Mock Alexander Polynomials

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Submitted: Mar 17, 2024; Accepted: Aug 27, 2025; Published: Nov 3, 2025

Abstract

In this paper, we construct Mock Alexander polynomials for starred links and linkoids in surfaces. These polynomials are defined as state summations on link or linkoid diagrams that satisfy $f = n$, where f denotes the number of regions and n denotes the number of crossings of diagrams. These new invariants are chirality and reversibility sensitive.

Mathematics Subject Classifications: 57K10, 57K12, 57K14

1 Introduction

This paper generalizes a state summation model for the Alexander-Conway polynomial of classical knots and links that we refer to as the 'FKT' (Formal Knot Theory) model [12]. The state sum depends upon the fact that, for a classical connected knot or link diagram, we have $R = V + 2$ where R is the number of regions in the diagram and V is the number of crossings. In the state sum, two regions are "starred" so that they do not participate in the state labelling. The structure of the summation is based on the remaining $F = (R-2)$ regions with $F = R-2 = V$. In the model for Alexander-Conway polynomial, it was necessary to choose two adjacent starred regions. In this new approach, we understand that invariants can arise from this state summation whenever $F = V$ where F is the number of unstarred regions, and V is the number of crossings in the diagram. We call the resulting invariants Mock Alexander Polynomials. Mock Alexander polynomials can be defined for classical knots and links, knotoids, linkoids, and knots and links in thickened surfaces of higher genus.

In following our dictum $F = V$, we can also have starred crossings in situations where the number of regions is smaller at the outset than the number of crossings. In all cases, of starred regions or starred crossings, the Reidemeister moves are restricted so that arcs

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cannot move across a star. In the case of linkoids, there is already the restriction that arcs cannot move across an endpoint of the linkoid. Thus it is natural to place stars in the vicinity of endpoints for knotoids.

The new Mock Alexander polynomials can detect chirality for many classical diagrams where the stars are in non-adjacent regions and for non-classical (endpoints not in the same region) knotoids and linkoids and diagrams on a surface of higher genus.

Furthermore, we show that the original state summation can be generalized to include an extra variable and that the resulting two-variable Mock Alexander polynomial is very strong at detecting certain forms or chirality (double mirror image chirality) for planar knotoids.

We now briefly present the organization of the paper. In Section 2 we recall the state-sum formula of the Alexander-Conway polynomial constructed in FKT [12].

Section 3 begins by a recollection of preliminary notions on linkoids. In Section 3.1 we study the relationship between the number of regions and crossings of a link or a linkoid diagram that lies in a closed, connected and orientable surface diagrams. In Section 3.2 we introduce starred link and linkoid diagrams, endowed with a star decoration either on their regions or crossings. Starred link or linkoid diagrams are considered up to star equivalence that is a restricted isotopy relation in the surface they lie that is defined far away from starred regions or crossings.

In Section 4 we generalize the ideas of Formal Knot Theory. We introduce a state-sum polynomial in variables W, B for starred links and linkoids lying in a surface in Section 4.1. In Section 4.1.1 we show that the state-sum polynomial admits a matrix permanent formulation. In Section 4.2 we exhibit the invariance condition on the variables of the state-sum polynomial up to star equivalence, and obtain invariants of starred links or linkoids that we call *Mock Alexander polynomials*. We also observe that the induced polynomial is an invariant of knotoids in S^2 when any knotoid diagram is considered to be endowed with a star at its region adjacent to its tail. In Section 4.2.1 we study the behavior of Mock Alexander polynomials of knotoids in S^2 under certain symmetries such as reversion, mirror symmetry and starred region exchange. Section 4.3 is devoted to the discussion of skein relations that Mock Alexander polynomials satisfy. We show that the Mock Alexander polynomial of a starred link in S^2 satisfies the skein relation at a non-separating crossing. We also show certain variations of the skein relation are satisfied by the Mock Alexander polynomial of a starred linkoid in S^2.

In Section 5 we study alternative approaches such as attaching handles to S^2 in specific ways to obtain link diagrams in genus two surfaces that satisfy the equality $f = n$ and define Mock Alexander polynomial for such link diagrams. In this section, we also study the Mock Alexander polynomial of the virtual closure of a knotoid in S^2.

Section 6 derives the most general vertex weights and indicates a 2-variable polynomial. We show that there are significant applications of the two variable polynomial to the detection of chirality and reversibility of knotoids.

Section 7 is a brief discussion of the states in terms of paths on the diagram, and sets the stage for a next paper, generalizing the Clock Theorem of the Formal Knot Theory.

2 Recalling FKT State Sums and the Alexander Polynomial

In this section we review of J. W. Alexander's original definition for his polynomial invariant of knots and links and its state-sum reformulation given in Formal Knot Theory [12].

The Alexander polynomial was defined by J. W. Alexander in his 1928 paper [2], as the determinant of a matrix associated with an oriented link diagram. Later in the paper, Alexander describes how the matrix whose determinant yields the polynomial is related to the fundamental group of the complement of the knot or link. It is sufficient here to say that the Alexander matrix is a presentation of the abelianization of the commutator subgroup of the fundamental group as a module over the group ring of the integers. Since Alexander's time, this relationship with the fundamental group has been understood very well. For more information about this point of view, see [6, 7, 13, 17].

Alexander's notation for generating the matrix associated to an oriented link diagram is as follows. At each crossing two dots are placed just to the left of the undercrossing arc, one before and one after the overcrossing arc at the crossing. Here one views the crossing so that the under-crossing arc is vertical and the over-crossing arc is horizontal. See Figure 1.

Four regions meet locally at a given crossing. Letting the regions be labeled by A, B, C, D, as shown in Figure 1.,Alexander associates the equation

$$xA - xB + C - D = 0$$

to that crossing. Here A, B, C, D proceed cyclically in the counter-clockwise direction around the crossing, starting at the top dot. In this way the two regions containing the dots give rise to the two occurrences of x in the equation. If some of the regions are the same at the crossing, then the equation is simplified by that equality. For example, if $A = D$ then the equation becomes $xA - xB + C - A = 0$. Each crossing in a diagram K gives an equation involving the regions of the diagram. Alexander associates a matrix M_K whose rows correspond to crossings of the diagram, and whose columns correspond to regions of the diagram. Each nodal equation gives rise to one row of the matrix where the entry for a given column is the coefficient of that column (understood as designating a region in the diagram) in the given equation. If R and R' are adjacent regions, let $M_K[R, R']$ denote the matrix obtained by deleting the corresponding columns of M_K. Finally, define the *Alexander polynomial* $\Delta_K(x)$ by the formula

$$\Delta_K(x) \doteq \det(M_K[R, R']).$$

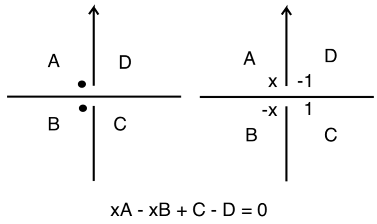


Figure 1: Alexander labeling.

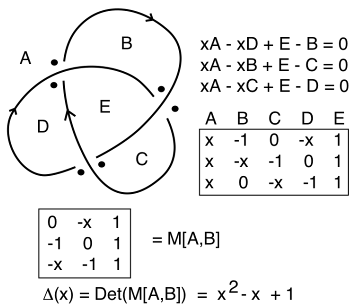


Figure 2: The Alexander Polynomial.

The notation $A \doteq B$ means that $A = \pm x^n B$ for some integer n. Alexander proves that his polynomial is well-defined, independent of the choice of adjacent regions and invariant under the Reidemeister moves up to $\doteq$. In Figure 2 we show the calculation of the Alexander polynomial of the trefoil knot using this method.
In this figure we show the diagram of the knot, the labelings and the resulting full matrix and the square matrix resulting from deleting two columns corresponding to a choice of adjacent regions. Computing the determinant, we find that the the Alexander polynomial of the trefoil knot is given by the equation $\Delta \doteq x^2 - x + 1$.

In 1983, the second author gave a reformulation of the Alexander polynomial as a state sum in his book titled Formal Knot Theory [12]. The state sum reformulation is based on combinatorial configurations on a knot or link diagram that are directly related to the expansion of the determinant that defines the Alexander polynomial. Here is a brief description of this reformulation.

Given an $n \times n$ square matrix $M = [M_{ij}]$, we consider the expansion formula for the determinant of M :

$$\det(M) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} M_{i\sigma(i)}.$$

Here the sum runs over all permutations of the index set $\{1, 2, \ldots, n\}$ and $sgn(\sigma)$ denotes the sign of a given permutation σ. In terms of the matrix, each product corresponds to a *choice* by each column of a single row such that each row is chosen exactly once. The order of rows chosen by the columns (taken in standard order) gives the permutation whose sign is calculated.

Consider our description of Alexander's determinant as given above. Each crossing is labeled with Alexander's dots so that we know that the four local quadrants at a crossing are each labeled with x, $-x$, 1 or -1. The matrix has one row for each crossing and one column for each region. Two columns corresponding to adjacent regions A and B are deleted from the full matrix to form the matrix $M[A, B]$, and we have the Alexander polynomial $\Delta_K(X) \doteq \det(M[A, B])$.

In the Alexander determinant expansion the *choice* of a row by a column corresponds to *a region choosing a crossing* in the link diagram. The only crossings that a region can choose giving a non-zero term in the determinant are the crossings in the boundary of the given region. Thus the terms in the expansion of $Det(M[A, B])$ are in one-to-one correspondence with decorations of the flattened link diagram (i.e. we ignore the over and under crossing structure) where each region (other than the two deleted regions corresponding to the two deleted columns in the matrix) labels one of its crossings. We call these labeled flat diagrams the *states* of the original link diagram. See Figure 3 for a list of the states of the trefoil knot. In this figure we show the states and the corresponding matrix forms with columns choosing rows that correspond to each state.

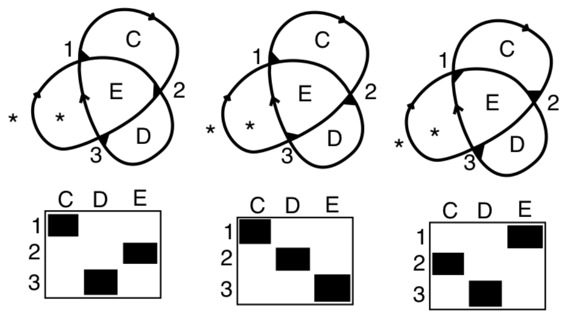


Figure 3: States with markers.

At this point we have almost a full combinatorial description of Alexander's determinant. The only thing missing is the permutation signs. Call a state marker a *black hole* if it labels a quadrant where both oriented segments point toward the vertex of the flattened link diagram. See Figure 4 for an illustration of a black hole.

Let S be a state of the diagram K. Consider the parity

$$(-1)^{b(S)}$$

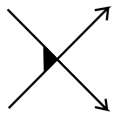


Figure 4: A black hole.

where $b(S)$ is the number of black holes in the state S. Then it turns out that up to one global sign ϵ depending on the ordering of nodes and regions, we have

$$(-1)^{b(S)} = \epsilon \ sgn(\sigma(S))$$

where $\sigma(S)$ is the permutation of crossings induced by the choice of ordering of the regions of the state. This gives a purely diagrammatic access to the sign of a state and allows us to write

$$\Delta_K(x) \doteq \sum_S < K|S > (-1)^{b(S)}$$

where S runs over all states of the diagram for a given choice of deleted adjacent regions, and $< K|S >$ denotes the product of the Alexander's crossing labels at the quadrants indicated by the state labels in the state S. Here we show the contributions of each state to a product of terms and in the polynomial we have followed the state summation by taking into account the number of black holes in each state. The most mysterious thing about this state sum is the agreement of the permutation signs for the determinant with the black hole parity signs. The proof of this correspondence follows from the Clock Theorem proved in Formal Knot Theory [12]. Thus we have a precise reformulation of the Alexander polynomial as a state summation.

In Figure 5 we illustrate the calculation of the Alexander polynomial of the trefoil knot using this state summation.

2.1 The Alexander-Conway Polynomial and its Generalizations

We can change the vertex weights of the state sum to the form shown in Figure 7, letting $z = W - W^{-1}$. This choice incorporates the sign for the black hole count directly into the weights and gives a direct state summation model for the Alexander-Conway polynomial of an oriented link diagram satisfying the Conway Skein relation $\nabla_{K_+} - \nabla_{K_-} = z\nabla_{K_0}$ for any triple of oriented link diagrams K_+, K_-, K_0 that differ from each other at a unique crossing as shown in Figure 6.

In fact, the Alexander-Conway polynomial of an oriented knot diagram is given as the permanent of the matrix associated to the knot diagram with this new labeling [12]. The state-sum formula for the Alexander-Conway polynomial is

$$\nabla_K(W) = \sum_S < K|S >$$

where S runs over all states of K.

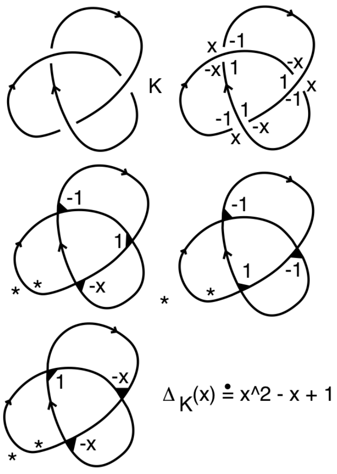


Figure 5: State-sum calculation of Alexander polynomial.

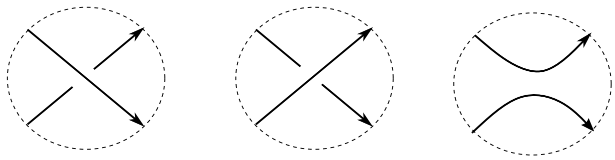


Figure 6: K_+, K_- and K_0, respectively from left to right.

3 Linkoids and starred universes

We begin with necessary definitions.

Definition 1. A *linkoid diagram* L in a closed, connected, oriented surface of genus g is an immersion of a number of unit intervals and circles in the surface. The immersion L has a finite number of double points, each of which is endowed with an under or over crossing information and called a *crossing* of L. The points corresponding to $L(0)$ and $L(1)$ are all distinct from each other and any crossing of L, and appear as *endpoints* of long (arc) components of L. An endpoint of a component that corresponds to $L(0)$ is called a *tail* and to $L(1)$ is called a *head* of L. Each long component of L is considered to be oriented from its tail to its head.

A *knotoid diagram* is a linkoid diagram that consists of only one long component

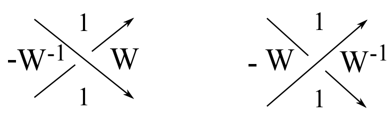


Figure 7: New labeling at crossings.

and a *knotoid multi-diagram* is a linkoid diagram that consists of a number of circular components and one long component.

Note that if every endpoint of every long component of a linkoid diagram L is assumed to lie at the boundary of a 2-disk then L is a tangle. Moreover, a linkoid diagram consisting of only circular components is clearly a link.

Throughout the paper, we assume the ambient manifolds that links and linkoids lie in as closed, connected, orientable surfaces.

Definition 2. Two linkoid diagrams in a surface are *equivalent* if they can be transformed to one other by a sequence of three Reidemeister moves RI, RII, RIII that take place away from the endpoints of diagrams and isotopy of the surface. It is forbidden to pull/push an endpoint across an arc. A *linkoid* is an equivalence class of linkoid diagrams given by the relation induced by Reidemeister moves.

Definition 3. The *universe* U_L of a linkoid or link diagram L in a surface is the graph that is obtained by replacing each crossing of L with a vertex. Each endpoint of L, if there is any, is also considered to be a vertex. Each semi-arc of L is regarded as an edge of U_L, and a region of L is a connected component of $\Sigma_g - U_L$. That is, a region of L corresponds to a face of U_L. A region that is enclosed with vertices and edges is a *bounded* region of L, otherwise it is called the *exterior* region of L.

Definition 4. If each region of a link or linkoid diagram L is homeomorphic to a disk then we say L is a *tight embedding* (or tightly embedded) in Σ_g. Note that every connected link or linkoid diagram in S^2 is a tight embedding.

Definition 5. A linkoid or link diagram in a surface is called *split* if its universe is disconnected.

Definition 6. A linkoid or link diagram in a surface is called *connected* if it is not split. Equivalently, a linkoid or link diagram is *connected* if its underlying universe is connected.

3.1 Admissible diagrams

We begin by characterizing the relation between the number of faces and the number of four-valent vertices of a link or linkoid universe that is tightly embedded in a surface.

Proposition 7. *Let U be the universe of a connected link or linkoid diagram with n crossings that is tightly embedded in a closed, connected and orientable surface Σ_g. We have the following.*

1. *If U is the universe of a link diagram then $f - n = 2 - 2g$, where f is the number of faces of U.*
2. *If U is the universe of a linkoid diagram, then $f - n = 2 - 2g - m$, where f is the number of faces of U, m is the number of long components of the linkoid diagram.*

Proof. It is an easy observation that a connected 4-regular graph with n vertices has $2n$ edges connecting the vertices. A connected linkoid universe U is a connected graph with n four-valent vertices and $2m$ pendant vertices where $2m$ is the number endpoints of the knotoid components of the linkoid. This gives us the following equality.

$$4n = 2e - 2m,$$

where e is the number of edges of the universe U_L.

By Euler's formula, we know that for a connected graph that is tightly embedded in Σ_g, $g \geqslant 0$,

$$v - e + f = 2 - 2g,$$

where v is the number of vertices, e is the number of edges, f is the number of faces of the graph.

By using the two equalities above, we can conclude the following. If U is the universe of a link diagram with n crossings then

$$n - 2n + f = 2 - 2g$$

and so

$$f - n = 2 - 2g.$$

If U is the universe of a linkoid diagram with n crossings and m knotoid components then

$$(n + 2m) - (2n + m) + f = 2 - 2g.$$

So,

$$f - n \ = \ 2 - 2g - m. \quad \square$$

Proposition 7 can be generalized to disconnected link or linkoid diagrams in S^2 or $\mathbb{R}^2$.

Proposition 8. *Let L be a disconnected link or linkoid diagram in S^2 or $\mathbb{R}^2$ with $k > 1$ disconnected components. Then we have the following.*

1. *If L is a link diagram then $f - n = k + 1$, where f is the number of regions and n is the number of crossings in L.*

2. *If L is a linkoid diagram with m knotoid components then $f - n = k - m + 1$.*

Proof. The generalized Euler formula for planar graphs is given as

$$v - e + f = 1 + k,$$

where k is the number of disconnected components of the graph. Let U is the universe of a link diagram in S^2 (or $\mathbb{R}^2$) with k disconnected components. We already know that $e = 2n$ for U, where n is the total number of crossings in L. Then, $f - n = 1 + k$ follows directly by substituting the formula. Note that when L is a knot diagram in S^2, $k = 1$ and so we obtain the equality $f - n = 2$, given in Proposition 7.

Let L be a linkoid diagram with n crossings, m knotoid components and k disjoint components. Substituting $v = n + 2m$, $e = 2n + m$ into the formula, we reach to the required equality $f - n = k - m + 1$. $\square$

Remark 9. We remind the reader that any tight embedding of a graph in a surface of genus $g > 0$ forces the graph to be connected [15]. We will be working with links and linkoid diagrams that are tight embeddings except in Section 5.

Definition 10. A tightly embedded link or linkoid diagram is called *admissible*, if $f = n$, where f denotes the number of faces and n denotes the number of vertices of its universe.

The following is deduced directly from Propositions 7 and 8.

Corollary 11. *i. A link diagram that is tightly embedded in a closed, connected, oriented genus g surface* Σ_g *is admissible if and only if the link diagram is connected and* $g = 1$.

ii. A linkoid diagram L is admissible if and only if it lies in S^2 *and the total number of knotoid components in L is one more than the number of its disjoint components. See Figure 9 where we picture an admissible linkoid diagram in* S^2 *with two disjoint components and three knotoid components.*

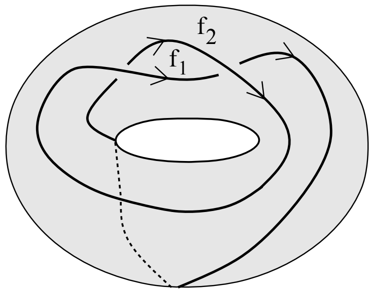


Figure 8: An admissible knot diagram in torus with two crossings and regions.

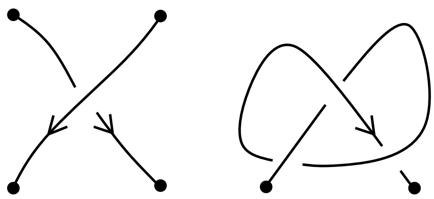


Figure 9: An admissible disconnected linkoid diagram with two disjoint components, consisting of three knotoids.

An RII move that takes place between two disjoint components of a split linkoid diagram with a number of disjoint components clearly increases or decreases the number of disjoint components of a split linkoid diagram in S^2 by one. This makes it possible that a sequence of RII moves turns a non-admissible split linkoid diagram into an admissible linkoid diagram. Here we study the case of split linkoid diagrams consisting of only two disjoint knotoid components. By Corollary 11 ii., it is true that one can always obtain an admissible linkoid diagram from a split linkoid diagram that consists of only two knotoid components by connecting the components by an RII move. We illlustrate examples of this in Figures 10 and 11 where disjoint components of split linkoid diagrams in S^2 are connected by RII moves.

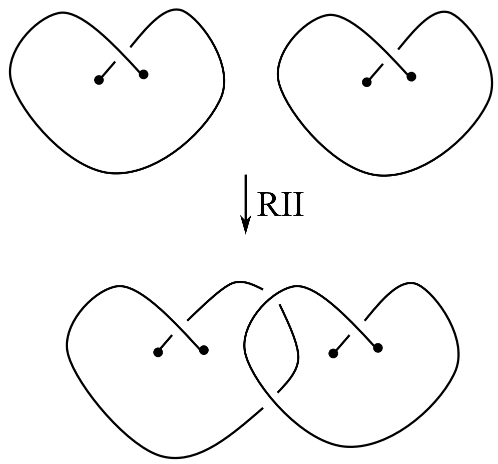


Figure 10: A split linkoid transformed into an admissable diagram.

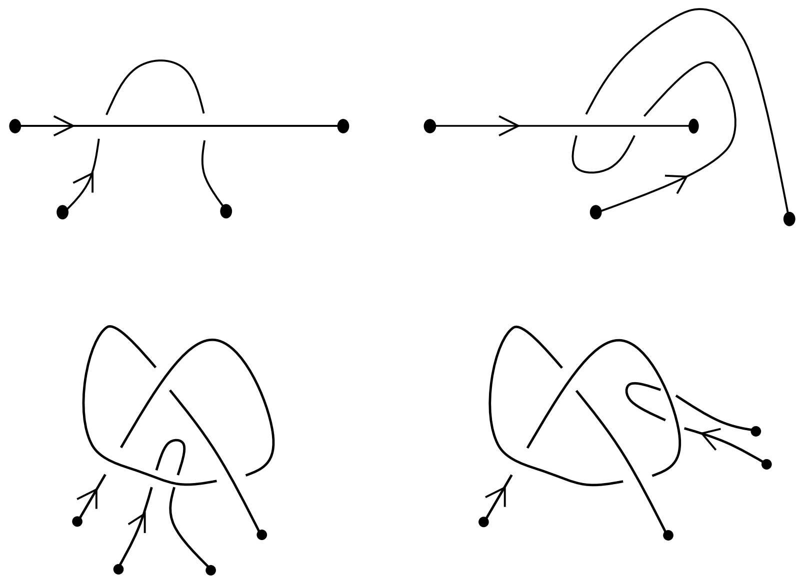


Figure 11: Two pairs of admissible linkoid diagrams on top and bottom obtained by RII moves.

Proposition 12. *Let L be a split linkoid diagram in S^2 that consists of only two knotoid components, denoted by α and β. Then, any two connected linkoid diagrams obtained from L by an RII move which pushes the component β under (or over) α are equivalent to each other through connected diagrams.*

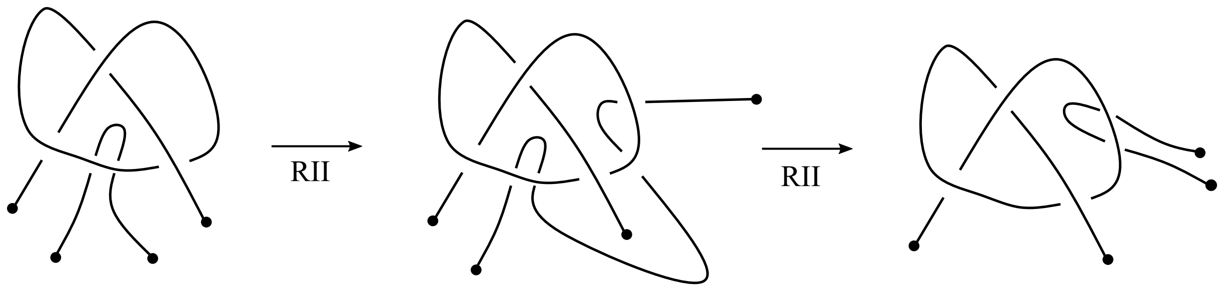


Figure 12: RII moves exchanging the connecting regions.

Proof. Consider any two possible ways to connect the components α and β by applying an RII move that pushes β under (or over) α. It is clear that the resulting connected diagrams can be taken to each other by a sequence of planar isotopy moves and two RII moves exchanging the connecting regions. During this transformation, connectivity is preserved. □

3.2 Starred links and linkoids

In this section, we introduce a decoration for link and linkoid diagrams lying in a closed, connected, orientable surface. The purpose for decorating a diagram will be to obtain the equality for the number of four-valent vertices and the number of regions in the corresponding decorated universe. This equality will induce a square region-vertex incidence matrix whose permanent gives Alexander type polynomials.

Let L be a non-admissible link or linkoid diagram in a surface of genus g. Let the equality $f = n + k$ hold, for some $k \in \mathbb{N}_{>0}$, where f is the number of regions and n is the number of crossings of L. We choose a collection of k regions of L and decorate each of them with a star. Regions endowed with a star are considered to be dismissed from L so that the equality $f = n$ holds for the resulting starred diagram. In a similar manner, if the number of crossings is larger than the number of regions of L, that is, if $n = f + k$ holds, for some $k \in \mathbb{N}_{>0}$, then we decorate a collection of k crossings of L with stars. A vertex that receives a star is considered to be dismissed from the universe of L so that in the resulting starred diagram, the number of regions is equal to the number of crossings.

Definition 13. A *starred* link or linkoid diagram in a surface is a link or linkoid diagram that is endowed with stars either on a number of its regions or crossings so that it satisfies the equality $f = n$. No unstarred region is allowed to have all the crossings on its boundary to be starred. We consider a link or linkoid diagram in a surface which already satisfies $f = n$ without any star addition as a starred diagram with zero stars. Figures 8 and 9 set examples for such diagrams.

Definition 14. Two starred link or linkoid diagrams in a surface are *star equivalent* if they are related to each other by Reidemeister moves that take place away from the starred regions or vertices and isotopy of the surface. The addition/deletion of a starred vertex or pulling a strand across a starred crossing or region is not allowed. In this way the number of starred regions or crossings is preserved. See Figure 13 for a list of moves that are not allowed for decorated diagrams.

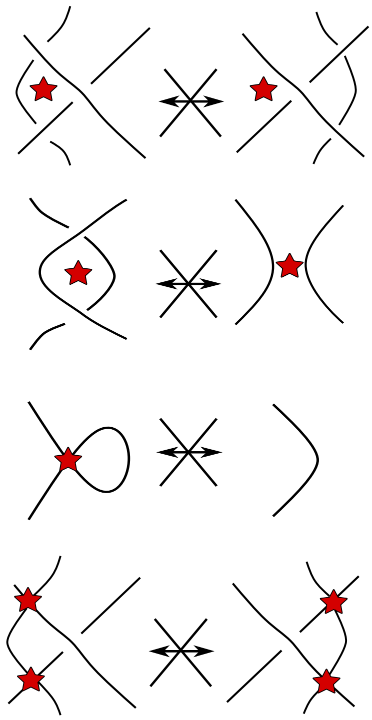


Figure 13: Moves that are not allowed for starred diagrams.

Definition 15. A *starred link* or *linkoid* in a surface is an equivalence class of star equivalent link or linkoid diagrams.

It is clear that the starred linkoid diagrams in Figure 14 are not star equivalent , since the number of regions endowed with a star is not equal.

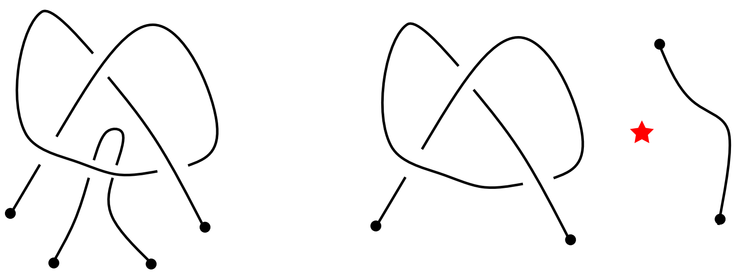


Figure 14: Two nonequivalent starred linkoid diagrams.

Note here that a starred knotoid in S^2 can also be considered as a generalized knotoid. Generalized knotoids were introduced in [1].

4 Mock Alexander polynomials for starred links and linkoids

4.1 A state-sum polynomial for starred links and linkoids

In this section we introduce states and a state-sum polynomal of starred link or linkoid diagrams.

Definition 16. Let L be an oriented starred link or linkoid diagram in a surface. A *state* of L is an assignment of exactly one marker to every region of L without any stars. Each marker assigned to.a region is placed at exactly one of the crossings without any stars that is incident to the region.

We illustrate a state of a starred knotoid diagram in S^2, with black state markers in Figure 15. The reader can find the collection of all states of the starred knotoid diagram in Figure 17.

If L is a starred link or linkoid diagram without any stars on its regions or crossings, then a state of L is obtained by endowing each face with a marker at exactly one of the crossings incident to the region that receives a marker.

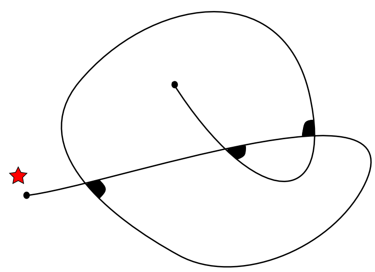


Figure 15: A state of a starred knotoid diagram.

The following observations can be derived easily from Proposition 7.

1. Let L be a connected link diagram in S^2. Since $f = n + 2$, we decorate the universe of L with two stars at any of its two regions, then endow each of the remaining regions with exactly one state marker at an incident crossing to obtain a state of L.

2. If L is a connected linkoid diagram in S^2 with two knotoid components, then its universe is admissible. A state of U is obtained directly by endowing U with exactly one state marker at each of its bounded regions, placed at a crossing incident to the region.

3. If L is connected linkoid diagram in S^2 with $m \geqslant 3$ knotoid components, then we have $f - n \leqslant -1$. In this case, endowing a collection of $f - n$ crossings with stars turns L into a starred linkoid diagram. A state of the resulting starred linkoid diagram is obtained by placing a state marker at each of its regions next to a crossing that is free of stars and incident to the region.

Figure 16: Labels at local regions incident a positive and negative crossing.

We consider the following labels given in Figure 16, at local regions incident to a crossing of L.

Definition 17. Let L be an oriented starred link or linkoid diagram and S denote a state of L. The *weight* of the state S, denoted by $< L \mid S >$ is defined to be the product of labels at the crossings that receive a marker in S.

Definition 18. Let L be an oriented starred link or linkoid diagram in a surface. We define a state-sum polynomial for L denoted by ∇_L as follows.

$$\nabla_L(W, B) = \sum_{S \in \mathcal{S}} < L \mid S >,$$

where $\mathcal{S}$ denotes the set of all states of L. We call ∇_L the *potential* of L.

Example 19. In Figure 17, we depict a starred knotoid diagram in S^2, named as K_1 and all of its states with their weights below. It can be easily verified that the potential of K_1 is $W^2 - WB + (W - B) + 1$.

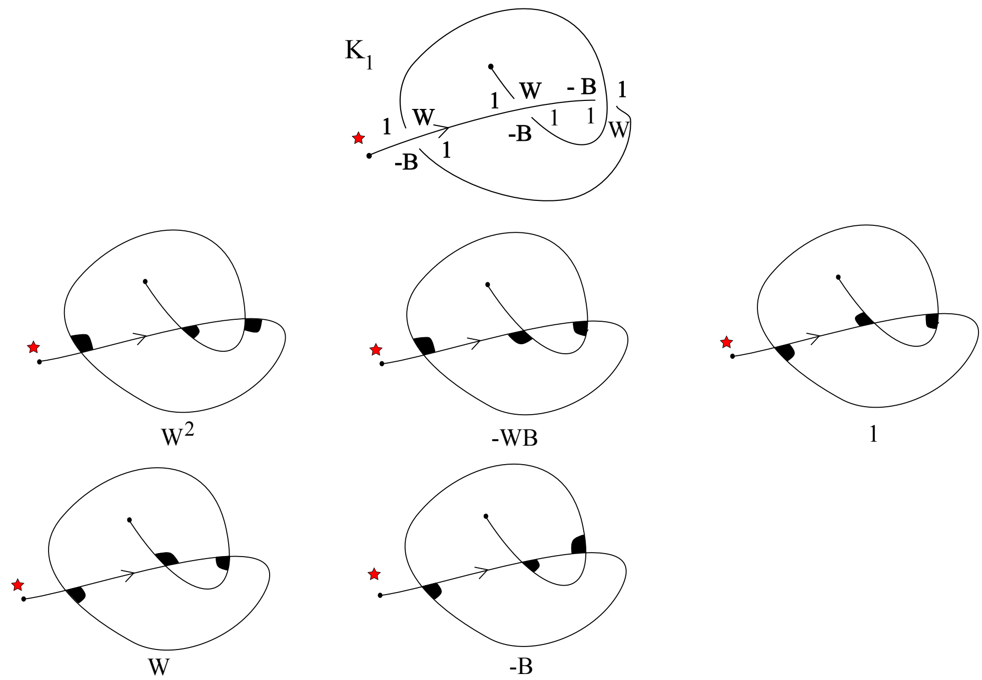


Figure 17: The starred knotoid diagram K_1 in S^2 and its states.

In Figure 18, we depict another starred knotoid diagram K_2 in S^2 (which is the same with K_1 except the starred region), with its states and state weights. We find that the potential of K_2 is $B^2 - WB + (W - B) + 1$.

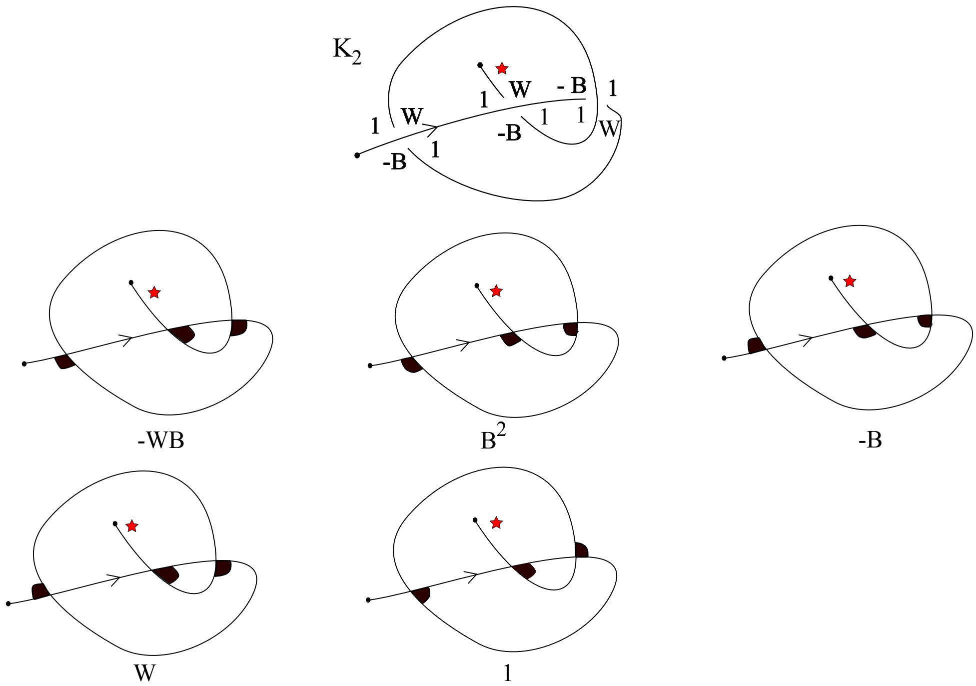


Figure 18: The starred knotoid diagram K_2 in S^2 and its states.

Notice that the potential of K_2 , can be obtained by replacing W by $-B$ and B by $-W$ in the potential of K_1 given in Figure 17. We have observed this symmetry in many other examples of pairs of starred knotoid diagrams in S^2 which differ from each other only for the starred regions as discussed above. We conjecture the following.

Conjecture 20. Let K be a knotoid diagram in S^2, and K_1, K_2 be two starred knotoid diagrams obtained from K by placing a star in the regions of K that are incident to the tail and the head of K, respectively. Then, the following holds.

$$\nabla_{K_1}(W, B) = \nabla_{K_2}(-B, -W).$$

At the time of writing this paper, a specialization of this conjecture, Conjecture 36 in Section 4.2.1, has been proven for the case when $W = B^{-1}$ [10]. Having a proof of a specialization of this conjecture is useful, but the verification of the whole conjecture is still an open project. The main results of the present paper are the construction of the state sum invariants and their properties.

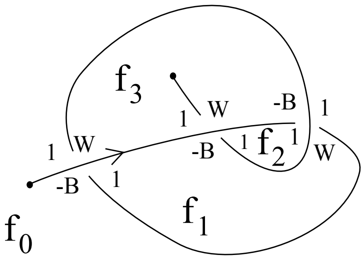
Note 21. We will show below that the potential of a starred link or linkoid diagram L is invariant with respect to the star equivalence with the condition $W = B^{-1}$. The reason that we first present the potential of L is that it is of graph theoretic interest on its own. This use of the potential function is supported by Conjecture 20.

4.1.1 A matrix formulation for the potential

Let L be an oriented link or linkoid diagram with n crossings, that is tightly embedded in a closed, connected, orientable surface of genus g. We enumerate each region and crossing of L with respect to a chosen starting point and the orientation on L. We consider the local region labeling at crossings of L, that we introduce in Figure 16.

L endowed with the local labels at its crossings can be represented by a matrix $M_L = [M_{ij}]$ whose rows and columns correspond crossings and regions of L, respectively, and consist of the corresponding label. Precisely, the ij^{th} entry of M_L is the sum of labels on the j^{th} region received at the i^{th} crossing. If the j^{th} region is not incident to the i^{th} region then the corresponding entry is 0. We call this matrix *potential* matrix of L.

The potential matrix of the knotoid diagram K is presented in Figure 19.



$$M_K = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cccc} f_0 & f_1 & f_2 & f_3 \\ 1-B & 1 & 0 & W \\ 0 & -B & 1 & 1+W \\ 1 & W & 1 & -B \end{array}$$

Figure 19: The potential matrix of K

Definition 22. Let L be a starred link or linkoid diagram with $n \geqslant 1$ crossings, obtained from a link or linkoid diagram in a surface by endowing a number of either regions or crossings of the link or linkoid diagram with stars. The *potential matrix* of L is the square matrix that is obtained by deleting each column or row from the potential matrix of the link or linkoid diagram which corresponds to a region or a crossing, respectively that receive a star. We denote the potential matrix of L by M_L.

Definition 23. Given an $n \times n$ matrix M_{ij}. The *permanent* of M, $Perm(M)$ is given by

$$Perm(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} M_{i\sigma(i)},$$

where the sum runs over the symmetric group S_n.

Proposition 24. *Let L be a starred link or linkoid diagram in a surface, with $n \geqslant 1$ crossings and regions. The potential of L, $\nabla_L(W, B)$ is equal to the permanent of the potential matrix M_L of L.*

Proof. Given an $n \times n$ matrix M_{ij}. In the permanent expansion of M_{ij}, for every $\sigma \in S_n$, the components of the product $\Pi_{i=1}^{n} M_{i\sigma(i)}$ corresponds to a single row choice by each column since σ is a bijection. For the potential matrix of L, this translates to a single choice of a vertex by each region of L. If L is a starred link diagram, then this correspondence implies that each product in the permanent sum are in fact a weight of a state that appears in the Mock Alexander polynomial of L.

Suppose now, L is a starred linkoid diagram with a region incident to an endpoint of a knotoid component of L and the crossing adjacent to the region do not admit a star. Then, the region incident to an endpoint receives two labels and the corresponding entry in the potential matrix is the sum of these labels. These labels contribute in state weights of two states of L that differ from each other at only the vertex (crossing) that receives the labels. Then, the sum of these state weights is equal to the permanent product involving entry corresponding to the vertex and the region incident to it receiving two labels.

By the arguments above, we see that each nonzero summand of the permanent of M_L is equal to a state weight or the sum of two state weights of L. Therefore, the permanent of M_L is equal to the potential of L. □

Example 25. Let K be the knotoid diagram given in Figure 19. Consider the starred diagram K_* obtained from K by endowing the region f_0 with a star. One can verify by direct calculation that the permanent of the potential matrix M_{K_*} is $W^2 - WB + (W - B) + 1$, and this coincides with the potential of K_1 calculated in Example 19.

4.2 An invariant of starred links and linkoids induced by the potential

Here, we investigate the conditions for the potential of a starred link or linkoid diagram to induce an invariant of starred links and linkoids.

Theorem 26. *The potential is an invariant of starred links and linkoids that lie in a surface of genus $g \geqslant 0$, when $WB = 1$.*

Definition 27. Let L be an oriented starred link or linkoid diagram. The potential of L is called the *Mock Alexander polynomial* of L when $W = B^{-1}$. Thus by Theorem 26, the Mock Alexander polynomial is an invariant of oriented starred links or linkoids in a surface of genus g.

Proof of Theorem 26. Let L be a starred link or linkoid diagram that lies in a surface. An RI move adds a crossing and a new region incident to the crossing to L or deletes a crossing and one of the regions that is incident to the crossing from L. Let us assume the first case. The region added after the move receives a state marker at the new crossing, as

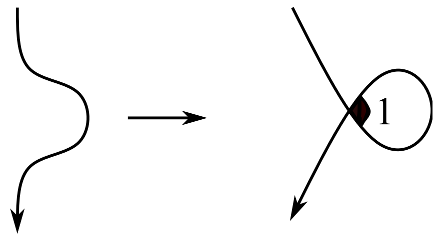


Figure 20: Invariance under an RI move.

shown in Figure 20, and this state marker appears in any state of the new diagram without any affect on the state markers of the former diagram. This implies that the states of the former and latter diagrams are in one-to-one correspondence with each other. We can see easily that the contribution of the state marker at the new crossing to the state weight is trivial regardless of the type of the crossing created. Therefore, the state-sum polynomial is not affected under this move. If a type I move deletes a crossing from the diagram then a region that is adjacent only to the deleted crossing is removed and since the weight of the state marker at the crossing is trivial by the directions of the strands at the crossing , the potential is not affected.

An RII adds or deletes two regions locally to a starred link or linkoid diagram. In Figure 21, we examine two cases of an RII move where the strands in the move sites are not adjacent to any endpoints (if the diagram is a linkoid), and no region incident to strands is endowed with a star. The possible local placements of the state markers at the crossings in the RII move are shown in Figure 21. Here each tensor product sign indicates that the regions with the tensor products receive the state markers at crossings that are not involved with the move. We deduce that the assumption $WB = 1$ gives the invariance under the first case of an RII move where the strands have the same direction. For the other case, where the strands are directed oppositely, the invariance is provided directly. Note here that the regions T and U may be the same region in the corresponding link or linkoid diagram. The local state configurations would be the same for this case as well. There if $WB = 1$, the invariance is provided.

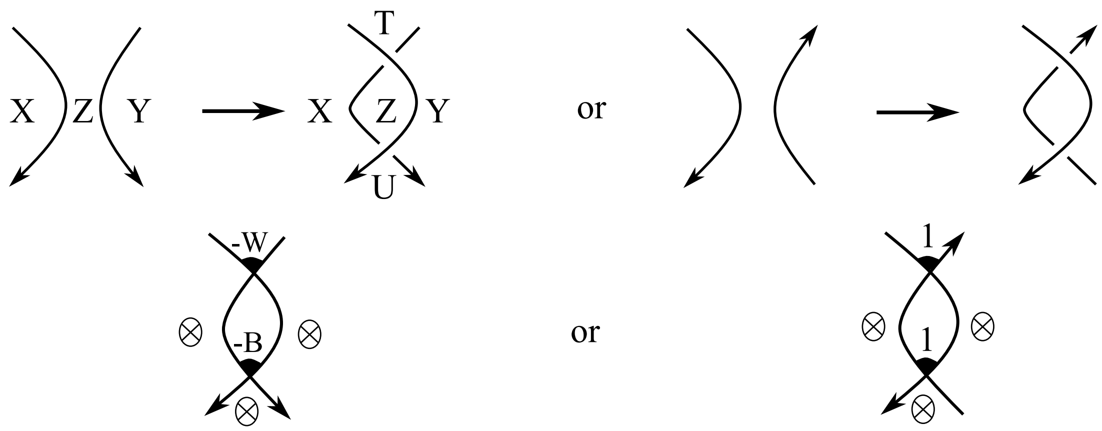


Figure 21: Invariance under an RII move.

Some other cases of an RII move applied on a starred link or linkoid diagram where the neighboring regions in an RII move site are endowed with stars, and the local state configurations corresponding to these cases, are depicted in Figure 22 and Figure 23. The invariance of the potential under the assumption $WB = 1$ can be verified easily by the figures for all of these cases. Here we examine in detail the behavior of the potential under the RII moves, depicted in Figure 23, where the strands in the move site are incident to endpoints of a starred linkoid diagram, and the verification for other cases of an RII move are left for the reader.

One of the strands in the RII move depicted on the top row of Figure 23, is assumed to be incident to the tail of the knotoid component, and the adjacent region that is labeled as X, is endowed with a star. Notice here that region X lies on both sides of the tail

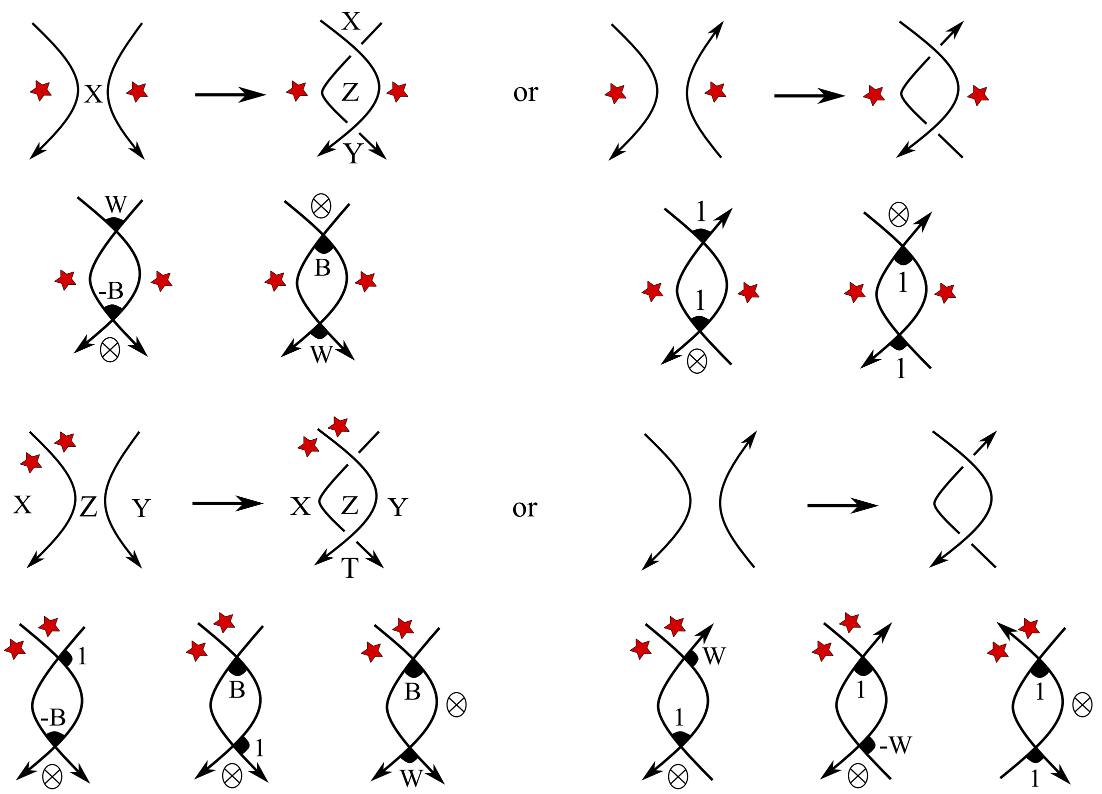


Figure 22: RII moves where the neighboring regions receive stars.

arc. The move adds two regions (labeled as T and Z) to the local portion of the diagram appearing after the move. We assume that the regions X and T are not the same region. There are in total three possible local placements of the state markers at the crossings in the RII move as in the previous cases. We picture these placements at the bottom row of Figure 23.

Notice that for every state s that involves the first (leftmost) configuration, there exists a unique state that involves the second (middle) configuration and coincides with s except for the state markers at the crossings in the move site. Then, it can be also be verified by the figure that for a state that involves the first configuration and contributes to the potential with $-BP$, where P denotes the product of weights of state markers outside the move site, there is a state that involves the second configuration and contributes with BP. Thus, the total contribution of states involving the first and the second configurations to the potential is zero. In the third local state configuration, the regions added by the move, namely the regions Z and T receive state markers and the state markers at the crossings outside the move site are not affected. Then, the contribution of a state that involves this configuration is WBP, where P holds for the product of state weights outside the move site in the state. Therefore, the potential is invariant under the type II move if $WB = 1$. For the second case, it is not hard to see that the contributions of the first two local state configurations cancel each other, and the invariance under this move follows directly, as the last local state contributes with trivially.

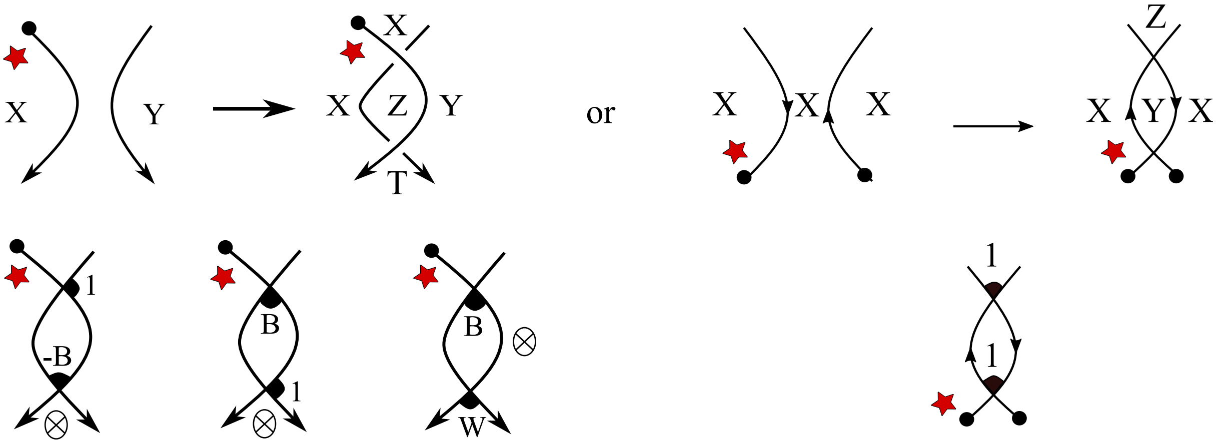


Figure 23: RII moves where strands are incident to endpoints and neighboring regions receive stars.

The regions labeled by X and T may be the same region only if the strands in the move belong to different components of the linkoid diagram. If X and T are the same region, then the region T is also the starred region and does not receive any state marker. So, the first two local state configurations at the crossings added by the move, are the only possible local configurations for states of the resulting diagram. where the region Y gets its marker at the crossings added by the move in these configurations. It can be verified from the figure that the contributions of these two local states multiply the product of weights of the remaining crossings of the diagram with $-B$ and B, respectively in every possible state. This means that the total sum of contributions, and so the potential of the starred linkoid diagram is zero.

In the case depicted on the right hand side of Figure 23, the strands in the move site are incident to the endpoints of a knotoid diagram. The endpoints apparently lie on the same region that is endowed with the star. Here the RII move adds two new regions, named as Y, Z to the knotoid diagram since the diagram is connected. There is only one possible configuration of state markers at the move site, as shown in the figure. The product of state weights in this local configuration is trivial (1) and the states of the knotoid diagrams before and after the move are in one-to-one correspondence since the state markers at the regions before the move are all outside of the move site and not affected by the move. Therefore, the potential of the knotoid diagram remains the same under the move.

There are various cases for an RIII move as well. We first study the invariance in a case depicted in Figure 24, where one of the strands appearing in the move site is adjacent to the tail of the diagram and the region incident to the tail is endowed with a star. It is not hard to observe that the regions of the diagram before the move takes place are in one-to-one correspondence with the regions of the diagram obtained after the move takes place, and there is only one possible placement of the state markers at the crossings that are in the move site before and after the move takes place. The move also has no effect on the state markers at the regions that lie outside of the move site. Without loss of

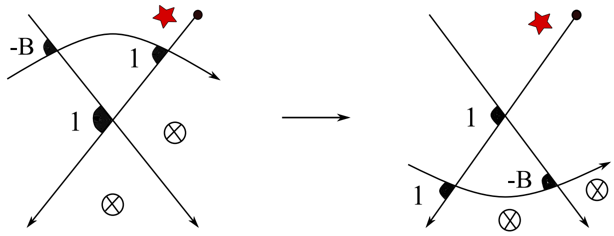


Figure 24: An RIII move.

generality, we can assume all flat crossings shown in the move site are positive crossings in the corresponding knotoid diagram. It can be verified by the figure that the product of weights of the crossings lying in the move sites before and after the move takes place, are both $-B$. Thus, the potential is preserved under this move.

Another case of an RIII move is given in Figure 25, where none of the regions adjacent to the crossings of the move site is starred and the strand adjacent to the tail is involved in the move. In addition to this, three of the local regions in the move site are occupied by tensor product signs. In this configuration, there are three possible placements of the state markers at the crossings in the move site before the move takes place. Assuming that all the crossings in the move site are positive crossings, the sum of the product of weights of the state markers in the local portions of the three states is $W^2 - W^2B - W$, as can be verified by the figure. After the move, we see that there are five possible placements of the state markers at the crossings of the move site. Summing up the product of weights of the markers in the move site among all these local states, we find that the contribution is $W^2 - B - WB + WB^2 + 1$. Therefore, the potential remains unchanged under this move if $WB = 1$.

A variation of the case depicted in Figure 25 where the region incident to the endpoint is not endowed with a star, is given in Figure 26. The invariance when $WB = 1$ can be verified directly by the figure. Figures 25 and 26 take care of all possible types of an RIII move where an endpoint is involved with the move site and the star of the diagram is far away from the endpoint.

The reader is directed to [12] for the details on possible cases of an RIII move that take place on a link diagram endowed with stars on a pair of adjacent regions. The verification of invariance can also be made similarly as in [12] for an RIII move that takes place on a link diagram that is endowed with stars on a pair of non-adjacent regions. □

Theorem 28. *[12] Let L be an oriented link diagram in S^2. The Alexander-Conway polynomial of L is equal the potential of a starred link diagram obtained by endowing a pair of adjacent regions of L with stars, under the assumption $W = B^{-1}$ on the region labels.*

Proof. We discuss the relation between the Alexander-Conway polynomial of an oriented link and the state-sum polynomial briefly in Section 1. See [12] for details. □

Recall that the Mock Alexander polynomial of a link or a linkoid is the potential when $W = B^{-1}$ (Definition 27). It follows from Proposition 24 that the Mock Alexander

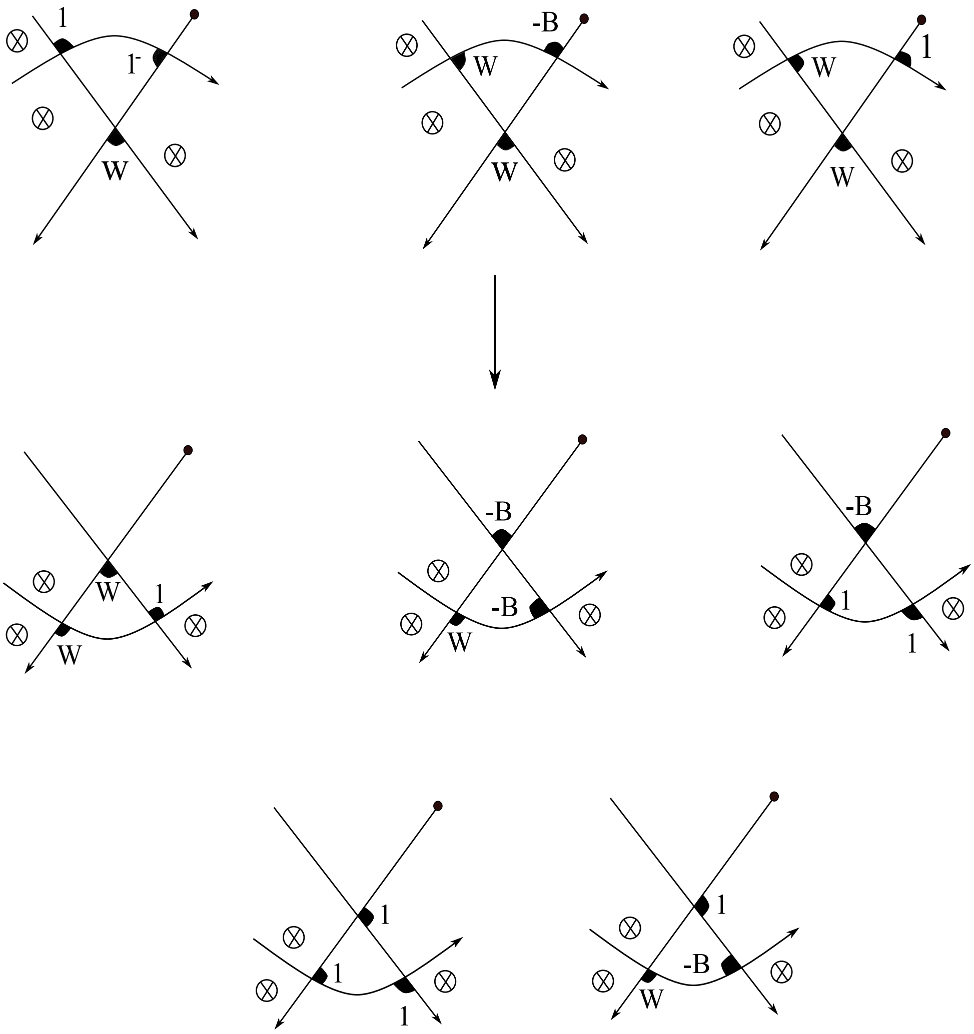


Figure 25: An RIII move.

polynomial of L, $\nabla_L(W)$ is the permanent of M_L, where M_L is the potential matrix of L with entries $0, 1, W^{\pm 1}$ or $1 \pm W^{\pm 1}$. In the following examples, $\nabla_L(W)$ is calculated as the permanent of the potential of L.

Examples. 1. Consider the horizontal trivial oriented 2-tangle T_0. We add $n \geqslant 1$ positive twists to T_0. Let K_n denote the numerator closure of the resulting tangle, endowed with stars at the interior regions added by the closure. It is clear that we obtain an oriented starred link diagram. See Figure 27 for the starred link diagram obtained in this way.

It can be verified from Figure 27 that there are exactly two states of K_n for every $n \geqslant 1$, and the contributions of these states to $\nabla_{K_n}(W)$ are W^n and $(-1)^n W^{-n}$, respectively. Therefore,

$$\nabla_{K_n}(W) = W^n + (-1)^n W^{-n},$$

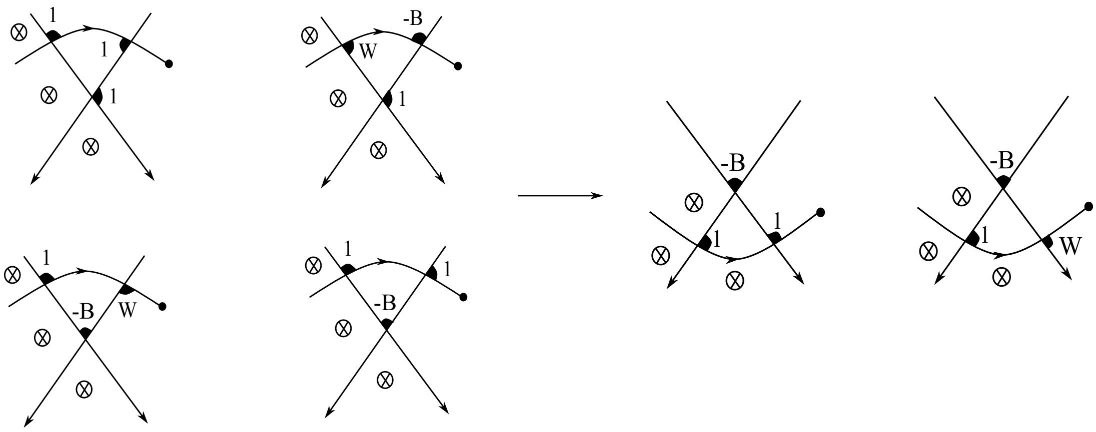


Figure 26: An RIII move

for every $n \geqslant 1$.

By direct calculation, one can also see that the following equality holds for every $n \geqslant 2$.

$$\nabla_{K_{n+1}}(W) = \nabla_{K_{n-1}} + (W - W^{-1})\nabla_{K_n}.$$

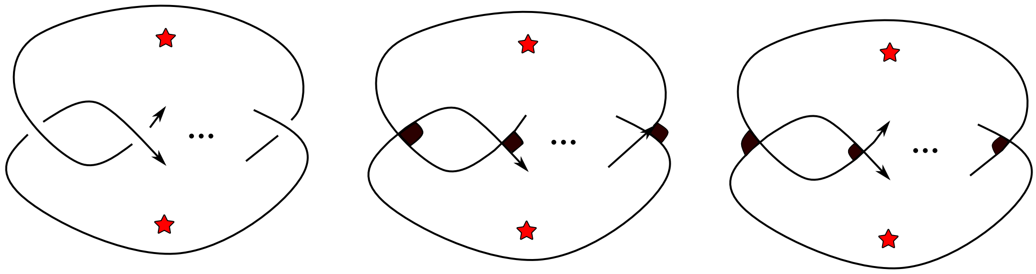


Figure 27: Starred link diagram K_n with nonadjacent stars, for some $n \geqslant 1$.

Let K be the starred trefoil knot with its two non-adjacent regions starred, as shown in Figure 28. Notice that K is star equivalent to K_3 discussed above. We verify that the permanent of M_K coincides with the Mock Alexander polynomial calculation of K_3.

2. Let K_n denote the spiral knot diagram with n crossings with its stars placed in twisted regions as illustrated in Figure 29 for $n = 2, 3$. The potential matrix of K_n, $n \geqslant 2$ is given as

$$M_{K_n} = \begin{bmatrix} W - W^{-1} & 1 & 0 & \dots & & & 0 \\ 1 & W - W^{-1} & 1 & 0 & \dots & & \\ 0 & 1 & W - W^{-1} & 1 & 0 & \dots & \\ \vdots & 0 & 1 & W - W^{-1} & 1 & 0 & \dots \\ 0 & \dots & & & 0 & 1 & W - W^{-1} \end{bmatrix}.$$

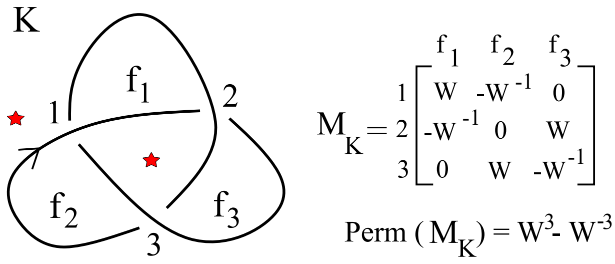


Figure 28: K_3 is a starred trefoil diagram.

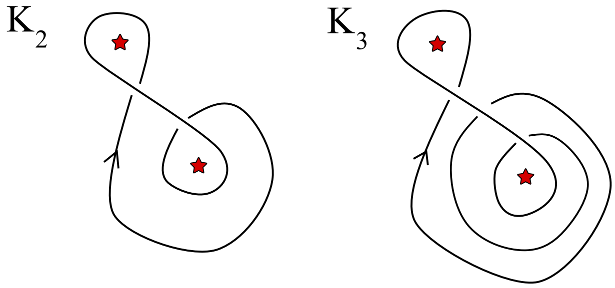


Figure 29: Starred knot diagrams in spiral form.

By direct calculation utilizing the permanent formula, we find

$$\nabla_{K_2}(W) = 1 + (W - W^{-1})^2$$

and

$$\nabla_{K_3}(W) = 2(W - W^{-1}) + (W - W^{-1})^3.$$

Assuming that $\nabla_{K_{-1}}(W) = 0$, $\nabla_{K_0}(W) = 1$ and $\nabla_{K_1}(W) = W - W^{-1}$, we have the following recursive relation for the Mock Alexander polynomial of K_n, for any $n \geqslant 1$,

$$\nabla_{K_{n+1}}(W) = (W - W^{-1})\nabla_{K_n}(W) + \nabla_{K_{n-1}}(W),$$

that follows directly from the form of the permanent matrix for the invariants.

The next theorem shows us how to define our invariants for diagrams with disjoint knot or knotoid components.

Theorem 29. *Let L be a split linkoid diagram in S^2 that consists of only two disjoint knotoid components.*

Let L_1 and L_2 be two admissible linkoid diagrams obtained from L by an RII move that connects the disjoint components by pushing the components over or under each other. Then,

$$\nabla_{L_1}(W) = \nabla_{L_2}(W).$$

Proof. The argument follows from Proposition 12 in Section 3.1 that tells any two admissible linkoid diagrams obtained by connecting L_1 and L_2 are equivalent to each other through connected diagrams. □

Note 30. By Theorem 29, we assume that the Mock Alexander polynomial of a split linkoid diagram L in S^2 consisting of only two knotoid components is the Mock Alexander polynomial of any connected diagram obtained by connecting the components of L by an RII move.

Let K be any knotoid diagram in S^2 with n crossings. It is apparent that K gives rise to $n+1$ different starred knotoid diagrams. One can also fix the placement of the star so that it is placed at the region adjacent to the tail of the knotoid diagram. Then, the region endowed with the star is uniquely determined for any knotoid diagram and the Mock Alexander polynomial of the resulting starred diagram is a polynomial assigned to the knotoid diagram.

Definition 31. Let K be a knotoid diagram in S^2. The Mock Alexander polynomial of K denoted by$\nabla^{\sharp}_{K}$, is defined to be the Mock Alexander polynomial of the starred diagram obtained from K by endowing the region adjacent to the tail with a star.

Example 32. We depict a simple knotoid diagram with two crossings in Figure 30. It

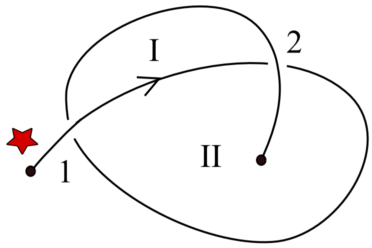


Figure 30: A knotoid diagram with two crossings.

is not hard to verify that the Mock Alexander polynomial of the knotoid diagram K, $\nabla^{\sharp}_{K}(W)$ is the permanent of $\begin{bmatrix} W & 1 \\ -W^{-1} & W+1 \end{bmatrix}$, which is equal to $W^2 + W - W^{-1}$.

Proposition 33. *The Mock Alexander polynomial* $\nabla^{\sharp}_{K}$ *is an invariant of knotoids in* S^2*.*

Proof. It follows from Theorem **??**. □

4.2.1 Reversal and mirror behavior of Mock Alexander polynomials

Proposition 34. *Let L be an oriented starred link or linkoid diagram in a surface and $\overline{L}$ be the starred diagram obtained from L by reversing the orientation on each of its components, without changing the placement of the star of L. We have the following relation between the Mock Alexander polynomials of L and $\overline{L}$.*

$$\nabla_{\overline{L}}(W) = \nabla_L(-W^{-1}).$$

Proof. The local labels assigned to local regions at a positive and a negative crossing of L and the corresponding crossings of $\overline{L}$, are shown in Figure 31. □

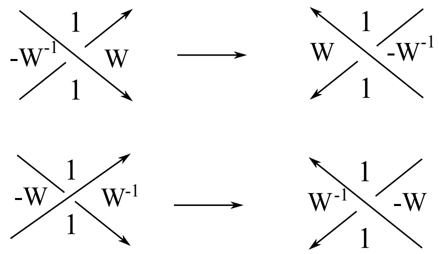


Figure 31: The effect of reversing the orientation on region labels.

Proposition 35. *Let L be an oriented starred link or linkoid diagram in a surface. The mirror image of L, L^* is obtained by only switching every crossing of L. We have the following relation between the Mock Alexander polynomials of L and L^*.*

$$\nabla_{L^*}(W) = \nabla_L(W^{-1}).$$

Proof. The change in local labels in the local regions adjacent to a crossing of L by switching the crossing, is shown in Figure 32. □

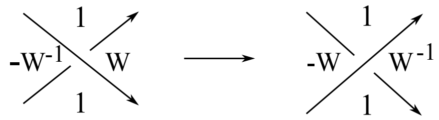


Figure 32: The change of region labels under mirror symmetry.

Here we present a topological specialization of Conjecture 20 for the Mock Alexander polynomial of a knotoid in S^2.

Conjecture 36. Let K be a knotoid diagram in S^2 with its tail lying in the exterior. Let K_{ext} denote the starred knotoid diagram with a star placed in the exterior region of K, and K_{in} denote the starred knotoid diagram with a star placed in the region where the head of K is located. Then, we have the following equality.

$$\nabla_{K_{ext}}(W) = \nabla_{K_{in}}(-W^{-1}).$$

Note that any knotoid diagram in S^2 can be represented by a diagram whose tail lies at the exterior region [18]. So, without loss of generality, we consider such representations of spherical knotoids. The proof of Conjecture 36 will appear in a joint paper by Wout Moltmaker, Jo-Ellis Monahan and the authors of the present paper.

The following corollary follows from Proposition 34 and Conjecture 36.

Corollary 37. *Let K be a knotoid diagram in S^2 and $\overline{K}$ be its reversion. Then,*

$$\nabla^\sharp_K(W) = \nabla^\sharp_{\overline{K}}(W).$$

4.3 Skein relations

Definition 38. Let L be a link or linkoid diagram in S^2. We call a crossing v of L *separating* (or *nugatory*) if at least one of the resolutions of v results in a split diagram. Note that in the classical case, that is if L is a classical link diagram, a nugatory crossing can be removed by isotopy. On the other hand, any crossing that is adjacent to an endpoint of a linkoid diagram is separating but not nugatory in the classical sense since it cannot be removed by an isotopy move. For example, the crossing of the linkoid diagram L_+ depicted in Figure 39, is a separating but not a nugatory crossing in the classical sense.

Theorem 39. *Let L_+, L_- and L_0 denote three starred link diagrams in S^2 that differ from each other at exactly one crossing v, as shown in Figure 33. Assume that all the regions that are incident to the crossing v are mutually distinct regions in L_+ and L_-. Then, the skein relation*

$$\nabla_{L_+}(W) - \nabla_{L_-}(W) = (W - W^{-1})\nabla_{L_0}(W).$$

holds at v if

i. none of the regions that is incident to the crossing is starred or,

ii. two adjacent regions that are incident to the crossing are starred or,

iii. both up and down regions that are incident to the crossing are starred.

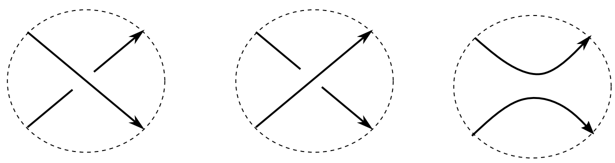


Figure 33: L_+, L_- and L_0 from left to right.

Proof. We begin by making the observation that L_+ and L_- admit the same universe. Let U denote the universe of L_+, L_-, and U_0 denote the universe of L_0. Assume first that none of the regions incident to v receives a star. This case is studied in [12] for classical link diagrams. We extend the argument for this case of linkoids.

Since the regions incident to v do not receive a star, the face of U_0 that is obtained by smoothing the crossing v in L_+ in the oriented way, does not receive a star. Let us call this region F. Let δ and δ_0 be the sets of all states of L_+ (also of L_-) and L_0, respectively. Consider the partition of δ_0 into two subsets, $\mathcal{L}$ and $\mathcal{R}$ where $\mathcal{L}$ denotes the collection of states where the state marker at the region F lies at a crossing that lies on the left hand side of the former crossing v, and $\mathcal{R}$ denotes the collection of states where the state marker at the region F lies at a crossing that is on the right hand side of v.

Consider now the partition of δ into the following four subsets: $\mathcal{L}_U$, where the state marker at v is placed at its left, $\mathcal{R}_U$ where the state marker at v is placed at its right, $\mathcal{U}_U$ where the state marker at v is placed up and $\mathcal{D}_U$ where the state marker is placed down.

There is a one-to-one correspondence between $\mathcal{R}_U$, where the crossing v receives its marker at its right and $\mathcal{L}$. Similarly, there is a one-to-one correspondence between $\mathcal{L}_U$ and $\mathcal{R}$. These correspondences result in the equalities in Figure 34 that induce the equalities in Figure 35 with the arguments above. The skein relation is obtained by subtracting the second equality in Figure 35 from the first one.

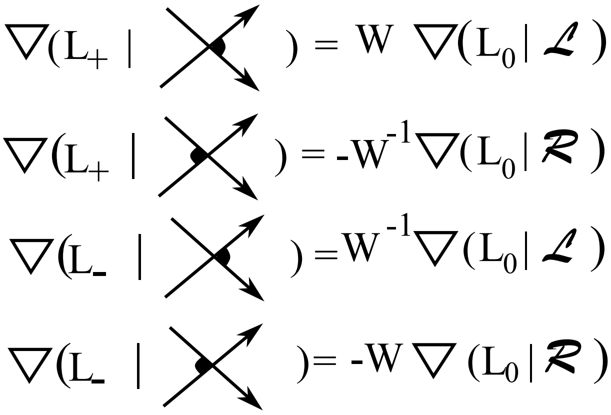


Figure 34: The relations induced by different placement of markers at v.

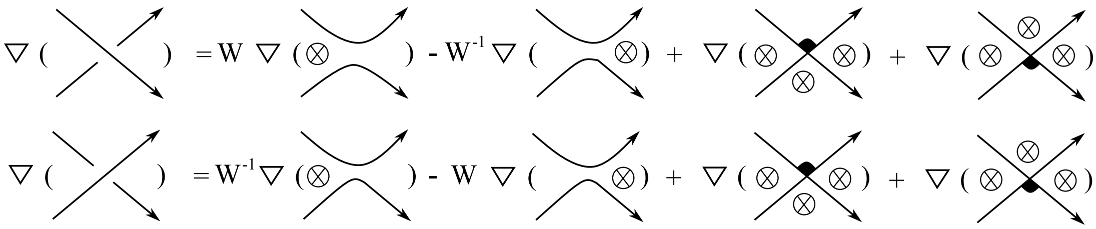


Figure 35: State expansions at crossings far away from starred regions.

Assume that *ii.* holds. That is, any two of the regions incident to v that are adjacent to each other, are starred. This implies the region F of L_0 receives a star. Without loss of generality, assume that the region that lies on left hand side of v in L_+ (and so, in L_-) is starred. We observe the equalities given in Figure 36 at v. By subtracting the second equality from the first one, we find the skein relation. (If the region on the right hand side of v is starred then similar equalities hold as in Figure 36, with the difference in the coefficients. Precisely, in the first (top) equality we see $-W^{-1}$ instead of W, and $-W$ in the second (bottom) equality instead of W^{-1}. From this, we can deduce the skein relation.

Now assume that *iii.* holds for v. The observation made for *i.* in Figure 34 directly applies to this case. Since there will be no state markers at top and bottom regions in any state of L_+ and L_-, in the skein state expansion at v we only observe the first two terms that are given on the right hand side of the equalities in Figure 35 for both types of crossings. This results in the skein relation. □

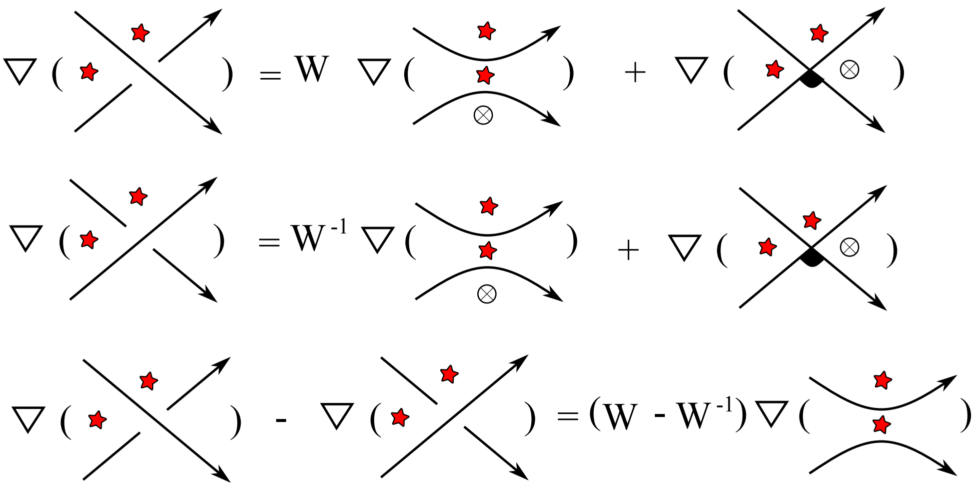


Figure 36: State expansions at crossings incident to starred regions.

Proposition 40. *Let L_+ be a starred link diagram in S^2 and v be the positive crossing of L_+ where L_+ differs both from L_- and L_0, as shown in Figure 33. Assume that the left hand side and the right hand side regions (not necessarily distinct regions) of v are starred. Then*

$$\nabla_{L_+}(W) = \nabla_{L_-}(W).$$

Proof. The collections of states of L_+ and L_- can both be partitioned into two subsets of states, one which v receives the state marker at the region that lies on top of v and another which v receives the state marker at the region that lies at the bottom of v, and the weight of a state marker at the top or the bottom of v is 1. Then the Mock Alexander polynomials of L_+ and L_- are equal. □

Proposition 41. *Let L_+ be a starred link diagram in S^2 with stars lying on a pair of adjacent regions. If the crossing v is nugatory, then*

$$\nabla_{L_+}(W) = \nabla_{L_-}(W).$$

Proof. This case is discussed in [12]. We illustrate L_+ and v in Figure 37 where K_1 and K_2 are the sublinks that can be obtained by separating L_+ at v.

The main observation that is utilized for the proof is the following. L_+ is assumed to be endowed with stars on a pair of adjacent regions. There is no state of L_+ that admits a state marker at v lying on the left or the right of v, so on the exterior region of L_+, as shown in the figure. The reason for this is that the stars lie on adjacent regions and a state where v receives a state market at its left (or at its right) forces the pair of stars to lie on a pair of interior regions of one of the sublinks that form L_+. This results in the corresponding sublink diagram, there are 3 occupied regions (two of them with stars and one of them is the marked exterior region) and if it has n crossings, then it has $n-1$ available regions to be starred, which is contradictory to the definition of state. Therefore, one of the stars is required to be placed on the left or right of the crossing v.

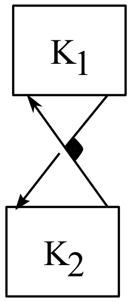


Figure 37: An impossible state marker placement at a nugatory crossing

From this observation, it follows that either the top or the bottom region that is incident to v receives a state marker whose weight is 1. Since this observation is also true for L_-, we get the equality $\nabla_{L_+}(W) = \nabla_{L_-}(W)$. □

Remark 42. By Proposition 41, the Alexander polynomial of a split link is assumed to be zero. See [12] for the discussion.

Proposition 43. *Let L_+ be a starred link diagram in S^2. Let v be the crossing where L_+, L_-, L_0 differ from each other. Assume that the crossing v is the unique nugatory crossing of L_+ and the stars lie on a pair of non-adjacent regions in L_+.*

i. If the stars lie on two interior regions that belong to only one of the sublinks of L_+, that is, the stars are on either K_1 or K_2 of Figure 37, or if one of the stars lie on the exterior region, then $\nabla_{L_+}(W) = \nabla_{L_-}(W)$.

ii. If the stars lie on two interior regions, one star on K_1 and the other star on K_2, then $\nabla_{L_+}(W) - \nabla_{L_-}(W) = (W - W^{-1})\nabla_{L_0}(W)$.

Proof. i. Assume first that the stars are placed on two nonadjacent interior regions of K_1. In this case, in any state of L_+, the exterior region of K_1 needs to get a state marker in one of the neighboring crossings of K_1. This forces the placement of the state marker at the nugatory crossing v is either up or down the crossing v. Thus, a state marker at the nugatory crossing contributes to ∇_{L_+} trivially. This is also true for L_-. Since the states of L_+ and L_- coincide, the equality follows. The equality is also true for the latter case: If the exterior region is occupied with a star, then the state marker placement at v is either up or down the crossing. The rest follows similarly.

ii. In this case, a state marker at the nugatory crossing v can be placed on any side of v (up, down, left or right). If the state marker lies at the right of v in a state of L_+ then its weight is W. Let P_1 be the sum of the product of weights of state markers in states where the state marker at v is at the right of v then the total contribution of all such states to ∇_{L_+} is WP_1. It is clear that the states where the state marker at v is at the right of v coincide with the states where the state marker at v is at the left of v, except for the state marker at v. Then, the total contribution of all states where the state marker at v is at the left of v, to ∇_{L_+} is $-W^{-1}P_1$. Let P_2 and P_3 denote the sums of the product of weights of state markers in states where the state marker at v is at the top and bottom of v, respectively. The weight of a state marker at the top or bottom of v is 1. We find that $\nabla_{L_+} = (W - W^{-1})P_1 + P_2 + P_3$.

States of L_+ clearly coincide with states of L_-. Similar arguments hold for L_- and we find that $\nabla_{L_-} = (W^{-1} - W)P_1 + P_2 + P_3$. Then $\nabla_{L_+}(W) - \nabla_{L_-}(W) = 2(W - W^{-1})P_1$.

On the other hand, L_0 is a split link diagram. Let $\tilde{L_0}$ denote the connected link diagram that is obtained from L_0 by an RII move. As shown in Figure 38, there are in total six possible placements of state markers at the crossings in the RII move site when none of the regions adjacent to the strands in the move site is endowed with a star. The states I, III, IV, VI of $\tilde{L_0}$ depicted in Figure 38 coincide with the states of L_+ (also with L_-) where the state marker at v is at the top or the bottom of v. Then, the total contributions of these states is $WP_2 - WP_2 + WP_3 - WP_3 = 0$. The states II and V contribute as $2P_1$, where P_1 is the sum of the product of weights of state markers in states of L_+ where the state marker at v is at the right (or left) of v. Then, $\nabla_{L_0}(W) = \tilde{L_0}(W) = 2P_1$. Therefore, $\nabla_{L_+}(W) - \nabla_{L_-}(W) = (W - W^{-1})\nabla_{L_0}(W)$. □

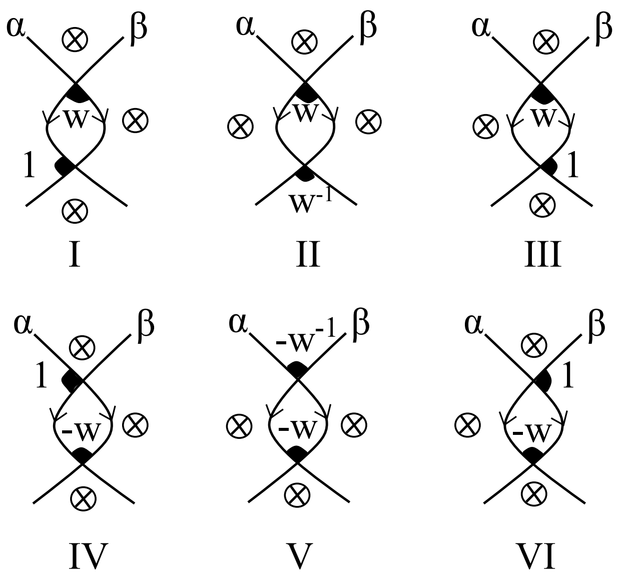


Figure 38: The local states in an RII move site connecting disjoint knots α, β.

Theorem 44. *Let L_+, L_- and L_0 denote three starred linkoid diagrams in S^2 which differ from each other at a unique crossing v, as shown in Figure 33. Then the skein relation*

$$\nabla_{L_+}(W) - \nabla_{L_-}(W) = (W - W^{-1})\nabla_{L_0}(W)$$

holds if v is not a separating crossing.

Proof. Let v be a crossing in L_+ that is not starred and not separating. Then either four of the regions that are adjacent to v are mutually distinct or two of them are the same regions if v is incident to an endpoint. But either of these cases, similar equalities that appear in the proof of Theorem 39 hold at v. Note that only one of the regions that are adjacent to v may be starred since L_+ is a linkoid diagram. The details of the proof are left to the reader. □

Remark 45. A linkoid diagram may have a separating crossing but still satisfies the skein relation at such crossing. See the linkoid diagram given in Figure 39 with one and only separating crossing. Notice that L_0 is a split linkoid diagram with two components and is not an admissible diagram. Transforming L_0 to an admissible diagram $\tilde{L}_0$ by an RII move is possible as in Figure 39. Then, direct computation shows that $\nabla_{\tilde{L}_0}(W) = 2$. Therefore, $\nabla_{L_0}(W) = 2$. One can verify easily that

$$\nabla_{L_+}(W) - \nabla_{L_-}(W) = 2(W - W^{-1}) = (W - W^{-1})\nabla_{L_0}(W).$$

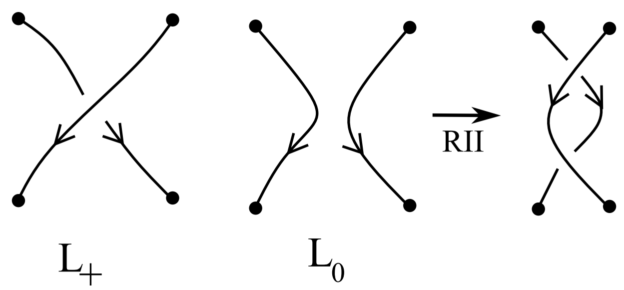


Figure 39: A linkoid diagram with a separable crossing and its resolution.

5 Diagrams on surfaces

5.1 Admissible link diagrams in higher genus surfaces

Let L be a connected link diagram that is tightly embedded in some surface of genus g. We know that the equality $f = 2 - 2g + n$ holds for L, where f denotes the number of regions and n denotes the number of crossings of L. In Section 3.1, we discuss decorating regions or crossings with stars to obtain the equality $f = n$ for the regions and crossings without stars. See Figure 40 for two examples, where we illustrate knot diagrams in a torus and double torus.

The number of regions of L can be also reduced by two by adding handles to S^2 that connect either a triple of regions or two distinct pairs of regions of L. The resulting link diagrams are admissible link diagrams in genus surface as it is described below. We first discuss adding handles to S^2 that connect a triple of regions of L.

Let τ denote $S^2 - \bigcup_{i=1}^{3} D_i$, where D_i is an open disk in S^2. We call τ a *trident surface.* Choose any three distinct regions in L and delete an open disk from each of these regions.

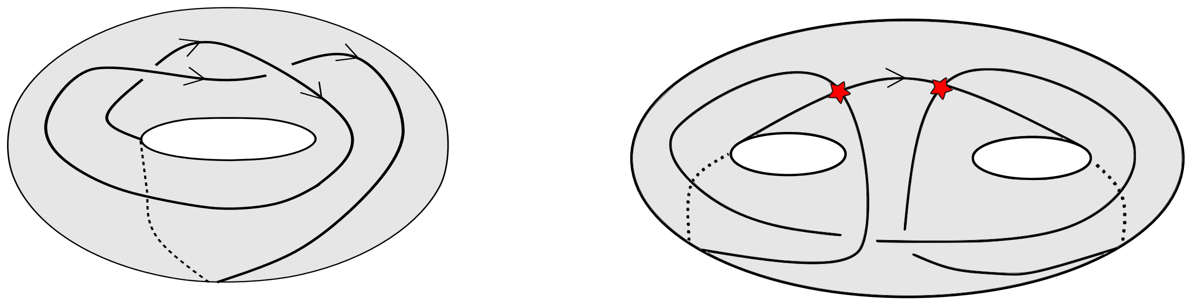


Figure 40: Admissible knots in a torus and a double torus

A trident surface is added to S^2 by gluing its boundary components to the boundaries of the deleted disks, as illustrated in Figure 41. As a result, L is regarded as a link diagram in a genus two surface. It is not hard to observe that the chosen triple of regions of L are connected by the trident surface and so represents the same region of L (named as T in the figure) that lies in the genus two surface. This implies the total number of regions in L that lies in the resulting genus two surface is equal the number of its crossings. That is, L becomes admissible in the genus two surface.

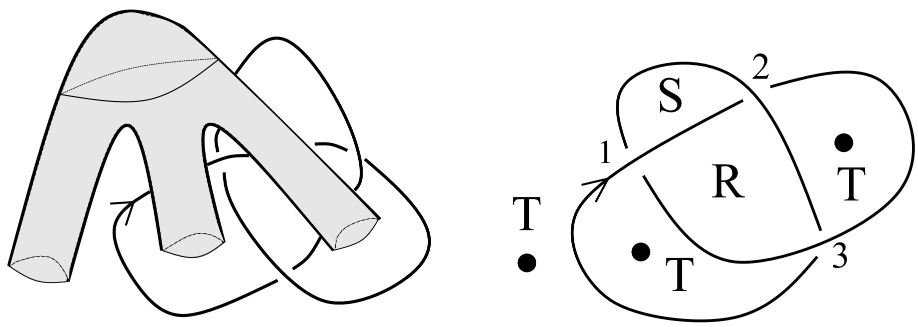


Figure 41: Attaching handles onto a triple of regions of the trefoil diagram

One another way to obtain an admissible link diagram in a genus two surface is to choose two distinct pairs of regions of L in S^2 (so in total four distinct regions), delete out an open disk from each of the regions and connect these regions pair-wisely by attaching two disjoint handles through the boundaries of the disks. The resulting link diagram is clearly an admissible link diagram in a genus two surface.

These constructions of admissible link diagrams in genus two surfaces bring us to the question of classification of admissible link diagrams in higher genus surfaces. We answer to the question by the following theorem but we shall first give a necessary definition.

Definition 46. Let L be a connected link diagram that is admissible in a surface of genus g. Assume that all extraneous handles have been removed from the surface. This means that all faces with one boundary component are disks, and that any face with multiple boundary components consists in handles attached to disks in such a way that removal of

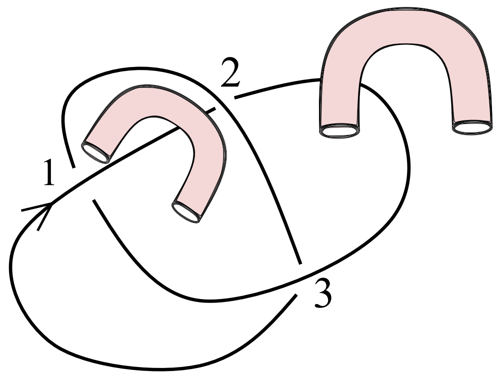


Figure 42: Attaching disjoint handles on pairs of distinct regions of the trefoil diagram

any handle will disconnect the region. We call such a surface a *minimal surface* for the link diagram L.

Theorem 47. *If L is an admissible link diagram that lies in a surface Σ_g of genus $g > 0$ and Σ_g is minimal for L, then $g = 1$ or $g = 2$. If $g = 2$, then the embedding can be obtained from a link diagram L in S^2 by adding some handles in either the form of three disk regions made into one region (attaching a trident surface) or two distinct pairs of disk regions made into two regions by connecting pairs by two disjoint handles.*

Proof. Assume L lies in its minimal genus surface in Σ_g and that we can reduce the genus by cutting handles. This will increase f by one for each handle we cut. We then obtain L in S_h where $h < g$ where it is a tight diagram with $f' < f$ faces and $f' > n$. By Euler's formula, we know that $f' - v = 2 - 2h$. Since $f' - v > 0$, we have $2 - 2h > 0$ and so $h < 1$, whence $h = 0$. Thus L in its minimal surface Σ_g is obtained from a diagram L in the 2-sphere S^2 by adding handles. We know that in S^2, $f' - v = 2$ and so we can only add two 1-handles. This yields the genus two case. If L is admissible in its minimal surface Σ_g, but there is no reduction by cutting handles, then all regions of L are disks (that is, L is tightly embedded in Σ_g and we have $f - v = 0 = 2 - 2g$ implies $g = 1$. □

5.2 Two variants of the domain of the Mock Alexander Polynomial

In this section we study the Mock Alexander polynomial over the domain of admissible link diagrams obtained by handle additions discussed in the previous section.

Definition 48. Let L be a classical link diagram in S^2 and $\tilde{L}$ be the corresponding link diagram in a genus two surface that is obtained by adding a trident surface to S^2 that connects a triple of regions in L. The *trident polynomial* of $\tilde{L}$, $T_{\tilde{L}}$ is defined to be the Mock Alexander polynomial of $\tilde{L}$.

Corollary 49 (Equivalent definition)**.** *The trident polynomial of $\tilde{L}$, is the permanent of the potential matrix of $\tilde{L}$.*

Remark 50. The link diagram L in S^2 can be considered as the planar representation of $\tilde{L}$ with the addition of nodes on the chosen triple of regions of L that are connected by two handles to obtain $\tilde{L}$. Let L_1 and L_2 be two connected (classical) link diagrams in S^2 with nodes on triples of regions for each. L_1 and L_2 represent the same link in the genus two surface obtained by attaching two handles to S^2 if and only if one can be transformed to the other by a sequence of Reidemeister moves and spherical isotopy moves that take place far from the regions with nodes.

It is more convenient to obtain the potential matrix of $\tilde{L}$ by using the planar representation, and so to compute the trident polynomial of $\tilde{L}$. The potential matrix of the trefoil diagram with columns determined by the regions R, S, T in order, is given as follows.

$$\begin{bmatrix} 1 & W & -W^{-1}+1 \\ 1 & -W^{-1} & W+1 \\ 1 & 0 & W-W^{-1}+1 \end{bmatrix}$$

The permanent calculation shows that $T_{\tilde{L}}(W) = 2(W^2 + W^{-2}) + 2(W - W^{-1}) - 2$.

Proposition 51. *The trident polynomial is an invariant of classical connected link diagrams that are embedded in a genus two surface obtained by adding a trident surface connecting three regions of classical link diagrams in S^2.*

Proof. This follows from the definition of the trident polynomial. □

In a similar manner, one can define the following variant of the Mock Alexander polynomial for admissible linkoid diagrams in a genus two surface, due to the second handle attaching construction of an admissible diagram in a genus two surface.

Definition 52. Let L be a classical link diagram in S^2 and $\tilde{L}$ be the corresponding link diagram in a genus two surface that is obtained by adding two disjoint handles to S^2 that connects two distinct pairs of regions of L. The *handle polynomial* of $\tilde{L}$, $T_{\tilde{L}}$ is defined to be the Mock Alexander polynomial of $\tilde{L}$.

Corollary 53 (Equivalent definition)**.** *The handle polynomial of $\tilde{L}$, is the permanent of the potential matrix of $\tilde{L}$.*

Proposition 54. *Let L be a classical link diagram in S^2 and $\tilde{L}$ be the corresponding link diagram in a genus two surface that is obtained by adding two disjoint handles to S^2 that connects two distinct pairs of regions of L. The handle polynomial is an invariant of $\tilde{L}$.*

Proof. This follows from the definition of the handle polynomial. □

Remark 55. 1. Link diagrams that lie in genus two surfaces obtained by handle addition techniques described in this section are not tight embeddings as regions of such link diagram contain genus.

2. A similar construction applies to knotoid diagrams in S^2 as follows. Choose a pair of regions in a knotoid diagram in S^2 and delete a pair of open disks out from these regions. Then attach a handle to S^2 over the boundaries of the deleted disks. In this way, one obtains a knotoid diagram immersed in torus which is an admissible diagram since two distinct regions in S^2 turn out to be the same region in the torus. Then, the Mock Alexander polynomial is an available invariant for knotoids immersed in torus. We plan to work on this observations further in a subsequent paper.

5.2.1 Virtual closure of knotoids

Let K be a knotoid diagram in S^2. The virtual closure of K is obtained by connecting two endpoints of K with a simple arc that may intersect K. Each intersection of the connecting arc with K is declared to be a virtual crossing so that the resulting closed diagram is a virtual knot diagram in S^2. The virtual closure of any knotoid diagram can be represented in a torus by attaching a handle to S^2 that holds the connecting arc realizing the closure. Figure 43 depicts the virtual closure of a knotoid diagram and its torus representation.

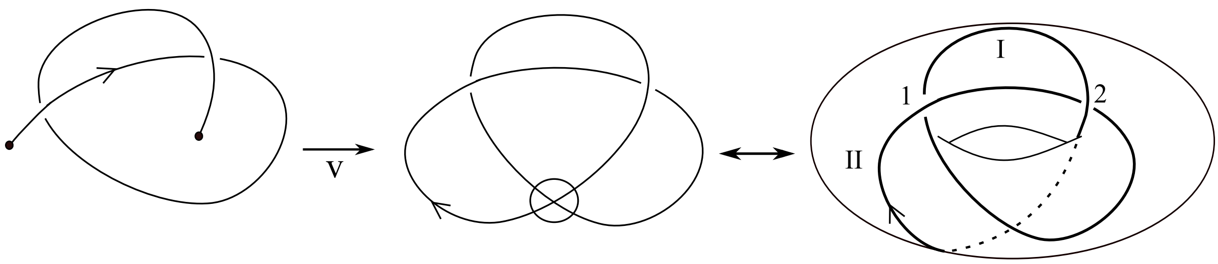


Figure 43: The virtual closure of a knotoid

The torus representation of the virtual closure of any knotoid diagram is an admissible diagram in the torus. The potential matrix of the resulting knot $v(K)$ in the torus, appearing in Figure 43 is given as $\begin{bmatrix} -W^{-1} & W+2 \\ W & 2-W^{-1} \end{bmatrix}$. By direct calculation of the permanent, we find that $\nabla_{v(K)}(W) = W^2 + W^{-2} + 2(W - W^{-1})$. Notice that in this case, the Mock Alexander polynomial of $v(K)$ is in fact, equal to $\nabla_{K_{ext}}(W) + \nabla_{K_{int}}(W)$. See Example 32 where we calculate $\nabla^{\sharp}_{K}(W) = \nabla_{K_{ext}}(W)$

Lemma 56. *Let K be a knotoid diagram in S^2 whose tail lies at the exterior region and $v(K)$ denote the torus representation of its virtual closure. Then*

$$\nabla_{v(K)}(W) = \nabla_{K_{ext}}(W) + \nabla_{K_{int}}(W).$$

Proof. Let $M_{K_{int}}$ and $M_{K_{ext}}$ be the potential matrices of K_{int} and K_{ext} with size $n \times n$, respectively. $M_{K_{int}}$ and $M_{K_{ext}}$ coincide except for one unique column which represents the exterior and the interior regions of K_{int} and K_{ext}, respectively. Without loss of generality, we can assume these columns are the n^{th} columns of the matrices. It is clear that the potential matrix of $M_{v(K)}$ consists of the same columns with $M_{K_{int}}$ and $M_{K_{ext}}$ except its n^{th} column is the sum of the n^{th} columns of $M_{K_{int}}$ and $M_{K_{ext}}$. From this, it follows that $\nabla_{v(K)}(W) = \nabla_{K_{ext}}(W) + \nabla_{K_{int}}(W)$. □

From Conjecture 36, we obtain the following equality.

Proposition 57. *Let K be a knotoid in S^2 and $v(K)$ its virtual closure.*

$$\nabla_{v(K)}(W) = \nabla^{\sharp}_K(W) + \nabla^{\sharp}_K(-W^{-1}).$$

Corollary 58. *The potential matrix of the virtual knot in torus, depicted in Figure 44 is,*

$$\begin{bmatrix} W & -W^{-1} & 2 \\ W^{-1} & W & 2 \\ 0 & 1 & 1+W-W^{-1} \end{bmatrix}.$$

It can be verified by direct permanent calculation that $\nabla_K(W) = 2W^2 + W - W^{-1} + W^{-2} - W^{-3} - 1$. ∇_K cannot be written as the sum $\nabla_K(W) + \nabla_K(-W^{-1})$ for some knotoid K, as the symmetric term for W^{-3}, $-W^3$ is missing in the polynomial. Thus, the virtual knot depicted in Figure 44 is not the virtual closure of a knotoid.

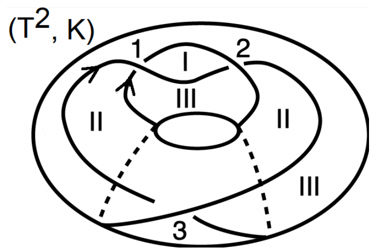


Figure 44: A virtual knot in T^2 which is not the virtual closure of a knotoid

Remark 59. It was shown in [9] that there are infinitely many virtual knots of virtual genus one obtained by virtualization and that do not lie in the image of the virtual closure map. This was shown by utilizing the surface bracket polynomial of virtual knots [5]. As Corollary 58 shows, $\nabla_{v(K)}$ can be also used to determine whether a virtual knot of virtual genus one is the virtual closure of a knotoid in S^2 or not.

The virtual closure map v is a well-defined map on the set of knotoids in S^2 to the set of virtual knots of virtual genus at most 1. This implies that any virtual knot invariant gives rise to an invariant of knotoids in S^2 via the virtual closure map [8]. Therefore, for every knotoid diagram K in S^2, the Mock Alexander polynomial of the torus representation of the virtual closure of K can be considered an invariant of K.

Definition 60. Let K be a knotoid diagram in S^2, and $v(K)$ be the virtual closure of K, represented in a torus. The virtual Alexander polynomial of K, ∇^v_K is defined to be the Mock Alexander polynomial of $v(K)$. For any knotoid K, ∇^v_K is an invariant of K.

Remark 61 (**A remark on starred link diagrams**). In Section 3.1, we have introduced starred link diagrams in surfaces as link diagrams admitting stars on a number their regions or crossings so that the diagrams satisfy the equality $f = n$. We now present the notion of a starred link diagram that lies in $\mathbb{R}^2$ in a more general sense.

Definition 62. A *region starred link diagram* in $\mathbb{R}^2$ is a classical link diagram with n crossings such that g of its regions, where $0 \leqslant g \leqslant n+2$, admit a number of stars. A region starred link diagram is not necessarily an admissible diagram and some of its regions may get multiple stars. We assume that two region starred link diagrams are *equivalent* if there is a sequence of star equivalence moves and planar isotopy moves transforming one to another.

A *region starred link* in $\mathbb{R}^2$ is an equivalence class of starred link diagrams in $\mathbb{R}^2$ taken up to star equivalence.

Let L be a link diagram in $\mathbb{R}^2$ with $g \geqslant 0$ of its regions are starred. L represents a unique link diagram embedded in genus g handlebody by considering L in $\mathbb{R}^2 \times I$, where I is the unit interval $[0,1]$, and drilling holes in $\mathbb{R}^2 \times I$ through the starred regions. Conversely, a link diagram in a genus g handlebody uniquely represents a starred link diagram in $\mathbb{R}^2$ by taking the projection of the link diagram onto $\mathbb{R}^2$, and representing each genus as a star in a region. This correspondence induces a bijection between the set of all region starred links in $\mathbb{R}^2$ and the set of links in a handlebody of genus g. This correspondence was initially utilized in [4] for the construction of the Jones polynomial of links in handlebodies.

With this correspondence the Mock Alexander polynomial can be considered as an invariant of links in handlebodies. Note that we consider multiple stars in a region of a starred link diagram as a single star on that region to calculate the Mock Alexander polynomial. Thus the Mock Alexander polynomial does not see the topological consequences of such multiple stars. On the other hand, the Jones polynomial via the bracket model for links in handlebodies can see such multiple stars when one uses the homology classes of the state loops. Comparison of these points of view will be the subject of another paper. This would extend the work done in [4].

The Mock Alexander polynomial can be also defined for link diagrams in a genus two handlebody via their representations in the plane that are region starred link diagrams.

6 A 2-variable generalization of the Mock Alexander polynomial

In this section we return to the roots of the state sum and find that it is possible in knotoid, linkoid and other non-classical cases, to add a new variable and obtain generalized Mock Alexander polynomials in two variables. These new polynomials are strong chirality and reversibility detectors for planar knotoids.

Consider a general labeling at crossings of a starred link or linkoid diagram with commuting variables $U, D, L, R, U', D', L', R'$, as given in Figure 45, and a generalized

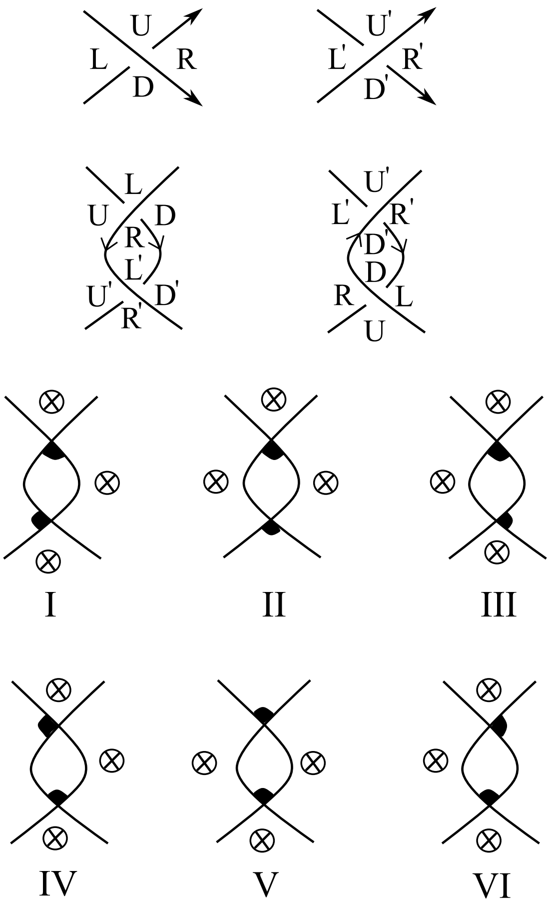


Figure 45: General labeling at crossings and state configurations at an RII move site.

state-sum with respect to such labeling. We obtain the following conditions on the labels, if we impose invariance under an RII move on the state-sum.

If the RII move is of first type depicted on the left hand side of the second row in Figure 45, then from states II and V we have,

$$RR^{'} = 1 = LL^{'}.$$

From the states I and IV, we have

$$RU^{'} + UL^{`} = 0.$$

From state III and VI, we have

$$RD^{`} + DL^{'} = 0.$$

If the RII move if the second type depicted on the right hand side of the second row in Figure 45, then from states II and V, we have

$$D^{'}U = 1 = U^{'}D.$$

From states I and IV, we have

$$RD' + L'D = 0.$$

From states III and VI, we have

$$D'L + R'D = 0.$$

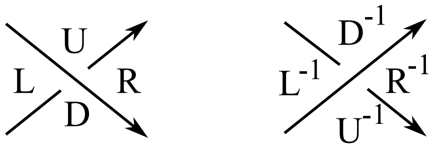


Figure 46: Necessary labeling for invariance under an RII move.

By substituting the new labels into equations, we find

$$R = -UDL^{-1}$$

$$L = -UDR^{-1}.$$

Then by direct calculation,

$$R = -(UD)^{-1}L^{-1}.$$

Therefore $(UD)^2 = 1$. We assume $UD = 1$, then the final labeling is as given in Figure 47. This labeling is a generalization of the one we assumed for the potential summation throughout the paper with $R = W$.

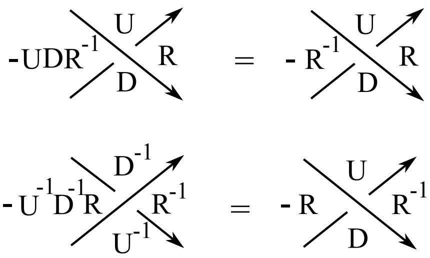


Figure 47: Final labels at crossings.

Let K be an oriented knot or knotoid diagram. Smoothing each crossing of K in the oriented way results in a number of oriented circles (namely, Seifert circles) and a long segment if K is a knotoid diagram. A Seifert circle that is oriented counterclockwise as a *positive* circle and a circle that is oriented clockwise as *negative* circle.

A generalization of the Mock Alexander polynomial of K with stars on a pair of its regions, can be defined as the state-sum over the labelings in Figure 47. Let $\nabla_K(W, D)$ denote this state sum. Let $Deg(K) = p(K) - n(K)$, where $p(K)$ is the number of positive Seifert circles in an oriented smoothing of K and $n(K)$ is the number of negative Seifert circles. Define $\tilde{\nabla}_K(W, D) = D^{Deg(K)}\nabla_K(W, D)$. Then one can verify that $\tilde{\nabla}$ is an invariant of planar links and planar linkoids by using the same techniques we used in Theorem 26.

We call $\tilde{\nabla}$ the *planar potential of K*. We note here that the planar potential of a knot or link diagram with adjacent stars is a multiple of $\nabla_K(W)$ (the original potential function in variable W) by $D^{Deg(K)}$, but is more complex for other cases.

In Figure 48 we depict a knotoid K, its double mirror image, $K!^*$ that is obtained taking the reflection of K through the vertical line in $\mathbb{R}^2$ and then taking the mirror image of the resulting knotoid by switching all the crossings) and its reverse K^{rev} that is obtained by reversing the direction on K. These knotoids can be distinguished from each other by the planar potential. More precisely, $Deg(K) = Deg(K!^*) = Deg(K^{rev})$ as the diagrams do not admit any Seifert circles in their oriented smoothings. $\tilde{\nabla}_K(W, D) = W^2 + WD - W^{-1}D$, $\tilde{\nabla}_{K!^*}(W, U) = W^2 + WU - W^{-1}U$, and $\tilde{\nabla}_{K^{rev}}(W, U) = B^2 + WU - BU$ where $U = D^{-1}$. Notice that $\nabla_K(W) = \nabla_{K!^*}(W)$ and also K and $K!^*$ admit the same Jones polynomial. Thus, $\tilde{\nabla}_K(W, D)$ is stronger than the Mock Alexander polynomial, and it can distinguish knotoids that the Jones polynomial cannot distinguish. This figure also indicates that K is not equivalent to its reverse and K^{rev} is also not equivalent to the double mirror image of K.

In Figure 49 we see another example of a planar knotoid for which we can distinguish it from its reverse, mirror image and reflection and also double mirror image by the two variable potential.

Precisely, $Deg(K) = Deg(K^*) = 1$ and $Deg(K^{rev}) = Deg(K!) = -1$. $\tilde{\nabla}_K(W, D) = D(W + D^{-1} - B) = D(W - B) + 1$, $\tilde{\nabla}_{K^{rev}}(W, D) = D^{-1}(W - B + D^{-1})$, $\tilde{\nabla}_{K^*}(W, D) = D(-W + B + D^{-1}) = D(B - W) + 1$, $\tilde{\nabla}_{K!}(W, D) = D^{-1}(-W + B + D) = D^{-1}(B - W) + 1$. Note that $K^{rev} = K!^*$. The reader can verify that the single variable potential and also the Jones polynomial cannot distinguish K from $K!^*$ and K^* from $K!$.

In this paper we have shown many variations of the one variable polynomial and significant applications of the two variable polynomial to mirror images and to reversibility. The many variations can also be used for the 2-variable polynomial, and these will be the subject of a further paper.

7 Discussion

The results of the present paper arise from generalizing the methods of [12]. This paper is the first of a series of papers on these generalizations, which will include wider versions of the Clock Theorem of Formal Knot Theory and other applications of these ideas.

All the invariants discussed in this paper are expressed by state summations that can be expressed as permanents of matrices associated with the diagram of a knot, link, linkoid or knotoid. In principle, permanents are of complexity higher than the polynomial time algorithms that compute determinants. At this time it remains an open problem to determine the complexity type of our permanent based invariants. This problem will be the subject of a paper subsequent to the present work. See [11] where complexity results about the Jones polynomial are proved.

Consider the starred knotoid diagram and its states depicted in Figure 17. Given a state of K, each crossing can be resolved in the direction of the state. This gives an *Eulerian trail* underlying K that is a path from one endpoint to another, visiting

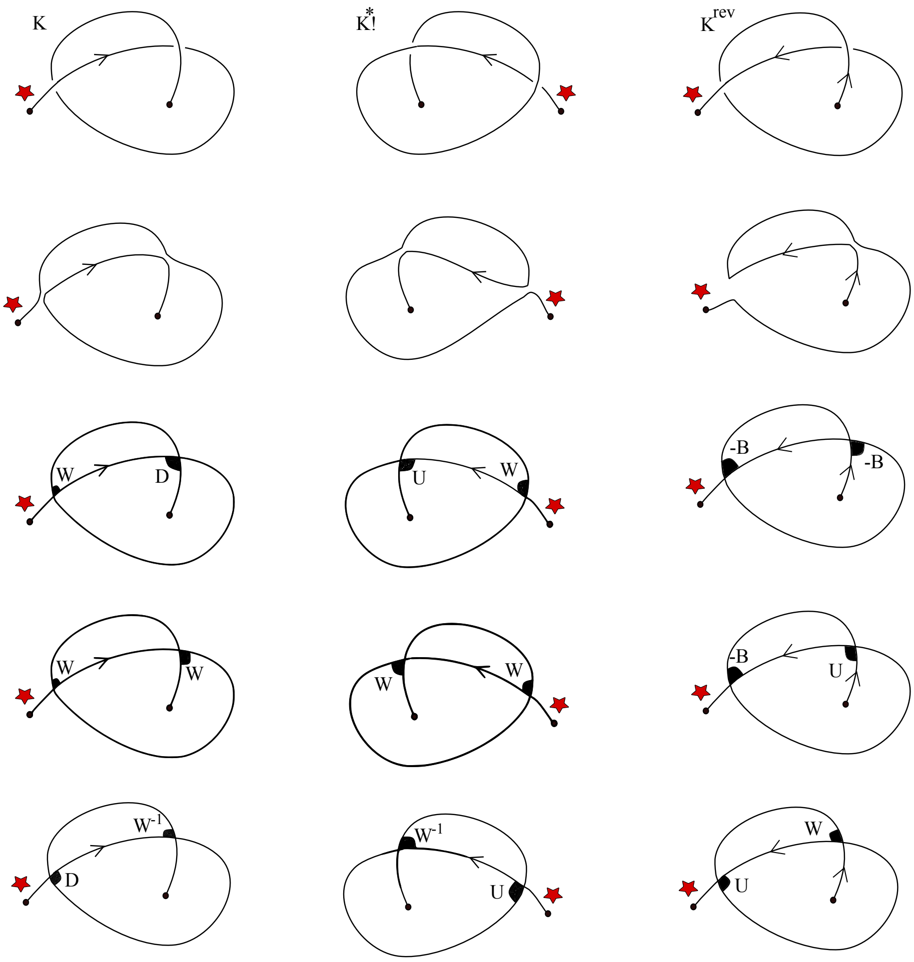


Figure 48: A knotoid K, its double mirror image $K!^*$ and its reverse K^{rev}, and their states

all edges of K exactly once. In fact, there is a one-to-one correspondence between the states of a starred knotoid diagram and the Eulerian trails underlying the graph of the knotoid diagrams. Moreover, each Eulerian trail determines a unique spanning tree of a generalized Tait graph of K. In Figure 50, we illustrate all Eulerian trails and the corresponding spanning trees (in blue) of K. There are versions of Heegaard Floer Link homology that are based on the correspondence between the Formal Knot Theory states and spanning trees of a link diagram. See [3, 14, 16]. We are in the process of using our generalizations of these states to study new versions of link homology for knotoids and their relatives as discussed in the present paper.

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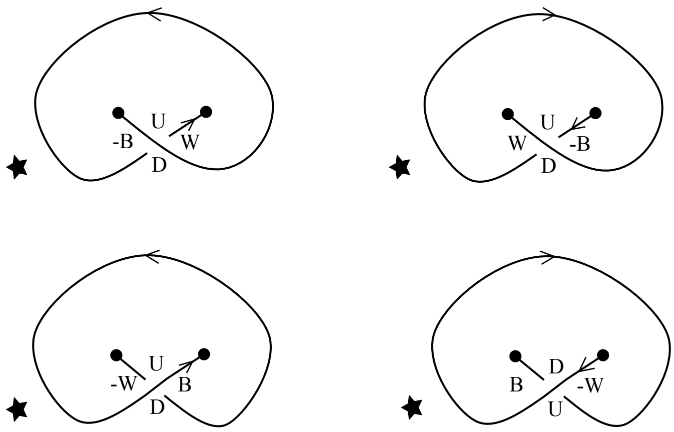

Figure 49: From leftmost top to rightmost bottom: A planar knotoid K, its reversal K^{rev}, mirror image K^{*} and its reflection $K!$

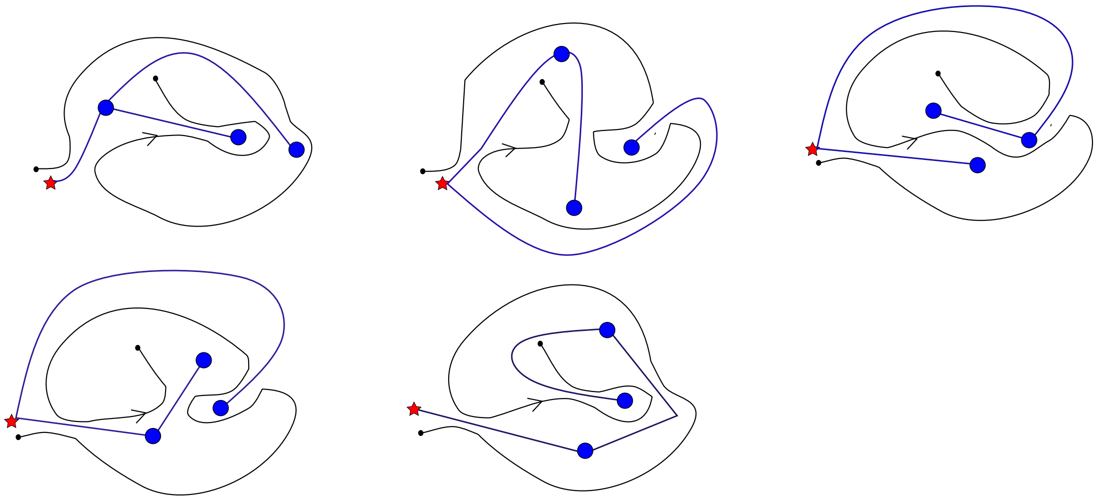

Figure 50: Euler trails and spanning trees determined by states of K.

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