{\sigma'+1, \alpha} \cdot \nu_{\sigma'}(n)$$

stabilises starting at some ordinal. But by (P1), $\nu_{\sigma'}(n)$ is zero for σ' large enough, so that the sequence indeed stabilises. Furthermore, it stabilises to:

$$\nu_{0,\alpha}(n) + \sum_{\sigma < \min\{\alpha, \tau\}} \lambda_{\sigma+1, \alpha} \cdot \nu_{\sigma}(n),$$

which also proves (P4).

It remains to check (P3). If $\sigma < \min\{\tau, \alpha\}$, then

$$\nu_{\tau, \alpha}(l(\nu_{\sigma})) = \lim_{\sigma' < \tau} \nu_{\sigma', \alpha}(l(\nu_{\sigma})) = \lim_{\sigma < \sigma' < \tau} \nu_{\sigma', \alpha}(l(\nu_{\sigma})) = \lim 0 = 0.$$

This closes the inductive verification of items (P1) to (P4).

Let now

$$\mathcal{B}' := \{\nu_{\alpha} | \alpha < \beta, \nu_{\alpha} \neq 0\}, \quad \mathcal{B}'_i = \{\nu_{\alpha} | \alpha < \beta_i, \nu_{\alpha} \neq 0\}.$$

Note that by (P1), each non-zero ν_{α} has a well-defined leading index $l(\nu_{\alpha})$, and $\nu_{\alpha} \neq \nu_{\beta}$ implies $l(\nu_{\alpha}) \neq l(\nu_{\beta})$. It follows that $\mathcal{B}'$ is at most countable. We can now check the following consequences of this process:

1. $\mathcal{B}'_i \subseteq \mathcal{C}_i$ and $\mathcal{B}' \subseteq \mathcal{C}$, which can be verified by induction using (P1), (P4) on ν_{τ} and since the spaces $\mathcal{C}_{\tau}$ and $\mathcal{C}$ are closed under thin sums.
2. $\mathcal{B}'_i$ spans $\mathcal{C}_i$ and $\mathcal{B}'$ spans $\mathcal{C}$ (i.e., the maps $A^{\mathcal{B}'_i} \rightarrow \mathcal{C}_i$ and $A^{\mathcal{B}'} \rightarrow \mathcal{C}$ are surjective). Indeed, eq. (5) yields (by setting $\alpha = \tau$)

$$\nu_{\tau} = \nu_{0,\tau} - \sum_{\sigma < \tau} \lambda_{\sigma+1, \tau} \cdot \nu_{\sigma},$$

so that we get

$$\nu_{0,\tau} = \nu_{\tau} + \sum_{\sigma < \tau} \lambda_{\sigma+1, \tau} \cdot \nu_{\sigma},$$

so that any element of $\mathcal{C}$ [resp. $\mathcal{C}_i$] is a thin sum of elements of $\mathcal{B}'$ [resp. $\mathcal{B}'_i$].

3. $\mathcal{B}'$ is thin, as a subset of $A^{\mathcal{B}}$, by the same argument as the proof of (P1) for τ limit.
4. $\mathcal{B}'$ is thin, as a subset of A^{EX} . Indeed, every $e \in EX$ appears in the support of a finite number of $b \in \mathcal{B}$, and every $b \in \mathcal{B}$ appears in the support of a finite number of elements of $\mathcal{B}'$ (by the point above).
5. The set $\mathcal{B}'$ is independent (i.e., the map $A^{\mathcal{B}'} \rightarrow \mathcal{C}$ is injective). If not, then there would exist a subset $\tilde{\beta} \subset \beta$ and non-zero coefficients $(\mu_{\sigma})_{\sigma \in \tilde{\beta}}$ such that $\sum_{\sigma \in \tilde{\beta}} \mu_{\sigma} \cdot \nu_{\sigma} = 0$. Let $\tau_0 := \min \tilde{\beta}$. Then $\sum_{\sigma \in \tilde{\beta}} \nu_{\sigma}(l(\nu_{\tau_0})) = \mu_{\tau_0}$, a contradiction. $\square$

5.2 Infinite dimensionality of $\mathcal{C}/\mathcal{C}_\infty$

Using lemma 14, we get a sufficient condition for the infinite dimensionality of the quotient $\mathcal{C}/\mathcal{C}_\infty$, as an A -vector space.

Lemma 15. *Let X be a locally finite, connected graph. If $\mathcal{C}_\infty \neq \mathcal{C}_i$ for all $i \in \mathbb{N}$, then $\mathcal{C}/\mathcal{C}_\infty$ is infinite-dimensional as a vector space.*

Proof. The condition $\forall i \in \mathbb{N} \mathcal{C}_\infty \neq \mathcal{C}_i$ means that for all i , there exists $i' > i$ with $\mathcal{C}_i \subsetneq \mathcal{C}_{i'}$; let $\{i_j\}_j$ be a sequence of indices with $\mathcal{C}_{i_j} \subsetneq \mathcal{C}_{i_{j+1}}$ for all j , and choose an element $b_j \in \mathcal{B}'_{i_j} - \mathcal{B}'_{i_{j-1}}$ for each j . Let $\{J_n\}_{n \in \mathbb{N}}$ be a partition of $\mathbb{N}$ into $\mathbb{N}$ subsets of cardinality $\mathbb{N}$, and

$$f_n := \sum_{j \in J_n} b_j, \quad n \in \mathbb{N}.$$

We claim that the family $\{f_n\}_n$ is linearly independent in the quotient, i.e., that no non-trivial sum thereof lies in $\mathcal{C}_\infty$. Let us verify the claim. Such a sum has the form

$$S = \sum_n \lambda_n f_n = \sum_n \sum_{j \in J_n} \lambda_n b_j,$$

(which is well-defined since the b_j s form a thin set). Assuming the sum lies in $\mathcal{C}_\infty$, hence in some $\mathcal{C}_i$, and we must thus get

$$S = \sum_{b \in \mathcal{B}'_i} \mu_b b,$$

so that

$$0 = S - S = \sum_n \sum_{j \in J_n} \lambda_n b_j - \sum_{b \in \mathcal{B}'_i} \mu_b b.$$

Grouping the coefficients belonging to a given $b \in \mathcal{B}'$ yields a sum

$$0 = \sum_{b \in \mathcal{B}'} \kappa_b b,$$

and each κ_b must then be zero, since $\mathcal{B}'$ is itself a basis. But, by construction of the f_n s, there are non-zero coefficients λ_n belonging to elements $b \in \mathcal{B}'_i$ for arbitrary high i , while the μ_b belong only to elements $b \in \mathcal{B}'_i$. This is a contradiction, and concludes the proof. $\square$

5.3 More on $\mathbb{Z}/2\mathbb{Z}$

The results obtained so far can be strengthened in the case of $A = \mathbb{Z}/2\mathbb{Z}$ by using compactness.

Lemma 16. *Let X be a locally finite, connected graph, and $A = \mathbb{Z}/2\mathbb{Z}$. If there exists r such that any cycle of X lies in $\mathcal{C}_r$, then $\mathcal{C} = \mathcal{C}_r$.*

Proof. Endow both A^{EX} and $A^{\mathfrak{C}}$ with the product topology, with A discrete; both spaces are compact. The set $A^{\mathfrak{C}_r}$ is closed in $A^{\mathfrak{C}}$, hence compact, and the restriction of $\mathcal{D} : A^{\mathfrak{C}} \rightarrow A^{EX}$ to $A^{\mathfrak{C}_r}$ is continuous. By assumption, for any cycle $c \in \mathfrak{C}$, there exists some $\phi_c \in A^{\mathfrak{C}_r}$ with $\mathcal{D}\phi_c = \mathcal{D}c$. Fix any element of $\mathcal{C}$ and any $\psi \in A^{\mathfrak{C}}$ with $\mathcal{D}\psi$ equal to this element. Let $\{c_i\}_{i \in \mathbb{N}}$ be an enumeration of the elements of $\mathfrak{C}$ appearing in ψ . Then, by construction,

$$\mathcal{D}\psi = \sum_i \mathcal{D}c_i = \lim_n \sum_{i=1}^n \mathcal{D}c_i.$$

One can then replace $\mathcal{D}c_i$ by $\mathcal{D}\phi_i$ and use linearity of $\mathcal{D}$:

$$= \lim_n \sum_{i=1}^n \mathcal{D}\phi_{c_i} = \lim_n \mathcal{D} \left(\sum_{i=1}^n \phi_{c_i} \right).$$

By compactness of $A^{\mathfrak{C}}$ and closure of $A^{\mathfrak{C}_r}$ therein, one may choose a subsequence (indexed by $(n_m)_m$) of $(\sum_{i=1}^n \phi_{c_i})_n$ converging to some element ϕ of $A^{\mathfrak{C}_r}$. Then:

$$= \lim_m \mathcal{D} \left(\sum_{i=1}^{n_m} \phi_{c_i} \right) = \mathcal{D} \left(\lim_m \sum_{i=1}^{n_m} \phi_{c_i} \right) = \mathcal{D}\phi,$$

now using continuity of $\mathcal{D}|_{A^{\mathfrak{C}_r}}$. □

We easily deduce the following.

Corollary 17. *Let X be a locally finite, connected graph, and $A = \mathbb{Z}/2\mathbb{Z}$. Then, the following are equivalent:*

1. *There exists some r such that $\mathcal{C} = \mathcal{C}_r$.*
2. *There exists some r such that $\mathcal{C}_\infty = \mathcal{C}_r$.*
3. *There exists some r such that for all $r' \geq r$, we have $\mathcal{C}_{r'} = \mathcal{C}_r$.*
4. *There exists some r such that any cycle of X lies in $\mathcal{C}_r$.*

Proof. Indeed, each downward implication is obvious, and lemma 16 allows closing the chain. □

Finally, putting together what we have so far:

Theorem 18. *Let X be a locally finite, connected graph and $A = \mathbb{Z}/2\mathbb{Z}$. If the quotient $\mathcal{C}/\mathcal{C}_\infty$ does not vanish, then it is infinite dimensional.*

Proof. $\mathcal{C}/\mathcal{C}_\infty \neq 0$ implies that $\mathcal{C} \neq \mathcal{C}_\infty$, and hence that none of the equivalent conditions of corollary 17 hold. It then suffices to apply lemma 15. □

So the case of $A = \mathbb{Z}_2$ is well-understood since all notions of “having large cycles” are equivalent. This is a priori also the case for any finite field $\mathbb{Z}_p$, but not otherwise. One might ask the following question:

Question 19. Assuming any cycle of length $\leq s$ lies in $\mathcal{C}_r(X, \mathbb{Z})$, does it hold that $\mathcal{C}_s(X, \mathbb{Z}) \subseteq \mathcal{C}_r(X, \mathbb{Z})$?

Without compactness, mimicking the argument for $A = \mathbb{Z}_2$ does not work. On the other hand, we did not manage to find a counterexample.

Another interesting question arises when we consider Cayley graphs of groups.

Question 20. It is relatively clear that a finitely presented group has $\mathbb{Z}_2$ -small cycles, since any cycle defines a relation, and all relations are made of a finite number of small relations. Does the converse hold ?

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