

Equatorial Flow Triangulations of Gorenstein Flow Polytopes

Benjamin Braun

Alvaro Cornejo

Submitted: Aug 26, 2024; Accepted: Sep 10, 2025; Published: Oct 17, 2025

© The authors. Released under the CC BY license (International 4.0).

Abstract

Generalizing work of Athanasiadis for the Birkhoff polytope and Reiner and Welker for order polytopes, in 2007 Bruns and Römer proved that any Gorenstein lattice polytope with a regular unimodular triangulation admits a regular unimodular triangulation that is the join of a special simplex with a triangulated sphere. These are sometimes referred to as equatorial triangulations. We apply these techniques to give purely combinatorial descriptions of previously-unstudied triangulations of Gorenstein flow polytopes. Further, we prove that the resulting equatorial flow polytope triangulations are usually distinct from the family of triangulations obtained by Danilov, Karzanov, and Koshevoy via framings. We find the facet description of the reflexive polytope obtained by projecting a Gorenstein flow polytope along a special simplex. Finally, we show that when a partially ordered set is strongly planar, equatorial triangulations of a related flow polytope can be used to produce new unimodular triangulations of the corresponding order polytope.

Mathematics Subject Classifications: 52B20, 52B12

1 Introduction

Given a finite directed acyclic graph (DAG) G , the *flow polytope* $\mathcal{F}_1(G)$ is the lattice polytope of non-negative flows on G of strength one. Flow polytopes have been the subject of intense recent study in geometric and algebraic combinatorics [2, 4, 5, 7, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24]. Flow polytopes are connected to many areas of mathematics; one example relevant to this work is that when the DAG G is strongly planar, it is known that the flow polytope of G is integrally equivalent to the order polytope of the poset with Hasse diagram given by the strongly planar dual of G [19].

A major tool used in the study of flow polytopes is the *DKK triangulation* induced by a framing on the edges of G , a regular unimodular triangulation introduced by Danilov,

Department of Mathematics, University of Kentucky, Lexington, KY, U.S.A.
(benjamin.braun@uky.edu, alvaro.cornejo@uky.edu).

Karzanov, and Koshevoy [10]. Mészáros, Morales, and Striker proved that when a strongly planar DAG G is equipped with the planar framing, the induced DKK triangulation is integrally equivalent to the canonical triangulation of the order polytope for the poset dual to G . DKK triangulations have been used extensively in the literature to study the structure of flow polytopes, dual graphs of their triangulations, and their volumes.

In this paper, we describe *equatorial flow triangulations*, a new family of regular unimodular triangulations for Gorenstein flow polytopes that arise as the join of a unimodular simplex and a simplicial sphere. The study of triangulations with this structure originated in work of Athanasiadis [1] for the Birkhoff polytope and Reiner and Welker [21] for order polytopes. In 2007, Bruns and Römer [9] proved that any Gorenstein lattice polytope with a regular unimodular triangulation admits a regular unimodular triangulation that is the join of a special simplex with a triangulated sphere. We apply this Bruns-Römer machinery to DKK triangulations arising from route decompositions of DAGs, producing equatorial flow triangulations of Gorenstein flow polytopes.

Our main contributions in this paper are the following.

1. In Theorem 26, we use route decompositions to give a new combinatorial characterization of DAGs yielding Gorenstein flow polytopes.
2. In Theorem 49, we give a purely combinatorial description of equatorial flow triangulations of a Gorenstein flow polytope.
3. In Theorem 50, we prove that if G has an inner vertex with indegree at least 3, then the equatorial flow triangulation is not a DKK triangulation.
4. As a corollary of the work of Bruns and Römer [9], a Gorenstein $\mathcal{F}_1(G)$ projects onto a reflexive polytope. In Section 5, we provide a combinatorial halfspace description of this reflexive polytope for an equatorial flow triangulation.
5. Finally, in Section 6, we prove that when G is strongly planar, a particular choice of route decomposition yields an equatorial flow triangulation that is integrally equivalent to the Reiner-Welker equatorial triangulation of the associated order polytope. We also use equatorial flow triangulations to produce new triangulations of Gorenstein order polytopes for strongly planar posets.

We begin with Section 2, which contains background on polytopes, triangulations, and Ehrhart theory.

2 Background

2.1 Lattice and Flow polytopes

Let G be a finite directed acyclic graph (DAG) with linearly ordered vertex set $\{s < 1 < 2 < \cdots < n < t\}$ such that if (i, j) is a directed edge in G , then $i < j$. We assume throughout this work that s is the unique source of G and t is the unique sink of G , and

all other vertices of G we call *inner vertices*. We denote by $\text{in}_G(v)$ the set of incoming edges at v , and by $\text{out}_G(v)$ the set of outgoing edges at v . A *route* is a directed path in G from s to t .

Definition 1. The *flow polytope* of G is

$$\mathcal{F}_1(G) := \left\{ x \in \mathbb{R}_{\geq 0}^{E(G)} : \sum_{e \in \text{in}_G(v)} x_e = \sum_{e \in \text{out}_G(v)} x_e \text{ for every inner vertex } v, \sum_{e \in \text{out}_G(s)} x_e = 1 \right\}.$$

Equivalently, $\mathcal{F}_1(G)$ is given by the convex hull of indicator vectors of routes in G . The *cone of non-negative flows* of G is

$$\mathcal{F}(G) := \left\{ x \in \mathbb{R}_{\geq 0}^{E(G)} : \sum_{e \in \text{in}_G(v)} x_e = \sum_{e \in \text{out}_G(v)} x_e \text{ for every inner vertex } v \right\}.$$

The dimension of $\mathcal{F}_1(G)$ is $|E| - n - 1$, where G has n inner vertices and E is the set of edges of G . Note that the facets of $\mathcal{F}_1(G)$ and $\mathcal{F}(G)$ are given by $x_e = 0$ for an edge $e \in E(G)$, but not all of these equations are facet defining. Towards identifying the facet defining hyperplanes, we define an edge to be *idle* if it is the only incoming or outgoing edge of an inner vertex. It is known [10, 23] that contracting an idle edge $e \in E(G)$ does not change the lattice-polyhedral structure of $\mathcal{F}_1(G)$ and $\mathcal{F}(G)$. Further, in a DAG with no idle edges, every edge yields a facet hyperplane $x_e = 0$. Hence, we will assume throughout this paper that we have no idle edges, and this can be done without losing information about $\mathcal{F}_1(G)$ and $\mathcal{F}(G)$.

A *framing* of G is a choice for each inner vertex v of linear orderings on $\text{in}_G(v)$ and $\text{out}_G(v)$. Danilov, Karzanov, and Koshevoy [10] proved that every framing F induces a regular unimodular triangulation of $\mathcal{F}_1(G)$, called the *DKK triangulation* induced by F , which we explain next. Assume that we have chosen a framing F of G , and note that all of the following definitions are dependent on F . If P is a route in G containing the vertex v , then we write Pv for the path from the source s to v following P , and similarly vP for the path from v to the sink t along P . We write vPw for the segment of P from v to w . We write $\text{Out}(v)$ for the set of partial routes in G starting at v and ending at t , and similarly $\text{In}(v)$ is partial routes from s to v . If e is less than f in the linear order for F on $\text{in}(v)$, we write $e \prec_{\text{in}(v)} f$, and similarly for $\text{out}(v)$.

Let P and Q be paths in $\text{Out}(v)$ that agree on the paths $P' \subset P$ and $Q' \subset Q$ that begin at v and end at w . Suppose further that the vertices following w on P and Q are distinct, and call them w_P and w_Q . We define an inequality between these paths by setting $P \prec_{\text{Out}(v)} Q$ if $(w, w_P) \prec_{F, \text{out}(w)} (w, w_Q)$, and similarly for $P \prec_{\text{In}(v)} Q$. Assume P and Q are routes that intersect at a common inner vertex v of G . We say P and Q are *in conflict*, also called *conflicting*, if $Pv \prec_{\text{In}(v)} Qv$ and $vQ \prec_{\text{Out}(v)} vP$. If P and Q are not conflicting at v , then they are *coherent* at v . P and Q are called *coherent* if they are coherent at every inner vertex v that is contained in both P and Q . Finally, a *clique* is a set of pairwise-coherent routes in G . When a route R in G is coherent with every other route in G , R is called *exceptional*.

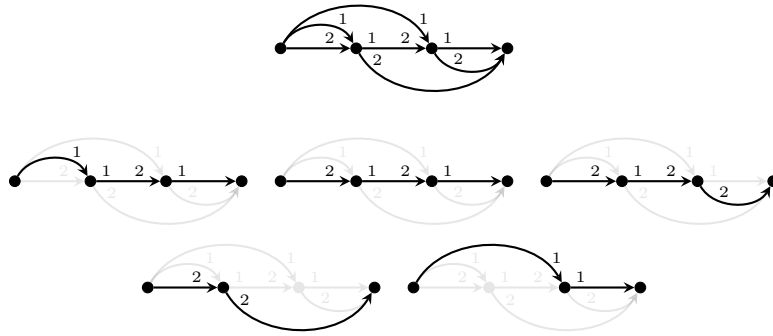


Figure 1: An example of a framed DAG where the edges are linearly ordered by $1 < 2$ and an example of coherent routes forming a maximal clique. The bottom two routes are the exceptional routes for this framed DAG.

Definition 2. Given a DAG G with framing F , the *DKK triangulation* of $\mathcal{F}_1(G)$ induced by F has one facet for each maximal clique, where the facet for a clique is the simplex given by the convex hull of the indicator vectors for the routes in the clique.

Recall that a lattice simplex $\text{conv}\{v_0, v_1, \dots, v_d\} \subset \mathbb{R}^d$ is *unimodular* if $\{v_1 - v_0, \dots, v_d - v_0\}$ is a lattice basis for $\mathbb{Z}^d$. Further, a triangulation of a lattice polytope is *unimodular* if every simplex in the triangulation is unimodular.

Theorem 3 (Danilov, Karzanov, and Koshevoy [10]). *For any finite DAG G and any framing F , the DKK triangulation of $\mathcal{F}_1(G)$ induced by F is a regular unimodular triangulation.*

Example 4. Figure 1 is an example of a framed DAG and coherent routes.

2.2 Face Enumeration and Ehrhart Theory

We next recall h -polynomials, Ehrhart series, and connections among these objects. Two invariants of interest for flow polytopes and their triangulations are h -vectors and Ehrhart h^* -vectors, defined as follows.

Definition 5. Given a finite simplicial complex T , let f_k denote the number of k -dimensional faces of T , where we define $f_{-1} := 1$. The *h -polynomial* of T is defined to be

$$h(T; z) := \sum_{k=-1}^{d-1} f_k z^{k+1} (1-z)^{d-1-k}.$$

The coefficient vector of $h(T; z)$ is the *h -vector* of T .

In the following definition, we rely on a result due to Ehrhart [12] that states the Ehrhart series of a lattice polytope is a rational function of the form given in the definition.

Definition 6. Given polytope P of dimension d with vertices in $\mathbb{Z}^n$, the *Ehrhart series* of P is

$$\text{Ehr}(P; z) := 1 + \sum_{t \geq 1} |tP \cap \mathbb{Z}^d| z^t = \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1 - z)^{d+1}}.$$

We call the numerator of this rational function the h^* -polynomial of P and its coefficient vector the h^* -vector of P .

Example 7. The Ehrhart series for the flow polytope of the DAG G given in Figure 1 is

$$\text{Ehr}(\mathcal{F}_1(G); z) = \frac{1 + 3z + z^2}{(1 - z)^5}$$

For a unimodular triangulation, the h -vector and h^* -vector are closely related; see Beck and Robins [3] for a textbook proof of the following result.

Theorem 8. *Given a unimodular triangulation T of a lattice polytope P , the h -vector of T is equal to the h^* -vector of P .*

We complete this subsection by recalling the definition of the codegree of a lattice polytope.

Definition 9. The *degree* of a d -polytope P is

$$\deg(P) := \max\{m \in \mathbb{Z}_{\geq 0} : h_m^* \neq 0\}.$$

The *codegree* of P is

$$\text{codeg}(P) := d + 1 - \deg(P).$$

The following theorem is a consequence of Ehrhart-Macdonald reciprocity [3, Theorem 4.5].

Theorem 10. *The codegree of a d -polytope P is the smallest integer k such that kP contains an interior lattice point.*

Example 11. The degree and codegree for the flow polytope of the DAG given in Figure 1 is $\deg(\mathcal{F}_1(G)) = 2$ and $\text{codeg}(\mathcal{F}_1(G)) = 4 + 1 - 2 = 3$. Hence, the dilate $3\mathcal{F}_1(G)$ contains an interior lattice point by Theorem 10.

2.3 Gorenstein Polytopes and Equatorial Triangulations

We next recall Gorenstein polytopes and their properties.

Definition 12. A rational pointed cone $C \subseteq \mathbb{R}^{d+1}$ is said to be *Gorenstein* if there exists an integer point $c \in \mathbb{Z}^{d+1}$ such that

$$c + (C \cap \mathbb{Z}^{d+1}) = C^\circ \cap \mathbb{Z}^{d+1}$$

where C° is the (relative) interior of C . We call c the *Gorenstein point* of C .

We can extend the definition of Gorenstein to lattice polytopes as follows.

Definition 13. A lattice polytope $P \subseteq \mathbb{R}^d$ is *Gorenstein* if

$$\text{cone}(P) := \{(x, t) \in \mathbb{R}^{d+1} : t \geq 0, x \in tP\}$$

is a Gorenstein cone. The polytope P is *reflexive* if the Gorenstein point in $\text{cone}(P)$ has final coordinate 1, i.e., is at height 1 in the cone over P .

Note that if P is d -dimensional, then P is reflexive if and only if P contains a unique interior lattice point v and there exists an integer matrix A such that

$$P - v = \{x \in \mathbb{R}^d : Ax \leq \mathbb{1}\}.$$

The following two propositions connect arbitrary Gorenstein polytopes to reflexive polytopes; for more details see the textbook treatment by Bruns and Herzog [8].

Proposition 14. A lattice polytope P is Gorenstein if and only if there exists some $r \in \mathbb{N}$ such that rP is reflexive. In this case, we say P is Gorenstein of index r and, in addition, r is the codegree of P .

Proposition 15. If $P \subseteq \mathbb{R}^d$ is a Gorenstein polytope with Gorenstein point $c \in \mathbb{R}^{d+1}$ for $\text{cone}(P)$, then $c = (v, r)$ where $r \in \mathbb{Z}_{\geq 1}$ is the index of P and v is the unique interior lattice point of rP .

A key idea in the theory of equatorial triangulations is that of a special simplex, defined as follows.

Definition 16. Given a d -dimensional lattice polytope P , a simplex $\Delta = \text{conv}\{v_1, \dots, v_k\}$ with $v_i \in P \cap \mathbb{Z}^d$ is *special* if $\Delta \cap F$ is a facet of Δ for all facets F of P .

For Gorenstein lattice polytopes, the following proposition provides a mechanism for finding special simplices.

Proposition 17 (Bruns and Römer [9]). If P is a Gorenstein polytope and there exist lattice points $\{v_1, \dots, v_k\} \subseteq P \cap \mathbb{Z}^d$ where $\{(v_1, 1), \dots, (v_k, 1)\}$ sum to the Gorenstein point of $\text{cone}(P)$, then $\text{conv}\{v_1, \dots, v_k\}$ forms a special simplex and is unimodular.

The notion of special simplices were first introduced by Athanasiadis [1] and later explored by Bruns and Römer [9] to show that Gorenstein polytopes with certain properties have unimodal h^* coefficients. In particular Bruns and Römer proved that Gorenstein polytopes with a regular, unimodular triangulation have such unimodality. In this setting, they abstractly define a triangulation given as a join of a special simplex and the boundary of a simplicial polytope, as stated in the results below.

Definition 18. Given a polytope P with a special simplex $\Delta = \text{conv}\{v_1, \dots, v_k\}$, define $\Gamma(P, \Delta)$ to be the polyhedral subcomplex of ∂P generated by faces of the form $\bigcap_{i=1}^k F_i$ where F_i is a facet of P such that $v_i \notin F_i$. We call the faces $\bigcap_{i=1}^k F_i$ described previously *generating faces* of $\Gamma(P, \Delta)$ since this complex is given by faces of the generating faces and their intersections. We call $\Gamma(P, \Delta)$ the *equatorial complex* of P with respect to Δ , or just equatorial complex if P and Δ are clear from context.

Theorem 19 (Bruns and Römer [9]). *If $\Delta \subseteq P$ is a special simplex of a Gorenstein polytope and S is a triangulation of $\Gamma(P, \Delta)$, then P is triangulated by the join $S * \Delta$, i.e., by the triangulation generated by $\text{conv}\{F \cup \Delta\}$ where F is a face of the triangulation S . Moreover, $S * \Delta$ is unimodular if S is unimodular and S is regular if it is the restriction to $\Gamma(P, \Delta)$ of a regular triangulation.*

Theorem 20 (Bruns and Römer [9]). *If $\Delta \subseteq P$ is a special simplex of a Gorenstein polytope P and S a regular unimodular triangulation of P , then the restriction $S|_{\Gamma(P, \Delta)}$ is the boundary complex of a simplicial polytope.*

Bruns and Römer also observed the following corollary.

Corollary 21. *If $\Delta \subseteq P$ is a special simplex of a Gorenstein polytope P and S a unimodular triangulation of $\Gamma(P, \Delta)$, then the h^* -polynomial of P and h -polynomial of S agree. In other words,*

$$h_0^* + h_1^*z + \cdots + h_d^*z^d = h(S; z).$$

3 Gorenstein Flow Polytopes and Route Decompositions

In this section, we provide a new characterization of DAGs with Gorenstein flow polytopes. Recall, that we assume G has no idle edges.

Definition 22. A DAG G is said to satisfy *degree equality* if $\text{indeg}_G(v) = \text{outdeg}_G(v)$ for all inner vertices v .

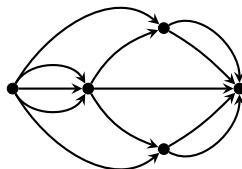


Figure 2: Example of a DAG with degree equality.

The following result gives a characterization of DAGs having a Gorenstein flow polytope.

Theorem 23 (von Bell et al. [23]). *The flow polytope $\mathcal{F}_1(G)$ is Gorenstein if and only if G satisfies degree equality.*

The following corollary shows that every flow polytope can be found as a face of a Gorenstein flow polytope.

Corollary 24. *Every flow polytope is a face of a Gorenstein flow polytope.*

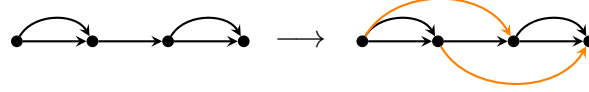


Figure 3: Example of adding new edges (in orange) to a DAG to satisfy degree equality.

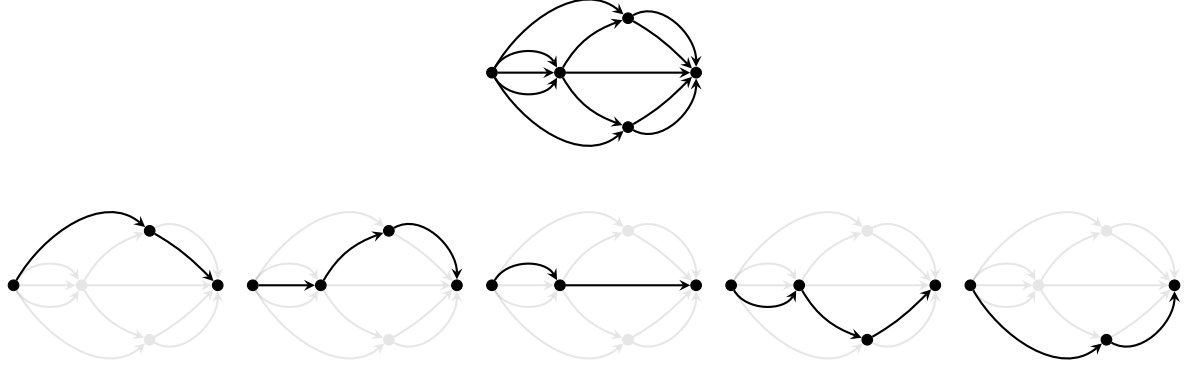


Figure 4: Example of a route decomposition for the DAG found in Figure 2.

Proof. Let G be a DAG with flow polytope $\mathcal{F}_1(G)$. For every inner vertex v of G , add edges either from the source s to v or from v to the sink t in order to create a DAG G' containing G that satisfies degree equality. See Figure 3 for an example. Then, $\mathcal{F}_1(G)$ is obtained as a face of $\mathcal{F}_1(G')$ by setting all the flow values on the newly added edges equal to zero. $\square$

Our goal in this section is to provide an alternative characterization of these DAGs using route decompositions, defined as follows.

Definition 25. A *route decomposition* $\mathcal{R}$ of a DAG G is a set of routes in G such that the edge set of G is the disjoint union of the edges of routes in $\mathcal{R}$.

See Figure 4 for an example of a route decomposition.

Theorem 26. A DAG G satisfies degree equality if and only if G admits a route decomposition.

Proof. The forward implication is proved in Proposition 33 below. For the reverse implication, let $v \in V(G)$ be an inner vertex. As v is an inner vertex, it has incoming and outgoing edges and thus must be incident to at least one route in our decomposition; let $\mathcal{S} \subseteq \mathcal{R}$ be the subset of our route decomposition containing routes incident to v . For each $e \in \text{in}_G(v)$, there exists a unique route $R_e \in \mathcal{S}$ since each edge is in some route of our decomposition and we have uniqueness as routes in $\mathcal{R}$ are edge disjoint. Similarly, given $R \in \mathcal{S}$, there exists a unique $e \in \text{in}_G(v)$ contained in R since G is acyclic. Thus, we have a bijection between edges in $\text{in}_G(v)$ and routes in $\mathcal{S}$ incident to v . We can make the same argument for $e' \in \text{out}_G(v)$ and therefore

$$\text{outdeg}_G(v) = |\mathcal{S}| = \text{indeg}_G(v),$$

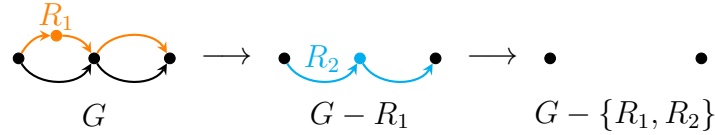


Figure 5: An example of deleting routes. The first route contains an inner vertex so it is deleted.

which proves that degree equality. $\square$

The remainder of this section develops the content required to prove Proposition 33, which establishes the forward direction of Theorem 26. For a DAG G and a route R from the source to the sink, we denote by $G - R$ the DAG with edge set $E(G) - E(R)$ and vertex set obtained by deleting from $V(G)$ any internal vertices that are incident only to edges in R . See Figure 5 for an example.

Proposition 27. *For a DAG G and a route R , the DAG $G - R$ is acyclic.*

Proof. If $G - R$ contains a cycle, then these edges are also in G and thus G contains a cycle, yielding a contradiction. $\square$

Proposition 28. *If G is a DAG satisfying degree equality, then $G - R$ also satisfies degree equality.*

Proof. Consider $v \in V(G - R)$ an inner vertex of $G - R$. Since v is not a source nor a sink in $G - R$, we have that v must also be an inner vertex of G . Thus, we have degree equality for v in G . Now consider the following two cases. If $v \notin V(R)$, then

$$\text{indeg}_{G-R}(v) = \text{indeg}_G(v) = \text{outdeg}_G(v) = \text{outdeg}_{G-R}(v),$$

hence we have degree equality of $G - R$ at v . Alternatively, if $v \in V(R)$, then the route R must use exactly one incoming edge and exactly one outgoing edge of v in G , since G is acyclic. Thus,

$$\text{indeg}_{G-R}(v) = \text{indeg}_G(v) - 1 = \text{outdeg}_G(v) - 1 = \text{outdeg}_{G-R}(v),$$

showing we have degree equality of $G - R$ at v . This proves that $G - R$ satisfies degree equality. $\square$

Lemma 29. *Suppose G is a DAG that satisfies degree equality. Then for any route R in G , and for any inner vertex v of $G - R$, there is a route in $G - R$ from s to t that is incident to v .*

Proof. Let v be an inner vertex of $G - R$. Thus, either $\text{indeg}_{G-R}(v) \geq 1$ or $\text{outdeg}_{G-R}(v) \geq 1$. Degree equality of $G - R$ follows from Proposition 28, and thus we have $\text{indeg}_{G-R}(v) = \text{outdeg}_{G-R}(v) \geq 1$. So there exists vertices $v_1, w_1 \in V(G - R)$ such that there are directed edges (v_1, v) and (v, w_1) in $G - R$. Consider v_1 . If $v_1 = s$, then we have a directed path

from s to v . Otherwise, v_1 is an internal vertex of G and degree equality of $G - R$ implies there exists a $v_2 \in V(G - R)$ such that we have a directed edge from (v_2, v_1) in $G - R$. We continue this process, constructing a sequence of $v_i \in V(G - R)$. We claim each v_i is distinct. If not, then we would have a directed cycle in $G - R$, which is a contradiction. Since G is finite, this process must end at s . The same argument holds to establish a directed path from w_j to t . Concatenating these two directed paths yields a route from s to t that is incident to v . $\square$

Corollary 30. *Suppose G is a DAG that satisfies degree equality. For any route R in G , if $G - R$ has internal vertices, then the underlying undirected graph is connected.*

Lemma 31. *For G a DAG that satisfies degree equality and has one source and one sink, $G - R$ is disconnected if and only if $G = R$.*

Proof. If $E(G - R) = \emptyset$, then by definition $G - R$ is disconnected as it consists of only the source and sink with no edges, establishing the reverse implication. For the forward implication, suppose $G - R$ is disconnected. By Corollary 30, we see $G - R$ has no internal vertices. Thus, any edges in G were in R , along with any internal vertices in G . $\square$

Lemma 32. *For G a DAG that satisfies degree equality, if $G - R$ is connected then it has the same unique source and sink as G .*

Proof. Suppose we have a second source $v \in V(G - R)$, i.e., we have $v \neq s$ and $\text{indeg}_{G-R}(v) = 0$. We claim $v \neq t$. Suppose for contradiction that $v = t$, so we have $\text{indeg}_{G-R}(t) = 0$ and we have

$$\text{outdeg}_{G-R}(t) = \text{outdeg}_G(t) = 0.$$

It follows that $G - R$ is disconnected which is a contradiction. Hence v is an inner vertex of G and satisfies $\text{outdeg}_{G-R}(v) = \text{indeg}_{G-R}(v) = 0$. This implies that $G - R$ is disconnected, but again by our assumption that $G - R$ is connected, this is a contradiction. Hence no such v can exist. A similar argument shows that there is a unique sink. $\square$

Proposition 33. *Every DAG G with one source and one sink that satisfies degree equality admits a route decomposition.*

Proof. Let R_1 be a route from the source to the sink in G . We have $G - R_1$ is acyclic by Proposition 27 and satisfies degree equality by Proposition 28. If $G - R_1 = \emptyset$ then we have a route decomposition. Otherwise, we have the same source s and sink t as G by Lemma 32. We also have at least one route from s to t , which come from the following two cases. Either we have an internal vertex and so by Lemma 29 we have a route. If we do not have an internal vertex, then we only have s and t and, as the graph is connected by Lemma 31, there must be at least one edge from s to t . So, let R_2 be a route from s to t using edges of $G - R_1$. Continue constructing $R_i \in G - \{R_1, \dots, R_{i-1}\}$ in this way. Since removing a route strictly decreases the number of edges of the previous step, and the number of edges of G is finite, this process must end. The result is a decomposition $G = \cup_{i=1}^k R_i$ for some $k \in \mathbb{N}$ and moreover these routes are edge disjoint since they are edge disjoint at every step of the construction. $\square$

Corollary 34. *The number of routes in a route decomposition $\mathcal{R}$ is $\text{outdeg}_G(s)$, where s is the source of G .*

Proof. Let $k = \text{outdeg}_G(s)$ and $\{e_i : i \in [k]\}$ be the edges out of s . Then since each edge of the graph is in some route from our decomposition, there exists a unique $R_i \in \mathcal{R}$ such that $e_i \in R_i$. Similarly, given $R \in \mathcal{R}$, it must use exactly one edge e out of s and no other route can use e as the routes have disjoint edge sets. This correspondence shows $|\mathcal{R}| = k$. $\square$

Remark 35. We can also partition the edge set of G when G has multiple sources and sinks by identifying all the sources and similarly identifying all the sinks.

4 Equatorial Flow Triangulations

Throughout this section, we assume that G is a DAG which has no idle edges and satisfies degree equality, hence has a Gorenstein flow polytope. First, note that the vector

$$\sum_{e \in E(G)} \chi_e = \mathbb{1} = (1, 1, \dots, 1) \in \mathbb{R}^{E(G)}$$

is the Gorenstein point for the flow polytope of G , where χ_e denotes the indicator vector with a 1 in the e -th position and 0's elsewhere. This follows from the observation by von Bell et al. [23] that if $\mathcal{F}_1(G)$ is Gorenstein, then it has Gorenstein point

$$\mathbb{1} \in \mathcal{F}(G) \cong \text{cone}(\mathcal{F}_1(G))$$

under this integral equivalence.

The following theorem provides a combinatorial method for producing special simplices in Gorenstein flow polytopes.

Theorem 36. *If G satisfies degree equality, then any route decomposition $\mathcal{R}$ corresponds to a special simplex in $\mathcal{F}_1(G)$ and the codegree of $\mathcal{F}_1(G)$ is $|\mathcal{R}|$.*

Proof. Given a route decomposition $\mathcal{R}$, the sum of indicator vectors of routes χ_R for $R \in \mathcal{R}$ gives

$$\sum_{R \in \mathcal{R}} \chi_R = \sum_{e \in E(G)} \chi_e,$$

since our route decomposition partitions the edge set. This is the Gorenstein point of $\mathcal{F}_1(G)$ and so by Proposition 17, we have that $\text{conv}\{\chi_R : R \in \mathcal{R}\}$ is a special simplex. By Theorem 10, we obtain the claimed value of the codegree. $\square$

We denote the special simplices corresponding to a route decomposition in the following manner.

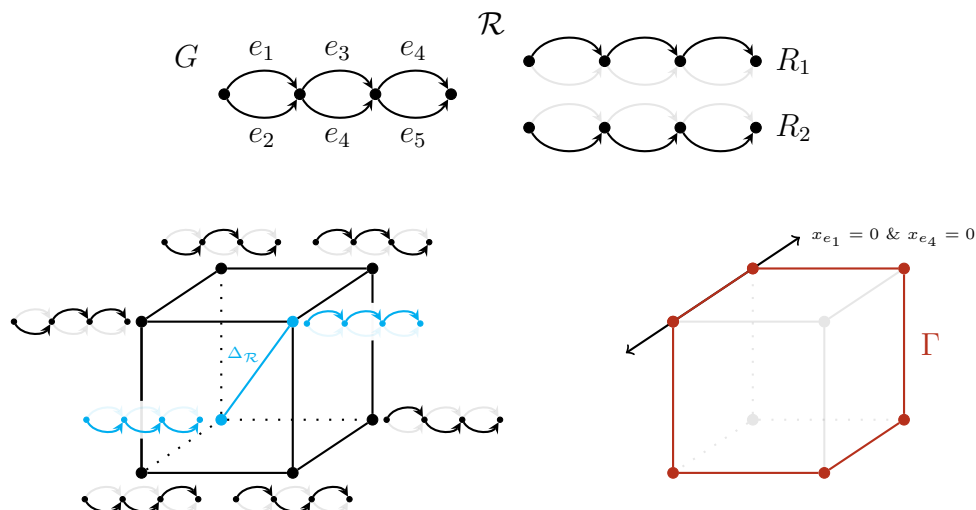


Figure 6: An example of the route simplex and the equatorial complex for the flow polytope of G with route decomposition $\mathcal{R}$.

Definition 37. Given a route decomposition $\mathcal{R}$, the corresponding special simplex

$$\Delta_{\mathcal{R}} = \text{conv}\{\chi_R : R \in \mathcal{R}\}$$

is called the *route simplex* for $\mathcal{R}$.

We now proceed to apply the Bruns-Römer theory from Section 2 in the context of Gorenstein flow polytopes. Our goal is to obtain a completely graph-theoretic description of the resulting equatorial flow triangulations.

Definition 38. Suppose we have a route decomposition $\mathcal{R}$ for a DAG G satisfying degree equality. Let Γ be the polyhedral subcomplex of $\partial\mathcal{F}_1(G)$ generated by the faces of the form $\bigcap_{R \in \mathcal{R}} F_R$ where each F_R is a facet of $\mathcal{F}_1(G)$ such that $\chi_R \notin F_R$. We call the faces $\bigcap_{R \in \mathcal{R}} F_R$ described previously *generating faces* of Γ since this complex is given by faces of the generating faces and their intersections. We call Γ the *equatorial complex* of the flow polytope.

Example 39. Consider the DAG and route decomposition $\mathcal{R} = \{R_1, R_2\}$ given in Figure 6. The route simplex and equatorial complex are shown in the same figure. Moreover, an example of a generating face is the intersection of the faces defined by $x_{e_1} = 0$ and $x_{e_4} = 0$ since χ_{R_1} satisfies $x_{e_1} = 1 \neq 0$ and χ_{R_2} satisfies $x_{e_4} = 1 \neq 0$. This gives an edge of the equatorial complex also shown in Figure 6. In addition, the empty face is a generating face since it can be seen as the intersection of the faces defined by $x_{e_1} = 0$ and $x_{e_2} = 0$.

Our next goal is to give a combinatorial criteria to determine when a set of routes are contained in a common face of the equatorial complex for $\mathcal{F}_1(G)$.

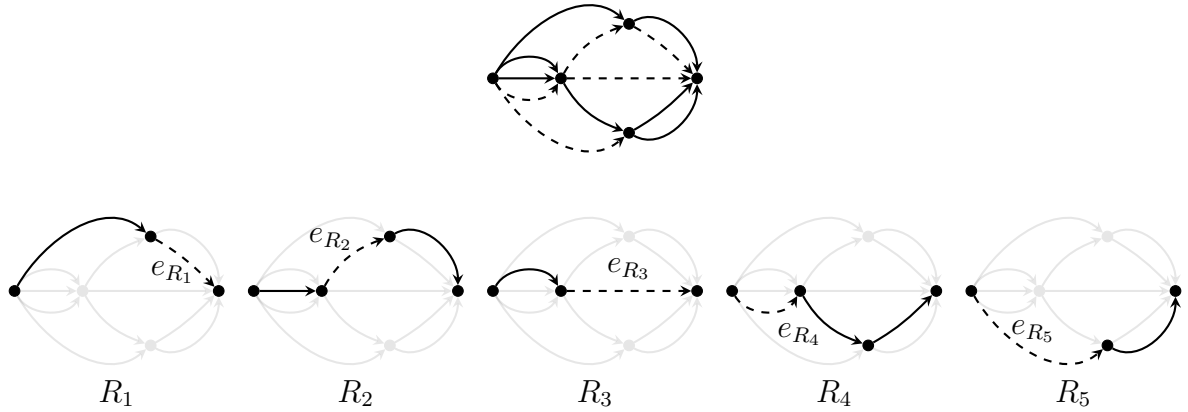


Figure 7: Given the route decomposition from Figure 4 with some indexing, the collection of dashed edges give a transversal $M = \{e_{R_1}, e_{R_2}, e_{R_3}, e_{R_4}, e_{R_5}\}$, and all routes of the DAG using only solid edges avoids M .

Definition 40. Given a route decomposition $\mathcal{R}$ of G , a set of edges M is a *transversal* of $\mathcal{R}$ if M consists of exactly one edge from each route of $\mathcal{R}$. Equivalently, a transversal is a system of distinct representatives for the routes in $\mathcal{R}$. We say a route in G *avoids the transversal* M if the route does not use the edges in M , i.e., the route carries zero flow on the edges in M .

Example 41. Figure 7 shows an example of a transversal of a route decomposition.

The next few results give combinatorial criteria for describing both general faces and facets of Gorenstein flow polytopes. Theorem 42 gives our first combinatorial criterion for when a collection of routes forms a common face of the equatorial complex for $\mathcal{F}_1(G)$. In particular, we will see that transversals characterize the faces of Γ . Note that for a transversal $M = \{e_R : R \in \mathcal{R}\}$ we denote by $\bigcap_{R \in \mathcal{R}} \{x_{e_R} = 0\}$ the intersection of facets corresponding to M .

Theorem 42. *Given a directed acyclic graph G with degree equality and a route decomposition $\mathcal{R}$, a collection of routes $\mathcal{S}$ form a face of Γ if and only if $\mathcal{S}$ is the set of all routes avoiding a union of transversals in $\mathcal{R}$.*

Proof. We first will expand on Definition 38. Facets of $\mathcal{F}_1(G)$ are given by $x_e = 0$ for $e \in E(G)$. Given a route $R \in \mathcal{R}$, since χ_R gives a flow of 1 on the edges it uses in G , then χ_R is not on the facets given by $\{x_e = 0 : e \in E(R)\}$. So, given $\{e_R \in E(R) : R \in \mathcal{R}\}$ to be any transversal, then by Definition 38, we have that

$$\bigcap_{R \in \mathcal{R}} \{x_{e_R} = 0\}$$

gives a generating face of the equatorial complex.

We will first show the backward direction. Let $\mathcal{S}$ be the set of all routes avoiding a union of transversals. Let one such transversal be $M = \{e_R \in E(R) : R \in \mathcal{R}\}$, then the

indicator vectors for routes avoiding M are the vertices of the generating face of Γ given by

$$\bigcap_{R \in \mathcal{R}} \{x_{e_R} = 0\}$$

since vertices of $\mathcal{F}_1(G)$ and hence Γ are routes. Similarly another transversal gives a generating face, and taking the intersection of all these generating faces is another face whose vertices are all routes avoiding the union of transversals. Thus $\mathcal{S}$ forms a face of Γ . For the forward direction, since our collection of routes form a face of the equatorial complex, the face with vertices $\mathcal{S}$ is the intersection of generating faces. Moreover each generating face is given by

$$\bigcap_{R \in \mathcal{R}} F_R$$

where F_R is a facet of $\mathcal{F}_1(G)$ such that $\chi_R \notin F_R$. We saw that facets of this form must use hyperplanes $\{x_e = 0 : e \in E(R)\}$. As this holds for each route in our route decomposition, this gives a collection $\{e_R \in E(R) : R \in \mathcal{R}\}$ which is a transversal for this generating face. Since routes in this generating face must give 0 for these edges, these routes avoid this transversal. As routes in $\mathcal{S}$ are all the vertices of the face formed by intersecting the generating faces, $\mathcal{S}$ is the set of all the routes which avoids the union of the transversals given by each generating face. $\square$

Corollary 43 gives an alternative combinatorial characterization of when routes share a face of the equatorial complex.

Corollary 43. *Given a directed acyclic graph G with degree equality, a collection of routes $\mathcal{S}$ lie on a common face of the equatorial complex if and only if there is no route in $\mathcal{R}$ contained in the union of edges of the routes in $\mathcal{S}$.*

Proof. We will prove the forward direction by contrapositive. Suppose that some $R \in \mathcal{R}$ is the union of edges in $\mathcal{S}$, i.e. $E(R) \subseteq \bigcup_{S \in \mathcal{S}} E(S)$. Then $\mathcal{S}$ must use all the edges of R and so we cannot avoid a transversal of $\mathcal{R}$ since we cannot avoid any edge from R . Thus, $\mathcal{S}$ does not lie on a common face by Theorem 42. For the backward direction, now suppose no route from our route decomposition $\mathcal{R}$ is in the union of edges of the routes in $\mathcal{S}$. Then for every route $R \in \mathcal{R}$, there must exist some edge $e_R \in E(G)$ not used by $\mathcal{S}$, otherwise we would find a route as a union of edges. This gives a transversal which routes in $\mathcal{S}$ avoid and so, by Theorem 42, these routes lie on a common face. $\square$

While Theorem 42 describes exactly when a collection of routes form a face of the equatorial complex, not all these faces are facets. Theorem 44 describes exactly when a transversal gives a facet of the equatorial complex.

Theorem 44. *Given a route decomposition of a directed acyclic graph G and a generating face F of Γ avoiding a transversal $M = \{e_R : R \in \mathcal{R}\}$, F is a facet of the equatorial complex Γ if and only for all inner vertices v of G there exists a route incident to v that does not use the edges in M .*

Proof. We will prove the forward implication by contrapositive. Suppose there exists an inner vertex $v \in V(G)$ such that all routes incident to v use an edge in the transversal M . Without loss of generality, suppose $R_1, \dots, R_{\text{indeg}_G(v)}$ are the routes in $\mathcal{R}$ using v . We will first show that the face of Γ given by $\{x_{e_R} : R \in \mathcal{R}\}$ is contained in a face where all incoming edges of v in G are in the transversal, specifically, the face of Γ given by $M' = \{e_R : R \in \mathcal{R} \setminus \{R_1, \dots, R_{\text{indeg}_G(v)}\}\} \cup \text{in}(v)$. Suppose we have a route S which avoids the edges in M . Then our route S would not be incident to v , as otherwise S would use an edge of M . This means that $E(S) \cap \text{in}(v) = \emptyset$, thus S also avoids the incoming edges of v , hence S also avoids the edges in M' . This shows our original face F is contained in the modified face defined by the transversal M' .

We will now show that the face of Γ given by M' is properly contained in another face and hence F is not a facet. Since G has no idle edges, we have $\text{indeg}_G(v) \geq 2$ and also $\text{outdeg}_G(v) \geq 2$. Moreover, for an incoming edge $e \in \text{in}(v)$ there must be some outgoing edge $f \in \text{out}(v)$ that lies on the same route, say R_i for some i . Define a new transversal $M'' = (M' \setminus \{e\}) \cup \{f\}$. This contains all the routes avoiding the original transversal M , since every route avoiding M is not incident to v . Moreover we claim this transversal contains a new route. We have a path given by the route R_i from s to v ; since $\text{outdeg}_G(v) \geq 2$, there must be another route in $\mathcal{R}$, say R_j , using an edge out of v different from f . We can concatenate at v the path from s to v following R_i with the path from v to t following R_j to create a new route that avoids M'' . Thus, F is properly contained in another face of Γ and hence is not a facet.

We will now prove the backward implication by contradiction. For a contradiction, suppose F avoiding the transversal M is not a facet, and thus there exists a face F' of Γ such that $F \neq F'$ and $F \subset F'$. Without loss of generality, F' is a generating face of Γ and so avoids a different transversal, say $\{e'_R : R \in \mathcal{R}\}$. Since we have a proper containment, there must exist a route $R' \in F'$ such that $R' \notin F$. Since routes in F avoid edges in M , there must exist an $S \in \mathcal{R}$ such that $e_S \in E(R')$ and $e'_S \notin E(R')$, since $R' \in F'$ for the same $S \in \mathcal{R}$. This means in particular that $e_S \neq e'_S$. We now claim there exists an inner vertex of indegree at least two in $V(S)$ appearing between e_S and e'_S . Since $e_S \neq e'_S$, there are at least three vertices and so the one between these edges in S must be an inner vertex. Since G has no idle edges this inner vertex must have indegree greater than or equal to two.

Having established our claim, suppose that e_S appears before e'_S in the route S (when the order is switched, the following argument is symmetric). Let v be an inner vertex with $\text{indeg}_G(v) = \text{outdeg}_G(v) \geq 2$ between these two edges. By our assumption, we have a route using v in G that avoids the edges in M . In particular we have a path from the source to v avoiding the edge e_S . We also have a path from v to the sink obtained by using the edges of S , which also avoids e_S since this edge appears before v in S . Concatenating these paths we get a route from our source to sink which avoids M and uses e'_S , hence the resulting route is contained in F but not in F' . This contradicts that $F \subset F'$, and hence F is a facet. $\square$

We next introduce a triangulation of Γ that uses as input only a route decomposition $\mathcal{R}$ of G with a linear ordering of the routes in $\mathcal{R}$.

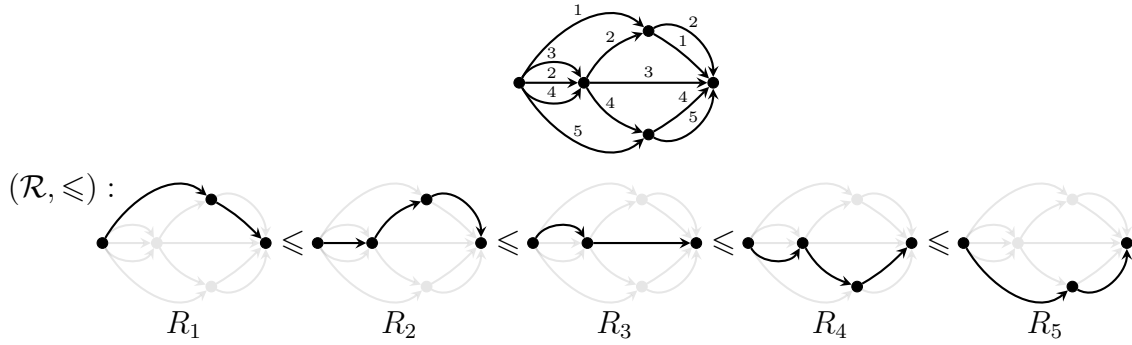


Figure 8: A route decomposition and framing.

Definition 45. Consider a route decomposition of G with a linear ordering $\leq$ given by

$$\mathcal{R} = \{R_1 < R_2 < R_3 < \cdots < R_k\}.$$

For an inner vertex v with indegree j that is incident to the routes $R_{i_1} < R_{i_2} < \cdots < R_{i_j}$, with incoming edges at v given by $e_{i_1}, \dots, e_{i_j}$, order the incoming edges according to the ordering of the routes, i.e., $e_{i_1} < \cdots < e_{i_j}$. We similarly order the outgoing edges at each vertex. This framing is called the *route decomposition framing* and yields by Theorem 3 a DKK triangulation T of $\mathcal{F}_1(G)$. The restriction $T|_\Gamma$ is denoted by T_{eq} .

Note that it is often useful to consider the incoming edges at v to be labeled $i_1, i_2, \dots, i_j$, and similarly for outgoing edges. A route decomposition and associated framing is given in Figure 8. Theorem 19 and Theorem 36 imply that the following triangulation exists and is well-defined.

Definition 46. Given a route decomposition with a linear ordering $(\mathcal{R}, \leq)$, we define the *equatorial flow triangulation* to be the triangulation of $\mathcal{F}_1(G)$ given by the join

$$T_{\text{eq}} * \Delta_{\mathcal{R}},$$

i.e., the triangulation obtained as the union of simplices by $\text{conv}\{F \cup \Delta_{\mathcal{R}}\}$ where F is a face of the triangulation T_{eq} .

The results of Bruns and Römer in Section 2 yield the following in the special case of flow polytopes.

Theorem 47. For a Gorenstein flow polytope $\mathcal{F}_1(G)$, the equatorial flow triangulation is regular and unimodular, and the equatorial sphere T_{eq} is the boundary complex of a simplicial polytope. Further, the h -vector of T_{eq} is the h^* -vector of $\mathcal{F}_1(G)$.

Proof. This follows immediately from Theorem 19, Theorem 20, and Corollary 21. $\square$

Remark 48. We can replace T_{eq} by any regular unimodular triangulation of Γ , say S , to obtain a regular unimodular triangulation of $\mathcal{F}_1(G)$ given by $S * \Delta_{\mathcal{R}}$. The usefulness and beauty of the equatorial flow triangulation is that it is defined entirely combinatorially by a route decomposition with a linear ordering.

A purely combinatorial description of the equatorial flow triangulation is the following.

Theorem 49. *For a directed acyclic graph G satisfying degree equality and $(\mathcal{R}, \leq)$ a route decomposition with a linear ordering, a collection of routes C is a simplex in T_{eq} if it satisfies precisely the following two conditions:*

1. *The routes in C form a clique in the DKK triangulation induced by $\leq$ on $\mathcal{R}$.*
2. *The routes in C avoid a transversal $e_1, \dots, e_k$ of $R_1, \dots, R_k \in \mathcal{R}$.*

Proof. From the equatorial flow triangulation, T_{eq} triangulates the equatorial complex Γ . In particular, routes which form a simplex lie on a common face of Γ and are a restriction of our DKK triangulation T to Γ . The first condition give routes which form a simplex from the DKK triangulation, while the second condition gives routes which lie on a common face by Theorem 42. $\square$

What is particularly remarkable is that if G satisfies degree equality and has a vertex of indegree 3 or greater, then the equatorial flow triangulation is not a DKK triangulation, i.e., is not induced by a framing.

Theorem 50. *If G has an inner vertex with indegree at least 3, then the equatorial flow triangulation $T_{\text{eq}} * \Delta_{\mathcal{R}}$ is not a DKK triangulation.*

Proof. Suppose v satisfies $\text{indeg}_G(v) \geq 3$, hence by degree equality $\text{outdeg}_G(v) \geq 3$. This implies that there exist at least three routes in $\mathcal{R}$ that are incident to v , without loss of generality suppose we have at least the routes R_1, R_2, R_3 . For the equatorial flow triangulation, R_1, R_2, R_3 are each adjacent to every other route, as they are in our route simplex. However, we will show that one of these three routes is in conflict with some other route, hence the equatorial flow triangulation is not a DKK triangulation. For $i \in [3]$, denote e_i to be the incoming edge at v and f_i to be the outgoing edge at v in R_i as we have a route decomposition. Since we have a framing, there exists permutations $\sigma, \pi \in S_3$ such that

$$e_{\sigma(1)} < e_{\sigma(2)} < e_{\sigma(3)}, \quad f_{\pi(1)} < f_{\pi(2)} < f_{\pi(3)}.$$

in our framing. There must exist a route R_j for some $j \in [3]$ such that $e_j = e_{\sigma(2)}$. We will now consider the following two cases. For the first case, suppose $f_j = f_{\pi(2)}$, then R_j conflicts with the route given by the concatenation of the paths $R_{\sigma(1)}v$ and $vR_{\pi(3)}$. For the second case, suppose $f_j \neq f_{\pi(2)}$ then either $f_j = f_{\pi(1)}$ or $f_j = f_{\pi(3)}$. If $f_j = f_{\pi(1)}$, then R_j conflicts with the route given by the concatenation of the paths $R_{\sigma(1)}v$ and $vR_{\pi(2)}$. If $f_j = f_{\pi(3)}$, then R_j conflicts with the route given by the concatenation of the paths $R_{\sigma(3)}v$ and $vR_{\pi(2)}$. In any case, R_j conflicts with a route, proving the claim. $\square$

Theorem 50 does not cover DAGs satisfying degree equality where all inner vertices have indegree 2. For such DAGs, the equatorial flow triangulation is not always the same triangulation as the DKK triangulation induced by the route decomposition framing, as shown by the following example that is equivalent to an example given by Reiner and Welker [21] via the flow and order polytope correspondence given in Section 6.

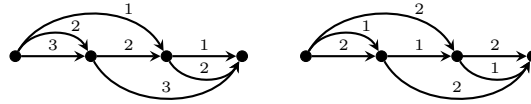


Figure 9: A DAG with a route decomposition for which the equatorial flow triangulation and route decomposition DKK triangulation are not the same. The DKK triangulation of the right DAG gives the same triangulation as the equatorial flow triangulation.

Example 51. For the left DAG given in Figure 9, in the equatorial flow triangulation, the route consisting of edges labeled by 2 is connected to every other route by an edge. However, in the DKK triangulation for the route decomposition framing, the all-2's route conflicts with the route along the spine labeled 321, and hence do not form an edge.

Even so, it is possible for the equatorial flow triangulation to be a DKK triangulation under a different framing. Consider the right DAG in Figure 9, then the DKK triangulation of this framing is the equatorial flow triangulation of the left DAG. The routes in our route decomposition are exceptional routes so for both triangulations these three routes appear in every maximal simplex. To form a maximal simplex for the equatorial flow triangulation, one can use Figure 10. Routes forming an edge of the quotient polytope in Figure 10 together with routes in our route decomposition gives our maximal simplices. These maximal simplices coincide with the DKK triangulation. It is not known whether equatorial flow triangulations are always DKK triangulations for some framing in the case of DAGs where all inner vertices have indegree 2.

5 A Projection of the Flow Polytope

In this section, our goal is to describe an explicit geometric realization of the equatorial complex for a Gorenstein flow polytope $\mathcal{F}_1(G)$ as the boundary of a reflexive polytope with the same h^* -polynomial as $\mathcal{F}_1(G)$. Specifically, we give both convex hull and (non-redundant) facet descriptions of the reflexive polytope. For an arbitrary polytope $P \subseteq \mathbb{R}^n$ and V a linear subspace of $\mathbb{R}^n$, the *quotient polytope* is

$$P/V := \{p + V : p \in P\} \subseteq \mathbb{R}^n/V.$$

In other words, the quotient polytope is the image of P under the canonical quotient map $\mathbb{R}^n \rightarrow \mathbb{R}^n/V$. In addition, the quotient polytope in $\mathbb{R}^n/V$ is linearly isomorphic to the image of P under any linear surjection $\mathbb{R}^n \rightarrow \mathbb{R}^{n-\dim V}$ with kernel V [1]. The following proposition is a mild restatement of results found in the work of Athanasiadis [1] and Reiner and Welker [21].

Proposition 52. *Given a Gorenstein polytope $P \subseteq \mathbb{R}^n$ and a special simplex Δ , let V be the linear space spanned by Δ . The boundary complex of the quotient polytope P/V is abstractly isomorphic to the equatorial complex $\Gamma(P, \Delta)$. Moreover, if we have a triangulation of P of the form $S * \Delta$, where S is a triangulation of Γ , then the boundary complex of P/V inherits a triangulation abstractly isomorphic to S .*

In particular, note that the facets of P/V are in bijection with the facets of the equatorial complex. Further, P/V is known to be a reflexive polytope [9].

We need to introduce some additional notation that will be used to describe the resulting reflexive polytope. Assume we have a DAG G (with no idle edges) satisfying degree equality with a route decomposition framing given by $R_1 < R_2 < \dots < R_k$. Given an edge $e \in E(G)$, let $\text{label}(e)$ be the index of the route $R \in \mathcal{R}$ which contains this edge. This is well defined as we have a route decomposition. As usual, the inner vertices are labeled by $[n]$.

Definition 53. Given an internal vertex $i \in [n]$, let $\text{inlevel}(i)$ denote the set of labels of the incoming edges $e \in \text{in}(i)$ at inner vertex i , i.e.

$$\text{inlevel}(i) := \{\text{label}(e) : e \in \text{in}(i)\}.$$

Additionally, for a subset of edges $D \subseteq E(G)$, define

$$\text{inlevel}(D, i) := \{\text{label}(e) : e \in \text{in}(i) \cap D\}$$

In particular, if we have a route R define $\text{inlevel}(R, i) := \text{inlevel}(E(R), i)$. Note that $\text{inlevel}(R, i)$ consists of at most a single element.

Similarly, define $\text{outlevel}(i)$ as the set of labels from the outgoing edges $e \in \text{out}_G(i)$, i.e.,

$$\text{outlevel}(i) := \{\text{label}(e) : e \in \text{out}(i)\}.$$

We also similarly define $\text{outlevel}(R, i)$ for a route R such that $i \in [n]$.

Definition 54. For each inner vertex $i \in [n]$ define $V^i := \mathbb{R}^{\text{indeg}(i)}$ to be the vector space with standard basis vectors $e_{j_1}^i, \dots, e_{j_{\text{indeg}(i)}}^i$ where $\{j_1, \dots, j_{\text{indeg}(i)}\} = \text{inlevel}(i)$ and where $j_1 < j_2 < \dots < j_{\text{indeg}(i)}$ under the natural ordering of $\mathbb{N}$. Note that we use the superscript to denote that this basis is with respect to inner vertex i .

Given this notation, we are now able to define a linear map φ with kernel given by the linear span of the special simplex formed by the indicator vectors of $\{R_1, \dots, R_k\}$. The map given in Definition 55 is a generalization of a map defined by Berggren [6] for flow polytopes of amply framed DAGs.

Definition 55. Define a linear map

$$\varphi : \mathbb{R}^{|E(G)|} \rightarrow V^1 \times V^2 \times \dots \times V^n$$

by extending linearly the map defined on a basis element $e_{(i,j)}$ for $(i, j) \in E(G)$ by

$$\varphi(e_{(i,j)}) := e_{\text{label}((i,j))}^j - e_{\text{label}((i,j))}^i$$

where if i or j is a source or a sink, we omit the corresponding summand.

Observe that for a route R in G , this definition of φ yields

$$\varphi(R) = \sum_{i \in [n] \cap V(R)} e_{\text{inlevel}(R, i)}^i - e_{\text{outlevel}(R, i)}^i. \quad (1)$$

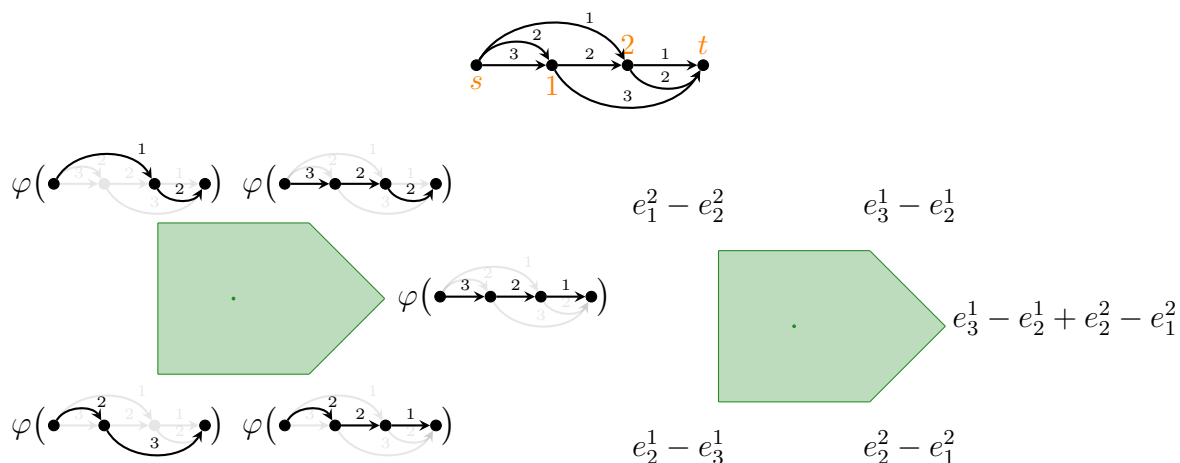


Figure 10: A DAG with a route decomposition given by the edge labels, vertices of the DAG in orange, and the quotient polytope $Q_{\mathcal{R}}$.

Definition 56. Given a DAG G satisfying degree equality and an ordered route decomposition $\mathcal{R}$, we define

$$Q_{\mathcal{R}} := \varphi(\mathcal{F}_1(G)).$$

Example 57. Figure 10 contains a DAG with a route decomposition and the corresponding polytope $Q_{\mathcal{R}}$ with vertices labeled by both routes S and vectors $\varphi(S)$. Figure 11 provides two examples of DAGs with three-dimensional projection polytopes $Q_{\mathcal{R}}$. Note that the left example in Figure 11 generalizes to give the type A root polytope as a quotient of a product of two simplices, i.e., suppose we have the standard basis vectors $e_k \in \mathbb{R}^n$ then the quotient polytope is given by $\text{conv}\{e_i - e_j : i \neq j \in [n]\}$.

The following lemma connects the geometric structure of $Q_{\mathcal{R}}$ with the equatorial complex.

Lemma 58. *The map φ vanishes on the linear subspace spanned by the route simplex. Thus, $Q_{\mathcal{R}}$ is a reflexive polytope whose face structure is given by the equatorial complex Γ and whose vertices are $\{\varphi(R) : R \notin \mathcal{R}\}$.*

Proof. That φ vanishes on a route $R_j \in \mathcal{R}$ follows from the fact that all the edge labels in R_j are j , and thus Equation (1) yields a telescoping sum that cancels to zero. The claims regarding the face structure and vertices of $Q_{\mathcal{R}}$ follow immediately from Proposition 52. $\square$

Having given a convex hull description of the reflexive projection $Q_{\mathcal{R}}$ of $\mathcal{F}_1(G)$ for a special simplex defined by a route decomposition $\mathcal{R}$, our next task is to give a hyperplane description of $Q_{\mathcal{R}}$. The following definition and theorem are the key tools we will use for this task.

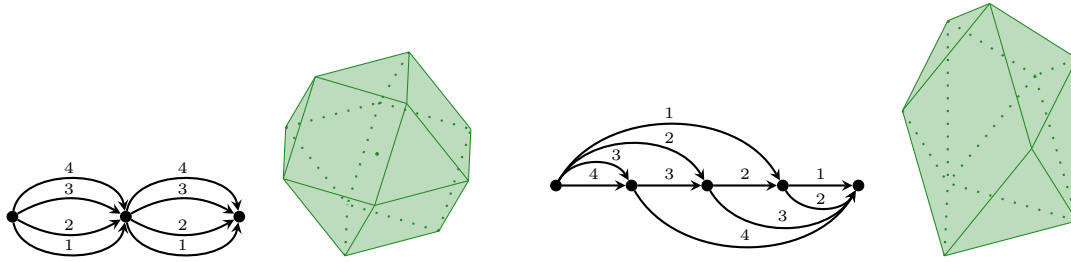


Figure 11: Examples of three dimensional quotient polytopes $Q_{\mathcal{R}}$ where the labels correspond to the route decompositions $\mathcal{R}$.

Definition 59. Given a transversal $\{e_R : R \in \mathcal{R}\}$, for every $R \in \mathcal{R}$ we have a partition of the edges $E(R)$ given by the following three sets

$$R^{\text{left}} \sqcup \{e_R\} \sqcup R^{\text{right}} = E(R)$$

where R^{left} are all the edges in G which the route R uses before the edge e_R and R^{right} are all the edges appearing in R after the edge e_R . Note that sometimes these sets can be empty.

Given a route S in our DAG G , we denote by $(\varphi(S))_l^i$ the l -th coordinate of $\varphi(S)$ in V^i .

Theorem 60. For a DAG G satisfying degree equality, a route decomposition $\mathcal{R}$ of G , and a transversal $M := \{e_R : R \in \mathcal{R}\}$, every route S satisfies

$$\sum_{i \in [n]} \sum_{\substack{R \in \mathcal{R}, \\ l \in \text{inlevel}(R^{\text{left}}, i)}} (\varphi(S))_l^i = 1 - |E(S) \cap M|. \quad (2)$$

In particular, for any choice of transversal M , every route S satisfies the linear inequality

$$\sum_{i \in [n]} \sum_{\substack{R \in \mathcal{R}, \\ l \in \text{inlevel}(R^{\text{left}}, i)}} x_l^i \leq 1.$$

Note that the edges in each R^{left} depend on the choice of M .

Proof. Suppose $S \in \mathcal{R}$, then it must use exactly one edge in the transversal. Moreover, the image $\varphi(S) = 0$ and thus

$$\sum_{i \in [n]} \sum_{\substack{R \in \mathcal{R}, \\ l \in \text{inlevel}(R^{\text{left}}, i)}} (\varphi(S))_l^i = 0 = 1 - |E(S) \cap M|.$$

Next, suppose $S \notin \mathcal{R}$. Since we have a route decomposition, S must exit the source following some $R \in \mathcal{R}$. Since $S \notin \mathcal{R}$, S must eventually switch to follow edges in a

different route in $\mathcal{R}$, and then possibly switch to follow another route in $\mathcal{R}$, and so forth. Hence, there is the following partition of S as

$$S = R_{i_1} v_1 R_{i_2} v_2 \cdots R_{i_{m-1}} v_{m-1} R_{i_m}$$

for some integer m where $m \geq 2$, where each $v_j \in [n]$ is an inner vertex where the edge into v_j agrees with $R_{i_j} \in \mathcal{R}$ and the edge out of v_j agrees with $R_{i_{j+1}} \in \mathcal{R}$ for $R_{i_j} \neq R_{i_{j+1}}$. Since S is not in our route simplex, at least one such v_j exists.

This partition of the edges in S implies that the image of S under φ is

$$\varphi(S) = \sum_{j=1}^{m-1} e_{i_j}^{v_j} - e_{i_{j+1}}^{v_j} = (e_{i_1}^{v_1} - e_{i_2}^{v_1}) + (e_{i_2}^{v_2} - e_{i_3}^{v_2}) + \cdots + (e_{i_{m-1}}^{v_{m-1}} - e_{i_m}^{v_{m-1}}),$$

since all other vertices in $[n]$ are either not on S and hence do not contribute to the sum or lie on S but S does not change levels at this vertex and hence there is no contribution for this vertex under the φ map. Because of this expression for $\varphi(S)$, we can simplify the left hand side of Equation (2) to only include the vertices v_j for $j \in [m-1]$. This simplification is given by

$$\sum_{i \in [n]} \sum_{\substack{R \in \mathcal{R}, \\ l \in \text{inlevel}(R^{\text{left}}, i)}} (\varphi(S))_l^i = \sum_{j \in [m-1]} \sum_{\substack{R \in \mathcal{R}, \\ l \in \text{inlevel}(R^{\text{left}}, v_j)}} (\varphi(S))_l^{v_j}. \quad (3)$$

Equation (3) also shows that we can compute all terms of this equation by considering each segment of S , i.e., on $R_{i_1} v_1$, on $v_j R_{i_{j+1}} v_{j+1}$ for all $j \in [m-1]$, and on $v_{m-1} R_{i_m}$, since the j -th term in the right-hand-side of Equation (3) is

$$\sum_{\substack{R \in \mathcal{R}, \\ l \in \text{inlevel}(R^{\text{left}}, v_j)}} (\varphi(S))_l^{v_j} = \begin{cases} (\varphi(S))_{i_j}^{v_j} & \text{if } i_j \in \text{inlevel}(R_{i_j}^{\text{left}}, v_j) \\ & \text{and } i_{j+1} \notin \text{inlevel}(R_{i_{j+1}}^{\text{left}}, v_j), \\ (\varphi(S))_{i_j}^{v_j} + (\varphi(S))_{i_{j+1}}^{v_j} & \text{if } i_j \in \text{inlevel}(R_{i_j}^{\text{left}}, v_j) \\ & \text{and } i_{j+1} \in \text{inlevel}(R_{i_{j+1}}^{\text{left}}, v_j), \\ (\varphi(S))_{i_{j+1}}^{v_j} & \text{if } i_j \notin \text{inlevel}(R_{i_j}^{\text{left}}, v_j) \\ & \text{and } i_{j+1} \in \text{inlevel}(R_{i_{j+1}}^{\text{left}}, v_j), \\ 0 & \text{otherwise.} \end{cases}$$

For instance, when considering v_1 , we can consider the first two segments $R_{i_1} v_1$ and $v_1 R_{i_2} v_2$ (or R_{i_m} if there are only two segments in the partition). Each of these make up the two possible values for the $j = 1$ term in the sum above, and hence determining the values on each segment will compute Equation (3).

We now use this setup to prove that Equation (2) holds. First consider $R_{i_1} v_1$. We have two cases. In the first case, suppose $e_{R_{i_1}} \notin E(R_{i_1} v_1)$. Then $E(R_{i_1} v_1) \subseteq R_{i_1}^{\text{left}}$ since we start this path from the source. Hence we have $(\varphi(S))_{i_1}^{v_1} = 1$ appearing in the right-hand sum for Equation (3). In the second case we have $e_{R_{i_1}} \in E(R_{i_1} v)$. Then $i_1 \notin \text{inlevel}(R_{i_1}^{\text{left}}, v_1)$

since the edge into v_1 will either be $e_{R_{i_1}}$ or be in $R_{i_1}^{\text{right}}$. So we have a 0 value, which we consider to be $1 - 1$.

We will now show that in the remaining segments, a 0 is contributed to the right-hand side of Equation (3) if the segment contains no edge from the transversal and a -1 is contributed if an edge for the transversal is contained in the segment. When combined with our analysis of the first segment, this will prove the equality in Equation (2). Consider $v_j R_{i_{j+1}} v_{j+1}$ for $j \in [m-1]$. We again consider two cases. For the first case, suppose $e_{R_{j+1}} \notin E(v_j R_{i_{j+1}} v_{j+1})$. Then we consider two subcases.

- In the first subcase, if $E(v_j R_{i_{j+1}} v_{j+1}) \subseteq R_{i_{j+1}}^{\text{left}}$, then $(\varphi(S))_{i_{j+1}}^{v_j} + (\varphi(S))_{i_{j+1}}^{v_{j+1}}$ must appear in our j -th and $j+1$ -st term of Equation (3). However, this simplifies to

$$(\varphi(S))_{i_{j+1}}^{v_j} + (\varphi(S))_{i_{j+1}}^{v_{j+1}} = (-e_{i_{j+1}}^{v_j})_{i_{j+1}}^{v_j} + (e_{i_{j+1}}^{v_{j+1}})_{i_{j+1}}^{v_{j+1}} = -1 + 1 = 0$$

by using our segment expression of $\varphi(S)$.

- In the second subcase, suppose $E(v_j R_{i_{j+1}} v_{j+1}) \not\subseteq R_{i_{j+1}}^{\text{left}}$. Since we know this segment does not contain $e_{R_{i_{j+1}}}$, then we must have $E(v_j R_{i_{j+1}} v_{j+1}) \subseteq R_{i_{j+1}}^{\text{right}}$. This means we have no terms associated to $(\varphi(S))_{i_{j+1}}^{v_j}$ or $(\varphi(S))_{i_{j+1}}^{v_{j+1}}$ and so there are no associated terms included in Equation (3), contributing 0.

Next consider the second case where $e_{R_{j+1}} \in E(v_j R_{i_{j+1}} v_{j+1})$. Then $i_{j+1} \in \text{inlevel}(R_{i_{j+1}}^{\text{left}}, v_j)$ and $i_{j+1} \notin \text{inlevel}(R_{i_{j+1}}^{\text{left}}, v_{j+1})$, since $e_{R_{i_{j+1}}}$ cannot appear in $R_{i_{j+1}}$ before v_j since it is used in $v_j R_{i_{j+1}} v_{j+1}$ and the edge of S into v_{j+1} must be either $e_{R_{i_{j+1}}}$ or in $R_{i_{j+1}}^{\text{right}}$. This means we have

$$(\varphi(S))_{i_{j+1}}^{v_j} = (-e_{i_{j+1}}^{v_j}) = -1$$

appearing for the j -th term of the right-hand side of Equation (3).

Our final step is to consider the last segment $v_{m-1} R_{i_m}$. Again we have two cases. In the first case, consider $e_{R_{i_m}} \notin E(v_{m-1} R_{i_m})$. We must have $E(v_{m-1} R_{i_m}) \subseteq R_{i_m}^{\text{right}}$ since S avoids our transversal edge and this segment goes to the sink. So, we do not get any additional contribution to the right-hand side of Equation (3). In the second case, suppose $e_{R_{i_m}} \in E(v_{m-1} R_{i_m})$. This implies that the edge of R_{i_m} into v_{m-1} must be in $R_{i_m}^{\text{left}}$ and hence $i_m \in \text{inlevel}(R_{i_m}^{\text{left}}, v_{m-1})$. So, we have the value

$$(\varphi(S))_{i_m}^{v_{m-1}} = (-e_{i_m}^{v_{m-1}})_{i_m}^{v_{m-1}} = -1$$

contributed to Equation (3) in the v_{m-1} -th summand.

Combining these observations shows that for the right-hand side of Equation (3), we start with a value of 1 coming from $R_{i_1} v_1$, and any time we have an edge from the transversal contained in a segment we add a value of -1 to this sum, while in the cases where we do not use a transversal edge, we do not change the value. Hence, we have established that

$$\sum_{i \in [n]} \sum_{\substack{R \in \mathcal{R}, \\ l \in \text{inlevel}(R^{\text{left}}, i)}} (\varphi(S))_l^i = 1 - |E(S) \cap M|. \quad \square$$

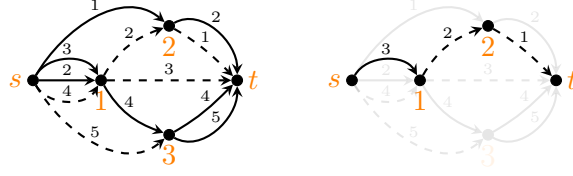


Figure 12: An example of a route and a DAG illustrating Equation (2).

Example 61. Consider the DAG with transversal and route decomposition given in Figure 12. For the route S following the edges labeled in the order 321, we have

$$\varphi(S) = e_3^1 - e_2^1 + e_2^2 - e_1^2,$$

and thus

$$\sum_{i \in [n]} \sum_{\substack{R \in \mathcal{R}, \\ l \in \text{inlevel}(R^{\text{left}}, i)}} (\varphi(S))_l^i = (\varphi(S))_3^1 + (\varphi(S))_2^1 + (\varphi(S))_1^2 = 1 - 1 - 1 = -1 = 1 - |E(S) \cap M|.$$

Combining our previous results, we are able to obtain the following halfspace description of $Q_{\mathcal{R}}$.

Theorem 62. For a DAG G satisfying degree equality, with a route decomposition $\mathcal{R}$ of G , the polytope $Q_{\mathcal{R}} = \varphi(\mathcal{F}_1(G))$ is full-dimensional in the subspace given by the intersection of $\sum_{j=1}^{\text{indeg}(i)} x_j^i = 0$ for all $i \in [n]$, with facets given by the following halfspaces, one for each transversal $M := \{e_R : R \in \mathcal{R}\}$ satisfying the conditions of Theorem (44):

$$H_M = \left\{ x \in V^1 \times \dots \times V^n \left| \sum_{i \in [n]} \sum_{\substack{R \in \mathcal{R}, \\ l \in \text{inlevel}(R^{\text{left}}, i)}} x_l^i = 1 \right. \right\}$$

Proof. Using Equation (1), it follows that $\varphi(\mathcal{F}_1(G))$ satisfies $\sum_{j=1}^{\text{indeg}(i)} x_j^i = 0$ for all $i \in [n]$. We will now show $\varphi(\mathcal{F}_1(G))$ is full-dimensional in this space. By Proposition 52 and Theorem 19, the equatorial triangulation is given by a simplex joined with a triangulated sphere. Suppose m is the dimension of the equatorial complex (as a sphere) and k is the out-degree of the source $k := \text{outdeg}(s)$. As our equatorial flow triangulation is given as a join, we have

$$\begin{aligned} \dim(\mathcal{F}_1(G)) &= \dim(\Delta_{\mathcal{R}}) + m + 1 \\ &= (k - 1) + m + 1 \\ &= k + m. \end{aligned}$$

Since $\dim(\mathcal{F}_1(G)) = |E(G)| - |\{\text{inner vertices of } G\}| - 1$ then we can compute m as

$$\begin{aligned} m &= |E(G)| - |\{\text{inner vertices of } G\}| - 1 - k \\ &= \left(\sum_{v \in V(G)} \text{outdeg}(v) \right) - |\{\text{inner vertices of } G\}| - 1 - k \\ &= \left(\sum_{v \in \text{inner } V(G)} \text{outdeg}(v) \right) + (\text{outdeg}(s) - k) - |\{\text{inner vertices of } G\}| - 1 \\ &= \left(\sum_{v \in \text{inner } V(G)} (\text{indeg}(v) - 1) \right) - 1. \end{aligned}$$

The second equality holds since the each edge $(i, j) \in E(G)$ appears exactly once as the outgoing edge of the left endpoint i . The third equality holds by removing the source term of the sum. This term also gives 0 when subtracting k . The last equality holds by degree equality and since we subtract the number of inner vertices and so can subtract 1 to each term of the sum. Since m is the dimension of the boundary of $Q_{\mathcal{R}}$ by proposition 52, then

$$\dim Q_{\mathcal{R}} = m + 1 = \sum_{v \in \text{inner } V(G)} (\text{indeg}(v) - 1)$$

which is the dimension of the space in $V^1 \times \cdots \times V^n$ given by the intersection of the hyperplanes $\sum_{j=1}^{\text{indeg}(i)} x_j^i = 0$ for all $i \in [n]$. Hence $Q_{\mathcal{R}}$ is full dimensional in this space.

By Proposition 52, the facets of $Q_{\mathcal{R}}$ correspond to the facets of the equatorial complex Γ . We know that each transversal satisfying the conditions of Theorem 44 yields a facet of the equatorial complex. Suppose we have H_M for some $M := \{e_R : R \in \mathcal{R}\}$ satisfying the conditions of Theorem 44. We claim any route S avoids the transversal M if and only if $\varphi(S) \in H_M$. If S avoids the transversal M , by Theorem 60, we have $\varphi(S) \in H_M$. Similarly, if $\varphi(S) \in H_M$, then S avoids M since otherwise by Theorem 60 the vector $\varphi(S)$ would not satisfy the defining equation of H_M . Thus, a route S is on the facet of Γ corresponding to M if and only if $\varphi(S)$ is on the hyperplane H_M , which by Theorem 60 is a supporting hyperplane of $Q_{\mathcal{R}}$. This completes the proof. $\square$

6 Triangulations of Order Polytopes for Strongly Planar Posets

In this section, we prove that for strongly planar DAGs with no idle edges, a particular equatorial flow triangulation is integrally equivalent to the equatorial triangulation of the associated order polytope defined by Reiner-Welker. Towards this goal, we first give the integral equivalence of flow polytopes and order polytopes when we have strongly planar posets and DAGs shown by Mészáros, Morales, and Striker. Second, we give the definitions and results of Reiner-Welker to see the structure of the equatorial triangulation of order polytopes, before proving our result.

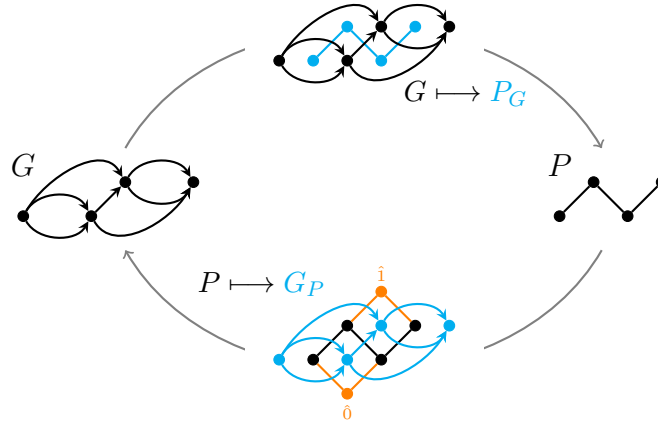


Figure 13: A strongly planar poset P and a strongly planar DAG G which are related by duality.

Definition 63. A DAG G is *strongly planar* if G is a planar graph with a planar realization such that the following two conditions hold. First, for every directed edge (i, j) of G , the x -coordinate of i is strictly less than the x -coordinate of j . Second, each edge (i, j) is embedded in the plane as the graph of a piecewise differentiable function of x .

Definition 64. A poset P is *strongly planar* if the Hasse diagram of P is a planar graph with a planar realization such that the following two conditions hold. First, for every directed edge (i, j) in the Hasse diagram, the y -coordinate of i is strictly less than the y -coordinate of j . Second, each edge (i, j) is embedded in the plane as the graph of a piecewise differentiable function of y .

The following definition was originally given by Mészáros, Morales, and Striker [19, Section 3.3].

Definition 65. Given a strongly planar DAG G , we define the *truncated dual graph* G^* to be a strongly planar realization of the dual graph of G with the vertex corresponding to the infinite region deleted. The graph G^* is thus the Hasse diagram of a strongly planar poset that we denote by P_G . Similarly, given a strongly planar poset P , let H be the Hasse diagram of $P \cup \{\hat{0}, \hat{1}\}$. We define G_P to be the strongly planar DAG obtained as the strongly planar dual of H .

Note that the definitions of planar DAG and strongly planar poset given in [19] are not sufficient for Theorem 71 to hold in general. Definitions 63 and 64 require additional structure for planar realizations of edges that are sufficient to establish Theorem 71 using the proofs given in [19].

Example 66. Figure 13 gives an example of a strongly planar DAG G and poset P . Moreover, these are related by duality and the corresponding constructions of G^* and H are given.

We next recall basic facts about order polytopes of posets.

Definition 67. Given a finite poset $P = \{t_1, \dots, t_n\}$, define the *order polytope* of P to be

$$\mathcal{O}(P) := \{(x_1, \dots, x_n) \in [0, 1]^{|P|} : x_i \leq x_j \text{ if } t_i \leq_P t_j\}.$$

Remark 68. Given a poset P , a map $f : P \rightarrow \mathbb{R}_{\geq 0}$ is *order preserving* if $f(t_i) \leq f(t_j)$ whenever $t_i \leq_P t_j$. The set of order preserving maps for P forms a polyhedral cone called the *order cone*, denoted K_P . This means the order polytope can be seen as bounding the codomain of maps in K_P to be in $[0, 1]$ since we can set $f(t_i) := x_i$ and similarly for x_j .

Definition 69. An *upper order ideal* (or *filter*) F is a subset of a poset P with the property that if $x \in F$ and $x \leq_P y$ for some $y \in P$, then $y \in F$. Let $J(P)$ denote the poset of upper order ideals ordered by inclusion.

The following proposition gives the vertex description of an order polytope.

Proposition 70. *Given a finite poset P , the vertices of the order polytope $\mathcal{O}(P)$ are characteristic functions of upper order ideals, i.e., for $F \subseteq P$ an upper order ideal the order preserving map $\chi_F : P \rightarrow \mathbb{R}_{\geq 0}$ defined by*

$$\chi_F(p) = \begin{cases} 1 & \text{if } p \in F, \\ 0 & \text{otherwise} \end{cases}$$

gives a vertex of $\mathcal{O}(P)$.

The following theorem is the key correspondence between flow and order polytopes in the strongly planar case.

Theorem 71 (Mészáros, Morales, and Striker [19]). *Given a strongly planar DAG G , the flow polytope $\mathcal{F}_1(G)$ is integrally equivalent to the order polytope of the strongly planar poset dual to G . Conversely, given a strongly planar poset P , the order polytope $\mathcal{O}(P)$ is integrally equivalent to the flow polytope of the strongly planar DAG G_P .*

As a consequence of Theorems 47 and 71, we have the following.

Lemma 72. *A strongly planar DAG G has a route decomposition if and only if P_G is graded.*

Proof. A strongly planar DAG G has a route decomposition if and only if $\mathcal{F}_1(G)$ is Gorenstein. By Theorem 71 this holds if and only if $\mathcal{O}(P_G)$ is Gorenstein which holds if and only if P_G is a graded poset. $\square$

The following theorem shows how our equatorial flow triangulations can be used to produce new equatorial triangulations of Gorenstein order polytopes for strongly planar posets, describable in purely graph theoretic language.

Theorem 73. *If G is a strongly planar DAG satisfying degree equality, then every route decomposition of G induces a unimodular triangulation of the corresponding order polytope $\mathcal{O}(P_G)$.*

In addition to the integral equivalence given in Proposition 77, Mészáros, Morales, and Striker give an equivalence of the canonical triangulation of order polytopes and a DKK triangulation given by a framing called the planar framing. The structure of the canonical triangulation is nicely given by the following definition and theorem.

Definition 74. The *order complex* of $J(P)$ is defined as the simplicial complex $\Delta(J(P))$ where each upper order ideal F of P defines a vertex and each chain $F_1 \subseteq \cdots \subseteq F_t$ of nested ideals defines a simplex.

Theorem 75. *The canonical triangulation of the order polytope is isomorphic as an abstract simplicial complex to the order complex of $J(P)$, i.e., chains of nested upper order ideals correspond to the simplices of the canonical triangulation.*

We will now define the framing which gives an equivalence of the DKK triangulation and the canonical triangulation for a strongly planar DAG.

Definition 76 (Mészáros, Morales, and Striker [19]). Consider a strongly planar DAG G , then for every inner vertex there is an order of the edges given by the planar embedding. At inner vertex j we can order the incoming and outgoing edges from top to bottom. More specifically, draw a small circle around the inner vertex such that each edge hits the circle once at a point and order the incoming and outgoing edges by decreasing y -coordinates of these points. This framing is called the *planar framing* of G (as it depends on the planar embedding of the DAG G).

Proposition 77 (Mészáros, Morales, and Striker [19]). *Consider a strongly planar DAG G . Under the integral equivalence map ϕ given by Mészáros, Morales, and Striker, the canonical triangulation of $\mathcal{O}(P_G)$ maps to a DKK triangulation of $\mathcal{F}_1(G)$ given by the planar framing*

A key observation for the proof of Mészáros, Morales, and Striker is given by the following proposition. It will also be useful later in this section in proving Theorem 89.

Proposition 78 (Mészáros, Morales, and Striker [19]). *Suppose we have a strongly planar DAG G and let ϕ be the integral equivalence of the order polytope and flow polytope given by Mészáros, Morales, and Striker. Given a vertex χ_F of $\mathcal{O}(P_G)$ for an upper order ideal F of P_G , we have $\phi(\chi_F)$ is the unit flow along a route in G separating F and $P_G - F$. Moreover, any route R in G separates some upper order ideal F and $P_G - F$.*

We now discuss the equatorial triangulation defined by Reiner-Welker [21] on order polytopes. The definitions used in this paper are order preserving, while Reiner and Welker used order reversing. However, this does not change their construction of the equatorial triangulation for the order polytope. We give the corresponding definitions here.

Let P be a graded poset with r rank sets, starting with rank 1. Let P_i denote the set of elements of P of rank i for $0 \leq i \leq r$ where we say $P_0 = \emptyset$.

Definition 79. An order preserving map $f : P \rightarrow \mathbb{R}_{\geq 0}$ is *rank-constant* if it is constant along ranks, in other words $f(p) = f(q)$ whenever $p, q \in P_i$ for some $i \in [r]$

Definition 80. An order preserving map $f : P \rightarrow \mathbb{R}_{\geq 0}$ is *equatorial* if $\min_{p \in P} f(p) = 0$ and for every $j \in [2, r]$ there exists a cover relation $p_{j-1} \prec_P p_j$ such that $p_{j-1} \in P_{j-1}$, $p_j \in P_j$, and $f(p_{j-1}) = f(p_j)$.

Definition 81. Suppose we have a chain of upper order ideals $F_1 \subset \cdots \subset F_t$. We say this is a *rank-constant chain* (resp. *equatorial chain*) if the sum $\chi_{F_1} + \cdots + \chi_{F_t}$ is rank-constant (resp. equatorial).

Example 82. The only rank constant upper order ideals are of the form

$$F_j^{rc} = \bigsqcup_{i > j} P_i$$

for $0 \leq j \leq r$ where $F_r^{rc} = \emptyset$.

Definition 83. Given a chain of ideals $F_1 \subset \cdots \subset F_t$, we say its *jumps* are

$$J_i = F_i - F_{i-1}$$

for $i \in [t+1]$ and where $F_0 = \emptyset$, $F_{t+1} = P$.

Theorem 84 (Reiner and Welker [21]). *A chain of nonempty upper order ideals $F_1 \subset \cdots \subset F_t$ is equatorial if and only if its jumps have the following property; for every $j \in [2, r]$ there exists $p_{j-1} \prec_P p_j$ with $p_{j-1} \in P_{j-1}$, $p_j \in P_j$ and a value $i \in [t+1]$ such that $p_{j-1}, p_j \in J_i$. In other words, between every pair of consecutive ranks there exists a cover relation which appears in the same jump.*

Definition 85. We define $\Delta_{eq}(P)$ as the subcomplex of the order complex $\Delta(J(P))$ whose faces are given by equatorial chains of non-empty ideals.

Remark 86. Reiner and Welker [21] define $\Delta_{eq}(P)$ as the equatorial complex which is different from our convention in defining Γ to be the equatorial complex. We will see later that $\Delta_{eq}(P)$ is in fact related to T_{eq} .

Reiner and Welker define the equatorial triangulation and show it has a nice structure given by a join.

Corollary 87 (Reiner and Welker [21]). *The equatorial triangulation of the order polytope $\mathcal{O}(P)$ is abstractly isomorphic to the simplicial join of $\sigma * \Delta_{eq}(P)$ where σ is the interior simplex spanned by the rank constant ideas $\{F_j^{rc}\}_{j=0}^r$.*

This join structure of the equatorial triangulation of the order polytope is very similar to the equatorial flow triangulations. In fact, Theorem 89 shows that the equatorial triangulation of the order polytope is equivalent to an equatorial flow triangulation, under the integral equivalence of flow polytopes and order polytopes for a strongly planar DAG.

Remark 88. Define the poset $\widehat{P}_G := P_G \cup \{\hat{0}, \hat{1}\}$. The integral equivalences of $\mathcal{F}_1(G)$ and $\mathcal{O}(P_G)$ given by Mészáros, Morales, and Striker [19] are defined as follows:

- Define $\varphi : \mathcal{F}_1(G) \rightarrow \mathcal{O}(P_G)$ by $fl \mapsto (f(x))_{x \in P_G}$ where $f : P_G \rightarrow \mathbb{R}_{\geq 0}$ is given by

$$f(x) = \sum_e fl(e)$$

where we sum over the edges $e \in G$ which correspond to edges from a chain from $\hat{0}$ to x in $\widehat{P}_G$ by duality. This works for any chain.

- Define $\phi : \mathcal{O}(P_G) \rightarrow \mathcal{F}_1(G)$ by $(f(x))_{x \in P_G} \mapsto fl$ where $fl : E(G) \rightarrow \mathbb{R}_{\geq 0}$ is given by

$$fl(e) = f(y) - f(x)$$

where the edge $e \in G$ corresponds to the cover relation $x \prec y$ in $\widehat{P}_G$ by duality and $f(\hat{1}) = 1, f(\hat{0}) = 0$.

Note that the map the map φ given by Mészáros, Morales, and Striker give in their paper is slightly stronger. In fact, the codomain of φ is all order preserving functions f of $P_G \cup \{\hat{0}, \hat{1}\}$ such that $f(\hat{0}) = 0$ and $f(\hat{1}) = 1$.

Theorem 89. *Suppose G is strongly planar and $\mathcal{F}_1(G)$ is Gorenstein, then the equatorial triangulation of $\mathcal{O}(P_G)$ corresponds to an equatorial flow triangulation under the integral equivalence of $\mathcal{F}_1(G)$ and $\mathcal{O}(P_G)$.*

Proof. Define the poset $\widehat{P}_G := P_G \cup \{\hat{0}, \hat{1}\}$. We will first show the rank constant simplex for the equatorial triangulation of $\mathcal{O}(P_G)$ is equivalent to the route simplex for a particular choice of route decomposition of $\mathcal{F}_1(G)$. We have P_G is graded by Lemma 72, say of rank r . Recall that $(P_G)_k$ denotes the set of elements of P_G of rank k for $1 \leq k \leq r$. Since G is strongly planar, each edge of our DAG is a piecewise differentiable function. So for each x -value between the x -values of s and t , there exists a largest y -value attained by some edge in G . By planarity, the collection of edges achieving these maximum values forms a route R_0 where all other edges in G have smaller y -values. By duality, the edges of R_0 correspond to the cover relations of $\widehat{P}_G$ between $\hat{1}$ and elements of $(P_G)_r$. Moreover, by Proposition 78, this gives the upper order ideal $\emptyset = F_r^{rc}$ of P_G . We can now consider $G - R_0$ and repeat the argument. This gives a route R_1 where all the edges correspond to the cover relations of elements between $(P_G)_r$ and $(P_G)_{r-1}$. Hence by Proposition 78, this gives the ideal $(P_G)_r = F_{r-1}^{rc}$ of P_G . We continue in this way, and the last route to remove will be the route R_r whose edges correspond to the cover relations between $(P_G)_1$ and $\hat{0}$ in $\widehat{P}_G$ which gives the ideal $P_G = F_0^{rc}$ of P_G . So our route decomposition and linear order $(\mathcal{R}, \leq)$ is given by $R_0 \leq R_1 \leq \dots \leq R_r$ which corresponds to the simplex given by rank constant ideals $F_r^{rc} \subset F_{r-1}^{rc} \subset \dots \subset F_0^{rc}$. In addition, the route decomposition framing given by $\mathcal{R}$ is equivalent to the planar framing as we will be ordering edges from the top down given by the planar embedding.

Let T be the route decomposition framing given by $(\mathcal{R}, \leq)$, i.e., the planar framing. We will now show that a simplex is in T_{eq} of the flow polytope $\mathcal{F}_1(G)$ if and only if the corresponding simplex under the integral equivalence to $\mathcal{O}(P_G)$ is in $\Delta_{eq}(P)$. A simplex in T_{eq} is a clique of routes under the planar framing which must also avoid a transversal by Theorem 49 and equivalence to the route decomposition framing. By Proposition 77, we have a clique of routes if and only if the corresponding vertices in order polytope $\mathcal{O}(P_G)$ give a chain of nested upper order ideals. Thus, we can assume coherency and it suffices to check that routes avoid a transversal if and only if for the corresponding ideals in P_G satisfy the equatorial condition and are non-empty to prove our main claim.

For the forward direction, suppose we have coherent routes $\{S_i\}_{i=1}^t$ which avoid a transversal $M = \{e_R \in E(G) : R \in \mathcal{R}\}$. Consider the ideals $\{\varphi(S_i)\}_{i=1}^t$ of P_G and let f_{R_i} denote the cover relation $b_i \prec a_i$ of $\widehat{P}_G$ corresponding to the edge $e_{R_i} \in M$. We will first show these ideals are nonempty. Consider e_{R_0} , we have a cover relation $b_0 \prec a_0 = \hat{1}$ in $\widehat{P}_G$, which gives

$$\varphi(S_i)(\hat{1}) - \varphi(S_i)(b_0) = \chi_{S_i}(e_{R_0}) + \varphi(S_i)(b_0) - \varphi(S_i)(b_0) = 0$$

by definition of φ for all i . Note in this case we are using the extended codomain of order preserving functions discussed in Remark 88. We have shown $1 = \varphi(S_i)(\hat{1}) = \varphi(S_i)(b_0)$. So $b_0 \in P_G$ is an element of all ideals $\{\varphi(S_i)\}_{i=1}^t$ and hence they are nonempty. We will now verify the equatorial condition. So for all $j \in [r-1]$, each $b_i \prec a_j$ is also a relation in P_G . Hence for all i we find the following property of φ . When considering the edge of the cover relation $b_j \prec a_j$, we have

$$\varphi(S_i)(a_j) - \varphi(S_i)(b_j) = \chi_{S_i}(e_{R_i}) + \varphi(S_i)(b_j) - \varphi(S_i)(b_j) = 0.$$

Hence, $\varphi(S_i)(a_j) = \varphi(S_i)(b_j)$. Since the edges of $R_j \in \mathcal{R}$ corresponds with the cover relations between consecutive ranks in P_G by prior observations, then we have cover relations f_{R_j} for all $j \in [r-1]$ between every pair of consecutive ranks of P_G . Moreover, when we consider the edge $e_{R_r} \in M$ this corresponds to a cover relation $b_r = \hat{0} \prec a_r$ for some $a_r \in P_G$. Hence

$$\varphi(S_i)(a_r) = \chi_{S_i}(e_{R_r}) = 0$$

meaning the minimum value of all $\varphi(S_i)$ is 0 and achieved at $a_r \in P_G$. Hence, $\sum_{i=1}^t \chi_{\varphi(S_i)}$ has minimum value 0 at a_r and is constant on f_{R_i} for $i \in [r-1]$ which shows $\{\varphi(S_i)\}_{i=1}^t$ is an equatorial chain of nonempty ideals.

For the backward direction, suppose we have an equatorial chain of nonempty upper order ideals $F_1 \subset \cdots \subset F_t$ of P_G . Since F_1 is nonempty and an upper order ideal, there exists $q_r \in F_1$ of the top rank. Define e_{R_0} to be the edge in G corresponding to the cover relation $q_r \prec \hat{1}$ of $\widehat{P}_G$. Moreover since we have a chain of ideals, $q_r \in \bigcap_{i=1}^t F_i$ and

$$\phi(F_i)(e_{R_0}) = 1 - \chi_{F_i}(q_r) = 0$$

for all $i \in [t]$. So the routes $\{\phi(F_i)\}_{i=1}^t$ avoid the edge e_{R_0} which is an edge in R_0 . Next, the equatorial condition gives $p_{j-1} \prec p_j$ in P_G for $j \in [2, r]$ where $p_k \in (P_G)_k$ such that

$\chi_{F_1} + \cdots + \chi_{F_t}$ is constant. By reindexing, we have for $k \in [r-1]$ the cover relations $p_{r-k} \prec p_{r-k+1}$ in P_G such that $\chi_{F_1} + \cdots + \chi_{F_t}$ is constant. Define the edge e_{R_k} in G to be the one dual to $p_{r-k} \prec p_{r-k+1}$ of P_G . By our construction of the route decomposition, we saw that the edges in G corresponding to cover relations between $(P_G)_{r-k}$ and $(P_G)_{r-k+1}$ are the edges of $R_k \in \mathcal{R}$. Theorem 84 implies that p_{r-k} and p_{r-k+1} must either both be elements of F_i or both not be elements of F_i for all $i \in [t]$. By definition of ϕ , the flow on the edge e_{R_k} for any $k \in [r-1]$ is

$$\phi(F_i)(e_{R_k}) = \chi_{F_i}(p_{r-k+1}) - \chi_{F_i}(p_{r-k}) = 0$$

by our previous observation of p_{r-k} and p_{r-k+1} . To define the edge e_{R_r} , since we have an equatorial chain we know the minimum value of $\chi_{F_1} + \cdots + \chi_{F_t}$ is 0. So there must exist an element in P_G of rank 1 that attains this value, otherwise this would imply the poset only has nonzero values as we have upper order ideals. Let this element be $q_1 \in P_G$ hence q_1 is not in any of our ideals. Define the edge $e_{R_r} \in G$ be the one dual to the relation $\hat{0} \prec q_1$, so then

$$\phi(F_i)(e_{R_0}) = \chi_{F_i}(q_1) - 0 = 0.$$

Thus, the routes $\{\phi(F_i)\}_{i=1}^t$ avoid the transversal given by $\{e_{R_j}\}_{j=0}^r$.

Therefore, we have shown T_{eq} is integrally equivalent to Δ_{eq} . This establishes an isomorphism of triangulations $\sigma * \Delta_{eq}(P)$ and $\Delta_{\mathcal{R}} * T_{eq}$, since both are joins of equivalent simplicial complexes. $\square$

For strongly planar graded posets, the equatorial flow triangulation is given by one route decomposition, but any route decomposition gives an equatorial flow triangulation. Hence, it is important to keep in mind that using equatorial flow triangulations we can find many different equatorial triangulations of order polytopes in the strongly planar case, and describe them combinatorially using DAGs.

Acknowledgements

Both authors were partially supported by the US National Science Foundation award DMS-1953785. The authors thank Jonah Berggren for sharing his preliminary results regarding projection maps of Gorenstein flow polytopes for amply framed DAGs, which inspired the contents of Section 5.

References

- [1] Christos A. Athanasiadis. Ehrhart polynomials, simplicial polytopes, magic squares and a conjecture of Stanley. *J. Reine Angew. Math.*, 583:163–174, 2005.
- [2] Welleda Baldoni and Michèle Vergne. Kostant partitions functions and flow polytopes. *Transform. Groups*, 13(3-4):447–469, 2008.
- [3] Matthias Beck and Sinai Robins. *Computing the continuous discretely*. Undergraduate Texts in Mathematics. Springer, New York, second edition, 2015. Integer-point enumeration in polyhedra, With illustrations by David Austin.

- [4] Carolina Benedetti, Rafael S. González D’León, Christopher R. H. Hanusa, Pamela E. Harris, Apoorva Khare, Alejandro H. Morales, and Martha Yip. The volume of the caracol polytope. *Sém. Lothar. Combin.*, 80B:Art. 87, 12, 2018.
- [5] Carolina Benedetti, Rafael S. González D’León, Christopher R. H. Hanusa, Pamela E. Harris, Apoorva Khare, Alejandro H. Morales, and Martha Yip. A combinatorial model for computing volumes of flow polytopes. *Trans. Amer. Math. Soc.*, 372(5):3369–3404, 2019.
- [6] Jonah Berggren. Flows on gentle algebras. [arXiv:2507.12688](https://arxiv.org/abs/2507.12688), 2025
- [7] Benjamin Braun and Ford McElroy. Volume inequalities for flow polytopes of full directed acyclic graphs *Comb. Theory*, 5(2):Paper No. 14, 2025.
- [8] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.
- [9] Winfried Bruns and Tim Römer. h -vectors of Gorenstein polytopes. *J. Combin. Theory Ser. A*, 114(1):65–76, 2007.
- [10] Vladimir I. Danilov, Alexander V. Karzanov, and Gleb A. Koshevoy. Coherent fans in the space of flows in framed graphs. In *24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012)*, volume AR of *Discrete Math. Theor. Comput. Sci. Proc.*, pages 481–490. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2012.
- [11] William T. Dugan, Maura Hegarty, Alejandro H. Morales, and Annie Raymond. Generalized Pitman-Stanley flow polytopes. *Sém. Lothar. Combin.*, 89B:Art. 80, 12, 2023.
- [12] Eugène Ehrhart. Sur les polyèdres rationnels homothétiques à n dimensions. *C. R. Acad. Sci. Paris*, 254:616–618, 1962.
- [13] Rafael S. González D’León, Christopher R. H. Hanusa, Alejandro H. Morales, and Martha Yip. Column-convex matrices, G -cyclic orders, and flow polytopes. *Discrete Comput. Geom.*, 70(4):1593–1631, 2023.
- [14] Kabir Kapoor, Karola Mészáros, and Linus Setiabrata. Counting integer points of flow polytopes. *Discrete Comput. Geom.*, 66(2):723–736, 2021.
- [15] B. V. Lidskiĭ. The Kostant function of the system of roots A_n . *Funktsional. Anal. i Prilozhen.*, 18(1):76–77, 1984.
- [16] Ricky I. Liu, Karola Mészáros, and Avery St. Dizier. Gelfand-Tsetlin polytopes: a story of flow and order polytopes. *SIAM J. Discrete Math.*, 33(4):2394–2415, 2019.
- [17] Ricky Ini Liu, Alejandro H. Morales, and Karola Mészáros. Flow polytopes and the space of diagonal harmonics. *Canad. J. Math.*, 71(6):1495–1521, 2019.
- [18] Karola Mészáros and Alejandro H. Morales. Volumes and Ehrhart polynomials of flow polytopes. *Math. Z.*, 293(3-4):1369–1401, 2019.
- [19] Karola Mészáros, Alejandro H. Morales, and Jessica Striker. On flow polytopes, order polytopes, and certain faces of the alternating sign matrix polytope. *Discrete Comput. Geom.*, 62(1):128–163, 2019.

- [20] Alejandro H. Morales and William Shi. Refinements and symmetries of the Morris identity for volumes of flow polytopes. *C. R. Math. Acad. Sci. Paris*, 359:823–851, 2021.
- [21] Victor Reiner and Volkmar Welker. On the Charney-Davis and Neggers-Stanley conjectures. *J. Combin. Theory Ser. A*, 109(2):247–280, 2005.
- [22] Konstanze Rietsch and Lauren Williams. Root polytopes, flow polytopes, and order polytopes. [arXiv:2406.15803](https://arxiv.org/abs/2406.15803), 2024.
- [23] Matias von Bell, Benjamin Braun, Kaitlin Bruegge, Derek Hanely, Zachery Peterson, Khrystyna Serhiyenko, and Martha Yip. Triangulations of flow polytopes, ample framings, and gentle algebras. *Selecta Math. (N.S.)*, 30(3):Paper No. 55, 2024.
- [24] Matias von Bell, Rafael S. González D’León, Francisco A. Mayorga Cetina, and Martha Yip. On framed triangulations of flow polytopes, the ν -Tamari lattice and Young’s lattice. *Sém. Lothar. Combin.*, 85B:Art. 42, 12, 2021.