Complement Avoidance in Binary Words

James Currie[a] L'ubomíra Dvořáková[b] Pascal Ochem[c]
Daniela Opočenská[b] Narad Rampersad[a] Jeffrey Shallit[d]

Submitted: Sep 21, 2022; Accepted: Sep 8, 2025; Published: Oct 17, 2025

Abstract

The complement $\overline{x}$ of a binary word x is obtained by changing each 0 in x to 1 and vice versa. We study infinite binary words $\mathbf{w}$ that avoid sufficiently large complementary factors; that is, if x is a factor of $\mathbf{w}$, then $\overline{x}$ is not a factor of $\mathbf{w}$. In particular, we classify such words according to their critical exponents.

Mathematics Subject Classifications: 68R15

1 Introduction

Let x be a finite nonempty binary word. We write $\overline{x}$ for the complementary word, image of the morphism that maps $0 \to 1$ and $1 \to 0$, and we write x^R for the reversal (mirror image) of x. We say y is a *factor* of a (one-sided) infinite word $\mathbf{w}$ if $\mathbf{w} = xy\mathbf{z}$ for a finite word x and an infinite word $\mathbf{z}$. In this paper, we are interested in the construction and properties of infinite binary words $\mathbf{w}$ *avoiding* complementary factors: that is, if x is a nonempty factor of $\mathbf{w}$, then $\overline{x}$ is not. This is not a new notion; for example, complement avoidance in de Bruijn words was studied by Sawada et al. [21].

Evidently it is impossible for an infinite word to avoid complementary factors of *all* lengths, except in the trivial cases $0^\omega = 000\cdots$ and $1^\omega = 111\cdots$. A natural question then poses itself: are there such infinite words if the set of exceptions is restricted in some way, say by length or by cardinality? And what is the repetition threshold of such infinite words? We now turn to repetitions.

We say that a finite word $w = w[1..n]$ has *period* $p \geqslant 1$ if $w[i] = w[i+p]$ for $1 \leqslant i \leqslant n-p$. The smallest period of a word w is called *the* period, and we write it as $\mathrm{per}(w)$.

[a]Department of Math/Stats, University of Winnipeg, 515 Portage Ave., Winnipeg, MB, R3B 2E9, Canada (j.currie,n.rampersad@uwinnipeg.ca).
[b]FNSPE Czech Technical University, Prague, Czech Republic (lubomira.dvorakova,opocedan@fjfi.cvut.cz).
[c]LIRMM, CNRS, Université de Montpellier, France (ochem@lirmm.fr).
[d]School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada (shallit@uwaterloo.ca).

The *exponent* of a finite word w, written $\exp(w)$ is defined to be $|w|/\operatorname{per}(w)$. For a real number α, we say a word (finite or infinite) is α-*free* if the exponent of all its nonempty factors is $< \alpha$. We say a word is α^+-*free* if the exponent of all its nonempty factors is $\leqslant \alpha$; here we regard α^+ as a kind of "extended real number" that is infinitesimally larger than α. A word that is 2-free is also called squarefree, and a word that is 3-free is also called cubefree. A word that is 2^+-free is also called overlap-free.

The *critical exponent* of a finite or infinite word x is the supremum, over all nonempty finite factors w of x, of $\exp(w)$; it is written $\operatorname{ce}(x)$. The *repetition threshold* for a language L of infinite words is defined to be the infimum, over all $x \in L$, of $\operatorname{ce}(x)$.

The repetition thresholds for various classes of words have been studied extensively. To name just a few classes, Dejean [8] determined the repetition threshold for all words over a 3-letter alphabet, and conjectured its value for larger alphabets. Her conjecture attracted a lot of attention, and was finally resolved by Rao [19] and Currie and Rampersad [5], independently.

Other classes that have been studied include the Sturmian words, studied by Carpi and de Luca [4, Prop. 15]; the palindromes, studied in [23]; the rich words, studied by Currie et al. [6]; the balanced words, studied by Rampersad et al. [18] and Dvořáková et al. [11]; and the complementary symmetric Rote words, studied by Dvořáková et al. [12]. Other related works include [14, 1, 13, 20, 17].

In this paper we study the repetition threshold for two classes of infinite words:

- CAL_ℓ, the binary words for which there is no length-ℓ word x such that both x and $\overline{x}$ appear as factors;
- CAN_n, the binary words for which there are at most n distinct words x such that both x and $\overline{x}$ appear as factors.

It turns out that there is an interesting and subtle hierarchy, depending on the values of ℓ and n.

Our work is very similar in flavor to that of [22], which found a similar hierarchy concerning critical exponents and sizes of squares avoided. The hierarchy for complementary factors, as we will see, however, is significantly more complex.

We will need the following famous infinite words.

- The Fibonacci word $\mathbf{f} = 01001010010010100101001 00\cdots$, fixed point of the morphism $0 \to 01$, $1 \to 0$. See, for example, [3].
- The word $\mathbf{p} = 01210210102102101021012 1\cdots$, fixed point of the morphism φ sending $0 \to 01$, $1 \to 21$, $2 \to 0$. This is sequence A287072 in the OEIS. Its properties were recently studied in [7] and morphic images of $\mathbf{p}$ appear in the context of binary words with few palindromes [10].

The paper is organized as follows. In Section 2, we introduce the class CAL_ℓ mentioned above and we establish the hierarchy alluded to previously. Section 3 does the same thing for the class CAN_n. In both cases we need some critical exponent calculations,

which are carried out in Section 4. Finally, in Section 5 we study finite words avoiding complementary factors and determine under what conditions there are exponentially many such words.

2 The class $\mathbf{CAL}_\ell$

In this section, we investigate the repetition threshold for the class CAL_ℓ, the binary words $\mathbf{w}$ with the property that if x is a length-ℓ factor of $\mathbf{w}$, then $\overline{x}$ is not. We will need the well-known Thue-Morse morphism μ, which maps $0 \to 01$ and $1 \to 10$.

Lemma 1. *Suppose* $\mathbf{x}$ *is an infinite binary word avoiding* $(7/3)$*-powers. Then* $\mathbf{x}$ *contains infinitely many (and hence, arbitrarily large) complementary factors.*

Proof. By a result of Karhumäki and Shallit [15], every infinite binary word avoiding e-powers for $e \leqslant \frac{7}{3}$ contains $\mu^n(0)$ as a factor for all $n \geqslant 1$. Such a word is of the form $\mu^{n-1}(01) = \mu^{n-1}(0)\mu^{n-1}(1)$, and these two terms are complementary factors of length 2^{n-1}. □

By Lemma 1, the lower limit on the repetition threshold is $\frac{7}{3}$. For $\ell = 1$ the only such words are 0^ω and 1^ω. For $\ell = 2$ the only such words are 0^ω, 1^ω, 10^ω, and 01^ω. Larger ℓ are handled in Theorem 3 below, but to prove it we first need to provide some terminology and a lemma from [16]. A morphism $f : \Sigma^* \to \Delta^*$ is called *q-uniform* if $|f(a)| = q$ for all $a \in \Sigma$, and is called *synchronizing* if for all $a, b, c \in \Sigma$ and $u, v \in \Delta^*$, if $f(ab) = uf(c)v$, then either $u = \varepsilon$ and $a = c$, or $v = \varepsilon$ and $b = c$. The following result is quoted almost verbatim from [16, Lemma 23]:

Lemma 2. *Let* $a, b \in \mathbb{R}$ *satisfy* $1 < a < b$. *Let* $\alpha \in \{a, a^+\}$ *and* $\beta \in \{b, b^+\}$. *Let* $h\colon \Sigma^* \to \Delta^*$ *be a synchronizing* q*-uniform morphism. Set*

$$t = \max\left(\frac{2b}{b-a}, \frac{2(q-1)(2b-1)}{q(b-1)}\right).$$

If $h(w)$ *is* β*-free for every* α*-free word* w *with* $|w| \leqslant t$*, then* $h(z)$ *is* β*-free for every* α*-free word* $z \in \Sigma^*$.

We will use this lemma as follows: through an exhaustive search, we find an appropriate uniform morphism[1] from $\{0,1,2\}^* \to \{0,1\}^*$, and then we apply this morphism to an arbitrary ternary squarefree word. Then we use the fact that there are uncountably many infinite ternary squarefree words, and exponentially many finite ternary squarefree words [24].

We are now ready to state and prove our result on avoiding complementary factors.

Theorem 3. *There exists an infinite* β^+*-free binary word containing no complementary factors of length* $\geqslant \ell$*, for the following pairs* (ℓ, β). *Moreover, this list of pairs is optimal.*

[1]The program that found the uniform morphisms in this paper is available at LINK_TO_LOCATION_ON_ELECJC_PAGE.

(a) $(3, 2+\alpha)$, *where* $\alpha = (1+\sqrt{5})/2$.

(b) $(5, 3)$

(c) $(7, \frac{8}{3})$

(d) $(8, \frac{5}{2})$

(e) $(11, \gamma)$, *where* $\gamma \doteq 2.4808627161472369$ *is the critical exponent of* $\mathbf{p}$.

(f) $(13, \frac{7}{3})$.

Proof.

(a) $(3, 2+\alpha)$ is achieved by any Sturmian word $\mathbf{x}$ with slope $[0, 3, 1, 1, 1, 1, \ldots] = (5-\sqrt{5})/10$. As is well-known, a Sturmian word has exactly $n+1$ factors of length n, and these factors are independent of the intercept of the Sturmian word. For intercept 0, the first 6 symbols are 001000, and so the four factors of $\mathbf{x}$ of length 3 are $001, 010, 100, 000$, and no complement of these words appears as a factor of $\mathbf{x}$.

On the other hand, we know from the proof of Proposition 15 of [4] that the critical exponent of $\mathbf{x}$ is $2+\alpha$. Thus, since we can choose the intercept of a Sturmian word to be any real in $[0, 1]$, there are uncountably many binary words in CAL_3 with critical exponent $2+\alpha$.

This is best possible, as will be shown in Theorem 16.

(b) $(5, 3)$ is achieved by applying the 17-uniform morphism h_1 defined by

$$\begin{aligned} 0 &\to 01000101000101001 \\ 1 &\to 01000101000100100 \\ 2 &\to 01000101000100010, \end{aligned}$$

to any ternary squarefree word $\mathbf{w}$. It is easy, by checking all squarefree words of length 5, to ensure that $h_1(\mathbf{w})$ contains no complementary factors of length $\geqslant 6$. To verify the 3^+-freeness of these words, we use Lemma 2 with $\alpha = 2$, $\beta = 3^+$, $q = 17$, $t = 6$, and check that the morphism is indeed synchronizing and that the image of every ternary squarefree word of length $\leqslant 6$ is 3^+-free. This gives uncountably many infinite binary words and exponentially many finite binary words with the desired avoidance property.

We use this same technique to verify the β^+-freeness of every word in this paper that is obtained with a uniform morphism.

To see that this result is optimal, backtracking easily shows that the longest word that contains no complementary factors of length $\geqslant 6$ and no cubes is of length 50, and one example is

$$00101001001101001001101001101001001101001101001001.$$

(c) $(7, \frac{8}{3})$ is achieved by applying the 36-uniform morphism

$$\begin{aligned} 0 &\to 001001010011001010010011001001010011 \\ 1 &\to 001001010010011001010011001010010011 \\ 2 &\to 0010010100100110010010100110010100 11, \end{aligned}$$

to any ternary squarefree word. This gives uncountably many infinite binary words and exponentially many finite binary words.

The longest word that contains no complementary factors of length $\geqslant 7$ and no $\frac{8}{3}$-powers is of length 51 and one example is

$$001001100100110010100110010011001010011001010010100.$$

(d) $(8, \frac{5}{2})$ is achieved by applying the morphism ξ, defined as follows

$$\begin{aligned} 0 &\to 01 \\ 1 &\to 0110 \\ 2 &\to 1, \end{aligned}$$

to the infinite word $\mathbf{p}$ defined in the introduction. We have

$$\xi(\mathbf{p}) = 0101101011001101100101100110110\cdots .$$

The proof that the critical exponent of $\xi(\mathbf{p})$ is $5/2$ is given in Section 4 starting from Section 4.6.

The longest binary word containing no complementary factors of length $\geqslant 10$ and no $\frac{5}{2}$-powers is of length 75 and one example is

$$001011010011001011010011011001011010011001011010011011001011010011011001100.$$

(e) $(11, \gamma)$ is achieved by the word $\psi(\mathbf{p})$, where the morphism ψ is defined as follows:

$$\begin{aligned} 0 &\to 011001 \\ 1 &\to 0 \\ 2 &\to 01101. \end{aligned}$$

We have

$$\psi(\mathbf{p}) = 0110010011010011001011010\cdots .$$

In Section 4 we show that the critical exponent of $\psi(\mathbf{p})$ is the same as that for $\mathbf{p}$.

For $\ell = 12$ and $\beta = \gamma$, the optimality is proved as follows. Let $z = 1001011001$. We can check that $\psi(\mathbf{p})$ avoids z and contains $\overline{z}$, z^R, and $\overline{z^R}$. Let x be the prefix of length 40 of $\psi(\mathbf{p})$. A computer check shows that there is no infinite $\frac{5}{2}$-free binary word with $\ell \leqslant 12$ that simultaneously avoids x, $\overline{x}$, x^R, and $\overline{x^R}$.

By symmetry, we consider a bi-infinite $\frac{5}{2}$-free binary word $\mathbf{w}$ with $\ell \leqslant 12$ that contains x. Let X be the set containing the complements of the factors of length 12 of x. Thus $\ell \leqslant 12$ means that $\mathbf{w}$ avoids X.

We compute the set S of factors f such that efg is $\frac{5}{2}$-free and avoids X and $|e| = |f| = |g| = 100$. We compute the set S' of factors of $\psi(\mathbf{p})$ of length 100. We verify that $S = S'$. This means that $\mathbf{w} = \psi(\mathbf{v})$ for some bi-infinite ternary word $\mathbf{v}$. Moreover, by considering the pre-images by ψ of $\mathbf{w} = \psi(\mathbf{v})$ and $\psi(\mathbf{p})$, this implies that $\mathbf{v}$ and $\mathbf{p}$ have the same set of factors of length $100/\max(|\psi(0)|, |\psi(1)|, |\psi(2)|) = 16$. In particular, $\mathbf{v}$ avoids the set

$$\{00, 11, 22, 20, 212, 0101, 02102, 121012, 01021010, 21021012102\}$$

mentioned in [7, Theorem 14].

Also, $\mathbf{v}$ is cube-free since $\mathbf{w}$ is $\frac{5}{2}$-free. By [7, Theorem 14], we know that $\mathbf{v}$ has the same set of factors as $\mathbf{p}$. Thus $\mathbf{w}$ has the same set of factors as $\psi(\mathbf{p})$. So the critical exponent of $\psi(\mathbf{p})$ is optimal for $\ell = 12$.

(f) $(13, \frac{7}{3})$ is achieved by applying the 69-uniform morphism

$$0 \to 001001100101101001100101100100110100110010110100110010011010011001011$$
$$1 \to 001001100101101001100100110100110010110100110010110010011010011001011$$
$$2 \to 001001100101101001100100110100110010110010011010011001011010011001011$$

to any ternary squarefree word. This gives uncountably many infinite binary words and exponentially many finite binary words.

By Proposition 1, if $\beta = \frac{7}{3}$, then there are arbitrarily long complemented words.

□

3 The class $\mathbf{CAN}_n$

One could also try to minimize the total number n of complemented words that appear. Obviously n has to be even. Recall that CAN_n denotes the set of binary words $\mathbf{w}$ for which there are at most n distinct words x such that both x and $\overline{x}$ appear as factors of $\mathbf{w}$. For $n = 0$ the only such infinite words are 0^ω and 1^ω. For $n = 2$ the only such infinite words are 0^ω, 1^ω, 01^ω, and 10^ω. For larger n the situation is summed up in the following theorem:

Theorem 4. *There exists an infinite β^+-free binary word having at most n complemented words, for the following pairs (n, β). Moreover, this list of pairs is optimal.*

(a) $(4, 2 + \alpha)$, *where* $\alpha = (1 + \sqrt{5})/2$

(b) $(8, 3)$

(c) $(24, \frac{8}{3})$

(d) $(36, \frac{5}{2})$

(e) $(64, \gamma)$, *where* $\gamma \doteq 2.4808627161472369$ *is the critical exponent of* $\mathbf{p}$.

(f) $(90, \frac{7}{3})$.

Proof. To prove existence, we use the same words as in Theorem 3. That is, for every $\beta \in \{2+\alpha, 3, \frac{8}{3}, \frac{5}{2}, \gamma, \frac{7}{3}\}$, the infinite β^+-free binary words given in Theorem 3 to achieve the pair (ℓ, β) also achieve the pair (n, β) in Theorem 4. It is not hard to count the complemented factors in such words since their length is less than ℓ. Now let us consider the optimality.

(a) For $4 \leqslant n \leqslant 6$ the optimal exponent we can avoid is $2+\alpha$. To see this note that any such word must avoid having both x and $\overline{x}$ as factors if $|x| \geqslant 4$, since taking non-empty prefixes of x and $\overline{x}$ gives at least 8 complemented factors. Hence, the smallest exponent that can be avoided is $2+\alpha$ by Theorem 16.

(b) For $8 \leqslant n \leqslant 22$ one can avoid 3^+-powers, and this is optimal. The longest word having at most 22 complemented words, and no cubes is of length 50 and one example is 00101001001101001001101001101001001101001101001001.

(c) For $24 \leqslant n \leqslant 34$ one can avoid $\frac{8}{3}^+$-powers, and this is optimal. The longest word having at most 34 complemented words and no $\frac{8}{3}$-powers is of length 51 and one example is 00100110010011001010011001001100101001100101001010100.

(d) For $36 \leqslant n \leqslant 62$ one can avoid $\frac{5}{2}^+$-powers and this is optimal. The longest word having at most 62 complemented words and no $\frac{5}{2}$-powers is of length 73 and one example is 00101101001100101101001101100101101001100101101001101100101101001101100101100110010.

(e) For $64 \leqslant n \leqslant 88$ the optimal exponent that can be avoided is γ. We use the proof of optimality of Theorem 3 (e) and replace the condition $\ell \leqslant 12$ (i.e., complemented words of length at most 12) by $n \leqslant 88$ (i.e., at most 88 complemented words).

(f) For $90 \leqslant n < \infty$, the optimal exponent that can be avoided is $\frac{7}{3}^+$. By Proposition 1, if $e = \frac{7}{3}$, then there are arbitrarily many complemented words. □

4 Critical exponent of $\boldsymbol{\xi}(\mathbf{p})$ and $\boldsymbol{\psi}(\mathbf{p})$

In this section, we compute the critical exponents of the words $\xi(\mathbf{p})$ and $\psi(\mathbf{p})$ introduced in the proofs of Theorem 3(d) and (e), respectively, using the lengths of their bispecial factors and their shortest return words.

Recall that $\mathbf{p}$ was defined in the introduction, and is the fixed point of the morphism φ that maps $0 \to 01$, $1 \to 21$, $2 \to 0$. The following characteristics of $\mathbf{p}$ are known (see [7]):

- The factor complexity (number of distinct length-n factors) of $\mathbf{p}$ is $2n+1$.
- The word $\mathbf{p}$ is not closed under reversal because 02 is a factor of $\mathbf{p}$, but 20 is not.
- The word $\mathbf{p}$ is uniformly recurrent because the morphism φ is primitive.

First recall the definitions of bispecial and return word. Let $\mathbf{u}$ be an infinite word and let $\mathcal{L}(\mathbf{u})$ denote the language of all finite factors of $\mathbf{u}$. Then $w \in \mathcal{L}(\mathbf{u})$ is called *left special* if $aw, bw \in \mathcal{L}(\mathbf{u})$ for two distinct letters a, b. A *right special* factor is defined analogously. The factor w is called *bispecial* if it is both left special and right special. A factor r of $\mathbf{u}$ is a *return word* to the factor w if $rw \in \mathcal{L}(\mathbf{u})$ and rw contains w exactly twice—once as a prefix and once as a suffix.

For a word u over an ordered d-letter alphabet Σ, we define its *Parikh vector* to be the vector of number of occurrences of each letter in u.

Theorem 5 ([9])**.** *Let* $\mathbf{u}$ *be a uniformly recurrent aperiodic sequence. Let* (w_n) *be a sequence of all bispecial factors ordered by their length. For every* $n \in \mathbb{N}$*, let* r_n *be a shortest return word to* w_n *in* $\mathbf{u}$*. Then*

$$\mathrm{ce}(\mathbf{u}) = 1 + \sup_{n\in\mathbb{N}} \left\{ \frac{|w_n|}{|r_n|} \right\} .$$

4.1 Bispecial factors in p

In order to determine bispecial factors in $\psi(\mathbf{p})$ and in $\xi(\mathbf{p})$, we first need to explore bispecial factors in $\mathbf{p}$. Let us start by examining the left special factors (LS). Observing the form of φ, we can see that every LS has left extensions either $\{0,2\}$, or $\{0,1\}$.

Observation 6. *Let* $v \in \mathcal{L}(\mathbf{p})$*,* $v \neq \varepsilon$*. Then*

- *v is LS such that $0v, 2v \in \mathcal{L}(\mathbf{p})$ if and only if v is a prefix of $1\varphi(w)$, where w is LS such that $0w, 1w \in \mathcal{L}(\mathbf{p})$.*
- *v is LS such that $0v, 1v \in \mathcal{L}(\mathbf{p})$ if and only if v is a prefix of $\varphi(w)$, where w is LS such that $0w, 2w \in \mathcal{L}(\mathbf{p})$.*

Second, we examine the right special factors (RS). By the definition of φ, every RS has right extensions either $\{1,2\}$, or $\{0,2\}$.

Observation 7. *Let* $v \in \mathcal{L}(\mathbf{p})$*,* $v \neq \varepsilon$*. Then*

- *v is RS such that $v1, v2 \in \mathcal{L}(\mathbf{p})$ if and only if v is a suffix of $\varphi(w)0$, where w is RS such that $w0, w2 \in \mathcal{L}(\mathbf{p})$.*
- *v is RS such that $v0, v2 \in \mathcal{L}(\mathbf{p})$ if and only if v is a suffix of $\varphi(w)$, where w is RS such that $w1, w2 \in \mathcal{L}(\mathbf{p})$.*

It follows from the form of left and right special factors that we have at most 4 possible kinds of nonempty bispecial factors in $\mathbf{p}$. The following statements are obtained when combining Observations 6 and 7.

Corollary 8. *Let $v \in \mathcal{L}(\mathbf{p}) \setminus \{\varepsilon\}$ be a bispecial factor in $\mathbf{p}$.*

1. *If $0v, 2v, v0, v2 \in \mathcal{L}(\mathbf{p})$, there exists w such that $v = 1\varphi(w)$ and $0w, 1w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
2. *If $0v, 1v, v1, v2 \in \mathcal{L}(\mathbf{p})$, there exists w such that $v = \varphi(w)0$ and $0w, 2w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*
3. *If $0v, 2v, v1, v2 \in \mathcal{L}(\mathbf{p})$, there exists w such that $v = 1\varphi(w)0$ and $0w, 1w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*
4. *If $0v, 1v, v0, v2 \in \mathcal{L}(\mathbf{p})$, there exists w such that $v = \varphi(w)$ and $0w, 2w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*

It follows from Corollary 8 that all bispecial factors can be constructed starting from the shortest ones in the following way: 1 is the shortest bispecial factor with left extensions $\{0, 2\}$ and right extensions $\{0, 2\}$. Applying the morphism φ, we obtain the bispecial factor $\varphi(1)0$ with left extensions $\{0, 1\}$ and right extensions $\{1, 2\}$. The second application of φ gives us the bispecial factor $1\varphi^2(1)\varphi(0)$ with left extensions $\{0, 2\}$ and right extensions $\{0, 2\}$. This process can be iterated infinitely many times

$$1 \to \varphi(1)0 \to 1\varphi^2(1)\varphi(0) \to \varphi(1)\varphi^3(1)\varphi^2(0)0 \to 1\varphi^2(1)\varphi^4(1)\varphi^3(0)\varphi(0) \to \cdots .$$

Similarly, when starting with the bispecial factor 10 with left extensions $\{0, 2\}$ and right extensions $\{1, 2\}$, we obtain, after application of φ, the bispecial factor $\varphi(10)$ with left extensions $\{0, 1\}$ and right extensions $\{0, 2\}$. After the second application of φ, we have the bispecial factor $1\varphi^2(10)0$ with left extensions $\{0, 2\}$ and right extensions $\{1, 2\}$. We continue in the same fashion and thus complete the list of all bispecial factors

$$10 \to \varphi(10) \to 1\varphi^2(10)0 \to \varphi(1)\varphi^3(10)\varphi(0) \to 1\varphi^2(1)\varphi^4(10)\varphi^2(0)0 \to \cdots .$$

The following statement is an immediate consequence of Corollary 8.

Proposition 9. *Let $w \in \mathcal{L}(\mathbf{p}) \setminus \{\varepsilon\}$. If w is a bispecial factor, then w has one of the following forms:*

$\mathcal{A}$)

$$w_A^{(n)} = 1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0) \qquad \text{for } n \geqslant 1.$$

If $n = 0$, then we set $w_A^{(0)} = 1$.

The Parikh vector of $w_A^{(n)}$ is the same as of the factor $1\varphi(012)\varphi^3(012)\cdots\varphi^{2n-1}(012)$.

$\mathcal{B}$)
$$w_B^{(n)} = \varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0 \qquad \text{for } n \geqslant 0.$$
The Parikh vector of $w_B^{(n)}$ *is the same as of the factor* $012\varphi^2(012)\varphi^4(012)\cdots\varphi^{2n}(012)$.

$\mathcal{C}$)
$$w_C^{(n)} = 1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0 \qquad \text{for } n \geqslant 0.$$
The Parikh vector of $w_C^{(n)}$ *is the same as of the factor* $01\varphi^2(01)\varphi^4(01)\cdots\varphi^{2n}(01)$.

$\mathcal{D}$)
$$w_D^{(n)} = \varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n+1}(0)\varphi^{2n-1}(0)\cdots\varphi(0) \qquad \text{for } n \geqslant 0.$$
The Parikh vector of $w_D^{(n)}$ *is the same as of the factor* $\varphi(01)\varphi^3(01)\cdots\varphi^{2n+1}(01)$.

4.2 The shortest return words in p

Let us derive the form of the shortest return words to all bispecial factors in $\mathbf{p}$. We will make use of them when solving the same problem for $\psi(\mathbf{p})$ and $\xi(\mathbf{p})$.

- The return words to ε are 0, 1, 2.
- The return words to 1 are 12, 102, 10.
- The return words to 10 are 10, 102, 1012. The shortest one is 10 and it is a prefix of all of them.
- The return words to $\varphi(1)0$ are $210 = \varphi(1)0$, 21010, 2101. The shortest one is 210 and it is a prefix of all of them.

Using this knowledge and the knowledge of how the bispecial factors can be constructed, we obtain the following observation for the shortest return words:

$\mathcal{A}$) The shortest return words to $w_A^{(0)} = 1$ are 12 and 10. The shortest return word to $w_A^{(n)}$, $n \geqslant 1$, has the same Parikh vector as the factor $\varphi^{2n-1}(012)$.

$\mathcal{B}$) The shortest return word to $w_B^{(n)}$, $n \geqslant 0$, has the same Parikh vector as the factor $\varphi^{2n}(012)$.

$\mathcal{C}$) The shortest return word to $w_C^{(n)}$, $n \geqslant 0$, has the same Parikh vector as the factor $\varphi^{2n}(01)$.

$\mathcal{D}$) The shortest return word to $w_D^{(n)}$, $n \geqslant 0$, has the same Parikh vector as the factor $\varphi^{2n+1}(01)$.

4.3 Bispecial factors in $\psi(\mathbf{p})$

Let us start with some simple observations. If a factor $v \in \mathcal{L}(\psi(\mathbf{p}))$ contains the factor 11001 or 1101, then we are able to express $v = x\psi(w)y$ uniquely, where $w \in \mathcal{L}(\mathbf{p})$, and x (resp., y) is a proper suffix (resp., proper prefix) of the image of some letter. Moreover, if v is a bispecial factor in $\psi(\mathbf{p})$, then w is a bispecial factor in $\mathbf{p}$.

Observation 10. *Let $v \in \mathcal{L}(\psi(\mathbf{p}))$ be a bispecial factor in $\psi(\mathbf{p})$ such that it contains* 11001 *or* 1101*. Then one of the following claims holds:*

1. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 01\psi(w)0110$ and $0w, 2w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*
2. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \psi(w)0$ and $0w, 1w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
3. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = 01\psi(w)0$ and $0w, 2w, w1, w2 \in \mathcal{L}(\mathbf{p})$.*
4. *There exists $w \in \mathcal{L}(\mathbf{p})$ such that $v = \psi(w)0110$ and $0w, 1w, w0, w2 \in \mathcal{L}(\mathbf{p})$.*

Corollary 11. *Let $v \in \mathcal{L}(\psi(\mathbf{p}))$. If v is a bispecial factor containing* 11001 *or* 1101*, then v has one of the following forms:*

$\mathcal{A}$)

$$v_A^{(n)} = 01\psi(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n-1}(0)\varphi^{2n-3}(0)\cdots\varphi(0))0110 \qquad \text{for } n \geqslant 1.$$

The Parikh vector of $v_A^{(n)}$ is the same as that of the factor

$$000111\psi(1\varphi(012)\varphi^3(012)\cdots\varphi^{2n-1}(012)).$$

$\mathcal{B}$)

$$v_B^{(n)} = \psi(\varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0)0 \qquad \text{for } n \geqslant 0.$$

The Parikh vector of $v_B^{(n)}$ is the same as that of the factor

$$0\psi(012\varphi^2(012)\varphi^4(012)\cdots\varphi^{2n}(012)).$$

$\mathcal{C}$)

$$v_C^{(n)} = 01\psi(1\varphi^2(1)\varphi^4(1)\cdots\varphi^{2n}(1)\varphi^{2n}(0)\varphi^{2n-2}(0)\cdots\varphi^2(0)0)0 \quad \text{for } n \geqslant 0.$$

The Parikh vector of $v_C^{(n)}$ is the same as that of the factor $001\psi(01\varphi^2(01)\varphi^4(01)\cdots\varphi^{2n}(01))$.

$\mathcal{D}$)

$$v_D^{(n)} = \psi(\varphi(1)\varphi^3(1)\cdots\varphi^{2n+1}(1)\varphi^{2n+1}(0)\varphi^{2n-1}(0)\cdots\varphi(0))0110 \qquad \text{for } n \geqslant 0.$$

The Parikh vector of $v_D^{(n)}$ is the same as that of the factor $0011\psi(\varphi(01)\varphi^3(01)\cdots\varphi^{2n+1}(01))$.

4.4 The shortest return words in $\psi(\mathbf{p})$

Knowing the Parikh vectors of the shortest return words to bispecial factors in $\mathbf{p}$ and using the unambiguous reconstruction of w from $\psi(w)$ when $\psi(w)$ contains 11001 or 1101, we obtain the following observation for the shortest return words:

$\mathcal{A}$) The shortest return word $r_A^{(n)}$ to $v_A^{(n)}$, $n \geqslant 1$, has the same Parikh vector as the factor $\psi(\varphi^{2n-1}(012))$.

$\mathcal{B}$) The shortest return word $r_B^{(n)}$ to $v_B^{(n)}$, $n \geqslant 0$, has the same Parikh vector as the factor $\psi(\varphi^{2n}(012))$.

$\mathcal{C}$) The shortest return word $r_C^{(n)}$ to $v_C^{(n)}$, $n \geqslant 0$, has the same Parikh vector as the factor $\psi(\varphi^{2n}(01))$.

$\mathcal{D}$) The shortest return word $r_D^{(n)}$ to $v_D^{(n)}$, $n \geqslant 0$, has the same Parikh vector as the factor $\psi(\varphi^{2n+1}(01))$.

4.5 Critical exponent of $\psi(\mathbf{p})$

Having determined the lengths of bispecial factors and of their shortest return words in $\psi(\mathbf{p})$, we can use Theorem 5 to compute the critical exponent of $\psi(\mathbf{p})$:

$$\mathrm{ce}(\psi(\mathbf{p})) = 1 + \max\{A, B, C, D, E\}$$

$$A = \sup\left\{\frac{|v_A^{(n)}|}{|r_A^{(n)}|} : n \geqslant 1\right\} = \sup\left\{\frac{|000111\psi(1\varphi(012)\varphi^3(012)\cdots\varphi^{2n-1}(012))|}{|\psi(\varphi^{2n-1}(012))|} : n \geqslant 1\right\};$$

$$B = \sup\left\{\frac{|v_B^{(n)}|}{|r_B^{(n)}|} : n \geqslant 0\right\} = \sup\left\{\frac{|0\psi(012\varphi^2(012)\varphi^4(012)\cdots\varphi^{2n}(012))|}{|\psi(\varphi^{2n}(012))|} : n \geqslant 0\right\};$$

$$C = \sup\left\{\frac{|v_C^{(n)}|}{|r_C^{(n)}|} : n \geqslant 0\right\} = \sup\left\{\frac{|001\psi(01\varphi^2(01)\varphi^4(01)\cdots\varphi^{2n}(01))|}{|\psi(\varphi^{2n}(01))|} : n \geqslant 0\right\};$$

$$D = \sup\left\{\frac{|v_D^{(n)}|}{|r_D^{(n)}|} : n \geqslant 0\right\} = \sup\left\{\frac{|0011\psi(\varphi(01)\varphi^3(01)\cdots\varphi^{2n+1}(01))|}{|\psi(\varphi^{2n+1}(01))|} : n \geqslant 0\right\};$$

$$E = \left\{\frac{|w|}{|r|} : w \text{ bispecial not containing 11001 or 1101, } r \text{ the shortest return word to } w\right\}.$$

4.5.1 Computation of E

As summarized in Table 4.5.1, for each bispecial w that does not contain 11001 and 1101 and its shortest return word r, we have $\frac{|w|}{|r|} \leqslant \frac{4}{3} \doteq 1.333$. Therefore $E = 1 + \frac{1}{3}$.

w	0	1	01	10	101	010	0110	1001
r	0	1	01	10	101	01011	011001	100

Table 1: Short bispecial factors w and their return words r

4.5.2 Computation of A and B

If we consider $a_n := |\psi(\varphi^n(012))|$, then we can see that a_n satisfies the recurrence relation $a_{n+1} = 2a_n - a_{n-1} + a_{n-2}$ with initial conditions $a_0 = 12, a_1 = 19$, and $a_2 = 32$.

This recurrence relation may be solved and we obtain

$$a_n = A_1\beta_1^n + B_1\lambda_1^n + C_1\lambda_2^n,$$

where

$$\beta_1 \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

are the roots of the polynomial $t^3 - 2t^2 + t - 1$, and

$$\begin{aligned} A_1 &= \frac{12|\lambda_1|^2 - 38\,\mathrm{Re}(\lambda_1) + 32}{|\beta_1 - \lambda_1|^2} \doteq 10.6175; \\ B_1 &= \frac{12\beta_1\lambda_2 - 19(\beta_1 + \lambda_2) + 32}{(\beta_1 - \lambda_1)(\lambda_2 - \lambda_1)} \doteq 0.6912 - 0.1330i; \\ C_1 &= \overline{B_1}. \end{aligned}$$

Let us first determine

$$A' := \limsup_{n\to+\infty} \frac{|v_A^{(n)}|}{|r_A^{(n)}|} = \limsup_{n\to+\infty} \frac{|000111\psi(1\varphi(012)\varphi^3(012)\cdots\varphi^{2n-1}(012))|}{|\psi(\varphi^{2n-1}(012))|}.$$

We have

$$\begin{aligned} A' &= \limsup_{n\to+\infty} \frac{6 + 1 + \sum_{k=1}^{n} a_{2k-1}}{a_{2n-1}} \\ &= \limsup_{n\to+\infty} \frac{7 + A_1\sum_{k=1}^{n}\beta_1^{2k-1} + B_1\sum_{k=1}^{n}\lambda_1^{2k-1} + C_1\sum_{k=1}^{n}\lambda_2^{2k-1}}{A_1\beta_1^{2n-1} + B_1\lambda_1^{2n-1} + C_1\lambda_2^{2n-1}} \\ &= \lim_{n\to+\infty} \frac{7 + A_1\beta_1\frac{\beta_1^{2n}-1}{\beta_1^2-1} + B_1\lambda_1\frac{\lambda_1^{2n}-1}{\lambda_1^2-1} + C_1\lambda_2\frac{\lambda_2^{2n}-1}{\lambda_2^2-1}}{A_1\beta_1^{2n-1} + B_1\lambda_1^{2n-1} + C_1\lambda_2^{2n-1}} = \frac{\beta_1^2}{\beta_1^2 - 1} \doteq 1.4809. \end{aligned}$$

Next we show that $A \leqslant A'$ and thus $A = A'$. We want to show for all $n \geqslant 1$ that

$$\frac{7 + A_1\sum_{k=1}^{n}\beta_1^{2k-1} + B_1\sum_{k=1}^{n}\lambda_1^{2k-1} + C_1\sum_{k=1}^{n}\lambda_2^{2k-1}}{A_1\beta_1^{2n-1} + B_1\lambda_1^{2n-1} + C_1\lambda_2^{2n-1}} \quad \leqslant^{?} \quad \frac{\beta_1^2}{\beta_1^2 - 1}$$

$$(\beta_1^2-1)\left(7+2\operatorname{Re}\left(B_1\lambda_1\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right)\right)+A_1\beta_1^{2n+1}-A_1\beta_1 \quad \leqslant^{?} \quad 2\beta_1^2\operatorname{Re}(B_1\lambda_1^{2n-1})+A_1\beta_1^{2n+1}$$

$$(\beta_1^2-1)\left(7+2\operatorname{Re}\left(B_1\lambda_1\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right)\right) \quad \leqslant^{?} \quad 2\beta_1^2\operatorname{Re}(B_1\lambda_1^{2n-1})+A_1\beta_1.$$

Now, on the one hand, for $n \geqslant 2$, we have

$$\begin{aligned}(\beta_1^2-1)\left(7+2\operatorname{Re}\left(B_1\lambda_1\frac{\lambda_1^{2n}-1}{\lambda_1^2-1}\right)\right) &\leqslant (\beta_1^2-1)\left(7+2|B_1||\lambda_1|\frac{|\lambda_1|^{2n}+1}{|\lambda_1^2-1|}\right)\\ &\leqslant (\beta_1^2-1)\left(7+2|B_1||\lambda_1|\frac{|\lambda_1|^{4}+1}{|\lambda_1^2-1|}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}2\beta_1^2\operatorname{Re}(B_1\lambda_1^{2n-1})+A_1\beta_1 &\geqslant A_1\beta_1-2\beta_1^2|B_1||\lambda_1|^{2n-1}\\ &\geqslant A_1\beta_1-2\beta_1^2|B_1||\lambda_1|^{3}.\end{aligned}$$

And if we substitute the values, we get

$$(\beta_1^2-1)\left(7+2|B_1||\lambda_1|\frac{|\lambda_1|^{4}+1}{|\lambda_1^2-1|}\right)\leqslant A_1\beta_1-2\beta_1^2|B_1||\lambda_1|^{3}.$$

For $n=1$, we get $\frac{7+a_1}{a_1}=1+\frac{7}{19}<A'$.

Therefore

$$A=A'=\frac{\beta_1^2}{\beta_1^2-1}.$$

Next, we use the same procedure for B.

$$\begin{aligned}B' &:= \limsup_{n\to+\infty}\frac{|v_B^{(n)}|}{|r_B^{(n)}|}\\ &= \limsup_{n\to+\infty}\frac{|0\psi(012\varphi^2(012)\varphi^4(012)\cdots\varphi^{2n}(012))|}{|\psi(\varphi^{2n}(012))|}\\ &= \limsup_{n\to+\infty}\frac{1+\sum_{k=0}^{n}a_{2k}}{a_{2n}}\\ &= \limsup_{n\to+\infty}\frac{1+A_1\sum_{k=0}^{n}\beta_1^{2k}+B_1\sum_{k=0}^{n}\lambda_1^{2k}+C_1\sum_{k=0}^{n}\lambda_2^{2k}}{A_1\beta_1^{2n}+B_1\lambda_1^{2n}+C_1\lambda_2^{2n}}\\ &= \frac{\beta_1^2}{\beta_1^2-1}.\end{aligned}$$

Now we show that $B = B'$, because for all $n \geqslant 0$ we have

$$\begin{aligned}
\frac{1 + A_1 \sum_{k=0}^{n} \beta_1^{2k} + B_1 \sum_{k=0}^{n} \lambda_1^{2k} + C_1 \sum_{k=0}^{n} \lambda_2^{2k}}{A_1\beta_1^{2n} + B_1\lambda_1^{2n} + C_1\lambda_2^{2n}} &\leqslant^{?} \frac{\beta_1^2}{\beta_1^2 - 1} \\
(\beta_1^2 - 1)\left(1 + 2\,\mathrm{Re}\left(B_1 \frac{\lambda_1^{2n+2} - 1}{\lambda_1^2 - 1}\right)\right) + A_1\beta_1^{2n+2} - A_1 &\leqslant^{?} 2\beta_1^2\,\mathrm{Re}(B_1\lambda_1^{2n}) + A_1\beta_1^{2n+2} \\
(\beta_1^2 - 1)\left(1 + 2\,\mathrm{Re}\left(B_1 \frac{\lambda_1^{2n+2} - 1}{\lambda_1^2 - 1}\right)\right) &\leqslant^{?} 2\beta_1^2\,\mathrm{Re}(B_1\lambda_1^{2n}) + A_1.
\end{aligned}$$

Now, on the one hand, for all $n \geqslant 2$, we have

$$\begin{aligned}
&(\beta_1^2 - 1)\left(1 + 2\,\mathrm{Re}\left(B_1 \frac{\lambda_1^{2n+2} - 1}{\lambda_1^2 - 1}\right)\right) \\
&\quad\leqslant (\beta_1^2 - 1)\left(1 + 2\,\mathrm{Re}\left(\frac{B_1}{1 - \lambda_1^2}\right) - 2\,\mathrm{Re}\left(B_1 \frac{\lambda_1^{2n+2}}{1 - \lambda_1^2}\right)\right) \\
&\quad\leqslant (\beta_1^2 - 1)\left(1 + 2\,\mathrm{Re}\left(\frac{B_1}{1 - \lambda_1^2}\right) + 2|B_1|\frac{|\lambda_1^{2n+2}|}{|1 - \lambda_1^2|}\right) \\
&\quad\leqslant (\beta_1^2 - 1)\left(1 + 2\,\mathrm{Re}\left(\frac{B_1}{1 - \lambda_1^2}\right) + 2|B_1|\frac{|\lambda_1|^6}{|1 - \lambda_1^2|}\right).
\end{aligned}$$

On the other hand,

$$2\beta_1^2\,\mathrm{Re}(B_1\lambda_1^{2n}) + A_1 \geqslant A_1 - 2\beta_1^2|B_1||\lambda_1|^4.$$

Substituting the values proves that

$$(\beta_1^2 - 1)\left(1 + 2\,\mathrm{Re}\left(\frac{B_1}{1 - \lambda_1^2}\right) + 2|B_1|\frac{|\lambda_1|^6}{|1 - \lambda_1^2|}\right) < A_1 - 2\beta_1^2|B_1||\lambda_1|^4.$$

For $n = 0$, we get $\frac{1+a_0}{a_0} = \frac{1+12}{12} < B'$. For $n = 1$, we get $\frac{1+a_0+a_2}{a_2} = 1 + \frac{1+12}{32} < B'$. The inequality $B \leqslant B'$ is proven, and therefore $B = B' = \frac{\beta_1^2}{\beta_1^2 - 1}$.

4.5.3 Computation of C and D

If we consider $c_n := |\psi(\varphi^n(01))|$, then we can see that c_n satisfies the same recurrence relation as a_n, i.e., $c_{n+1} = 2c_n - c_{n-1} + c_{n-2}$ with initial conditions $c_0 = 7, c_1 = 13$, and $c_2 = 25$.

This recurrence relation can be solved and we obtain

$$c_n = A_2\beta_1^n + B_2\lambda_1^n + C_2\lambda_2^n,$$

where

$$\beta_1 \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

are the roots of the polynomial $t^3 - 2t^2 + t - 1$, and

$$\begin{aligned} A_2 &= \frac{7|\lambda_1|^2 - 26\,\mathrm{Re}(\lambda_1) + 25}{|\beta_1 - \lambda_1|^2} \doteq 8.0149; \\ B_2 &= \frac{7\beta_1\lambda_2 - 13(\beta_1 + \lambda_2) + 25}{(\beta_1 - \lambda_1)(\lambda_2 - \lambda_1)} \doteq -0.5075 + 0.6315i; \\ C_2 &= \overline{B_2}. \end{aligned}$$

The proof of $C' = D' = \frac{\beta_1^2}{\beta_1^2 - 1}$ proceeds in a similar fashion as that for B' (resp., A'). To complete the proof, we need to show firstly $C \leqslant C'$, i.e., for all $n \geqslant 0$ that

$$\begin{aligned} \frac{3 + A_2 \sum_{k=0}^{n} \beta_1^{2k} + B_2 \sum_{k=0}^{n} \lambda_1^{2k} + C_2 \sum_{k=0}^{n} \lambda_2^{2k}}{A_2\beta_1^{2n} + B_2\lambda_1^{2n} + C_2\lambda_2^{2n}} &\leqslant^{?} \frac{\beta_1^2}{\beta_1^2 - 1} \\ (\beta_1^2 - 1)\left(3 + 2\,\mathrm{Re}\left(B_2\frac{\lambda_1^{2n+2} - 1}{\lambda_1^2 - 1}\right)\right) &\leqslant^{?} 2\beta_1^2\,\mathrm{Re}(B_2\lambda_1^{2n}) + A_2. \end{aligned}$$

Now, we have for all $n \geqslant 2$ that

$$\begin{aligned} (\beta_1^2 - 1)\left(3 + 2\,\mathrm{Re}\left(B_2\frac{\lambda_1^{2n+2} - 1}{\lambda_1^2 - 1}\right)\right) &\leqslant (\beta_1^2 - 1)\left(3 + 2\,\mathrm{Re}\left(\frac{B_2}{1 - \lambda_1^2}\right) + 2|B_2|\frac{|\lambda_1|^6}{|1 - \lambda_1^2|}\right); \\ 2\beta_1^2\,\mathrm{Re}(B_2\lambda_1^{2n}) + A_2 &\geqslant A_2 - 2\beta_1^2|B_2||\lambda_1|^4. \end{aligned}$$

Substituting the values proves that

$$(\beta_1^2 - 1)\left(3 + 2\,\mathrm{Re}\left(\frac{B_2}{1 - \lambda_1^2}\right) + 2|B_2|\frac{|\lambda_1|^6}{|1 - \lambda_1^2|}\right) < A_2 - 2\beta_1^2|B_2||\lambda_1|^4.$$

For $n = 0$, we get $\frac{3+c_0}{c_0} = 1 + \frac{3}{7} < C'$. For $n = 1$, we get $\frac{3+c_0+c_2}{c_2} = 1 + \frac{3+7}{25} < C'$. The inequality $C \leqslant C'$ is proven. Hence $C = C' = \frac{\beta_1^2}{\beta_1^2 - 1}$.

Finally, we need to prove $D \leqslant D'$, i.e., for all $n \geqslant 1$, that

$$\begin{aligned} \frac{4 + A_2 \sum_{k=1}^{n} \beta_1^{2k-1} + B_2 \sum_{k=1}^{n} \lambda_1^{2k-1} + C_2 \sum_{k=1}^{n} \lambda_2^{2k-1}}{A_2\beta_1^{2n-1} + B_2\lambda_1^{2n-1} + C_2\lambda_2^{2n-1}} &\leqslant^{?} \frac{\beta_1^2}{\beta_1^2 - 1} \\ (\beta_1^2 - 1)\left(4 + 2\,\mathrm{Re}\left(B_2\lambda_1\frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right)\right) &\leqslant^{?} 2\beta_1^2\,\mathrm{Re}(B_2\lambda_1^{2n-1}) + A_2\beta_1. \end{aligned}$$

Using the same estimates as for A and $n \geqslant 2$, it remains to check the following inequality

$$(\beta_1^2 - 1)\left(4 + 2|B_2||\lambda_1|\frac{|\lambda_1|^4 + 1}{|\lambda_1^2 - 1|}\right) \leqslant A_2\beta_1 - 2\beta_1^2|B_2||\lambda_1|^3.$$

The inequality holds for given values. For $n = 1$, we get $\frac{4+c_1}{c_1} = 1 + \frac{4}{13} < D'$. Consequently, $D = D'$.

We have shown that $A = B = C = D = \frac{\beta_1^2}{\beta_1^2 - 1}$ and $E = 1 + \frac{1}{3} < A$. In other words, we have proved the following theorem.

Theorem 12. *The critical exponent of $\psi(\mathbf{p})$ equals*

$$\mathrm{ce}(\psi(\mathbf{p})) = 1 + \frac{\beta_1^2}{\beta_1^2 - 1} = 2 + \frac{1}{\beta_1^2 - 1} \doteq 2.4808627161472369.$$

Let us emphasize that the critical exponents of $\mathbf{p}$ and $\psi(\mathbf{p})$ are the same [7].

4.6 Bispecial factors in $\xi(\mathbf{p})$

Now we turn to $\xi(\mathbf{p})$. Let us start with some simple observations. If a factor $v \in \mathcal{L}(\xi(\mathbf{p}))$ contains the factor 00 or 1011 or 11010, then we are able to express $v = x\xi(w)y$ uniquely, where $w \in \mathcal{L}(\mathbf{p})$ and x (resp., y) is a proper suffix (resp., proper prefix) of an image of some letter. Moreover, if v is a bispecial factor in $\xi(\mathbf{p})$, then w is a bispecial factor in $\mathbf{p}$.

Observation 13. *Let $v \in \mathcal{L}(\xi(\mathbf{p}))$ be a bispecial factor such that it contains 00 or 1011 or 11010. Assume $v \neq 1\xi(1) = 10110$ and $v \neq 1\xi(10) = 1011001$. Then there exists $w \in \mathcal{L}(\mathbf{p})$ such that either $v = \xi(w)$ and w is a bispecial factor with left extensions $\{0, 1\}$, or $v = 01\xi(w)$ and w is a bispecial factor with left extensions $\{0, 2\}$. Moreover, the length of the shortest return word to v in $\xi(\mathbf{p})$ equals $|\xi(r)|$, where r is the shortest return word to w in $\mathbf{p}$.*

4.7 Critical exponent of $\xi(\mathbf{p})$

Using Observation 13, the lengths of bispecial factors and their shortest return words in $\xi(\mathbf{p})$ may be derived in an analogous way as in the case of $\psi(\mathbf{p})$. Next, we can use Theorem 5 to compute the critical exponent of $\xi(\mathbf{p})$:

$$\begin{aligned}
\mathrm{ce}(\xi(\mathbf{p})) &= 1 + \max\{A, B, C, D, E\} \\
A &= \sup\left\{ \frac{|01\xi(1\varphi(012)\varphi^3(012)\cdots\varphi^{2n-1}(012))|}{|\xi(\varphi^{2n-1}(012))|} : n \geqslant 1 \right\}; \\
B &= \sup\left\{ \frac{|\xi(012\varphi^2(012)\varphi^4(012)\cdots\varphi^{2n}(012))|}{|\xi(\varphi^{2n}(012))|} : n \geqslant 0 \right\}; \\
C &= \sup\left\{ \frac{|01\xi(01\varphi^2(01)\varphi^4(01)\cdots\varphi^{2n}(01))|}{|\xi(\varphi^{2n}(01))|} : n \geqslant 0 \right\}; \\
D &= \sup\left\{ \frac{|\xi(\varphi(01)\varphi^3(01)\cdots\varphi^{2n+1}(01))|}{|\xi(\varphi^{2n+1}(01))|} : n \geqslant 0 \right\}; \\
E &= \max\left\{ \frac{|w|}{|r|} : w \text{ short bispecial and } r \text{ the shortest return word to } w \right\},
\end{aligned}$$

where we say that w is a *short bispecial factor* if w does not contain 00, 1011, 11010 or $w = 1\xi(1)$ or $w = 1\xi(10)$.

4.7.1 Computation of E

Thus, we have $E = \frac{3}{2}$.

w	0	1	01	10	101	0110	01101	$1\xi(1)$	$1\xi(10)$
r	0	1	01	10	10	011	011010110	$1\xi(1)$	$1\xi(10)1^{-1}$

4.7.2 Computation of A, B, C, D

If we consider $x_n := |\xi(\varphi^n(012))|$, then we can see that x_n satisfies the recurrence relation $x_{n+1} = 2x_n - x_{n-1} + x_{n-2}$ with initial conditions $x_0 = 7, x_1 = 13$, and $x_2 = 24$.

This recurrence relation may be solved and we obtain

$$x_n = X_1\beta_1^n + Y_1\lambda_1^n + Z_1\lambda_2^n,$$

where

$$\beta_1 \doteq 1.75488, \quad \lambda_1 \doteq 0.12256 + 0.74486i, \quad \lambda_2 = \overline{\lambda_1}$$

are the roots of the polynomial $t^3 - 2t^2 + t - 1$, and

$$\begin{aligned} X_1 &= \frac{7|\lambda_1|^2 - 26\,\mathrm{Re}(\lambda_1) + 24}{|\beta_1 - \lambda_1|^2} \doteq 7.704; \\ Y_1 &= \frac{7\beta_1\lambda_2 - 13(\beta_1 + \lambda_2) + 24}{(\beta_1 - \lambda_1)(\lambda_2 - \lambda_1)} \doteq -0.352 + 0.291i; \\ Z_1 &= \overline{Y_1}. \end{aligned}$$

We want to show for all $n \geqslant 1$ that

$$\frac{6 + X_1\sum_{k=1}^n \beta_1^{2k-1} + Y_1\sum_{k=1}^n \lambda_1^{2k-1} + Z_1\sum_{k=1}^n \lambda_2^{2k-1}}{X_1\beta_1^{2n-1} + Y_1\lambda_1^{2n-1} + Z_1\lambda_2^{2n-1}} \quad \leqslant^? \quad \frac{\beta_1^2}{\beta_1^2 - 1} < \frac{3}{2}$$

$$\begin{gathered} (\beta_1^2 - 1)\left(6 + 2\,\mathrm{Re}\left(Y_1\lambda_1\frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right)\right) + X_1\beta_1^{2n+1} - X_1\beta_1 \quad \leqslant^? \\ 2\beta_1^2\,\mathrm{Re}(Y_1\lambda_1^{2n-1}) + X_1\beta_1^{2n+1} \end{gathered}$$

$$(\beta_1^2 - 1)\left(6 + 2\,\mathrm{Re}\left(Y_1\lambda_1\frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right)\right) \quad \leqslant^? \quad 2\beta_1^2\,\mathrm{Re}(Y_1\lambda_1^{2n-1}) + X_1\beta_1.$$

Now, on one hand, for $n \geqslant 7$ we have

$$(\beta_1^2 - 1)\left(6 + 2\,\mathrm{Re}\left(Y_1\lambda_1\frac{\lambda_1^{2n} - 1}{\lambda_1^2 - 1}\right)\right) \leqslant (\beta_1^2 - 1)\left(6 + 2|Y_1||\lambda_1|\frac{|\lambda_1|^{14} + 1}{|\lambda_1^2 - 1|}\right).$$

On the other hand, we have

$$2\beta_1^2\,\mathrm{Re}(Y_1\lambda_1^{2n-1}) + X_1\beta_1 \geqslant X_1\beta_1 - 2\beta_1^2|Y_1||\lambda_1|^{13}.$$

And if we substitute the values, we get

$$(\beta_1^2 - 1)\left(6 + 2|Y_1||\lambda_1|\frac{|\lambda_1|^{14} + 1}{|\lambda_1^2 - 1|}\right) \leqslant X_1\beta_1 - 2\beta_1^2|Y_1||\lambda_1|^{13}.$$

n	1	2	3	4	5	6
A_n	1.4615	1.4524	1.4766	1.4785	1.4803	1.4806

Table 2: The first 6 elements of A.

The first 6 values from the set A are given in Table 2. We can see that they are all smaller than $\frac{3}{2}$. The inequality therefore holds for all $n \geqslant 1$.

We have shown that $A < \frac{3}{2} = E$. Using the same estimates, we can show that also the remaining values B, C, D are smaller than $\frac{3}{2}$. In other words, we have proved the following theorem.

Theorem 14. *The critical exponent of $\xi(\mathbf{p})$ equals $\frac{5}{2}$.*

5 Complexity threshold

Notice that we used two different constructions of infinite words in the proof of Theorem 3. When possible, that is, in cases (b), (c), and (f), we use morphic images of arbitrary ternary square-free words, thus showing that exponentially many binary words have the considered property. Otherwise, in cases (a), (d), and (e), every bi-infinite binary word with the considered property has the same set of factors as one or a few morphic words.

In this section, we consider the latter case and we relax the constraints on the critical exponent or the complementary factors in order to get exponential factor complexity.

Let h be the morphism that maps $0 \mapsto 0$ and $1 \mapsto 01$.

Lemma 15. *Every bi-infinite binary 4-free word avoiding pairs of complementary factors of length 4 and $\{1001001, 0110110\}$ has the same set of factors as either $h(\mathbf{f})$ or $\overline{h(\mathbf{f})}$.*

Proof. We compute the set S of factors y such that there exists a binary 4-free word xyz avoiding pairs of complementary factors of length 4 and $\{1001001, 0110110\}$ and such that $|x| = |y| = |z| = 100$. We observe that S consists of the (disjoint) union of the factors of length 100 of $h(\mathbf{f})$ and $\overline{h(\mathbf{f})}$. Let $\mathbf{w}$ be a bi-infinite binary 4-free word avoiding pairs of complementary factors of length 4 and $\{1001001, 0110110\}$. By symmetry, we suppose that $\mathbf{w}$ contains 00, so that $\mathbf{w}$ has the same factors of length 100 as $h(\mathbf{f})$. So $\mathbf{w} \in \{0, 01\}^\omega$, that is, $\mathbf{w} = h(\mathbf{v})$ for some bi-infinite binary word $\mathbf{v}$. Since $\mathbf{w}$ is 4-free, so is $\mathbf{v}$. Moreover, by considering the pre-images by h of $\mathbf{w} = h(\mathbf{v})$ and $h(\mathbf{f})$, this implies that $\mathbf{v}$ and $\mathbf{f}$ have the same set of factors of length $100/\max(|h(0)|, |h(1)|) = 50$. In particular, $\mathbf{v}$ avoids $\{11, 000, 10101\}$. By [2, Thm. 8], $\mathbf{v}$ has the same set of factors as $\mathbf{f}$. Thus $\mathbf{w}$ has the same set of factors as $h(\mathbf{f})$. □

Theorem 16. *Every bi-infinite binary $\frac{11}{3}$-free word that contains no pair of complementary factors of length 4 has the same set of factors as $h(\mathbf{f})$ or the same set of factors as $\overline{h(\mathbf{f})}$.*

Proof. Suppose that such a word $\mathbf{w}$ contains 1001001.

First, suppose that $\mathbf{w}$ contains 11. Since $\mathbf{w}$ avoids $0110 = \overline{1001}$, $\mathbf{w}$ contains 111. Since $\mathbf{w}$ avoids the 4-power 1111, $\mathbf{w}$ contains 01110. Since $\mathbf{w}$ avoids $1101 = \overline{0010}$, $\mathbf{w}$ contains 011100. Symmetrically, since $\mathbf{w}$ avoids $1011 = \overline{0100}$, $\mathbf{w}$ contains 0011100. So $\mathbf{w}$ contains both 0011 and 1100, a contradiction. Thus, $\mathbf{w}$ avoids 11.

Now, suppose that $\mathbf{w}$ contains 101. Since $\mathbf{w}$ avoids 11, $\mathbf{w}$ contains 01010. So $\mathbf{w}$ contains both 0101 and 1010, a contradiction. Thus, $\mathbf{w}$ avoids 101.

Since $\mathbf{w}$ contains 1001001 and avoids 11, $\mathbf{w}$ contains 010010010. Since $\mathbf{w}$ contains 010010010 and avoids 101, $\mathbf{w}$ contains 00100100100. This is a contradiction, since $00100100100 = (001)^{11/3}$.

So $\mathbf{w}$ avoids 1001001. By symmetry, $\mathbf{w}$ avoids 0110110.

By Lemma 15, $\mathbf{w}$ has the same set of factors as either $h(\mathbf{f})$ or $\overline{h(\mathbf{f})}$. □

By contrast, there exist exponentially many binary $\frac{11}{3}^+$-free words with no pair of complementary factors of length 4.

Theorem 17. *The image of any ternary squarefree word by the 31-uniform morphism*[2]

$$\begin{aligned} 0 &\to 0010001001001000100100100010010 \\ 1 &\to 0010001001001000100100100010001 \\ 2 &\to 0010001001001000100100010010010 \end{aligned}$$

is a $\frac{11}{3}^+$-free word containing only $0, 1, 01$, and 10 as complementary factors.

Let ρ be the morphism defined as follows

$$\begin{aligned} 0 &\to 01100101101 \\ 1 &\to 0110010 \\ 2 &\to 011001. \end{aligned}$$

Using the technique of Theorem 3 (d), we can show that $\rho(\mathbf{p})$ is a $\frac{5}{2}^+$-free word with no pair of complementary factors of length 8 and exactly 40 complementary factors.

Theorem 18. *Every bi-infinite binary $\frac{29}{11}$-free word that contains no pair of complementary factors of length 8 has the same set of factors as either $\xi(\mathbf{p})$, $\overline{\xi(\mathbf{p})}$, $\xi(\mathbf{p})^R$, $\overline{\xi(\mathbf{p})^R}$, $\rho(\mathbf{p})$, $\overline{\rho(\mathbf{p})}$, $\rho(\mathbf{p})^R$, or $\overline{\rho(\mathbf{p})^R}$.*

Proof. First, we show the following. If $\mathbf{w}$ is a bi-infinite cube-free binary word and every factor of length 21 of $\mathbf{w}$ is also a factor of $\xi(\mathbf{p})$, then $\mathbf{w}$ has the same set of factors as $\xi(\mathbf{p})$.

We compute the set S of factors f such that there exists $u = efg$, where u is cube-free, every factor of length 21 of u is a factor of $\xi(\mathbf{p})$, and $|e| = |f| = |g| = 500$. We verify that every factor $f \in S$ is a factor of $\xi(\mathbf{p})$.

[2]We remind the reader that the program used to find the uniform morphisms in this paper, including this section, is available at LINK_TO_LOCATION_ON_ELECJC_PAGE.

This means that $\mathbf{w} = \xi(\mathbf{v})$ for some bi-infinite ternary word $\mathbf{v}$. Moreover, by considering the pre-images by ξ of $\mathbf{w} = \xi(\mathbf{v})$ and $\xi(\mathbf{p})$, this implies that $\mathbf{v}$ and $\mathbf{p}$ have the same set of factors of length $500/\max(|\xi(0)|, |\xi(1)|, |\xi(2)|) = 125$. In particular, $\mathbf{v}$ has the same set of factors as $\mathbf{p}$ by [7, Theorem 14].

Similarly, we show that if $\mathbf{w}$ is a bi-infinite cube-free binary word and every factor of length 63 of $\mathbf{w}$ is also a factor of $\rho(\mathbf{p})$, then $\mathbf{w}$ has the same set of factors as $\rho(\mathbf{p})$.

Finally, we compute the set X of factors f such that there exists $u = efg$, where u is $\frac{29}{11}$-free, u contains no pair of complementary factors of length 8, $|f| = 200$, and $|e| = |g| = 80$. We verify that every factor $f \in X$ is a factor of either $\xi(\mathbf{p})$, $\overline{\xi(\mathbf{p})}$, $\xi(\mathbf{p})^R$, $\xi(\mathbf{p})^R$, $\rho(\mathbf{p})$, $\overline{\rho(\mathbf{p})}$, $\rho(\mathbf{p})^R$, or $\overline{\rho(\mathbf{p})^R}$. Since $\frac{29}{11}$-free words are cube-free, Theorem 18 follows by the two previous results and the symmetries by complement and reversal. □

By contrast, there exist exponentially many binary $\frac{29}{11}^+$-free words containing no pair of complementary factors of length 8 and exactly 36 complementary factors.

Theorem 19. *The image of any ternary squarefree word by the 84-uniform morphism*

$$0 \to 100101100100110010100101100101001011001001100101100101001011001001100101100100110010$$
$$1 \to 100101100100110010100101100101001011001001100101100101001011001001100101001011001001$$
$$2 \to 100101100100110010100101100101001011001001100101100101001011001001100101100101101001$$

is a $\frac{29}{11}^+$-free word with no pair of complementary factors of length 8 and exactly 36 complementary factors.

We also get exponentially many words if we allow complementary factors of length 8.

Theorem 20. *The image of any ternary squarefree word by the 31-uniform morphism*

$$0 \to 0010100110010011001010011001011$$
$$1 \to 0010100101100101001100101001011$$
$$2 \to 0010011001011001010010110010011$$

is a $\frac{5}{2}^+$-free word with no pair of complementary factors of length 9 and exactly 40 complementary factors.

References

[1] G. Badkobeh and M. Crochemore. Finite-repetition threshold for infinite ternary words. In P. Ambrož, Š. Holub, and Z. Masáková, editors, *WORDS 2011*, volume 63 of *Elec. Proc. Theor. Comput. Sci.*, pages 37–43. Open Publishing Association, 2011.

[2] A. Baranwal, J. D. Currie, L. Mol, P. Ochem, N. Rampersad, and J. Shallit. Antisquares and critical exponents. *Discrete Math. & Theoret. Comput. Sci.*, 25(2):DMTCS–25–2–23, 2023.

[3] J. Berstel. Fibonacci words—a survey. In G. Rozenberg and A. Salomaa, editors, *The Book of L*, pages 13–27. Springer-Verlag, 1986.

[4] A. Carpi and A. de Luca. Special factors, periodicity, and an application to Sturmian words. *Acta Informatica*, 36:983–1006, 2000.

[5] J. Currie and N. Rampersad. A proof of Dejean's conjecture. *Math. Comp.*, 80:1063–1070, 2011.

[6] J. D. Currie, L. Mol, and N. Rampersad. The repetition threshold for binary rich words. *Discrete Math. & Theoret. Comput. Sci.*, 22(1):DMTCS–22–1–6, 2020.

[7] J. D. Currie, P. Ochem, N. Rampersad, and J. Shallit. Properties of a ternary infinite word. *RAIRO Inform. Théor. App.*, 57:1, 2023.

[8] F. Dejean. Sur un théorème de Thue. *J. Combin. Theory. Ser. A*, 13:90–99, 1972.

[9] F. Dolce, L. Dvořáková, and E. Pelantová. On balanced sequences and their critical exponent. *Theoret. Comput. Sci.*, 939:18–47, 2023.

[10] L. Dvořáková, P. Ochem, and D. Opočenská. Critical exponent of binary words with few distinct palindromes. *Electronic J. Combinatorics*, 31(2):#P2.29, 2024.

[11] L. Dvořáková, D. Opočenská, E. Pelantová, and A. M. Shur. On minimal critical exponent of balanced sequences. *Theoret. Comput. Sci.*, 922:158–169, 2022.

[12] L. Dvořáková, K. Medková, and E. Pelantová. Complementary symmetric Rote sequences: the critical exponent and the recurrence function. *Discrete Math. & Theoret. Comput. Sci.*, 22(1):DMTCS–22–1–20, 2020.

[13] F. Fiorenzi, P. Ochem, and E. Vaslet. Bounds for the generalized repetition threshold. *Theoret. Comput. Sci.*, 412:2955–2963, 2011.

[14] L. Ilie, P. Ochem, and J. Shallit. A generalization of repetition threshold. *Theoret. Comput. Sci.*, 345:359–369, 2005.

[15] J. Karhumäki and J. Shallit. Polynomial versus exponential growth in repetition-free binary words. *J. Combin. Theory. Ser. A*, 105(2):335–347, 2004.

[16] L. Mol, N. Rampersad, and J. Shallit. Extremal overlap-free and extremal β-free binary words. *Electronic J. Combinatorics*, 27(4):#P4.42, 2020.

[17] H. Mousavi and J. Shallit. Repetition avoidance in circular factors. In M.-P. Béal and O. Carton, editors, *DLT 2013*, volume 7907 of *Lecture Notes in Computer Science*, pages 384–395. Springer-Verlag, 2013.

[18] N. Rampersad, J. Shallit, and É. Vandomme. Critical exponents of infinite balanced words. *Theoret. Comput. Sci.*, 777:454–463, 2019.

[19] M. Rao. Last cases of Dejean's conjecture. *Theoret. Comput. Sci.*, 412:3010–3018, 2011.

[20] A. V. Samsonov and A. M. Shur. On Abelian repetition threshold. *RAIRO Inform. Théor. App.*, 46:147–163, 2012.

[21] J. Sawada, B. Stevens, and A. Williams. De Bruijn sequences for the binary strings with maximum density. In N. Katoh and A. Kumar, editors, *WALCOM 2011*, volume 6552 of *Lecture Notes in Computer Science*, pages 182–190. Springer-Verlag, 2011.

[22] J. Shallit. Simultaneous avoidance of large squares and fractional powers in infinite binary words. *Internat. J. Found. Comp. Sci.*, 15:317–327, 2004.

[23] J. Shallit. Minimal critical exponents for palindromes. arXiv:1612.05320, 2016.

[24] A. Shur. Growth properties of power-free languages. *Comput. Sci. Rev.*, 6:187–208, 2012.