# Revisiting Extremal Graphs Having No Stable Cutsets

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**Abstract**

Confirming a conjecture posed by Caro, it was shown by Chen and Yu that every graph $G$ with $n$ vertices and at most $2n-4$ edges has a stable cutset, which is a stable set of vertices whose removal disconnects the graph. Le and Pfender showed that all graphs with $n$ vertices and $2n-3$ edges without stable cutset arise from recursively gluing together triangles and triangular prisms along an edge or triangle. Le and Pfender's proof contains a gap, which we fill in the present article.

**Mathematics Subject Classifications:** 05C40, 05C69

## 1 Introduction

We consider only finite, simple, and undirected graphs and refer to [9] for further notational details. A *stable cutset* in a graph $G$ is a stable set $S$ of vertices of $G$ for which $G-S$ is disconnected. By an elegant inductive argument, Chen and Yu [3] showed the following result confirming a conjecture by Caro.

**Theorem 1** (Chen and Yu [3]). *If $G$ is a graph with $n$ vertices and at most $2n-4$ edges, then $G$ contains a stable cutset.*

Le and Pfender [9] gave an elegant structural characterization of the graphs $G$ of order $n$ with $2n-3$ edges that do not contain a stable cutset, cf. Theorem 4 below. Our present goal is to fill a gap in the original proof given by Le and Pfender for Theorem 4. Stable cutsets were considered in a number of publications concerning structural refinements, algorithmic complexity, tractable cases, fixed parameter tractability, and their relation to perfect graphs [1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 13].

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## 2 Graphs $G$ with $2n(G) - 3$ edges and no stable cutset

We recall definitions and results from [9], formulate the main result Theorem 4, and provide a proof, in which we also explain the gap in the original proof.

If $G$ and $H$ are two graphs and $G \cap H$ is isomorphic to $K_2$ or $K_3$, then $G \cup H$ is said to arise from $G$ and $H$ by an *edge identification* or a *triangle identification*, respectively. Le and Pfender [9] define the class $\mathcal{G}_{sc}$ of graphs recursively as follows:

- $K_3, \overline{C_6} \in \mathcal{G}_{sc}$.
- If $G, H \in \mathcal{G}_{sc}$ and $G \cap H$ is isomorphic to $K_2$ or $K_3$, then $G \cup H \in \mathcal{G}_{sc}$.

They already observe that $H$ may be restricted to $\left\{K_3, \overline{C_6}\right\}$ without changing $\mathcal{G}_{sc}$.

For a positive integer $k$, let $[k]$ denote the set of positive integers at most $k$.

A *generating sequence* for a graph $G$ is a sequence $(G_1, \ldots, G_k)$ for some positive integer $k$ such that

- for every $i \in [k]$, the graph $G_i$ is isomorphic to $K_3$ or $\overline{C_6}$,
- for every $i \in [k-1]$, the graph $G_{\leqslant i} \cap G_{i+1}$ is isomorphic to $K_2$ or $K_3$, where $G_{\leqslant i} = G_1 \cup \ldots \cup G_i$, and
- $G = G_{\leqslant k}$.

A simple inductive argument using the recursive definition of $\mathcal{G}_{sc}$ implies that every graph with a generating sequence belongs to $\mathcal{G}_{sc}$. Le and Pfender's mentioned observation is that every graph in $\mathcal{G}_{sc}$ has a generating sequence. We need the following strengthening of this statement.

**Lemma 2.** *Every graph $G$ in $\mathcal{G}_{sc}$ has a generating sequence $(G_1, \ldots, G_k)$. Furthermore, for every $i \in [k]$, the graph $G$ has a generating sequence $(H_1, \ldots, H_k)$ with $H_1 = G_i$.*

*Proof.* The proof is by induction on the order of $G$ using the original recursive definition of $\mathcal{G}_{sc}$. If $G$ is isomorphic to $K_3$ or $\overline{C_6}$, then $(G_1) = (G)$ is a generating sequence for $G$, and the second statement is trivially true.

Now, let $G = G^{(1)} \cup G^{(2)}$ be such that $G^{(1)}, G^{(2)} \in \mathcal{G}_{sc}$ are proper subgraphs of $G$ and $G^{(1)} \cap G^{(2)}$ is isomorphic to $K_2$ or $K_3$, that is, the two graphs $G^{(1)}$ and $G^{(2)}$ share exactly two or three vertices that form a clique in both graphs. By induction, the graph $G^{(1)}$ has a generating sequence $\left(G_1^{(1)}, \ldots, G_k^{(1)}\right)$ and the graph $G^{(2)}$ has a generating sequence $\left(G_1^{(2)}, \ldots, G_\ell^{(2)}\right)$. Let $i \in [\ell]$ be such that the edge or triangle $G^{(1)} \cap G^{(2)}$ is a subgraph of $G_i^{(2)}$; the existence of such an index $i$ follows immediately from the definition of $\mathcal{G}_{sc}$. By the second statement, the graph $G^{(2)}$ has a generating sequence $\left(H_1^{(2)}, \ldots, H_\ell^{(2)}\right)$ with $H_1^{(2)} = G_i^{(2)}$. Now, the sequence $\left(G_1^{(1)}, \ldots, G_k^{(1)}, H_1^{(2)}, \ldots, H_\ell^{(2)}\right)$ is a generating sequence $(G_1, \ldots, G_{k+\ell})$ for $G$.

For the second statement, it remains to show that, for every $j \in [k+\ell]$, the graph $G$ has a generating sequence starting with $G_j$. If $j \in [k]$, then, by the second statement, the graph $G^{(1)}$ has a generating sequence $\left(H_1^{(1)}, \ldots, H_k^{(1)}\right)$ with $H_1^{(1)} = G_j^{(1)} = G_j$, and the sequence $\left(H_1^{(1)}, \ldots, H_k^{(1)}, H_1^{(2)}, \ldots, H_\ell^{(2)}\right)$ is a generating sequence for $G$ starting with $G_j$. Now, let $j \in [k+\ell] \setminus [k]$. By the second statement, the graph $G^{(2)}$ has a generating sequence $\left(I_1^{(2)}, \ldots, I_\ell^{(2)}\right)$ with $I_1^{(2)} = H_{j-k}^{(2)} = G_j$. Similarly as above, there is some $i \in [\ell]$ such that the edge or triangle $G^{(1)} \cap G^{(2)}$ is a subgraph of $G_i^{(1)}$. By the second statement, the graph $G^{(1)}$ has a generating sequence $\left(H_1^{(1)}, \ldots, H_k^{(1)}\right)$ with $H_1^{(1)} = G_i^{(1)}$. Now, the sequence $\left(I_1^{(2)}, \ldots, I_\ell^{(2)}, H_1^{(1)}, \ldots, H_k^{(1)}\right)$ is a generating sequence for $G$ starting with $G_j$. This completes the proof. □

The definition of $\mathcal{G}_{sc}$ easily implies that the only possible induced cycles of graphs in this class are $C_3$ and $C_4$. Readers acquainted with the notion of a *tree decomposition* might like to think about the graphs in $\mathcal{G}_{sc}$ in terms of this notion. In fact, the definition easily implies that a graph $G$ belongs to $\mathcal{G}_{sc}$ if and only if it has a tree decomposition where each bag induces $K_3$ or $\overline{C_6}$ and the intersection of two adjacent bags is either empty or induces a $K_2$ or a $K_3$.

From the main result of Chen and Yu [3], Le and Pfender [9] deduce the following.

**Corollary 3** (Le and Pfender, Corollary 3 in [9])**.** *Let $G$ be a graph of order $n$ with at most $2n-4$ edges and let $x$ be a vertex of $G$. Unless $x$ is the unique cut vertex in $G$, the graph $G$ has a stable cutset not containing $x$.*

The following is the main result from [9].

**Theorem 4** (Le and Pfender, Theorem 5 in [9])**.** *If $G$ is a graph of order $n$ with at most $2n-3$ edges, then $G$ has a stable cutset or belongs to $\mathcal{G}_{sc}$.*

*Proof.* For a proof by contradiction as in [9], we assume that $G$ is a counterexample of minimum order $n$. The following properties of $G$ are deduced in [9], where we use the same numbering of the claims as in [9]:

**Claim 6.** *$G$ has exactly $2n-3$ edges.*

**Claim 7.** *Every vertex of $G$ lies in a triangle.*

**Claim 8.** *$G$ contains no $K_2$-cutset or $K_3$-cutset.*

**Claim 9.** *$G$ is 3-connected.*

**Claim 10.** *$G$ contains no 3-edge matching cut, which is an edge cut with three edges that is also a matching.*

**Claim 11.** *$G$ contains no $K_4^-$.*

**Claim 12.** *For every two non-adjacent vertices $x$ and $y$, we have $|N_G(x) \cap N_G(y)| \leqslant 2$.*

**Claim 13.** *$G$ contains no $P_3$-cutset.*

The gap in the argument lies in the proof of the following claim.

**Claim 14.** *In every triangle, at least two vertices belong to other triangles as well.*

*Proof of Claim 14.* For a proof by contradiction, we assume that $xy_0z_0$ is a triangle in $G$ and that $y_0$ and $z_0$ lie in no other triangles in $G$. Since $(N_G(y_0) \cup N_G(z_0)) \setminus \{y_0, z_0\}$ is not a stable cutset, there are neighbors $y_1$ of $y_0$ and $z_1$ of $z_0$ such that $y_1$ and $z_1$ are adjacent. By Claim 12, the vertices $y_1$ and $z_0$ are the only common neighbors of $y_0$ and $z_1$. Let the graph $G'$ arise from $G$ by identifying the vertices $y_0$ and $z_1$ to form the vertex $v$. The order of $G'$ is $n-1$ and its size is $2(n-1)-3$. Since every stable cutset in $G'$ is also a stable cutset in $G$, it follows that $G'$ has no stable cutset. Now, the choice of $G$ implies that $G' \in \mathcal{G}_{sc}$.

> *Explanation of the gap:*
> *At this point, Le and Pfender correctly show that $G'$ does not contain a 3-edge matching cut. From that they incorrectly deduce that $G'$ can be built by starting with a triangle and recursively gluing on triangles along an edge, that is, that $G'$ is a so-called 2-tree. Clearly, edge or triangle identifications with copies of $\overline{C_6}$ during the construction of $G'$ create 3-edge matching cuts in intermediate graphs. Nevertheless, subsequent further identifications in the construction of $G'$ can eliminate these cuts.*

By Lemma 2, the graph $G'$ has a generating sequence $\mathcal{S} = (G'_1, \ldots, G'_\ell)$ such that $G'_1$ contains the triangle $xvz_0$. Possibly by inserting the triangle $xvz_0$ within $\mathcal{S}$ before $G'_1$, we may assume that $G'_1$ equals the triangle $xvz_0$. If $G'_{i+1}$ is isomorphic to $\overline{C_6}$ and $G'_{\leqslant i} \cap G'_{i+1}$ is isomorphic to $K_2$, then we may assume that edge $ab$ common to $G'_{\leqslant i}$ and $G'_{i+1}$ belongs to the 3-edge matching cut of $G'_{i+1}$; otherwise, the edge $ab$ belongs to a triangle $abc$ in $G'_{i+1}$ with $c \notin V(G'_{\leqslant i})$, and we can replace $G'_{i+1}$ within the generating sequence $\mathcal{S}$ by $abc, G'_{i+1}$, that is, we consider the alternative generating sequence $(G'_1, \ldots, G'_i, abc, G'_{i+1}, \ldots, G'_\ell)$, where we first form an edge identification along $ab$ with the triangle $abc$ and then a triangle identification along $abc$ with $G'_{i+1}$. Subject to these restrictions, we assume that the generating sequence $\mathcal{S}$ is chosen in such a way that the smallest index $k$ with $y_1 \in V(G'_k)$ is as small as possible.

By Claim 8, the vertex $v$ is involved in every edge or triangle identification within $\mathcal{S}$, more precisely, the vertex $v$ belongs to each graph $G'_i$ within $\mathcal{S}$.

**Claim 14a.** *The graph $G'_{\leqslant k}$ does not have a stable set $X_k$ with the following properties:*

- *$X_k$ contains $y_1$ and $z_0$.*
- *$X_k$ does not contain $v$.*

- *If a cycle $C$ in $G'_{\leqslant k} - X_k$ contains $v$, then, in the graph $G$, the two neighbors of $v$ on $C$ are adjacent to $z_1$ and non-adjacent to $y_0$.*

*Proof of Claim 14a.* For a proof by contradiction, we assume the existence of such a set $X_k$. By an inductive argument along the generating sequence starting at $G'_{\leqslant k}$, which is the first graph containing the two vertices $z_0$ and $y_1$, we show that $X_k$ can be extended to sets $X_k \subseteq X_{k+1} \subseteq \ldots \subseteq X_\ell$ such that, for every $i \in [\ell] \setminus [k-1]$, the set $X_i$ has analogous properties, that is,

- $X_i$ is a stable set in $G'_{\leqslant i}$.
- $X_i$ contains $y_1$ and $z_0$.
- $X_i$ does not contain $v$.
- If a cycle $C$ in $G'_{\leqslant i} - X_i$ contains $v$, then, in the graph $G$, the two neighbors of $v$ on $C$ are adjacent to $z_1$ and non-adjacent to $y_0$.

For $i = k$, the statement is our assumption.

Now, let $i > k$. If $G'_i$ is a triangle $vab$, where $v$ and $b$ belong to $G'_{\leqslant i-1}$, then

$$X_i = \begin{cases} X_{i-1}, & \text{if } b \in X_{i-1} \text{ and} \\ X_{i-1} \cup \{a\}, & \text{otherwise} \end{cases}$$

has the desired properties. If $G'_i$ is isomorphic to $\overline{C_6}$ with the two triangles $uvw$ and $abc$ and the 3-edge matching cut $\{au, bv, cw\}$, where $v$ and $b$ belong to $G'_{\leqslant i-1}$, then

$$X_i = \begin{cases} X_{i-1} \cup \{w\}, & \text{if } b \in X_{i-1} \text{ and} \\ X_{i-1} \cup \{a, w\}, & \text{otherwise} \end{cases}$$

has the desired properties. Finally, if $G'_i$ is isomorphic to $\overline{C_6}$ with the two triangles $uvw$ and $abc$ and the 3-edge matching cut $\{au, bv, cw\}$, where $u$, $v$, and $w$ belong to $G'_{\leqslant i-1}$, then $X_i = X_{i-1} \cup \{b\}$ has the desired properties. Note that in the final case, if $X_{i-1}$ contains neither $u$ nor $w$, then, in the graph $G$, these two vertices are adjacent to $z_1$ and non-adjacent to $y_0$. This completes the inductive argument.

Now, the set $X_\ell$ is also a stable set in the graph $G$ containing $y_1$ and $z_0$ and not containing $y_0$ and $z_1$. Suppose, for a contradiction, that $y_0$ and $z_1$ lie in the same component of $G - X_\ell$. Since the two common neighbors of $y_0$ and $z_1$ belong to $X_\ell$, a path in $G - X_\ell$ between $y_0$ and $z_1$ has length at least three, and the vertex $v$ lies on a cycle $C$ in $G' - X_\ell$ such that, in the graph $G$, one of the two neighbors of $v$ on $C$ is adjacent to $y_0$ and the other one of the two neighbors of $v$ on $C$ is adjacent to $z_1$, which is a contradiction. Hence, the set $X_\ell$ is a stable cutset in $G$, which is a contradiction and completes the proof of the subclaim. □

If the vertex $z_0$ is involved in any edge identification within $\mathcal{S}$, then the edge must be $vz_0$ and, in the graph $G$, the set $\{y_0, z_0, z_1\}$ is a $P_3$-cutset, contradicting Claim 13. Hence, the vertex $z_0$ is not involved in any edge identification within $\mathcal{S}$.

Our next goal is to construct an induced path $P : y_1y_2 \dots y_{p-1}y_p$ in $G'_{\leqslant k} - v$ starting in the neighbor $y_1$ of $v$, ending in $(y_{p-1}, y_p) = (x, z_0)$, and containing all neighbors of $v$ in $G'_k$. We construct this path inductively following the generating sequence backwards from $G'_k$ down to $G'_1$ starting in $y_1$. Our construction will ensure that, for every induced cycle $vabc$ of length 4 in $G'_{\leqslant k}$, the path $P$ either contains $abc$ as a subpath or there are two further vertices $b'$ and $b''$ such that $P$ contains $ab'b''c$ as a subpath.

Suppose that, for some $i \in [k] \setminus \{1\}$, we have already constructed an induced path $y_1 \dots y_j$ starting in $y_1$ such that

- the set $\{y_1, \dots, y_{j-1}\}$ is a subset of $V(G'_{\leqslant k}) \setminus V(G'_{\leqslant i})$,
- the set $\{y_1, \dots, y_{j-1}\}$ contains all neighbors of $v$ in $G'_{\leqslant k}$ that do not belong to $G'_{\leqslant i}$,
- $y_j$ is a neighbor of $v$,
- $y_j$ is the only vertex of $y_1 \dots y_j$ that belongs to $G'_i$, and
- $y_j \notin G'_{\leqslant i-1}$.

Initially, this holds for $i = k$ and $j = 1$, and the inductive construction is such that these properties are maintained. If $G'_i$ is a triangle $vab$, where $v$ and $b$ belong to $G'_{\leqslant i-1}$ and $y_j = a$, then the choice of the generating sequence $\mathcal{S}$ implies that the vertex $b$ belongs to $G'_{i-1}$ but not to $G'_{\leqslant i-2}$, where $G'_{\leqslant 0}$ is the empty graph. Now, setting $y_{j+1} = b$ has the desired properties (for $i$ replaced by $i - 1$). If $G'_i$ is isomorphic to $\overline{C_6}$ with the two triangles $uvw$ and $abc$ and the 3-edge matching cut $\{au, bv, cw\}$, where $v$ and $b$ belong to $G'_{\leqslant i-1}$ and $y_j = u$, then the choice of the generating sequence $\mathcal{S}$ implies that the vertex $b$ belongs to $G'_{i-1}$ but not to $G'_{\leqslant i-2}$. Now, setting $y_{j+1} = w$, $y_{j+2} = c$, and $y_{j+3} = b$ has the desired properties (for $i$ replaced by $i - 1$). See Figure 1 for an illustration.

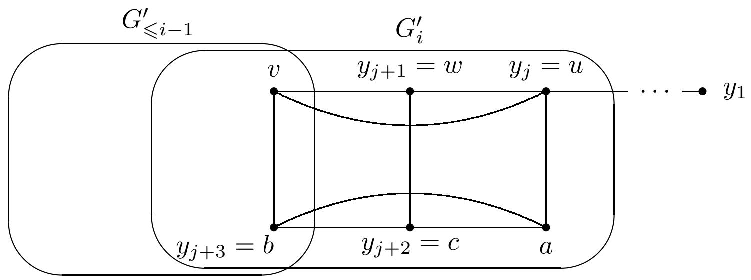


Figure 1: Definition of $P$ for an edge identification with $\overline{C_6}$.

Finally, if $G'_i$ is isomorphic to $\overline{C_6}$ with the two triangles $uvw$ and $abc$ and the 3-edge matching cut $\{au, bv, cw\}$, where $u$, $v$, and $w$ belong to $G'_{\leqslant i-1}$ and $y_j = b$, then the choice of the generating sequence $\mathcal{S}$ implies that at least one of the two vertices $u$ and $w$, say $u$, belongs to $G'_{i-1}$ but not to $G'_{\leqslant i-2}$. Now, setting $y_{j+1} = a$ and $y_{j+2} = u$ has the desired properties (for $i$ replaced by $i - 1$). See Figure 2 for an illustration.

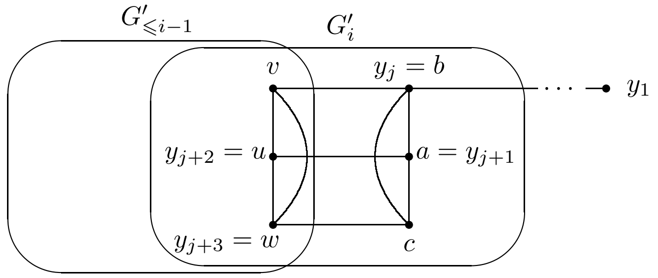


Figure 2: Definition of $P$ for a triangle identification with $\overline{C_6}$.

Note that in this final case, the next step of the construction of $P$ ensures $y_{j+3} = w$. If $i = 1$, then, since $z_0$ is not involved in any edge identification, we may assume that $y_j = x$. Now, setting $\ell = j + 1$ and $y_\ell = z_0$ yields $P$ as desired. This completes the construction of $P$.

**Claim 14b.** *$p$ is even.*

*Proof of Claim 14b.* Suppose, for a contradiction, that $p$ is odd. Let $X_k$ be the set $\{y_1, y_3, y_5, \dots, y_p\}$. By the construction of $P$, the set $X_k$ is stable, contains $y_1$ and $y_p = z_0$, and does not contain $v$. In order to obtain a contradiction to Claim 14a, we show that no cycle in $G'_{\leqslant k} - X_k$ contains $v$, that is, the last condition on $X_k$ in Claim 14a is void. In fact, suppose that $C$ is a cycle in $G'_{\leqslant k} - X_k$ that contains $v$. Let $C'$ be an induced cycle that contains $v$ with $V(C') \subseteq V(C)$. Recall that the only induced cycles in graphs from $\mathcal{G}_{sc}$ are triangles and induced $C_4$s. Since $P$ is induced and contains all neighbors of $v$, the set $X_k$ intersects every triangle in $G'_{\leqslant k}$ that contains $v$. Hence, the cycle $C'$ is an induced $C_4$, say $vabc$. If $P$ contains $abc$ as a subpath, then $X_k$ contains $a$ or $b$, and, hence, intersects $V(C')$. Hence, there are two further vertices $b'$ and $b''$ such that $P$ contains $ab'b''c$ as a subpath. By construction, the set $X_k$ contains $a$ or $c$, that is, also in this case the set $X_k$ intersects $V(C')$. Altogether, assuming the existence of a cycle in $G'_{\leqslant k} - X_k$ that contains $v$ leads to a contradiction, and the set $X_k$ contradicts Claim 14a. Hence, it follows that $p$ is even. ☐

**Claim 14c.** *Each $G'_i$ in the generating sequence $\mathcal{S}' = (G'_1, \dots, G'_k)$ is isomorphic to $K_3$.*

*Proof of Claim 14c.* First, suppose that the generating sequence $\mathcal{S}' = (G'_1, \dots, G'_k)$ of $G'_{\leqslant k}$ involves an edge identification with $\overline{C_6}$. Let $i \in [k]$ be the largest index such that $G'_i$ is isomorphic to $\overline{C_6}$ and $G'_{\leqslant i-1} \cap G'_i$ is the edge $vb$. Let $j$ be the smallest index with $y_j \in V(G'_i)$. See Figure 1 for an illustration. If $j$ is odd, then let $X_k = \{y_1, y_3, \dots, y_j\} \cup \{y_{j+3}, y_{j+5}, \dots, y_p\}$, and if $j$ is even, then let $X_k = \{y_1, y_3, \dots, y_{j+1}\} \cup \{a\} \cup \{y_{j+4}, y_{j+6}, \dots, y_p\}$. Again, the set $X_k$ is stable, contains $y_1$ and $y_p = z_0$, and does not contain $v$. As before we establish a contradiction to Claim 14a by showing that no cycle in $G'_{\leqslant k} - X_k$ contains $v$. Clearly, the set $X_k$ intersects every triangle in $G'_{\leqslant k}$ that contains $v$. The two induced $C_4$s in $G'_{\leqslant k}$ that contain $v$ and are contained in $G'_i$ intersects $X_k$ by construction. As before it follows that the remaining induced $C_4$s in $G'_{\leqslant k}$ that contain $v$ and are not contained in $G'_i$ all intersect $X_k$. Altogether, the set $X_k$ intersects every triangle and induced $C_4$ in $G'_{\leqslant k}$ that contains $v$, which implies that no cycle in $G'_{\leqslant k} - X_k$ contains $v$. It follows that the set $X_k$ contradicts Claim 14a. Hence, the generating sequence $\mathcal{S}'$ involves no edge identification with $\overline{C_6}$.

Next, suppose that the generating sequence $\mathcal{S}' = (G'_1, \dots, G'_k)$ of $G'_{\leqslant k}$ involves a triangle identification with $\overline{C_6}$. Let $i \in [k]$ be the largest index such that $G'_i$ is isomorphic to $\overline{C_6}$ and $G'_{\leqslant i-1} \cap G'_i$ is the triangle $uvw$. Let $j$ be the smallest index with $y_j \in V(G'_i)$. See Figure 2 for an illustration. Recall that $y_{j+3} = w$ in this case. If $j$ is odd, then let $X_k = \{y_1, y_3, \dots, y_j\} \cup \{y_{j+3}, y_{j+5}, \dots, y_p\}$, and if $j$ is even, then let $X_k = \{y_1, y_3, \dots, y_{j-1}\} \cup \{c\} \cup \{y_{j+2}, y_{j+4}, \dots, y_p\}$. Again, the set $X_k$ is stable, contains $y_1$ and $y_p = z_0$, does not contain $v$, and intersects every triangle as well as every induced $C_4$ in $G'_{\leqslant k}$ that contains $v$, contradicting Claim 14a as before. Hence, the generating sequence $\mathcal{S}'$ involves no triangle identification with $\overline{C_6}$. □

Claim 14c implies that $G'_{\leqslant k}$ arises from the path $P$ by adding $v$ as a universal vertex. Claim 14c implies that $k = p - 1$ is odd.

**Claim 14d.** *In $G$ the vertex $y_0$ is adjacent to each vertex in $\{y_1, y_3, \dots, y_{p-1}\}$.*

*Proof of Claim 14d.* Suppose, for a contradiction, that $y_0$ is not adjacent to $y_j$ for some odd index $j$ in $[p]$. Since $y_0$ is adjacent to $y_1$ and $y_{p-1} = x$, this implies $p \geqslant 6$. Choosing $j$ as the smallest odd index such that $y_0$ is not adjacent to $y_j$, we obtain that the vertices in $\{y_1, y_3, \dots, y_{j-2}\}$ are all adjacent to $y_0$. Since $xy_0z_0$ is the only triangle in $G$ that contains $y_0$, the vertex $y_{j-1}$ is adjacent to $z_1$ and non-adjacent to $y_0$. Let $X_k = \{y_1, y_3, \dots, y_{j-2}\} \cup \{y_{j+1}, y_{j+3}, \dots, y_p\}$. In view of the structure of $G'_{\leqslant k}$, the only cycle $C$ in $G'_{\leqslant k} - X_k$ that contains $v$ is the triangle $vy_{j-1}y_j$, which satisfies the last condition on $X_k$ from Claim 14a. Hence, the set $X_k$ contradicts Claim 14a, which completes the proof. □

Since $xy_0z_0$ is the only triangle in $G$ that contains $y_0$, Claim 14d implies that $z_1$ is adjacent to each vertex in $\{y_2, y_4, \dots, y_p\}$.

**Claim 14e.** $p = 4$.

*Proof of Claim 14e.* Suppose, for a contradiction, that $p \geqslant 6$. Let the graph $G''$ arise from $G$ by identifying the vertices $z_0 = y_p$ and $y_1$ to form a vertex $v''$. Similarly, as for $G'$, it follows that $G'' \in \mathcal{G}_{sc}$. Nevertheless, the graph $G''$ contains an induced cycle $v''y_0y_{p-3}y_{p-2}z_1v''$ of length 5, contradicting $G'' \in \mathcal{G}_{sc}$. □

At this point, the subgraph of $G$ induced by $\{x, y_0, z_0, y_1, z_1, y_2\} = \{y_0, z_1\} \cup V(P)$ is isomorphic to $\overline{C_6}$ with the two triangles being $xy_0z_0$ and $y_1y_2z_1$. Let $G'''$ arise from $G$ by identifying the vertices of $P : y_1y_2xz_0$ to form a vertex $v'''$. The order of $G'''$ is $n - 3$ and its size is at most $(2n-3) - 7 = 2(n-3) - 4$. If $v'''$ is not the only cut vertex in $G'''$, then, by Corollary 3, the graph $G'''$ has a stable cutset not containing $v'''$, which is also a stable cutset in $G$. Hence, it follows that $v'''$ is the only cut vertex in $G'''$. Since $v$ is involved in every edge or triangle identification within $\mathcal{S}$, it follows that all vertices added by the identifications with $G'_{k+1}, \ldots, G'_\ell$ belong to the same component of $G''' - v'''$ as $y_0$ or $z_1$. This implies that $G''' - v'''$ has exactly two components, one component $C_0$ containing $y_0$ and the other component $C_1$ containing $z_1$. Furthermore, for every $i \in [\ell] \setminus [k]$, all vertices in $V(G'_i) \setminus V(G'_{\leqslant i-1})$ belong to either $C_0$ or $C_1$. Since $G$ is not isomorphic to $\overline{C_6}$, it follows that $\ell > k$. In the graph $G'_{\leqslant k+1}$, the set $X = V(G'_{\leqslant k}) \cap V(G'_{k+1})$ is a $K_2$-cutset or a $K_3$-cutset. If $V(G'_{k+1}) \setminus V(G'_{\leqslant k})$ lies in $C_0$, then let $X' = (X \setminus \{v\}) \cup \{y_0\}$, and if $V(G'_{k+1}) \setminus V(G'_{\leqslant k})$ lies in $C_1$, then let $X' = (X \setminus \{v\}) \cup \{z_1\}$. In the subgraph of $G$ induced by $\{x, y_0, z_0, y_1, z_1, y_2\} \cup \big(V(G'_{k+1}) \setminus \{v\}\big)$, the set $X'$ is a stable cutset of order 2 or a $K_2$-cutset or a $K_3$-cutset or a $P_3$-cutset. Furthermore, since, for every $i \in [\ell] \setminus [k]$, all vertices in $V(G'_i) \setminus V(G'_{\leqslant i-1})$ belong to either $C_0$ or $C_1$, the set $X'$ is still a cutset in $G$, which contradicts the choice of $G$, Claim 8, and Claim 13.

This final contradiction completes the proof of Claim 14. □

At this point the entire proof can be finished exactly as in [9]. □


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