Minimal Cellular Resolutions of Path Ideals

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Abstract

In this paper, we prove that the path ideals of both paths and cycles have minimal cellular resolutions. Specifically, these minimal free resolutions coincide with the Barile-Macchia resolutions for paths, and their generalized counterparts for cycles. Furthermore, we identify edge ideals of cycles as a class of ideals that lack a minimal Barile-Macchia resolution, yet have a minimal generalized Barile-Macchia resolution.

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1 Introduction

It has been a powerful approach to associate a combinatorial object with an algebraic one and study its algebraic properties via combinatorics [13, 14, 17, 18, 19]. In the spirit of this approach, the algebraic objects of interest in this paper are path ideals, while the combinatorial counterparts are graphs. Specifically, our focus is on studying the path ideals of graphs and their minimal free resolutions by leveraging the underlying structure of the graphs. Central to this work is the use of (generalized) Barile-Macchia resolutions from [10]. Such resolutions fall under the umbrella of Morse resolutions, a class of cellular resolutions first introduced by Batzies and Welker in [6], and they generalize the minimal free resolution constructed in [5] for edge ideals of forests. These resolutions are obtained using homogeneous acyclic matchings, a concept from discrete Morse theory. In addition to the Barile-Macchia resolution, other examples of Morse resolutions have been introduced in the literature. One recent example is the *pruned resolutions* from [3].

Path ideals, first introduced by Conca and De Negri in [12], have been studied for their algebraic properties [1, 2, 4, 8, 9]. Path ideals can be seen as a generalization of edge ideals, which have been of significant interest in recent years. Let $G = (V, E)$ be a finite,

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simple graph with the vertex set $V = \{x_1, \dots, x_n\}$. Associating the vertices of G with the variables in the polynomial ring $R = \Bbbk[x_1, \dots, x_n]$, where $\Bbbk$ is any field, the edge ideal of G is then generated by monomials of the form $x_i x_j$ for every $\{x_i, x_j\} \in E$. In essence, edge ideals arise from monomials corresponding to edges in G, which are inherently paths of length 1. Extending this idea, ideals generated by monomials corresponding to paths of a specified length in G are called the path ideals of G.

In this paper, our goal is to construct minimal resolutions for path ideals of paths and cycles. While there is literature discussing and providing explicit formulas for the (graded) Betti numbers of these ideals [2, 15], no construction has yet been provided for their minimal resolution. We achieve this by working with (generalized) Barile-Macchia resolutions from [10], thereby expanding the class of ideals for which these resolutions are minimal.

Our two main results are:

1. We establish that path ideals of paths have the *bridge-friendly* property (Theorem 19). This property ensures the minimality of Barile-Macchia resolutions as described in [10]. Thus, we determine that the path ideals of paths admit a minimal Barile-Macchia resolution. From this, we derive formulas for their projective dimension and graded Betti numbers, recovering results from [2] and [8] (Corollary 28, Theorem 29).

2. We shift our focus to path ideals of cycles. In this context, Barile-Macchia resolutions are not always minimal. One instance is the edge ideal of a 9-cycle, $I_2(C_9)$, which, as indicated in [10], does not have any minimal Barile-Macchia resolution. Nonetheless, we prove that path ideals of cycles have a minimal generalized Barile-Macchia resolution (Theorem 46).

Our results on minimal cellular resolutions of path ideals for paths and cycles generalize the findings from [3], where it is shown that edge ideals of paths and cycles have minimal pruned resolutions.

Both Barile-Macchia and generalized Barile-Macchia resolutions are induced by homogeneous acyclic matchings, called Barile-Macchia and generalized Barile-Macchia matchings, respectively. At the heart of the Barile-Macchia matching construction is the comparison of least common multiples of subsets of the minimal generating set $\mathcal{G}(I)$ of a monomial ideal I, with respect to a total order on $\mathcal{G}(I)$. An algorithm for producing Barile-Macchia matchings was introduced by the first two authors in [10]. In addition, `MorseResolutions` `Macaulay2` package dedicated to Barile-Macchia resolutions were introduced by the first two authors and O'Keefe in [11]. While the generalized version adopts the same foundational principle, it extends to multiple total orders on $\mathcal{G}(I)$ as discussed in Section 4. For further details, refer to [10].

An important concept in relating homogeneous acyclic matchings and free resolutions is *critical cells*: These are subsets of $\mathcal{G}(I)$ that remain untouched by a homogeneous acyclic matching of I. In [6], it was shown that these cells are in one-to-one correspondence with the ranks of free modules from (generalized) Barile-Macchia resolutions. Thus, in this paper, we focus on characterizing critical cells of path ideals using *bridges, gaps* and *true gaps* – simple yet powerful concepts rooted in the graph's structure as introduced in [10]. We use these notions to produce minimal free resolutions of path ideals of paths and cycles.

A key observation concerning the critical cells of path ideals of paths is that two distinct critical cells have different least common multiples. This observation proves useful in constructing a critical cell of maximum size, which in turn allows us to deduce a formula for the projective dimension of path ideals of paths. Additionally, this insight is helpful in identifying all critical cells of path ideals of cycles. Such cells consists of critical cells of path ideals of induced paths as well as critical cells whose least common multiple has the largest multidegree – for a cycle on n vertices, it would be $x_1 \cdots x_n$. Consequently, in the paper's final section, our study primarily focuses on identifying the critical cells of the latter type for cycles.

The structure of this paper is as follows. In Section 2, we revisit essential concepts and results relevant to Morse resolutions, as well as Barile-Macchia matchings and resolutions. In Section 3, we explore path ideals of paths. We start this section by delving into the characterizations of bridges, gaps, and true gaps specific to paths. With these characterizations in hand, we obtain the bridge-friendliness and affirm the minimality of the associated Barile-Macchia resolution. This paves the way for us to introduce formulas for the projective dimension and to provide a recursive formula for graded Betti numbers. Lastly, in Section 4, we turn our attention to path ideals of cycles, offering both a review of and insights into the application of generalized Barile-Macchia resolutions. After characterizing the bridges, gaps, and true gaps, we verify the minimality of their generalized Barile-Macchia resolutions by drawing upon our earlier findings on path ideals.

2 Preliminaries

2.1 Morse resolutions

Let I be a monomial ideal in the polynomial ring $R = \Bbbk[x_1, \ldots, x_N]$ with a minimal generating set denoted by $\mathcal{G}(I) = \{m_1, \ldots, m_n\}$. We associate to I the Taylor simplex X. The vertices of X correspond to the generators of I, whereas its cells are labeled by the least common multiple of the labels of their incident vertices. We also associate a directed graph $G_X = (V, E)$ with this structure, where V denotes the cells of X, or equivalently, the subsets of $\mathcal{G}(I)$. The edge set E consists of directed edges of the form (σ, σ') where $\sigma' \subseteq \sigma$ and $|\sigma'| = |\sigma| - 1$. For any subset A of E, we define G_X^A as the directed graph having vertex set V and edge set

$$E(G_X^A) = (E \setminus A) \cup \{(\sigma', \sigma) \mid (\sigma, \sigma') \in A\}.$$

Essentially, G_X^A is derived from G_X by reversing the direction of edges belonging to A.

Central to our discussion on a Morse resolution of I is the notion of homogeneous acyclic matchings from discrete Morse theory, a concept we revisit below.

Definition 1. A subset $A \subseteq E$ is called a **homogeneous acyclic matching** of I if the following conditions hold:

1. (*matching*) Each cell appears in, at most, one edge of A.
2. (*acyclicity*) The graph G_X^A does not contain a directed cycle.
3. (*homogeneity*) For any edge (σ, σ') in A, we have $\mathrm{lcm}(\sigma) = \mathrm{lcm}(\sigma')$.

A cell σ that does not appear in any edge of A is called an A-**critical** cell of I. In contexts where there is no ambiguity, we simply refer to it as *critical*.

Recall that X is a $\mathbb{Z}^N$-graded complex that induces a free resolution $\mathcal{F}$ where $\mathcal{F}_r$ is the free R-module with a basis indexed by all cells of cardinality r. The differentials $\partial_r \colon \mathcal{F}_r \to \mathcal{F}_{r-1}$, are defined as

$$\partial_r(\sigma) = \sum_{\substack{\sigma' \subseteq \sigma, \\ |\sigma'| = r-1}} [\sigma : \sigma'] \frac{\mathrm{lcm}(\sigma)}{\mathrm{lcm}(\sigma')} \sigma'.$$

where $[\sigma : \sigma']$ denotes the coefficient of σ' in the boundary of σ and is either 1 or -1. The complex $\mathcal{F}$ is called the *Taylor resolution* of R/I.

Morse resolutions are refinements of the Taylor resolution. Each homogeneous acyclic matching yields a Morse resolution, and these resolutions may coincide. To precisely define the differentials of a Morse resolution, we need to introduce some additional terminology.

Given a directed edge $(\sigma, \sigma') \in E(G^A)$, set

$$m(\sigma, \sigma') = \begin{cases} -[\sigma' : \sigma] & \text{if } (\sigma', \sigma) \in A, \\ [\sigma : \sigma'] & \text{otherwise.} \end{cases}$$

A *gradient path* $\mathcal{P}$ from σ_1 to σ_t is a directed path

$$\mathcal{P} \colon \sigma_1 \to \sigma_2 \to \cdots \to \sigma_t$$

in G_X^A. Similarly, set $m(\mathcal{P}) = m(\sigma_1, \sigma_2) \cdot m(\sigma_2, \sigma_3) \cdots m(\sigma_{t-1}, \sigma_t)$.

We are now ready to recall Morse resolutions of monomial ideals.

Theorem 2. *[6, Proposition 2.2, Proposition 3.1, Lemma 7.7] Let A be a homogeneous acyclic matching of I. Then A induces a cellular resolution $\mathcal{F}_A$, where:*

- *$(\mathcal{F}_A)_r$ is the free R-module with a basis indexed by all critical cells of cardinality r.*
- *The differentials $\partial_r^A : (\mathcal{F}_A)_r \to (\mathcal{F}_A)_{r-1}$ are defined by:*

$$\partial_r^A(\sigma) = \sum_{\substack{\sigma' \subseteq \sigma, \\ |\sigma'| = r-1}} [\sigma : \sigma'] \sum_{\substack{\sigma'' \text{ is critical}, \\ |\sigma''| = r-1}} \sum_{\mathcal{P} \text{ is a gradient path from } \sigma' \text{ to } \sigma''} m(\mathcal{P}) \frac{\mathrm{lcm}(\sigma)}{\mathrm{lcm}(\sigma'')} \sigma''.$$

The resulting (cellular) free resolution $\mathcal{F}_A$ is called the Morse resolution *of R/I associated to A. Furthermore, $\mathcal{F}_A$ is minimal if for any two A-critical cells σ and σ' with $|\sigma'| = |\sigma| - 1$, we have $\mathrm{lcm}(\sigma) \neq \mathrm{lcm}(\sigma')$.*

2.2 Barile-Macchia matchings and Barile-Macchia resolutions.

In this subsection, we recall Barile-Macchia matchings and resolutions. To ensure that this paper is self-contained and accessible, we present relevant definitions and results from [10] that are instrumental to our discussions.

Given Theorem 2, the key to producing a Morse resolution of R/I is finding a homogeneous acyclic matching of I. However, systematically crafting such a matching for any monomial ideal is a challenging task. A recent development in this direction is the Barile-Macchia algorithm, as presented in [10, Theorem 2.8], which produces a homogeneous acyclic matching. In the context of [10], a matching arising from this algorithm is called a *Barile-Macchia matching*, and the resulting resolution is called a *Barile-Macchia resolution*.

Below, we recall some of the terminology useful for discussing Barile-Macchia resolutions. Throughout this section, we fix a total order $(\succ_I)$ on $\mathcal{G}(I)$. For simplicity, we write $S \setminus s$ (resp, $S \cup s$) instead of $S \setminus \{s\}$ (resp, $S \cup \{s\}$) where S is a set and $s \in S$ (resp, $s \notin S$). Additionally, throughout our discussion, we use the terms "subsets of $\mathcal{G}(I)$" and "cells of I" interchangeably. By cells of I, we refer to cells of the corresponding Taylor simplex X. Lastly, when we express a cell σ as $\sigma = \{m_{i_1}, \dots, m_{i_k}\}$, we assume that $m_{i_1} \succ_I m_{i_2} \succ_I \cdots \succ_I m_{i_k}$.

Definition 3. Let σ be a subset of $\mathcal{G}(I)$. A monomial m is called a **bridge** of σ if $m \in \sigma$ and removing m from σ does not change the lcm, i.e., $\operatorname{lcm}(\sigma \setminus m) = \operatorname{lcm}(\sigma)$. If σ has a bridge, the notation $\operatorname{sb}(\sigma)$ denotes the smallest bridge of σ with respect to $(\succ_I)$. We set $\operatorname{sb}(\sigma) = \emptyset$ if σ has no bridges.

To fully understand the terms and context we discuss, in Algorithm 1 we recall the Barile-Macchia algorithm as outlined in [10, Algorithm 2.9].

Algorithm 1 Barile-Macchia Algorithm

Input: A total order $(\succ_I)$ on $\mathcal{G}(I)$.
Output: Set of directed edges A in G_X.

$A \leftarrow \emptyset$
$\Omega \leftarrow \{$all subsets of $\mathcal{G}(I)$ with cardinality at least 3$\}$
while $\Omega \neq \emptyset$ **do**
    Pick $\sigma \in \Omega$ with maximal cardinality
    Remove $\{\sigma, \sigma \setminus \operatorname{sb}(\sigma)\}$ from Ω
    if $\operatorname{sb}(\sigma) \neq \emptyset$ **then**
        Add edge $(\sigma, \sigma \setminus \operatorname{sb}(\sigma))$ to A
for all distinct edges (σ, τ) and (σ', τ') in A with $\tau = \tau'$ **do**
    if $\operatorname{sb}(\sigma') \succ_I \operatorname{sb}(\sigma)$ **then**
        Remove (σ', τ') from A
    else
        Remove (σ, τ) from A
return A

The Barile-Macchia Algorithm systematically constructs a matching in G_X based on the concept of bridges within subsets on $\mathcal{G}(I)$ and a fixed total order $(\succ_I)$ of $\mathcal{G}(I)$. As it was shown in [10, Theorem 2.11], this process always produces a homogeneous acyclic matching.

Definition 4. Given a Barile-Macchia matching A of I:

1. For any edge (σ, τ) in the final A from Algorithm 1, the cell σ is called **type-2** while the cell τ is called **type-1**.

2. During the execution of Algorithm 1, if a directed edge (σ, τ) is added to A, the cell σ is called **potentially-type-2**. It is important to note that this edge may not persist in the final A produced by Algorithm 1.

Remark 5. If a subset of $\mathcal{G}(I)$ has a bridge, then it is either type-1 or potentially-type-2.

In earlier discussions, we emphasized the one-to-one correspondence between the ranks of the free R-modules in a Barile-Macchia resolution of R/I and the A-critical cells of I. Here, A represents the Barile-Macchia matching of I with respect to $(\succ_I)$. Thus, delving into the nature of A-critical cells, along with the remaining cells, is crucial for a deeper comprehension of the Barile-Macchia resolution of R/I. It is important to point out that critical cells of I consists of those cells either left out of A during the Barile-Macchia Algorithm or added initially but later excluded in the final refinement of the algorithm. We name these critical cells as follows:

Definition 6. Let σ be a subset of $\mathcal{G}(I)$. If σ is never added to A in any of the steps throughout Algorithm 1, it is called **absolutely critical**. If σ is potentially-type-2 but not type-2 (initially added to A but removed in the final A produced by Algorithm 1), we call it **fortunately critical**.

The following concepts from [10] are useful in characterizing critical and non-critical cells of I.

Definition 7. [10, Definition 2.19] Let $m, m' \in \mathcal{G}(I)$ and σ be a subset of $\mathcal{G}(I)$.

1. We say that m **dominates** m' if and only if $m \succ_I m'$.

2. The monomial m is called a **gap** of σ if $m \notin \sigma$, and $\operatorname{lcm}(\sigma \cup m) = \operatorname{lcm}(\sigma)$.

3. The monomial m is called a **true gap** of σ if it is a gap of σ and if $\sigma \cup m$ has no new bridges dominated by m. The last condition is equivalent to the following: if m' is a bridge of $\sigma \cup m$ such that $m \succ_I m'$, then m' is a bridge of σ.

In [10], the first two authors characterized type-1, potentially-type-2, and type-2 cells in terms of bridges and true gaps. We focus primarily on the characterization of potentially-type-2 cells as this is the type of cell that we encounter in our proofs. Recall that, if σ is potentially-type-2, then it follows from the definition of potentially-type-2 that σ has at least one bridge.

Theorem 8. *[10, Theorem 2.24 (b)] A cell σ is potentially-type-2 if and only if* $\min_{\succ_I}(\sigma)$ *is a bridge where*

$$\min_{\succ_I}(\sigma) = \min_{\succ_I}\{\textit{bridges of } \sigma,\ \textit{true gaps of } \sigma\}.$$

In this case, we have $\min_{\succ_I}(\sigma) = \mathrm{sb}(\sigma)$.

Based on the characterizations of type-1 and potentially-type-2 cells from [10], we have the following characterization of absolutely critical cells which is also the statement of [10, Corollary 2.28]:

Corollary 9. *A cell is absolutely critical if and only if it has no bridges and no true gaps.*

In [10], the first two authors introduced a pivotal class of ideals called "bridge-friendly" for analyzing the minimality of Barile-Macchia resolutions. As established in [10, Theorem 2.29], bridge-friendliness is a sufficient condition for an ideal to have a minimal Barile-Macchia resolution. We revisit the definition of this class of ideals using the concept of absolutely critical cells.

Definition 10. [10, Definition 2.27] A monomial ideal I is called **bridge-friendly** if, for some total order $\succ_I$ on $\mathcal{G}(I)$, every potentially-type-2 cell is of type-2 with respect to $(\succ_I)$. Equivalently, all A-critical cells of I are absolutely critical. Here, A represents the Barile-Macchia matching of I with respect to $(\succ_I)$.

Theorem 11. *[10, Theorem 2.29] If I is bridge-friendly, then R/I has a minimal Barile-Macchia resolution.*

In the next chapter, we study class of ideals that are bridge-friendly.

3 Minimal free resolutions of path ideals of paths

In this section, our primary goal is to investigate the bridge-friendliness and, consequently, the minimal Barile-Macchia resolutions of path ideals of paths.

Fix two integers p and n. Consider a path L on the vertices $\{x_1, \ldots, x_{n+p-1}\}$. Let $R = \Bbbk[x_1, \ldots, x_N]$ with $N = n + p - 1$. The ***p*-path ideal** of L, denoted as $I_p(L_N)$, is generated by monomials in R corresponding to paths on p vertices along L. Explicitly, we have:

$$I_p(L_N) = (x_1x_2\cdots x_p,\ x_2x_3\cdots x_{p+1},\ \cdots,\ x_nx_{n+1}\cdots x_{n+p-1}).$$

Remark 12. Path ideals can be viewed as an extension of edge ideals of graphs. Specifically, the 2-path ideal of a graph coincides with its edge ideal.

The set of minimal generators of the p-path ideal of L is $\{m_1, m_2, \ldots, m_n\}$ and we denote this set by $\mathcal{G}$. So,

$$\mathcal{G} = \{m_1, m_2, \ldots, m_n\}$$

where $m_i := x_ix_{i+1}\cdots x_{i+p-1}$ for each $1 \leqslant i \leqslant n$. Fix a total order $(\succ)$ on $\mathcal{G}$ such that

$$m_1 \succ m_2 \succ \cdots \succ m_n.$$

Throughout the rest of this chapter, our focus is on the monomial ideal $I_p(L_N)$. In particular, we examine its Barile-Macchia matching and resolution with respect to the aforementioned total order. For ease of readability, we introduce the following notation, which is consistently employed throughout the paper.

Definition 13. Fix a variable x_i where $1 \leqslant i \leqslant N$. Let M_i denote the set of all monomials in $\mathcal{G}$ that are divisible by x_i where

$$M_i = \{m_j : \ \max\{1, i-p+1\} \leqslant j \leqslant \min\{i, n\}\}.$$

This means

$$M_i = \begin{cases} \{m_1, \dots, m_i\} & \text{if } i < p, \\ \{m_{i-p+1}, \dots, m_{i-1}, m_i\} & \text{if } p \leqslant i \leqslant n, \\ \{m_{i-p+1}, \dots, m_n\} & \text{if } i > n. \end{cases}$$

For a monomial $m \in R$, we denote its support by $\operatorname{supp}(m)$. Recall that this is the set of all variables dividing m. For instance, we have $\operatorname{supp}(m_i) = \{x_i, \dots, x_{i+p-1}\}$ for $m_i \in \mathcal{G}$.

We begin our analysis with the following lemma, which serves as a foundational tool for the classification of bridges, gaps, and true gaps of a cell.

Lemma 14. *Let σ be a cell of $I_p(L_N)$ such that $m_i \notin \sigma$ for some $m_i \in \mathcal{G}$. Assume that* $\operatorname{lcm}(\sigma)$ *is divisible by m_i. Then there exist monomials $m_j \in \sigma \cap M_i$ and $m_k \in \sigma \cap M_{i+p-1}$ with $1 \leqslant j < i < k \leqslant n$ such that $k - j \leqslant p$.*

Proof. First note that $1 < i < n$ since $\operatorname{lcm}(\sigma)$ is divisible by m_i but $m_i \notin \sigma$. Consider the set of all monomials in $\mathcal{G}$ that are divisible by one of the variables in $\operatorname{supp}(m_i)$. This set can be expressed as the union $M_i \cup M_{i+p-1}$ where

$$M_i = \{m_q, \dots, m_i\} \text{ and } M_{i+p-1} = \{m_i, \dots, m_\ell\}$$

with $q = \max\{1, i-p+1\}$ and $\ell = \min\{i+p-1, n\}$ as in Definition 13. Then, for any $m_s \in M_i$ and $m_t \in M_{i+p-1}$, we have $s \leqslant i \leqslant t \leqslant i+p-1$. Lastly, note that if $m_s \in M_i$, then $m_{s+p} \notin M_i$ which implies that $i < s+p$.

Since m_i divides $\operatorname{lcm}(\sigma)$ but $m_i \notin \sigma$, there exist monomials $m_s \in \sigma \cap M_i$ and $m_t \in \sigma \cap M_{i+p-1}$ with $s < i < t$. Let j be the largest index of a monomial in $\sigma \cap M_i$ and k be the smallest index of a monomial in $\sigma \cap M_{i+p-1}$. In other words, pick the monomials m_j and m_k in σ that are closer to m_i from either direction. Observe that m_i divides $\operatorname{lcm}(m_j, m_k)$. Otherwise, there exists $m_s \in \sigma \cap M_i$ with $j < s$ or $m_t \in \sigma \cap M_{i+p-1}$ with $t < k$ which contradicts to either maximality of j or minimality of k.

Lastly, we show $k - j \leqslant p$. If $k - j > p$, then x_{j+p} does not divide $\operatorname{lcm}(m_j, m_k)$. However, x_{j+p} divides m_i since $i+1 \leqslant j+p < k \leqslant i+p-1$ from the first paragraph. This leads to a contradiction as m_i divides $\operatorname{lcm}(m_j, m_k)$. Thus, we must have $k - j \leqslant p$. $\square$

In the following result, we provide a characterization of bridges and gaps of a cell. It is important to note that neither m_1 nor m_n can be a gap or a bridge of any cell.

Proposition 15. *Let σ be a cell of $I_p(L_N)$. For a monomial $m_i \in \mathcal{G}$ with $1 < i < n$, the following statements hold:*

1. *the monomial m_i is a bridge of σ if and only if $m_i \in \sigma$ and there exist monomials $m_j \in \sigma \cap M_i$ and $m_k \in \sigma \cap M_{i+p-1}$ such that $j < i < k \leqslant n$ and $k - j \leqslant p$.*

2. *the monomial m_i is a gap of σ if and only if $m_i \notin \sigma$ and there exist monomials $m_j \in \sigma \cap M_i$ and $m_k \in \sigma \cap M_{i+p-1}$ such that $j < i < k \leqslant n$ and $k - j \leqslant p$.*

Proof. If m_i is a bridge of σ, then application of Lemma 14 to $\sigma \setminus m_i$ results with the desired conditions. For the other direction, if the conditions are met, then m_i divides $\operatorname{lcm}(m_j, m_k)$. Since $\operatorname{lcm}(m_j, m_k)$ divides $\operatorname{lcm}(\sigma)$, the monomial m_i must be a bridge of σ. Characterization of gaps in part (2) follows similarly. □

Remark 16. When m_i is a bridge or a gap of a cell σ, there can be several (m_j, m_k) pairs of m_i where m_j and m_k are as in the statement of Proposition 15. However, there is only one (m_j, m_k) pair if m_j and m_k are chosen to be the closest to m_i as in the proof of Lemma 14.

Next, we present a characterization of true gaps in terms of these (m_j, m_k) pairs. In particular, we show that $m_k = m_{k'}$ for any two pairs (m_j, m_k) and $(m_{j'}, m_{k'})$ of a true gap as in Proposition 15. So, there is a unique such m_k for true gaps.

Proposition 17. *Let σ be a cell of $I_p(L_N)$. Consider a monomial $m_i \in \mathcal{G}$ for $1 < i < n$ such that m_i does not dominate any bridges of σ. Then m_i is a true gap of σ if and only if the following statements hold:*

(a) The monomial m_i is a gap of σ.

(b) There is only one monomial in $\sigma \cap M_{i+p-1}$.

(c) If $i + p \leqslant n$, then $m_{i+p} \notin \sigma$.

Proof. Recall the assumption that m_i does not dominate any bridges of σ. In addition, recall the following result from [10, Proposition 2.21] which will be useful in the proof: m_i is a true gap of σ that does not dominate any bridges of σ if and only if m_i is a gap of σ and $\operatorname{sb}(\sigma \cup m_i) = m_i$.

For the forward direction, suppose m_i is a true gap of σ. It then follows from [10, Proposition 2.21] that $\operatorname{sb}(\sigma \cup m_i) = m_i$.

(a) By the definition of a true gap, m_i is a gap of σ.

(b) It follows from part (a) and Proposition 15 (2) that there exists a monomial $m_k \in \sigma \cap M_{i+p-1}$ for $i < k$. If there exists another monomial $m_t \in \sigma \cap M_{i+p-1}$, then $i < k, t \leqslant i + p - 1$ where the last inequality is due to Definition 13. We may assume $k < t$ (otherwise, switch k and t in the following arguments). This means $\operatorname{supp}(m_k) \subseteq \operatorname{supp}(m_i) \cup \operatorname{supp}(m_t)$ which is equivalent to m_k divides $\operatorname{lcm}(m_i, m_t)$. Since $\operatorname{lcm}(m_i, m_t)$ divides $\operatorname{lcm}(\sigma \cup m_i)$, the monomial m_k is a bridge of $\sigma \cup m_i$ where $m_i \succ m_k$. This contradicts $\operatorname{sb}(\sigma \cup m_i) = m_i$. So, we have $\sigma \cap M_{i+p-1} = \{m_k\}$.

(c) Suppose $i + p \leqslant n$. Let m_k be the monomial from part (b). If $m_{i+p} \in \sigma$, then $\mathrm{supp}(m_k) \subseteq \mathrm{supp}(m_i) \cup \mathrm{supp}(m_{i+p})$ since $i < k < i + p$. This implies that m_k is a bridge of $\sigma \cup m_i$ where $m_i \succ m_k$, a contradiction as $\mathrm{sb}(\sigma \cup m_i) = m_i$. Hence, $m_{i+p} \notin \sigma$.

For the other direction, assume (a), (b), and (c) hold. For contradiction, suppose that m_i is not a true gap of σ. Let $m_t := \mathrm{sb}(\sigma \cup m_i)$. Then, by [10, Proposition 2.21], we have $m_i \succ m_t$ which implies that $i < t$ and $m_t \in \sigma$. Although m_t is a bridge of $\sigma \cup m_i$, notice that m_t cannot be a bridge of σ since m_i does not dominate any bridges of σ by the initial assumption.

Since m_t is a bridge of $\sigma \cup m_i$, by Proposition 15 (1), there are monomials $m_{j_t}, m_{k_t} \in \sigma \cup m_i$ such that $j_t < t < k_t \leqslant n$ and $k_t - j_t \leqslant p$. Since m_t is not a bridge of σ, either $j_t = i$ or $k_t = i$. Given $i < t$, it follows that $j_t = i$. Consequently, we have $i < t < k_t \leqslant i + p$ which implies that $m_t \in \sigma \cap M_{i+p-1}$. Since $m_{k_t} \in \sigma$, it follows from (b) that $m_{k_t} \notin \sigma \cap M_{i+p-1}$, meaning that $k_t \geqslant i + p$. This means $k_t = i + p \leqslant n$, which in turn implies $m_{k_t} = m_{i+p} \in \sigma$, contradicting (c). Therefore, m_i must indeed be a true gap of σ. $\square$

Example 18. Consider the 3-path ideal of an 8-path,

$$I = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6, x_5x_6x_7, x_6x_7x_8).$$

Consider the subset $\sigma = \{m_1, m_4, m_6\}$. It has no bridges, and its gaps are m_2, m_3, and m_5. Using Proposition 17, we identify which among these gaps are true gaps.

- The monomial m_2 is a true gap of σ. This is confirmed by: (a) the observation that $m_2 \notin \sigma$ and $m_1, m_4 \in \sigma$, with the difference in their indices satisfying $4 - 1 \leqslant 3$; (b) within the set $M_4 = \{m_2, m_3, m_4\}$, only m_4 belongs to σ; and (c) the monomial $m_{2+3} = m_5$ is not in σ.
- The monomial m_3 is not a true gap of σ. This is due to the failure of part (c) of Proposition 17 (3), given that $m_6 \in \sigma$.
- The monomial m_5 is a true gap of σ based on: (a) $m_5 \notin \sigma$ and both m_4 and m_6 are in σ, satisfying $6 - 4 \leqslant 3$; (b) from the set $M_7 = \{m_5, m_6\}$, only m_6 is in σ. Furthermore, (c) is not applicable since $8 > 6$, and thus, $m_{5+3} = m_8$ does not exist.

In what follows, we show that path ideals of paths are bridge-friendly.

Theorem 19. *The path ideal $I_p(L_N)$ is bridge-friendly, and its Barile-Macchia resolution is minimal.*

Proof. It suffices to show that $I_p(L_N)$ is bridge-friendly by Theorem 11. Recall from [10, Lemma 2.33] that $I_p(L_N)$ is bridge-friendly if and only if, for any potentially-type-2 cell σ (should it exist), there is no monomial $m \in \mathcal{G}$ such that m is a true gap of $\sigma \setminus \mathrm{sb}(\sigma)$ and $\mathrm{sb}(\sigma) \succ m$.

If $I_p(L_N)$ has no potentially-type-2 cells, then, by definition of bridge-friendly, there is nothing to prove. So, we may assume $I_p(L_N)$ has a potentially-type-2 cell, say σ. By definition, $\mathrm{sb}(\sigma)$ exists. Since $\mathrm{sb}(\sigma) \neq m_n$ as m_n cannot be a bridge of σ, there exists a monomial $m_i \in \mathcal{G}$ satisfying $\mathrm{sb}(\sigma) \succ m_i$. It is possible to have $m_i = m_n$.

Our goal is to show that m_i cannot be a true gap of $\sigma \setminus \mathrm{sb}(\sigma)$. We prove this by contradiction. If m_i is a true gap of $\sigma \setminus \mathrm{sb}(\sigma)$, we verify in the next paragraph that m_i is a true gap of σ. Then, since $\mathrm{sb}(\sigma) \succ m_i$, σ has a bridge dominating a true gap, contradicting Theorem 8 (b) as σ is potentially-type-2. Thus, no such m_i exists. Therefore, the path ideal $I_p(L_N)$ is bridge-friendly.

To complete the proof, suppose m_i is a true gap of $\sigma \setminus \mathrm{sb}(\sigma)$. Since $\mathrm{sb}(\sigma) \succ m_i$, the monomial m_i does not dominate any bridges of σ. Thus, we can apply Proposition 17 to $\sigma \setminus \mathrm{sb}(\sigma)$ and conclude that

(a) m_i is a gap of σ;

(b) there is only one monomial m_k in $\sigma \cap M_{i+p-1}$ since $\mathrm{sb}(\sigma) \succ m_i \succ m_k$;

(c) if $i + p \leqslant n$, then $m_{i+p} \notin \sigma \setminus \mathrm{sb}(\sigma) \iff m_{i+p} \notin \sigma$ since $m_{i+p} \neq \mathrm{sb}(\sigma)$. This is because $m_i \succ m_{i+p} = \mathrm{sb}(\sigma)$ contradicts to $\mathrm{sb}(\sigma) \succ m_i$.

Hence, m_i is a true gap of σ by Proposition 17. □

In the subsequent discussion, our goal is to demonstrate that for every multidegree m, i.e., a monomial in R, there exists at most one critical cell σ such that $\mathrm{lcm}(\sigma) = m$. Establishing the uniqueness of this critical cell for each multidegree allows us to compute both the projective dimension and Betti numbers of the path ideal via its Barile-Macchia resolution. To pave the way for this claim, we first introduce several auxiliary lemmas.

Recall from Corollary 9 that the critical cells of $I_p(L_N)$ have no bridges and no true gaps, a consequence of being bridge-friendly.

Lemma 20. *Let σ be a critical cell of $I_p(L_N)$. Let $a, b \in \mathbb{N}$ such that $p < a \leqslant b - p - 1$ and*

$$M = M_a \cup \cdots M_{b-p-1} = \{m_{a-p+1}, m_{a-p+2}, \dots, m_{b-p-1}\}.$$

Assume that $\sigma \cap M = \{m_{a-p+1}\}$. Then the following statements hold:

1. $\sigma \cap \{m_{a-2p+1}, \dots, m_{a-p-1}\} = \emptyset$.

2. $\mathrm{lcm}(\sigma)$ *is divisible by x_{a-p} if and only if σ contains m_{a-p}.*

Proof. (1) Suppose $\sigma \cap \{m_{a-2p+1}, \dots, m_{a-p-1}\} \neq \emptyset$. Under this assumption, we show that m_{a-p} is either a bridge or a true gap of σ which leads to a contradiction by Corollary 9 since σ is a critical cell of a bridge-friendly ideal.

Since we assumed (1) fails, there exists a monomial m_j in $\sigma \cap \{m_{a-2p+1}, \dots, m_{a-p-1}\}$. Notice that $m_j \in \sigma \cap M_{a-p}$ and $m_{a-p+1} \in \sigma \cap M_{a-1}$ where $j < a - p < a - p + 1$ with $(a - p + 1) - j \leqslant p$. By setting $i = a - p$ and $k = a - p + 1$, it follows from Proposition 15 that the monomial m_{a-p} is either a bridge or a gap of σ.

If m_{a-p} is a bridge of σ, we are done. Now, assume m_{a-p} is a gap of σ. Since σ is a critical cell, it has no bridges which means m_{a-p} does not dominate any bridges of σ. This allows us to use Proposition 17 to conclude that m_{a-p} is a true gap of σ, completing the proof of (1). The following is a verification of Proposition 17 (a)-(c):

(a) m_{a-p} is a gap of σ by our assumption.

(b) Observe that $M_{a-1} \setminus \{m_{a-p}\} \subseteq M$ where $M_{a-1} = \{m_{a-p}, m_{a-p+1}, \ldots, m_{a-1}\}$. Since $m_{a-p} \notin \sigma$ and $\sigma \cap M = \{m_{a-p+1}\}$, we have $\sigma \cap M_{a-1} = \{m_{a-p+1}\}$.

(c) Since $a \leqslant n$, we need to show $m_a \notin \sigma$. This follows from $m_a \in M$ and $\sigma \cap M = \{m_{a-p+1}\}$.

(2) It is immediate that $m_{a-p} \in \sigma$ implies x_{a-p} divides $\mathrm{lcm}(\sigma)$. On the other hand, if $\mathrm{lcm}(\sigma)$ is divisible by x_{a-p}, then $\sigma \cap M_{a-p} \neq \emptyset$. This means $\sigma \cap M_{a-p} = \{m_{a-p}\}$ by part (1). Thus, we have $m_{a-p} \in \sigma$. □

Lemma 21. *For a critical cell σ of $I_p(L_N)$, let $b \in \mathbb{N} \cup \{\infty\}$ such that $\{i : x_i \mid \mathrm{lcm}(\sigma)$ and $i \leqslant b-p-1\} \neq \emptyset$. Define b_1 as:*

$$b_1 := \max\{i : x_i \mid \mathrm{lcm}(\sigma) \text{ and } i \leqslant b-p-1\}.$$

Assume $\sigma \cap \{m_{b-2p+1}, \ldots, m_{b-p-1}\} = \emptyset$ when $b \in \mathbb{N}$. Then $\sigma \cap M = \{m_{b_1-p+1}\}$ and $b_1 \geqslant p$ where

$$M = M_{b_1} \cup M_{b_1+1} \cup \cdots \cup M_{b-p-1} = \{m_{b_1-p+1}, m_{b_1-p+2}, \ldots, m_{b-p-1}\}.$$

Proof. We analyze the two possible cases for b separately.

Case 1: $b = \infty$. In this case, we have $M = M_{b_1} \cup M_{b_1+1} \cup \cdots M_n$. By the definition of b_1, it is clear that b_1 is finite. Given this, $\mathrm{lcm}(\sigma)$ is not divisible by x_k for any integer $k \geqslant b_1+1$. Thus, $\sigma \cap (M \setminus \{m_{b_1-p+1}\}) = \emptyset$. Moreover, as x_{b_1} divides $\mathrm{lcm}(\sigma)$, we have $\sigma \cap M_{b_1} \neq \emptyset$ which guarantees that $\sigma \cap M = \{m_{b_1-p+1}\}$ and $b_1 \geqslant p$.

Case 2: $b \in \mathbb{N}$. It follows from the definition that $b_1 \leqslant b-p-1$. If $b_1 < b-p-1$, then $\mathrm{lcm}(\sigma)$ is divisible by x_{b_1} but it is not divisible by any of the variables among $\{x_{b_1+1}, \ldots, x_{b-p-1}\}$. This means $\sigma \cap M_{b_1} \neq \emptyset$ but $\sigma \cap (M \setminus M_{b_1}) = \emptyset$. Thus, $\sigma \cap M = \{m_{b_1-p+1}\}$ and $b_1 \geqslant p$.

If $b_1 = b-p-1$, then the assumption $\sigma \cap \{m_{b-2p+1}, \ldots, m_{b-p-1}\} = \emptyset$ becomes $\sigma \cap (M_{b_1} \setminus \{m_{b_1-p+1}\}) = \emptyset$ where $M = M_{b_1}$. Since x_{b_1} divides $\mathrm{lcm}(\sigma)$, we have $\sigma \cap M_{b_1} \neq \emptyset$. Hence, m_{b_1-p+1} is the only monomial in $\sigma \cap M$. □

Building upon the preceding lemmas, we introduce a sequence of results that are key to understanding the monomials $m \in \mathcal{G}$ that are in a given critical subset σ. We first introduce a new terminology and a few immediate observations that will be used in the next few results.

Definition 22. For a critical cell σ of $I_p(L_N)$, define a sequence $(b_0, b_1, b_2, \ldots)$, called the **σ-sequence**, by setting $b_0 = \infty$ and, for $j \geqslant 1$,

$$b_j := \max\{i : \ x_i \mid \mathrm{lcm}(\sigma) \text{ and } i \leqslant b_{j-1} - p - 1\}.$$

If no such x_i exists, set $b_j = -\infty$.

Example 23. Consider the 3-path ideal of an 8-path where $n = 6$ and $p = 3$:

$$I = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6, x_5x_6x_7, x_6x_7x_8).$$

and the cell $\sigma = \{m_1, m_2, m_5, m_6\}$. One can use Proposition 15 and Proposition 17 to show σ is a critical cell of I. Computation of the σ-sequence $(b_1, b_2, \ldots)$ relies on $\mathrm{lcm}(\sigma) = x_1x_2 \cdots x_8$. By the set-up, we have $b_0 = \infty$ and

$$\begin{aligned} b_1 &= \max\{i : \ x_i \mid \mathrm{lcm}(\sigma)\} = 8 \\ b_2 &= \max\{i : \ x_i \mid \mathrm{lcm}(\sigma) \text{ and } i \leqslant b_1 - 4 = 4\} = 4 \end{aligned}$$

Since $\{i : \ x_i \mid \mathrm{lcm}(\sigma) \text{ and } i \leqslant b_2 - 4 = 0\} = \emptyset$, we have $b_3 = -\infty$.

As the above example indicates, each σ-sequence is finite. We discuss this more in detail in the following observation.

Observation 24. *Let σ be a critical cell of $I_p(L_N)$.*

(a) Observe that the σ-sequence $(b_0, b_1, b_2, \ldots)$ is finite. To see this, first note that b_1 is the largest index of a variable in the support of $\mathrm{lcm}(\sigma)$*. Next, notice that the sequence $(b_0, b_1, b_2, \ldots)$ strictly decreases after b_1 since $b_j \leqslant b_{j-1} - p - 1$ for $j \geqslant 2$. Since* $\mathrm{supp}(\mathrm{lcm}(\sigma))$ *is finite, the σ-sequence eventually reaches $-\infty$ after finitely many steps, i.e., there exists an $\ell \geqslant 1$ for which*

$$\{i : \ x_i \mid \mathrm{lcm}(\sigma) \textit{ and } i \leqslant b_{\ell-1} - p - 1\} \neq \emptyset,$$

$$\{i : \ x_i \mid \mathrm{lcm}(\sigma) \textit{ and } i \leqslant b_\ell - p - 1\} = \emptyset.$$

This means b_ℓ is finite and $b_{\ell+1} = -\infty$.

From now on, we write the σ-sequence as a finite sequence $(b_0, b_1, \ldots, b_\ell)$ where each $b_i \in \mathbb{N}$ is non-zero.

(b) Since $b_j \leqslant b_{j-1} - p - 1$ for each $1 \leqslant j \leqslant \ell$, we have

$$(\ell - j)p + (\ell - j + 1) \leqslant b_j < \cdots < b_1 \leqslant n + p - 1.$$

Now, we can determine which monomials in $\mathcal{G}$ are part of a critical cell σ, using the σ-sequence as our key tool.

Proposition 25. *Let σ be a critical cell of $I_p(L_N)$ with its σ-sequence $(b_0, b_1, \ldots, b_\ell)$. Let*

$$\begin{aligned} M_{b_j,b_{j-1}} &:= M_{b_j} \cup M_{b_j+1} \cup \cdots \cup M_{b_{j-1}-p-1} \\ &= \{m_{b_j-p+1}, m_{b_j-p+2}, \ldots, m_{b_{j-1}-p-1}\}. \end{aligned}$$

Then, we have $\sigma \cap M_{b_j,b_{j-1}} = \{m_{b_j-p+1}\}$ when $1 \leqslant j \leqslant \ell$. Moreover, $m_k \notin \sigma$ for $k \leqslant b_\ell - p - 1$.

Proof. We use induction on j where $1 \leqslant j \leqslant \ell$. The base case $j = 1$ is covered in Lemma 21.

For a fixed $j < \ell$, suppose the statement holds for each $k \in \{1, \ldots, j\}$, that is, $\sigma \cap M_{b_k,b_{k-1}} = \{m_{b_k-p+1}\}$. Since $\sigma \cap M_{b_j,b_{j-1}} = \{m_{b_j-p+1}\}$ by the induction hypothesis and $p < b_j \leqslant b_{j-1} - p - 1$, we can apply Lemma 20 to obtain

$$\sigma \cap \{m_{b_j-2p+1}, \ldots, m_{b_j-p-1}\} = \emptyset.$$

The quality above and the fact that $\{i : \ x_i \mid \operatorname{lcm}(\sigma) \text{ and } i \leqslant b_j - p - 1\} \neq \emptyset$ allow us to utilize Lemma 21. Then, we conclude that $\sigma \cap M_{b_{j+1},b_j} = \{m_{b_{j+1}-p+1}\}$, as desired.

For the final part, note that $\operatorname{lcm}(\sigma)$ is not divisible by any variable x_k with $k \leqslant b_\ell - p - 1$. This implies that $\sigma \cap M_k = \emptyset$ for any such k. Hence, $m_k \notin \sigma$ for $k \leqslant b_\ell - p - 1$. □

Proposition 26. *If $\operatorname{lcm}(\sigma) = \operatorname{lcm}(\sigma')$ for two critical cells σ and σ' of $I_p(L_N)$, then $\sigma = \sigma'$.*

Proof. Let σ and σ' be two critical cells of $I_p(L_N)$ where $(b_0, b_1, \ldots, b_\ell)$ is the σ-sequence and $(b'_0, b'_1, \ldots, b'_k)$ be the σ'-sequence. Recall that the value of each b_i and b'_j are determined solely by the least common multiples. Since $\operatorname{lcm}(\sigma) = \operatorname{lcm}(\sigma')$, we must have $(b_0, b_1, \ldots, b_\ell) = (b'_0, b'_1, \ldots, b'_k)$.

Next, notice that $\mathcal{G}$ can be written as the disjoint union of the following three sets:

$$\mathcal{G} = \{m_1, \ldots, m_{b_\ell-p}\} \sqcup \left(\bigcup_{j=1}^{\ell} M_{b_j,b_{j-1}} \right) \sqcup \left(\bigcup_{j=1}^{\ell} \{m_{b_j-p}\} \right).$$

It follows from Proposition 25 that none of the monomials in the first set are contained in σ or σ'. Moreover, the only monomials in the second set that appear in both σ and σ' are those of the form m_{b_j-p+1} for each $1 \leqslant j \leqslant \ell$. For the last set, we apply Lemma 20 (2) and conclude that m_{b_j-p} is contained in the critical cell τ if and only if x_{b_j-p} divides $\operatorname{lcm}(\tau)$, where $\tau \in \{\sigma, \sigma'\}$ for $1 \leqslant j \leqslant \ell$. Therefore, $\sigma = \sigma'$. □

From the preceding proposition, we deduce that the least common multiples of distinct critical cells of $I_p(L_N)$ are different. This particularly implies the following information on its multi-graded Betti numbers.

Corollary 27. *For any monomial m and any integer i, we have*

$$\beta_{i,m}(R/I_p(L_N)) = \begin{cases} 1 & \textit{if there is a critical subset of size } i \textit{ with } \operatorname{lcm} = m, \\ 0 & \textit{otherwise.} \end{cases}$$

From the characterization of critical cells in Proposition 25 and the insights from the proof of Proposition 26, we can deduce the projective dimension of $I_p(L_{n+p-1})$.

Corollary 28. *Let n be expressed as $n = (p+1)q + s$ where $0 \leqslant s \leqslant p$. The projective dimension of $R/I_p(L_{n+p-1})$ is given by:*

$$\operatorname{pdim}(R/I_p(L_{n+p-1})) = \begin{cases} 2q & \text{if } s = 0, \\ 2q+1 & \text{if } s = 1, \\ 2q+2 & \text{otherwise.} \end{cases}$$

Proof. Among every collection of $p+1$ consecutive monomials, at most two can be in a critical cell by Proposition 25. Namely, among the monomials $m_{b_j-p}, m_{b_j-p+1}, \ldots, m_{b_j}$ for $1 \leqslant j \leqslant \ell$, we can have at most $m_{b_j-p}, m_{b_j-p+1} \in \sigma$. This allows us to construct cells of maximal cardinality as follows:

- when $s = 0$, set $\sigma_1 = \{m_p, m_{p+1}, \ldots, m_{n-p-2}, m_{n-p-1}, m_{n-1}, m_n\}$ where $|\sigma_1| = 2q$,
- when $s = 1$, set $\sigma_2 = \{m_1, m_{p+1}, m_{p+2}, \ldots, m_{n-p-2}, m_{n-p-1}, m_{n-1}, m_n\}$ where $|\sigma_2| = 2q+1$,
- when $2 \leqslant s \leqslant p$, set $\sigma_3 = \{m_{s-1}, m_s, m_{p+s}, m_{p+s+1}, \ldots, m_{n-p-2}, m_{n-p-1}, m_{n-1}, m_n\}$ where $|\sigma_3| = 2q+2$.

One can verify that each of these cells σ_1, σ_2 and σ_3 are absolutely critical as neither of them have bridges or true gaps by Proposition 15 and Proposition 17. Consequently, we can derive the maximal cardinality of a critical cell for each case, thus obtaining the projective dimension. $\square$

A formula for the projective dimension of the path ideal of a path was also given in [2]. The formula from [2] matches ours. However, while we express the projective dimension based on the number of minimal generators of the p-path ideal, [2] does so using the length of the path.

We also recover the recursive formula for graded Betti numbers from [8]. This formula was utilized in [2] to provide explicit calculations for the projective dimension and regularity of path ideals of paths and cycles.

Theorem 29. *For all indices r, d, we have*

$$\begin{aligned}\beta_{r,d}\big(R/I_p(L_n)\big) = \beta_{r,d}\big(R/I_p(L_{n-1})\big) &+ \beta_{r-1,\,d-p}\big(R/I_p(L_{n-(p+1)})\big) \\ &+ \beta_{r-2,\,d-(p+1)}\big(R/I_p(L_{n-(p+1)})\big).\end{aligned}$$

Proof. By Corollary 27, $\beta_{r,d}(R/I_p(L_n))$ counts the critical cells of cardinality r and degree d. To derive our desired expression, it suffices to partition the set of critical subsets of cardinality r and degree d in an appropriate way. Recall that $\mathcal{G}(I_p(L_n)) = \{m_1, \ldots, m_{n-p+1}\}$ and neither m_1 nor m_{n-p+1} can be a bridge or a true gap of any cell of $I_p(L_n)$. Consider a cell σ of $I_p(L_n)$. The following three scenarios for σ complete the proof:

Case 1: Suppose $m_{n-p+1} \notin \sigma$. In this case, σ is a cell of both $I_p(L_n)$ and $I_p(L_{n-1})$. Our goal is to show the following: σ is a critical cell of $I_p(L_n)$ if and only if it is a critical cell of $I_p(L_{n-1})$.

Notice that any critical cell of $I_p(L_n)$ is automatically a critical cell of $I_p(L_{n-1})$. Conversely, any critical cell of $I_p(L_{n-1})$ must also be a critical cell of $I_p(L_n)$, since m_{n-p+1} cannot be a true gap–indeed, it cannot be a gap at all. This completes the proof.

Case 2: Suppose $m_{n-p+1} \in \sigma$ but $m_{n-p} \notin \sigma$. Our goal is to show the following: σ is a critical cell of $I_p(L_n)$ if and only if $\sigma \backslash \{m_{n-p+1}\}$ is a critical cell of $I_p(L_{n-(p+1)})$. Recall that $\mathcal{G}(I_p(L_{n-(p+1)})) = \{m_1, \ldots, m_{n-2p}\}$ and neither m_1 nor m_{n-2p} can be a bridge or true gap of any cell of $I_p(L_{n-(p+1)})$.

Suppose σ is a critical cell of $I_p(L_n)$. Let $\tau = \sigma \backslash \{m_{n-p+1}\}$. We first show that τ is a cell of $I_p(L_{n-(p+1)})$, i.e. there exists no $m_j \in \sigma$ for $n-2p < j < n-p$. If there is at least one such $m_j \in \sigma$, then m_{n-p} is a true gap of σ by Proposition 17 as we explain in the following steps: (a) $m_j, m_{n-p+1} \in \sigma$ and $(n-p+1)-j \leqslant p$; (b) $\sigma \cap M_{n-1} = \{m_{n-p+1}\}$; (c) $n > n-p+1$. Since σ is critical, it cannot have a true gap. Thus, $m_j \notin \sigma$.

Next, we show that τ is critical. Note that τ has no bridges; otherwise, σ would have a bridge, which is impossible since σ is critical. If τ has a true gap, say m_i, then m_i cannot dominate any bridges of τ since it has none. Then, we apply Proposition 17 to m_i and τ and conclude that m_i is a true gap of σ, a contradiction. Thus, τ is indeed critical. We verify Proposition 17 (a)-(c) for m_i and σ below:

(a) there exists $m_j \in \tau \cap M_i \subset \sigma \cap M_i$ and $m_k \in \tau \cap M_{i+p-1} \subset \sigma \cap M_{i+p-1}$ such that $j < i < k \leqslant n-2p$ and $k-j \leqslant p$. So, Proposition 17 (a) holds for m_i and σ.

(b) $\tau \cap M_{i+p-1} = \{m_k\} = \sigma \cap M_{i+p-1}$ since $k \leqslant i+p-1 < n-p-1$. This means Proposition 17 (b) holds for m_i and σ.

(c) Since $i < n-2p$, we have $i+p < n-p+1$. We need $m_{i+p} \notin \sigma$ to verify Proposition 17 (c) for m_i and σ. It suffices to show $m_{i+p} \notin \tau$ since $i+p < n-p+1$. If $i+p \leqslant n-2p$, then $m_{i+p} \notin \tau$ Proposition 17 (c) since m_i is a true gap of τ. If $i+p > n-2p$, then $m_{i+p} \notin \tau$ since τ is a cell of $I_p(L_{n-(p+1)})$. So, $m_{i+p} \notin \tau$.

Now, suppose τ is a critical cell of $I_p(L_{n-(p+1)})$. Let $\delta := \tau \cup \{m_{n-p+1}\}$ and observe that δ is a cell of $I_p(L_n)$. Note that δ has no bridges. If it has a bridge m_i, then $m_i \neq m_{n-p+1}$ and $m_i \in \tau$. Since $(n-p+1)-j \geqslant p+1$ for each $m_j \in \tau$ as $1 \leqslant j \leqslant n-2p$, m_i must be a bridge of τ, contradiction. If δ has a true gap, say m_i, then m_i does not dominate any bridges of δ. So, we can apply Proposition 17 to m_i and δ to conclude that m_i is a true gap of τ which leads to a contradiction. Thus, δ is critical. We verify Proposition 17 (a)-(c) for m_i and τ below:

(a) there exists $m_j \in \tau \cap M_i$ and $m_k \in \delta \cap M_{i+p-1}$ such that $j < i < k \leqslant i+p-1$ and $k-j \leqslant p$. Notice that $m_k \in \tau$ because if $k = n-p+1$, then $(n-p+1)-j \geqslant p+1$. So, $i < k \leqslant n-2p$ which means $m_i \in \tau$ and Proposition 17 (a) holds for m_i and τ.

(b) $\delta \cap M_{i+p-1} = \{m_k\} = \tau \cap M_{i+p-1}$ by part (b). So, Proposition 17 (b) holds for m_i and τ.

(c) If $i + p \leqslant n - 2p < n - p + 1$, then $m_{i+p} \notin \delta$ implies that $m_{i+p} \notin \tau$. This means Proposition 17 (c) holds for m_i and τ.

Case 3: Suppose both m_{n-p+1} and m_{n-p} are in σ. Our goal is to show the following: σ is a critical cell of $I_p(L_n)$ if and only if $\sigma \backslash \{m_{n-p+1}, m_{n-p}\}$ is a critical cell of $I_p(L_{n-(p+1)})$.

Its proof is almost identical to the proof of Case 2 and we only highlight the differences for the reader. If σ is a critical cell of $I_p(L_n)$, let $\tau := \sigma \backslash \{m_{n-p+1}\}$. As in Case 2, τ is a cell of $I_p(L_{n-(p+1)})$; otherwise, m_{n-p} is a bridge of of σ by Proposition 15, which is not possible. The rest follows similarly as in this part of Case 2 and this completes the proof that τ is a critical cell of $I_p(L_{n-(p+1)})$.

Let τ be a critical cell of $I_p(L_{n-(p+1)})$. Then $\delta := \tau \cup \{m_{n-p}, m_{n-p+1}\}$ is a cell of $I_p(L_n)$. We prove it is also a critical cell following the steps in the proof of Case 2. The only difference is in part (a) where we need to consider the possibility of $m_k = m_{n-p}$. We recommend the reader to follow along part (a) of Case 2 to keep track of indexes that are referenced here. First notice that $j < i \leqslant n - 2p + 1$ since $k \neq n - p + 1$. If $m_k = m_{n-p}$, then $k - j \geqslant p$ as $j \leqslant n - 2p$. Since $k - j \leqslant p$ by Proposition 17 (a), we must have $k - j = p$, indicating that $j = n - 2p$ and $i = n - 2p + 1$. Then $m_{i+p} = m_{n-p+1} \in \delta$ which contradicts Proposition 17 (c) as we assumed m_i is a true gap of δ. Thus, $m_k \neq m_{n-p}$ which means $m_k \in \tau$. The rest of the proofs follows similarly to that of Case 2. ☐

Barile-Macchia resolutions are cellular, i.e., they are supported on CW complexes. We first remark that the dimension of the CW complex that supports the minimal resolution of a monomial ideal equals its projective dimension. In general, the minimal resolution of any monomial ideal of projective dimension 1 is supported on a tree [16, Theorem 1]. In fact, in the cases where the path ideals of paths have projective dimension 1, their minimal resolutions are supported on paths, which can be shown using the techniques that will be employed in the next example. In what follows, we provide an example where the path ideal of a path has projective dimension 2.

Example 30. Consider the path ideal $I = I_p(L_{2p+1})$ under the total order

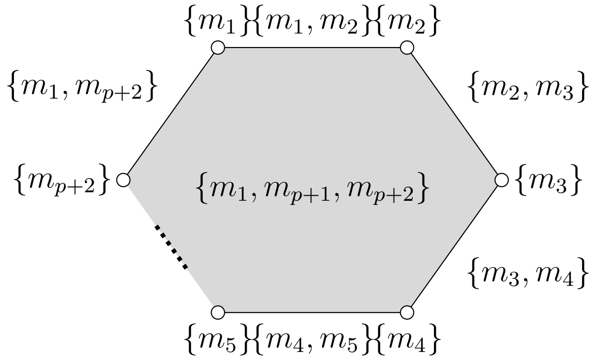
$$m_1 \succ m_2 \succ \cdots \succ m_{p+2}.$$

For any subset $\sigma = \{m_{i_1}, \ldots, m_{i_k}\}$, each element of σ (except m_{i_1} and m_{i_k}) is a bridge of σ due to Proposition 15 (1). Consequently, $m_{i_{k-1}}$ emerges as the smallest bridge of σ. Therefore, the Barile-Macchia matching of I with respect to $(\succ)$, denoted by A, is achieved by removing the penultimate element at every iteration. Note that there is only one cell (of cardinality of at least 3) with no bridges: $\{m_1, m_{p+1}, m_{p+2}\}$. This results in the following list of all critical cells of I:

$$\begin{gathered}\emptyset, \{m_1\}, \ldots, \{m_{p+2}\},\\ \{m_1, m_2\}, \{m_2, m_3\}, \ldots, \{m_{p+1}, m_{p+2}\}, \{m_{p+2}, m_1\},\\ \{m_1, m_{p+1}, m_{p+2}\}.\end{gathered}$$

Given two distinct critical cells, σ and σ', their least common multiples are different by Proposition 26. This uniqueness ensures that the Barile-Macchia resolution of I is minimal by Theorem 2.

By discrete Morse theory, there exists a cellular complex that supports this minimal free resolution, and it has $(p+2)$ many 0-cells, $(p+2)$ many 1-cells, and one 2-cell. One obvious object that fits this description is the following $(p+2)$-gon:



Denote this $(p+2)$-gon by Δ. It follows from [7, Lemma 2.2] that Δ supports a free resolution of R/I if, for each monomial m, the subcomplex $\Delta[m] := \{\sigma \in \Delta : \operatorname{lcm}(\sigma) \mid m\}$ is either empty or contractible. Notice that any subcomplex $\Delta[m]$ is either empty, a line segment, or the $(p+2)$-gon itself for each squarefree monomial m. Since the last two objects are both contractable, Δ supports a free resolution of R/I.

4 Minimal free resolutions of path ideals of cycles

In this section, we turn our attention to path ideals of cycles, demonstrating that these ideals have minimal cellular resolutions. While in the preceding section we derived this outcome for path ideals of paths through identifying a minimal Barile-Macchia resolution of R/I with respect to a specific total order on $\mathcal{G}(I)$, this approach falls short for path ideals of cycles. For instance, when we consider the edge ideal of a 9-cycle, it has no minimal Barile-Macchia resolutions as pointed out in [10, Remark 4.23].

In our investigation of path ideals of cycles, we transition our focus towards the generalized Barile-Macchia resolutions. These are Morse resolutions and can be considered as an extension of the Barile-Macchia resolutions, introduced in [10]. The crux of these resolutions lies in utilizing a collection of total orders on $\mathcal{G}(I)$, instead of one. For the reader's convenience, we restate the construction of generalized Barile-Macchia resolutions from [10] along with a theorem stating that they induce cellular free resolutions. First, recall that G_X denotes the directed graph obtained from the Taylor complex of I.

Theorem 31. *[10, Theorem 5.19] For a monomial $u \in R$, let G_u be the induced subgraph of G_X on the vertices $\sigma \subseteq \mathcal{G}(I)$ where $\operatorname{lcm}(\sigma) = u$. Consider a total order $(\succ_u)$ on $\mathcal{G}(I)$ for each monomial $u \in R$.*

Let A be the union of all A_u, where A_u is the collection of directed edges obtained by applying the Barile-Macchia Algorithm to G_u with respect to $(\succ_u)$ for each monomial $u \in R$. Then, A is a homogeneous acyclic matching of I. The Morse resolution induced by A is called the ***generalized Barile-Macchia*** *resolution of R/I with respect to $(\succ_u)_{u\in R}$.*

Consider a cycle C_n on n vertices $\{x_1, \dots, x_n\}$. Let $R = \Bbbk[x_1, \dots, x_n]$. The p-path ideal of C_n, denoted as $I_p(C_n)$, is generated by monomials in R corresponding to paths on p vertices along C_n where

$$I_p(C_n) = (x_1 \cdots x_p, \dots, x_n \cdots x_{p-1}).$$

Let $m_i = x_i \cdots x_{i+p-1}$ for each $1 \leqslant i \leqslant n$ and consider the indices modulo n. For the remainder of this section, we denote the minimal generating set of $I_p(C_n)$ by $\mathcal{G}$ where

$$\mathcal{G} = (m_1, \dots, m_n).$$

We restrict our attention to $2 \leqslant p \leqslant n$ and begin our discussion with an observation on the construction of the set A, as outlined in Theorem 31, leveraging our previous findings on path ideals of paths.

Observation 32. *Let A be a homogeneous acyclic matching constructed in accordance with Theorem 31 for the ideal $I_p(C_n)$. To obtain a generalized Barile-Macchia resolution of $R/I_p(C_n)$, it is necessary to determine its A-critical cells.*

1. *The following is immediate from the definition of A:*

$$\{A\text{-critical cells}\} = \bigcup_{u:\text{ monomial in } R} \{A_u\text{-critical cells}\}.$$

2. *Assuming $u \neq x_1 \cdots x_n$, remember that each vertex of G_X is a cell of $I_p(C_n)$, namely, a subset of $\mathcal{G}$. For an induced subgraph G_u of G_X, its vertex set $V(G_u)$ is empty or every vertex of G_u is a subset of $\mathcal{G}(J)$ where J is the path ideal of some path. When $V(G_u)$ is non-empty, Proposition 26 assures the existence of a total order $(\succ)$ on $\mathcal{G}(J)$ that allows for precisely one A-critical cell.*

In light of the above observation, our attention is on the A_u-critical cells where $u = x_1 \cdots x_n$ to derive a minimal generalized Barile-Macchia resolution of $R/I_p(C_n)$.

For the remainder of this section, let $u = x_1 \cdots x_n$. In addition, let A_u and G_u be as in Theorem 31 and adopt the following total order $(\succ)$ for u:

$$m_1 \succ m_2 \succ \cdots \succ m_n.$$

Our primary objective is to demonstrate that all A_u-critical cells have the same cardinality. Consequently, the resulting generalized Barile-Macchia resolution is minimal by Theorem 2 and Observation 32.

As a preliminary step, we examine the structure of $\sigma \in V(G_u)$. Recall that $\operatorname{lcm}(\sigma) = u$ for each vertex σ in G_u. For the remainder of this section, we express σ based on the total order $(\succ)$. In other words, when $\sigma = \{m_{i_1}, \dots, m_{i_t}\}$ we have $m_{i_1} \succ \cdots \succ m_{i_t}$; equivalently, $i_1 < \cdots < i_t$.

Proposition 33. *Let $\sigma = \{m_{i_1}, \ldots, m_{i_t}\}$ be a vertex of G_u. Then the distance between consecutive elements of σ is at most p.*

Proof. Suppose that either $i_{j+1} - i_j > p$ for some $j \in \{1, \ldots, t-1\}$ or $i_1 + n - i_t > p$. In the first case, we have $m_{i_j} = x_{i_j} \cdots x_{i_j+p-1}$ and $m_{i_{j+1}} = x_{i_{j+1}} \cdots x_{i_{j+1}+p-1}$ where $i_{j+1} > i_j + p$. This implies that x_{i_j+p} does not divide $\mathrm{lcm}(\sigma) = u = x_1 \cdots x_n$ which is not possible. The other case follows similarly by arguing x_{i_t+p} does not divide u, leading to contradiction. $\square$

Unlike paths in the previous section, every critical cell contains m_n for cycles.

Lemma 34. *The monomial m_n is contained in each A_u-critical cell.*

Proof. Let σ be an A_u-critical cell. If $m_n \notin \sigma$, then m_n is a gap of σ since $\mathrm{lcm}(\sigma) = \mathrm{lcm}(\sigma \cup m_n)$. Indeed, any $m \notin \sigma$ is a gap due to same reasoning. Notice that this means m_n is a bridge of $\sigma \cup m_n$ and we have $\mathrm{sb}(\sigma \cup m_n) = m_n$ since m_n is the smallest monomial among those in $\mathcal{G}$ with respect to $(\succ)$. It follows from Theorem 8 that $\sigma \cup m_n$ is a potentially-type-2 cell. In fact, this cell is type-2 since there is no cell τ such that $\tau \setminus \mathrm{sb}(\tau) = \sigma$ with $m_n \succ \mathrm{sb}(\tau)$. Then, the directed edge $(\sigma \cup m_n, \sigma)$ is contained in A_u which means σ cannot be a critical cell, a contradiction. Thus, $m_n \in \sigma$. $\square$

Definition 35. Similar to Definition 13 for paths, define M_i to be the collection of all monomials in $\mathcal{G}$ that are divisible by x_i for each $1 \leqslant i \leqslant n$. Note that $|M_i| = p$ for each value of i and

$$M_i = \begin{cases} \{m_{i-p+1}, \ldots, m_i\} & \text{if } i \geqslant p, \\ \{m_{(n+i)-p+1}, \ldots, m_n, m_1, \ldots, m_i\} & \text{if } 1 \leqslant i < p. \end{cases}$$

In the subsequent discussion, we analyze the vertices of G_u and describe their bridges, gaps, and true gaps. Analogous to the path case, we can classify the bridges in a similar manner. The proof is omitted since it directly follows from Lemma 14, keeping in mind that indices are now considered modulo n. First, we consider gaps as their classification is immediate.

Proposition 36. *Let $\sigma \in V(G_u)$. A monomial m_i is a gap of σ if and only if $m_i \notin \sigma$.*

Proof. Since $\mathrm{lcm}(\sigma) = x_1 \cdots x_n$, adding m_i to σ does not change the least common multiple for any $m_i \notin \sigma$. So, any m_i that is not contained in σ is a gap. $\square$

Proposition 37. *Let $\sigma \in V(G_u)$. Then the monomial m_i is a bridge of σ if and only if $m_i \in \sigma$ and there exist monomials m_j and m_k in $\sigma \setminus m_i$ for $1 \leqslant j < k \leqslant n$ such that the distance between these two monomials along m_i is at most p, i.e., $k - j \leqslant p$ when $j < i < k$ and $(j + n) - k \leqslant p$ when $j < k < i < j + n$.*

Proof. If m_i is a bridge of σ, one can apply the steps in the proof of Lemma 14 to $\sigma \setminus m_i$ (while keeping in mind that indices are now modulo n) and obtain consecutive monomials m_j and m_k in $\sigma \setminus m_i$ where either $j < i < k$ or $k < j < i < k + n$ (as shown in Figure 1).

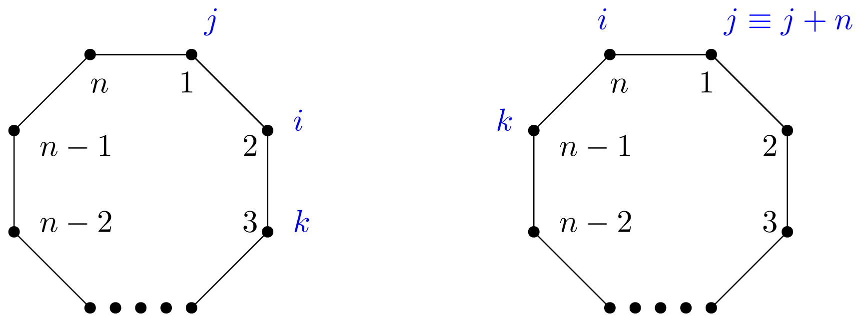


Figure 1: Only possible orderings of i, j, k along C_n: $j < i < k$ (left) and $j < k < i < j+n$ (right).

Since $\sigma \setminus m_i$ is still a vertex of G_u, distance between consecutive elements of $\sigma \setminus m_i$ is at most p by Proposition 33. For the other direction, if the conditions are met, $\sigma \setminus m_i$ is still a vertex of G_u and m_i divides $\operatorname{lcm}(m_j, m_k)$. Since $\operatorname{lcm}(m_j, m_k)$ divides $\operatorname{lcm}(\sigma \setminus m_i)$, the monomial m_i is a bridge of σ. $\square$

Remark 38. When we use Proposition 37 in the rest of the paper, in order to make clear which monomials serve as m_j, m_i, m_k, we might refer to them as a triple (m_j, m_i, m_k) where m_i is the bridge in this triple. It is possible to have $j < i < k$ or $k < j < i < k+n$ for this triple. Here, m_j is closer to m_n from the left and m_k is closer to m_n from the right.

We now turn our attention to characterizing the true gaps for vertices in G_u that are A_u-critical.

Proposition 39. *Let $\sigma \in V(G_u)$ be an A_u-critical cell and let $m_i \in \mathcal{G}$. Assume m_i does not dominate any bridge of σ. Then m_i is a true gap of σ if and only if the following statements hold:*

(a) $m_i \notin \sigma$.

(b) If there exists a monomial $m_k \in \sigma \cap M_{i+p-1}$, then $\sigma \cap M_{i+p-1} = \{m_k\}$ and $m_{i+p} \notin \sigma$.

(c) If $1 \leqslant i \leqslant p-1$ and $\sigma \cap \{m_1, \ldots, m_{i-1}\} = \emptyset$, then $\sigma \cap M_i = \{m_n\}$ and $m_{n+i-p} \notin \sigma$.

Proof. Suppose m_i does not dominate any bridges of σ. Recall from Lemma 34 that $m_n \in \sigma$ as σ is an A_u-critical cell. Lastly, as in the proof of Proposition 17, it is useful to recall [10, Proposition 2.21]: m_i is a true gap of σ that does not dominate any bridges of σ if and only if m_i is a gap of σ and $\operatorname{sb}(\sigma \cup m_i) = m_i$.

For the forward direction, assume m_i is a true gap of σ. So, $\operatorname{sb}(\sigma \cup m_i) = m_i$ from the previous paragraph. First, it is immediate that m_i is a gap of σ which means $m_i \notin \sigma$ by Proposition 36. So, (a) holds. For (b), if there are two monomials $m_k, m_t \in \sigma \cap M_{i+p-1}$, then we can reorder the vertices of C_n so that $i < k, t \leqslant i+p-1$. As in the proof of Proposition 17 (b), we can assume $k < t$ and conclude that m_t is a bridge of $\sigma \cup m_i$ where $m_i \succ m_t$, which contradicts to $\operatorname{sb}(\sigma \cup m_i) = m_i$. For the second part of (b), notice that, if $m_{i+p} \in \sigma$, there exists $m_j \in \sigma$ where $i < j < i + p$ by Proposition 33 (by reordering vertices, if needed). This means m_j is a bridge of $\sigma \cup m_i$ with $m_i \succ m_j$, a contradiction since $\mathrm{sb}(\sigma \cup m_i) = m_i$. Thus, $m_{i+p} \notin \sigma$.

Now, to verify (c), assume that $1 \leqslant i \leqslant p-1$ and $\sigma \cap \{m_1, \ldots, m_{i-1}\} = \emptyset$. Given that $m_n \in M_i$ for the stated range of i, we have $m_n \in \sigma \cap M_i$. If there exists another monomial $m_t \in \sigma \cap M_i$ with $t \notin \{1, \ldots, i\}$, the monomial m_n would emerge as a bridge for $\sigma \cup m_i$ by Proposition 37 for the triple (m_t, m_n, m_i) since $(n + i) - p + 1 < t$. This implies that $\mathrm{sb}(\sigma \cup m_i) = m_n$, a contradiction. Hence, $\sigma \cap M_i = \{m_n\}$. For the last segment of (c), if we assume $m_{n+i-p} \in \sigma$, it results with m_n as a bridge of $\sigma \cup m_i$ by Proposition 37 for the triple (m_{n+i-p}, m_n, m_i). This would imply that $\mathrm{sb}(\sigma \cup m_i) = m_n$, which is, yet again, contradictory to $\mathrm{sb}(\sigma \cup m_i) = m_i$, thereby confirming that $m_{n+i-p} \notin \sigma$.

For the reverse direction, assume (a), (b), and (c) hold. We argue by contradiction and assume that m_i is not a true gap of σ. Then there exists a monomial $m_t \in \sigma$ that is a bridge of $\sigma \cup m_i$ but is not a bridge of σ, with $m_i \succ m_t$. Furthermore, by Proposition 37, there exist monomials m_{j_t} and m_{k_t} in $(\sigma \setminus m_t) \cup m_i$ such that $k_t - j_t \leqslant p$ when $j_t < t < k_t$ or $(j_t + n) - k_t \leqslant p$ otherwise.

Note that either $j_t = i$ or $k_t = i$ since m_t is not a bridge of σ. As in the proof Proposition 37, we may assume that m_{j_t} and m_{k_t} are consecutive monomials in $(\sigma \setminus m_t) \cup m_i$ that are closest to m_t. If $j_t < t < k_t$, we must have $j_t = i$ since $m_i \succ m_t$. The reasoning for this case mirrors the corresponding portion of Proposition 17, eventually leading to a contradiction.

For the remaining case (2), we have:

$$j_t < k_t < t < j_t + n \leqslant k_t + p \leqslant n + p. \tag{$\star$}$$

Since $i < t$, either $k_t = i$ or $j_t = i$. If $k_t = i$, then $m_{j_t} \in \sigma$. It follows from ($\star$) that $m_t \in \sigma \cap M_{i+p-1}$. Then by (b) and the fact that $m_{j_t} = m_{j_t+n} \in \sigma$, we have $j_t + n = i + p$ which means $m_{j_t+n} = m_{i+p} \in \sigma$, a contradiction to (b).

For the final case, we can assume $j_t = i$ and we have $m_{k_t} \in \sigma$ in this case. Next, we verify that conditions of (c) are satisfied. Notice that $\sigma \cap \{m_1, \ldots, m_{i-1}\} = \emptyset$ since j_t and k_t are chosen to be the closest monomials to m_t in $(\sigma \setminus m_t) \cup m_i$. In addition, $i < p$ as a consequence of $i + n \leqslant k_t + p \leqslant n + p$ from ($\star$). Then (c) holds, namely, $\sigma \cap M_i = \{m_n\}$ where $M_i = \{m_{(n+i)-p+1}, \ldots, m_n, m_1, \ldots, m_i\}$. Notice that $m_t \in \sigma \cap M_i$ since $(i + n) - p \leqslant k_t < t \leqslant n$. Thus, $m_t = m_n$ and $m_{k_t} = m_{n+i-p} \in \sigma$, a contradiction to (c). Therefore, m_i is indeed a true gap of σ. □

Having addressed the true gaps of A_u-critical cells in the preceding proposition, our focus now shifts to the distinct classes of critical cells introduced in Definition 6: the absolutely critical and the fortunately critical cells. As noted in Corollary 9, the absolutely critical cells are uniquely characterized by the lack of both bridges and true gaps. On the other hand, the fortunately critical cells stand out. Their first element serves as their smallest bridge while still having no true gaps.

Lemma 40. *Let $\sigma \in V(G_u)$ be an A_u-critical cell. If σ is fortunately critical, then:*

1. *The smallest bridge of* σ *satisfies* $\mathrm{sb}(\sigma) \succ m_p$.

2. *If* $\sigma = \{m_{i_1}, \ldots, m_{i_t}\}$, *then* $\mathrm{sb}(\sigma) = m_{i_1}$.

Furthermore, no monomial from the set $\{m_p, m_{p+1}, \ldots, m_n\}$ *serves as a bridge or a true gap of* σ *(irrespective of whether* σ *is fortunately or absolutely critical).*

Proof. Let σ be a vertex in G_u that is fortunately critical. Then σ is potentially-type-2 and, by definition, there exists another vertex σ' such that $\sigma \setminus \mathrm{sb}(\sigma) = \sigma' \setminus \mathrm{sb}(\sigma')$ and $\mathrm{sb}(\sigma) \succ \mathrm{sb}(\sigma')$.

Since σ is potentially-type-2, every true gap of σ dominates $\mathrm{sb}(\sigma)$ by Theorem 8. So, $\mathrm{sb}(\sigma')$ cannot be a true gap of σ as $\mathrm{sb}(\sigma) \succ \mathrm{sb}(\sigma')$. Then there exists a monomial $m_i \in \sigma$ for which $\mathrm{sb}(\sigma \cup \mathrm{sb}(\sigma')) = m_i$, with $\mathrm{sb}(\sigma') \succ m_i$ while m_i is not a bridge of σ. Note as well that $m_i \in \sigma'$ and it is not a bridge of σ'. Before proceeding further, we first state our goal: to show that $m_i = m_n$.

By Proposition 37, there exist monomials m_j and m_k in $\sigma \cup \mathrm{sb}(\sigma')$ with $1 \leqslant j < k \leqslant n$ and $i \notin \{j, k\}$ such that

(a) $k - j \leqslant p$ when $j < i < k$, or

(b) $(j + n) - k \leqslant p$ when $j < k < i < j + n$.

Note that σ and σ' overlap at every element except $\mathrm{sb}(\sigma)$ and $\mathrm{sb}(\sigma')$. In other words,

$$\sigma \setminus \{\mathrm{sb}(\sigma), \mathrm{sb}(\sigma')\} = \sigma' \setminus \{\mathrm{sb}(\sigma), \mathrm{sb}(\sigma')\}.$$

Inevitably, we have $\{\mathrm{sb}(\sigma), \mathrm{sb}(\sigma')\} = \{m_j, m_k\}$. Otherwise, m_i is a bridge of σ and σ', which is not possible.

Notice that $j < i < k$ is not possible since $\mathrm{sb}(\sigma) \succ \mathrm{sb}(\sigma') \succ m_i$. So, we must have $j < k < i < j + n$ as in (b) above. Consequently, $m_j = \mathrm{sb}(\sigma)$ and $m_k = \mathrm{sb}(\sigma')$. Since $m_n \in \sigma \cup \mathrm{sb}(\sigma')$, we conclude that m_n is a bridge of $\sigma \cup \mathrm{sb}(\sigma')$ from Proposition 37 by using the triple (m_k, m_n, m_{j+n}) and (b). This means $m_i = m_n$.

1. The proof of $\mathrm{sb}(\sigma) \succ m_p$ follows from the following simple observation: the inequality $j + n \leqslant k + p \leqslant n + p$ from (b) implies that $j < p$. Moreover, it is not possible to have $j = p$, as this would imply $k = n$, leading to $\mathrm{sb}(\sigma') = m_n$, which is a contradiction. Thus, we conclude that $j < p$, which is equivalent to $m_j = \mathrm{sb}(\sigma) \succ m_p$.

2. Let $\sigma = \{m_{i_1}, \ldots, m_{i_t}\}$ where $i_1 < \cdots < i_t$. Our goal is to demonstrate $\mathrm{sb}(\sigma) = m_{i_1}$, i.e. $i_1 = j$. On the contrary, suppose there is a monomial m_s in σ such that $m_s \succ \mathrm{sb}(\sigma) = m_j$. Then, the monomial m_n is a bridge of σ' from Proposition 37 by using the triple (m_k, m_n, m_{s+n}) where each monomial is in σ' and $(s+n) - k < p$ by (b) above. This posits a contradiction since $\mathrm{sb}(\sigma') = m_k \succ m_n$. Thus, $\mathrm{sb}(\sigma) = m_{i_1}$.

For the final part of the statement, consider a vertex σ in G_u. If σ is absolutely critical, it lacks both bridges and true gaps, by Corollary 9, thereby satisfying the given statement. When σ is fortunately critical, the statement remains valid due to $\mathrm{sb}(\sigma) \succ m_p$, and the fact that each true gap of σ dominates $\mathrm{sb}(\sigma)$ by Theorem 8. □

Our primary objective is to comprehensively identify every element within an A_u-critical cell. We begin our identification with a series of observations and initiate the process by pinpointing specific values of j for which $m_j \in \sigma$.

Lemma 41. *Let $\sigma = \{m_{i_1}, \ldots, m_{i_t}\}$ be an A_u-critical cell where $i_1 < \cdots < i_t$. Then*

(a) $i_t = n$, and m_n is not a bridge for σ.

(b) $i_{t-1} = n - k$ for some $1 \leqslant k \leqslant p$.

(c) $i_{t-2} < n - p$.

(d) $i_1 = p - k + 1$ where k is given as in (b).

We need to make assumptions on the possible values of t. For instance, to consider (b), we require $t \geqslant 2$; for (c), we need $t \geqslant 3$, and so on. However, since these assumptions are clear from the context, we omit them.

Proof. Assume that $\sigma = \{m_{i_1}, \ldots, m_{i_t}\}$ is an A_u-critical cell.

(a) We begin by noting that $m_{i_t} = m_n$ as per Lemma 34. If σ is absolutely critical, then it has no bridges by Corollary 9. On the other hand, for a fortunately critical σ, Lemma 40 forces that $\mathrm{sb}(\sigma) = m_{i_1}$. Given that $m_{i_1} \succ m_n$, it is evident that m_n is not a bridge for σ.

(b) Recall from Proposition 33 that $i_t - i_{t-1} \leqslant p$. Then, $i_{t-1} = n - k$ for some $1 \leqslant k \leqslant p$.

(c) For the sake of contradiction, suppose $m_{n-j} \in \sigma$ for $1 \leqslant k < j \leqslant p$. Then, m_{n-k} is a bridge for σ from Proposition 37 by using the triple (m_{n-j}, m_{n-k}, m_n). However, this contradicts the nature of σ, as it is either absolutely critical (hence having no bridges) or fortunately critical (where $\mathrm{sb}(\sigma) = m_{i_1}$). So, $i_{t-2} < n - p$.

(d) Our initial step is to derive $p - k + 1 \leqslant i_1 \leqslant p$. The upper bound $i_1 \leqslant p$ is a direct consequence of Proposition 33 since $(i_1 + n) - i_t \leqslant p$. To establish the lower bound, if there exists an $m_i \in \sigma$ for $i \leqslant p - k$, then m_n is a bridge of σ from Proposition 37 by using the triple $(m_{n-k}, m_n, m_{n+p-k})$. This assertion, however, yields a contradiction by (a). Thus, $p - k + 1 \leqslant i_1 \leqslant p$.

If $k = 1$, then it is immediate that $i_1 = p$, satisfying the statement of (d). For $2 \leqslant k \leqslant p$, it remains to show $i_1 \leqslant p - k + 1$. A significant insight here is that m_{n-k+1} cannot be a true gap of σ. This is immediate when σ is absolutely critical as it has no true gaps. Suppose σ is fortunately critical. Then $\mathrm{sb}(\sigma) = m_{i_1}$ by Lemma 40 (2) and every true gap of σ dominates $\mathrm{sb}(\sigma)$ by Theorem 8 as σ is potentially-type-2. Since $\mathrm{sb}(\sigma) = m_{i_1} \succ m_{n-k} \succ m_{n-k+1}$, then m_{n-k+1} cannot be a true gap of σ. This means $\sigma \cup m_{n-k+1}$ has a bridge $m_i \in \sigma$ such that $m_{n-k+1} \succ m_b$. Since the only monomial in σ that is dominated by m_{n-k+1} is m_n, we have $m_i = m_n$. Using Proposition 37 for the triple $(m_{n-k+1}, m_n, m_{n+i_1})$, we conclude that $i_1 \leqslant p - k + 1$. ☐

To identify other elements of σ, we examine them in relation to the possible values of k. Here, $n-k$ is the penultimate element of σ when $1 \leqslant k \leqslant p$. The next two lemmas address the $k=1$ case and the $2 \leqslant k \leqslant p$ case separately due to nuanced variations in their proofs. Together, these lemmas give a complete overview of all A_u-critical cells in Proposition 44.

Lemma 42. *Let $\sigma = \{m_{i_1}, \ldots, m_{i_t}\}$ be an A_u-critical cell with $i_{t-1} = n-1$. Then, the distance between consecutive elements of σ alternates between 1 and p. Specifically,*

$$\sigma = \{m_p, \ldots, m_{n-(p+1)}, m_{n-(p+1)}, m_{n-1}, m_n\}.$$

Moreover, we have $p \equiv n$ or $n-1 \pmod{p+1}$.

Proof. First note that $i_1 = p$ by Lemma 41. Then, σ is an absolutely critical cell by Lemma 40.

The main idea of the proof revolves around retracing our steps from i_{t-1}, pinpointing the preceding indices of elements in σ until we arrive at $i_1 = p$. A recurring and instrumental point from Proposition 33 is: $i_j - i_{j-1} \leqslant p$ for $m_{i_{j-1}}, m_{i_j} \in \sigma$. We reference this result as $(\star)$ throughout the proof.

Recall from Lemma 41 (c) that $i_{t-2} \leqslant n-1-p$, indicating $i_{t-1} - i_{t-2} \geqslant p$ as $i_{t-1} = n-1$. This bound coupled with $(\star)$ yields to $i_{t-1} - i_{t-2} = p$. So, the first step of our analysis is complete. If $i_{t-2} = i_1$, we are done; otherwise, we move on to i_{t-3}.

Next, we show that $i_{t-2} - i_{t-3} = 1$ which is equivalent to obtaining $m_{n-p-2} \in \sigma$. For the sake of contradiction, suppose $m_{n-p-2} \notin \sigma$. Since σ is absolutely critical, m_{n-p-2} cannot be a true gap of σ. This means $\sigma \cup m_{n-p-2}$ has a bridge $m_b \in \sigma$ such that $m_{n-p-2} \succ m_b$ and m_b is not a bridge of σ. Then, by Proposition 37, there exists $m_s \in \sigma$ such that the distance between m_s and m_{n-p-2} along m_b is at most p. Given that $m_{n-p-2} \succ m_b$, we have one of the following scenarios by Proposition 37:

1. $s - (n-p-2) \leqslant p$ when $1 \leqslant n-p-2 < b < s \leqslant n$, or
2. $s + n - (n-p-2) \leqslant p$ when $1 \leqslant s < n-p-2 < b \leqslant n$.

The latter case is not possible because it implies that $s \leqslant -2$. Thus, (1) holds. In this case, we have $m_b = m_{n-1}$ and $m_s = m_n$ since $m_b, m_s \in \sigma$. Then, (1) results with $p+2 \leqslant p$, a contradiction. Thus, we conclude that $m_{n-p-2} \in \sigma$. So, $i_{t-3} = n-p-2$. If $i_{t-3} = i_1$, our task is complete; otherwise, our focus shifts to i_{t-4}.

Towards showing $i_{t-3} - i_{t-4} = p$, recall that $i_{t-3} - i_{t-4} \leqslant p$ by $(\star)$, or equivalently, $i_{t-4} \geqslant n - 2(p+1)$. If $m_{n-2(p+1)} \notin \sigma$, then $m_{i_{t-3}}$ is a bridge of σ by Proposition 37 since $i_{t-2} - i_{t-4} \leqslant p$. This leads to a contradiction because σ is absolutely critical. Hence, $i_{t-4} = n - 2(p+1)$. If $i_{t-4} = i_1$, our investigation is complete; otherwise, we continue in the same fashion for i_{t-5}.

Subsequent distances between consecutive elements of σ can be concluded by employing similar arguments in an alternating way until we reach $i_1 = p$. This results in the congruence $p \equiv n$ or $n-1 \pmod{p+1}$, concluding the proof. □

Lemma 43. *Let $\sigma = \{m_{i_1}, \ldots, m_{i_t}\}$ be an A_u-critical cell with $i_{t-1} = n - k$ for some $2 \leqslant k \leqslant p$. Then, the distance between consecutive elements of σ alternates between k and $(p+1) - k$. Specifically,*

$$\sigma = \{m_{p-k+1}, \ldots, m_{n-k-(p+1)}, m_{n-(p+1)}, m_{n-k}, m_n\}$$

Moreover, we have $p - k + 1 \equiv n$ or $n - k \pmod{p+1}$.

Proof. The idea behind this proof is similar to that of Lemma 42. Recall that $i_{t-2} \leqslant n - p - 1$ and $i_1 = p - k + 1$ by Lemma 41 (c) and (d). So, our first goal is to verify $i_{t-2} = n - p - 1$.

Assume, for the sake of contradiction, that $m_{n-p-1} \notin \sigma$. We then observe that m_{n-p-1} cannot be a true gap of σ. This is evident if σ is absolutely critical because it lacks true gaps. If σ is fortunately critical, then $\mathrm{sb}(\sigma) = m_{i_1}$ as stated in Lemma 40 and any true gap of σ dominates $\mathrm{sb}(\sigma) = m_{i_1}$ by Theorem 8. If m_{n-p-1} is a true gap of σ, then we must have $\sigma = \{m_{n-k}, m_n\}$ and it cannot have any bridges, a contradiction. So, m_{n-p-1} cannot be a true gap of σ. This means $\sigma \cup m_{n-p-1}$ has a bridge $m_b \in \sigma$ such that $m_{n-p-1} \succ m_b$ and m_b is not a bridge of σ. Using Proposition 37, there exists a monomial $m_s \in \sigma$ such that the distance between m_s and m_{n-p-1} along m_b is at most p. In other words, given that $m_{n-p-1} \succ m_b$, we have

1. $s - (n - p - 1) \leqslant p$ when $1 \leqslant n - 1 - p < b < s \leqslant n$ or

2. $s + n - (n - p - 1) \leqslant p$ when $1 \leqslant s < n - 1 - p < b \leqslant n$.

The latter is not possible since it implies $s < -1$. Thus, (1) holds. Since $m_b, m_s \in \sigma$, we have $m_b = m_{n-k}$ and $m_s = m_n$. Using these in (1) results with $p + 1 < p$, a contradiction. Hence, $i_{t-2} = n - 1 - p$. If $i_{t-2} = i_1$, our investigation concludes. Suppose $i_{t-2} \neq i_1$.

The next step is to show $i_{t-3} = n - k - (p+1)$. We first claim that $m_{i_{t-2}}$ cannot be a bridge of σ. If it is a bridge of σ, then $\mathrm{sb}(\sigma) = m_{i_1}$ by Lemma 40 because σ must be fortunately critical. This means $i_1 = i_{t-2}$, a contradiction to our assumption. Since $m_{i_{t-2}}$ cannot be a bridge of σ, we have $i_{t-3} \leqslant n - k - (p+1)$ by Proposition 37. So, it suffices to show $m_{n-k-(p+1)} \in \sigma$.

For the sake of contradiction, suppose $m_{n-k-(p+1)} \notin \sigma$. If $m_{n-k-(p+1)}$ is a true gap of σ, then σ is potentially-type-2 and $m_{n-k-(p+1)} \succ \mathrm{sb}(\sigma) = m_{i_1}$ by Theorem 8 and Lemma 40. This can happen only when $i_1 = i_{t-2}$, a contradiction. Thus, $m_{n-k-(p+1)}$ is not a true gap of σ. This means $\sigma \cup m_{n-k-(p+1)}$ has a bridge $m_b \in \sigma$ such that $m_{n-k-(p+1)} \succ m_b$ and m_b is not a bridge of σ. Then, by Proposition 37, there exists $m_s \in \sigma$ such that the distance between $m_{n-k-(p+1)}$ and m_s along m_b is at most p. In other words, given that $m_{n-k-(p+1)} \succ m_b$, we have

1. $s - (n - k - (p+1)) \leqslant p$ when $1 \leqslant n - k - (p+1) < b < s \leqslant n$ or

2. $s + n - (n - k - (p+1)) \leqslant p$ when $1 \leqslant s < n - k - (p+1) < b \leqslant n$

One can verify that (2) cannot happen. Then, by (1), the monomial m_s is either m_{n-k} or m_n since $m_s \in \sigma$. Using either value of s in (1) results with $p+1 \leqslant p$, a contradiction. Therefore, $i_{t-3} = n-k-(p+1)$. If $i_{t-3} = i_1$, the process terminates here. If $i_{t-3} \neq i_1$, one can repeat the above arguments for the next steps until reaching $i_1 = p-k+1$. This results in the congruence $p-k+1 \equiv n$ or $n-k \pmod{p+1}$, concluding the proof. □

Below, we describe all A_u-critical cells. This proposition serves as the centerpiece of this chapter's main result.

Proposition 44. *Let $n = (p+1)q + r$ where $0 \leqslant r \leqslant p$. Then, we have:*

1. *For $r = 0$, the only A_u-critical cells are the cells σ_i, where*

$$\sigma_i := \{m_j \mid j \equiv i \pmod{p+1}\}$$

   for each $1 \leqslant i \leqslant p$. Moreover, each σ_i is absolutely critical.

2. *For $r \neq 0$, the only A_u-critical cell is*

$$\tau_r := \{m_j \mid j \geqslant r, j \equiv r, 2r \pmod{p+1}\}.$$

   Moreover, the cell τ_r is absolutely critical if and only if $2r > p$.

Proof. Let $\sigma = \{m_{i_1}, \ldots, m_{i_t}\}$ be an A_u-critical cell with $t \geqslant 2$. To identify all A_u-critical cells, we first address the immediate case where $n = p+1$. It is evident that $t \geqslant 3$ is not possible, as this would lead to $\mathrm{sb}(\sigma) = m_n$, a contradiction by Lemma 41 (a). Consequently, σ must be of the form $\sigma_i = \{m_i, m_n\}$ for all $1 \leqslant i \leqslant p$. A straightforward check confirms that every such σ_i is absolutely critical. Having settled this case, we proceed under the assumption $n > p+1$.

Recall from Lemma 42 and Lemma 43 that the distance between consecutive elements of σ alternates between k and $(p+1)-k$. Specifically, for $1 \leqslant k \leqslant p$, we have

$$\sigma = \{m_{p-k+1}, \ldots, m_{n-k-(p+1)}, m_{n-(p+1)}, m_{n-k}, m_n\}$$

where $p-k+1 \equiv n$ or $n-k \pmod{p+1}$. It is important to clarify that the distance between the first and last elements of σ is not considered.

Observe that the conditions $p-k+1 \equiv n \pmod{p+1}$ and $n-k \equiv 0 \pmod{p+1}$ are equivalently expressed as $n \equiv p+1-k \pmod{p+1}$ and $n \equiv 0 \pmod{p+1}$, respectively. Consequently, the cell σ can be expressed as follows:

$$\sigma = \begin{cases} \{m_{(p+1-k)+i(p+1)}, m_{(i+1)(p+1)} \ : \ 0 \leqslant i \leqslant q-1\} & \text{if } r = 0, \\ \{m_{r+i(p+1)}, m_{2r+i(p+1)} \ : \ 0 \leqslant i \leqslant q-1\} \cup \{m_n\} & \text{if } r \neq 0, \end{cases}$$

where the former corresponds to σ_{p+1-k} for $1 \leqslant k \leqslant p$ and the latter corresponds to τ_r in the statement of the proposition. One can verify that each σ_{p+1-k} is absolutely critical by noticing that they have no bridges and no true gaps for $1 \leqslant k \leqslant p$. Furthermore, one can verify that τ_r is absolutely critical if and only if $2r > p$. □

The conclusions drawn below directly arise from the descriptions of A_u-critical cells given in Proposition 44.

Corollary 45. *Let $\sigma \in V(G_u)$ be a critical cells of $\mathcal{G}$. Then*

$$|\sigma| = \begin{cases} 2q & \text{if } n \equiv 0 \pmod{(p+1)}, \\ 2q+1 & \text{otherwise.} \end{cases}$$

In particular, all A_u-critical cells have the same cardinality.

We now present the main theorem of this section, which follows immediately from the preceding corollary, and the description of the differentials of the corresponding generalized Barile-Macchia resolution in Theorem 2.

Theorem 46. *A generalized Barile-Macchia resolution of $I_p(C_n)$ is minimal.*

We conclude this section with a brief discussion. In [10], the first two authors introduced and examined Barile-Macchia resolutions, which are cellular and independent of $\operatorname{char}(\Bbbk)$. Though effective for many classes of ideals, this method does not always produce a minimal free resolution. Specifically, the edge ideal of a 9-cycle, $I_2(C_9)$, cannot be minimally resolved by a Barile-Macchia resolution, as highlighted in [10].

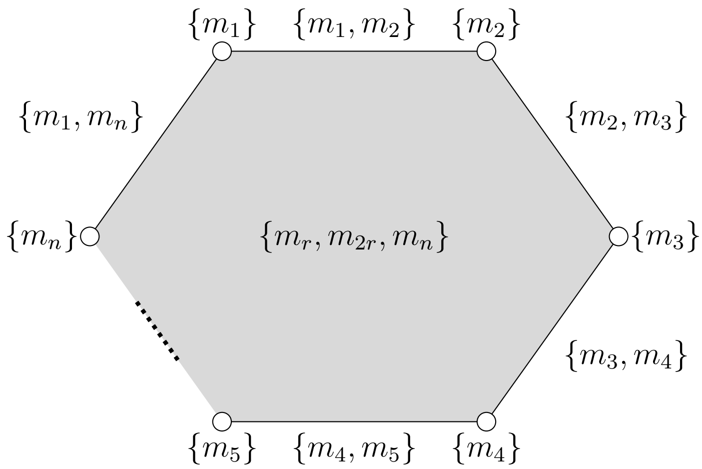
Corollary 47. *Edge ideals of cycles have cellular minimal free resolutions.*

Echoing the approach of the preceding section, we conclude with an example presenting a CW complex that supports minimal free resolutions of the path ideals of cycles, where the projective dimension is equal to 2.

Example 48. Consider the cycle C_n where $n = (p+1) + r$ for $1 \leqslant r \leqslant p$. We can list all critical cells of $I_p(C_n)$ using Proposition 44 as follows:

$$\emptyset, \{m_1\}, \ldots, \{m_{p+2}\}, \{m_1, m_2\}, \{m_2, m_3\}, \ldots, \{m_{n-1}, m_n\}, \{m_n, m_1\}, \{m_r, m_{2r}, m_n\}.$$

The following n-gon, denoted by Δ, is in correspondence with the critical cells of $I_p(C_n)$:



As in Example 30, all restricted subcomplexes of Δ are either empty, a line segment, or the n-gon itself, which are all acyclic. Thus, Δ support the minimal free resolution of $R/I_p(C_n)$ by [7, Lemma 2.2].

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