

Exponential odd-distance sets under the Manhattan metric

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Abstract

We construct a set of 2^n points in $\mathbb{R}^n$ such that all pairwise Manhattan distances are odd integers, which improves the recent linear lower bound of Golovanov, Kuvavskii and Sagdeev. In contrast to the Euclidean and maximum metrics, this shows that the odd-distance set problem behaves very differently to the equilateral set problem under the Manhattan metric. Moreover, all coordinates of the points in our construction are integers or half-integers, and we show that our construction is optimal under this additional restriction.

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1 Introduction

An *equilateral set* in a metric space is a set of points in which all pairwise distances are the same. Determining the largest equilateral set in a metric space has been well-studied and the case of $\mathbb{R}^n$ equipped with the ℓ_p -norm¹ $\|\cdot\|_p$ has received significant attention. Denoting the size of the largest equilateral set in $\mathbb{R}^n$ with the ℓ_p -norm by $e_p(n)$, Table 1 gives a summary of the best bounds known for $p \in \{1, 2, \infty\}$.

An old conjecture of Kusner [5] states that $e_p(n) = n + 1$ for all integers $1 < p < \infty$ and $e_1(n) = 2n$. For progress on Kusner's conjecture, see for example [1, 7, 8].

Graham, Rothschild and Straus [4] introduced the following relaxation of an equilateral set. An *odd-distance set* in a metric space is a set of points in which all pairwise distances

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¹A point $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ has *ℓ_p -norm* $\|\mathbf{v}\|_p := (\sum_i |v_i|^p)^{1/p}$ for $1 \leq p < \infty$ and has *ℓ_∞ -norm* $\|\mathbf{v}\|_\infty := \max_i |v_i|$.

	Lower bound		Upper bound	
$e_1(n)$	$2n$	cross polytope	$cn \log n$	Alon and Pudlák [1]
$e_2(n)$	$n + 1$	unit simplex	$n + 1$	folklore
$e_\infty(n)$	2^n	unit hypercube	2^n	Petty [6]

Table 1: The size of the largest equilateral sets in $(\mathbb{R}^n, \|\cdot\|_p)$.

are odd integers. We denote the size of the largest odd-distance set in $\mathbb{R}^n$ equipped with the ℓ_p -norm by $\text{odd}_p(n)$. In a normed space, any equilateral set can be scaled to a set where all pairwise distances are 1, and so $\text{odd}_p(n) \geq e_p(n)$. Table 2 gives a summary of the best bounds known for $\text{odd}_p(n)$. The bounds for $\text{odd}_2(n)$ were proved in the original paper of Graham, Rothschild and Straus [4], who in fact determined the exact value of $\text{odd}_2(n)$ (it is $n + 1$ except when $n \equiv -2 \pmod{16}$). The other bounds are due to a recent paper of Golovanov, Kupavskii and Sagdeev [3].

	Lower bound		Upper bound	
$\text{odd}_1(n)$	$(\frac{7}{3} - o(1))n$	[3]	$(4 + o(1))n! n \log n$	[3]
$\text{odd}_2(n)$	$n + 1$	unit simplex	$n + 2$	[4]
$\text{odd}_\infty(n)$	2^n	unit hypercube	2^n	[3]

Table 2: The size of the largest odd-distance sets in $(\mathbb{R}^n, \|\cdot\|_p)$.

From these results we see that relaxing from the equilateral to the odd-distance problem makes very little difference for the ℓ_2 and ℓ_∞ -norms. The case of the ℓ_1 -norm is less clear: the lower bound of Golovanov, Kupavskii and Sagdeev [3] for $\text{odd}_1(n)$ is larger than Kusner's conjectured upper bound for $e_1(n)$, but only by a constant multiplicative factor, and it is still smaller than the upper bound proven by Alon and Pudlák [1].

In this note, we significantly improve the lower bound for $\text{odd}_1(n)$, showing that there are odd-distance sets of size 2^n under the ℓ_1 -norm. Together with the upper bound for $e_1(n)$ of Alon and Pudlák [1], this confirms in a strong sense that the equilateral and odd-distance problems behave very differently for the ℓ_1 -norm.

Theorem 1. *For all positive integers n , $\text{odd}_1(n) \geq 2^n$. Moreover, there is an ℓ_1 -odd-distance set $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^n$ of size 2^n .*

It is not possible to replace the half-integer lattice $(\frac{1}{2} \cdot \mathbb{Z})^n$ by the integer lattice $\mathbb{Z}^n$ in the statement of Theorem 1. Indeed, there is not even an odd-distance set of size 3 in $\mathbb{Z}^n$: every integer x satisfies $|x| \equiv x \pmod{2}$ and so, for any integer points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^n$, we have that $\|\mathbf{a} - \mathbf{b}\|_1 + \|\mathbf{b} - \mathbf{c}\|_1 \equiv \|\mathbf{a} - \mathbf{c}\|_1 \pmod{2}$.

As a corollary of our result, we obtain a lower bound on the chromatic number of the odd-distance graph on $(\mathbb{R}^n, \|\cdot\|_1)$. This is the infinite graph with vertex set $\mathbb{R}^n$ where two vertices are adjacent whenever their distance in the ℓ_1 -norm is an odd integer. Theorem 1 immediately implies that any proper colouring of this graph must use at least 2^n colours.

The best upper bound is the same as the upper bound on $\text{odd}_1(n)$ obtained by Golovanov, Kupavskii and Sagdeev [3]. This is in stark contrast with the case of the Euclidean distance, where Davies [2] recently proved that any proper colouring must use infinitely many colours, even in dimension 2.

We further show that Theorem 1 is best possible in the case that all coordinates are half-integers.

Proposition 2. *Let $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^n$ be an ℓ_1 -odd-distance set. Then $|\mathcal{P}| \leq 2^n$.*

As a last result, we also note that, when considering odd-distance sets in the ℓ_1 -norm, we may restrict the coordinates to be elements of the ring $\mathbb{Z}[\frac{1}{2}]$ of dyadic rationals (that is, rational numbers whose denominators are powers of 2).

Proposition 3. *For any ℓ_1 -odd-distance set $\mathcal{P} \subseteq \mathbb{R}^n$, there exists an ℓ_1 -odd-distance set $\mathcal{Q} \subseteq (\mathbb{Z}[\frac{1}{2}])^n$ of the same size as $\mathcal{P}$.*

2 Proofs

To prove the lower bound on $\text{odd}_1(n)$, we will inductively construct an odd-distance set of size 2^n . We will achieve this by splitting in two a coordinate of each point of an odd-distance set in $\mathbb{R}^n$. By doing this in two distinct ways, we replace each point with two points in $\mathbb{R}^{n+1}$. This construction relies on the following simple lemma.

Lemma 4. *For every $x \in \frac{1}{2} \cdot \mathbb{Z}$, there exists an odd-distance set $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^2$ of size 2 such that the coordinates of each point sum to x and all coordinates are at least $x/2 - 1/2$ and at most $x/2 + 1/2$.*

Proof. If $x \in \mathbb{Z}$, let

$$\mathcal{P} = \left\{ \left(\frac{x}{2} + \frac{1}{2}, \frac{x}{2} - \frac{1}{2} \right), \left(\frac{x}{2}, \frac{x}{2} \right) \right\}.$$

Otherwise, if $x \in \mathbb{Z} + \frac{1}{2}$, let

$$\mathcal{P} = \left\{ \left(\frac{x}{2} + \frac{1}{4}, \frac{x}{2} - \frac{1}{4} \right), \left(\frac{x}{2} - \frac{1}{4}, \frac{x}{2} + \frac{1}{4} \right) \right\}.$$

Clearly, in both cases $\mathcal{P}$ satisfies the claim. □

We may now prove our main result. Our strategy is to start with an odd-distance set $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^n$ and first space out the points in $\mathcal{P}$ in the first dimension. We achieve this by translating the first coordinate of each point by some even integer while maintaining their order in the first dimension and the odd-distance property. Then we replace each first coordinate by the odd-distance set in $(\frac{1}{2} \cdot \mathbb{Z})^2$ given by Lemma 4. Because the first dimension was sufficiently spaced out in advance, this guarantees that the relative order of the points in the two new dimensions will be the same as their relative order in the old dimension. As a result, the new set will be an odd-distance set in $(\frac{1}{2} \cdot \mathbb{Z})^{n+1}$ of size $2|\mathcal{P}|$.

Theorem 1. *For all positive integers n , $\text{odd}_1(n) \geq 2^n$. Moreover, there is an ℓ_1 -odd-distance set $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^n$ of size 2^n .*

Proof. We will prove, by induction on n , that $(\frac{1}{2} \cdot \mathbb{Z})^n$ contains an ℓ_1 -odd-distance set of size 2^n for all $n \geq 1$. For $n = 1$, the set $\mathcal{P} = \{0, 1\}$ suffices. Now, suppose $\mathcal{P}$ is an odd-distance set in $(\frac{1}{2} \cdot \mathbb{Z})^n$ of size 2^n . We label the points of $\mathcal{P}$ as $\mathbf{p}_1, \dots, \mathbf{p}_{2^n}$ so that the first coordinate is increasing. That is, for all $i \in [2^n]$ we may write $\mathbf{p}_i = (a_i, \mathbf{v}_i)$, where $a_1 \leq a_2 \leq \dots \leq a_{2^n}$ and the vectors $\mathbf{v}_i$ are in $(\frac{1}{2} \cdot \mathbb{Z})^{n-1}$.

For each $i \in [2^n]$, let $b_i := a_i + 2i$ and $\mathbf{q}_i := (b_i, \mathbf{v}_i)$. Note that $\mathcal{Q} := \{\mathbf{q}_1, \dots, \mathbf{q}_{2^n}\}$ is also an ℓ_1 -odd-distance set in $(\frac{1}{2} \cdot \mathbb{Z})^n$. Indeed, for $1 \leq i < j \leq 2^n$,

$$\begin{aligned} \|\mathbf{q}_j - \mathbf{q}_i\|_1 &= (b_j - b_i) + \|\mathbf{v}_j - \mathbf{v}_i\|_1 \\ &= (a_j - a_i) + \|\mathbf{v}_j - \mathbf{v}_i\|_1 + 2(j - i) \\ &= \|\mathbf{p}_j - \mathbf{p}_i\|_1 + 2(j - i), \end{aligned}$$

which is odd.

By Lemma 4, for each $i \in [2^n]$ there is an odd-distance set $\mathcal{C}_i = \{\mathbf{c}_i^{(1)}, \mathbf{c}_i^{(2)}\} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^2$ where the coordinates of each point sum to b_i and all coordinates are between $b_i/2 - 1/2$ and $b_i/2 + 1/2$. For each $i \in [2^n]$, let $\mathbf{q}_i^{(1)} := (\mathbf{c}_i^{(1)}, \mathbf{v}_i)$ and $\mathbf{q}_i^{(2)} := (\mathbf{c}_i^{(2)}, \mathbf{v}_i)$. Finally, let $\mathcal{Q}' := \{\mathbf{q}_1^{(1)}, \mathbf{q}_1^{(2)}, \dots, \mathbf{q}_{2^n}^{(1)}, \mathbf{q}_{2^n}^{(2)}\}$. Clearly, since all the b_i are distinct, $\mathcal{Q}'$ is a subset of $(\frac{1}{2} \cdot \mathbb{Z})^{n+1}$ of size 2^{n+1} . It remains to show that it is an ℓ_1 -odd-distance set.

First note that, for all $i \in [2^n]$,

$$\|\mathbf{q}_i^{(1)} - \mathbf{q}_i^{(2)}\|_1 = \|\mathbf{c}_i^{(1)} - \mathbf{c}_i^{(2)}\|_1 + \|\mathbf{v}_i - \mathbf{v}_i\|_1 = \|\mathbf{c}_i^{(1)} - \mathbf{c}_i^{(2)}\|_1,$$

which is odd as $\mathcal{C}_i$ is an odd-distance set. Now fix $1 \leq i < j \leq 2^n$ and consider the distance between $\mathbf{q}_i^{(r)}$ and $\mathbf{q}_j^{(s)}$, where $r, s \in \{1, 2\}$. Recall, from our choice of the b_i , that $b_j \geq b_i + 2$. Since the coordinates of $\mathbf{c}_i^{(r)}$ are at most $b_i/2 + 1/2$ and the coordinates of $\mathbf{c}_j^{(s)}$ are at least $b_j/2 - 1/2$, both coordinates of $\mathbf{c}_i^{(r)}$ are bounded from above by both coordinates of $\mathbf{c}_j^{(s)}$. Since the sum of the coordinates of $\mathbf{c}_i^{(r)}$ is b_i and the sum of the coordinates of $\mathbf{c}_j^{(s)}$ is b_j , it follows that

$$\|\mathbf{c}_j^{(s)} - \mathbf{c}_i^{(r)}\|_1 = b_j - b_i.$$

Thus,

$$\|\mathbf{q}_j^{(s)} - \mathbf{q}_i^{(r)}\|_1 = \|\mathbf{c}_j^{(s)} - \mathbf{c}_i^{(r)}\|_1 + \|\mathbf{v}_j - \mathbf{v}_i\|_1 = (b_j - b_i) + \|\mathbf{v}_j - \mathbf{v}_i\|_1 = \|\mathbf{q}_j - \mathbf{q}_i\|_1,$$

which is odd since $\mathcal{Q}$ is an odd-distance set, as required. $\square$

If we restrict a point set to having half-integer coordinates, we show that our construction is optimal. We prove this with a simple pigeonhole argument.

Proposition 2. *Let $\mathcal{P} \subseteq (\frac{1}{2} \cdot \mathbb{Z})^n$ be an ℓ_1 -odd-distance set. Then $|\mathcal{P}| \leq 2^n$.*

Proof. Let $\phi: \mathcal{P} \rightarrow \{0, 1\}^n$ be defined by setting $\phi(\mathbf{p}) := (\mathbb{1}_{\{p_i \in \mathbb{Z} + 1/2\}})_{i \in [n]}$.

First, we note that there cannot be two points $\mathbf{p}, \mathbf{q} \in \mathcal{P}$ with $\|\phi(\mathbf{p})\|_1 \not\equiv \|\phi(\mathbf{q})\|_1 \pmod{2}$. Indeed, in such a case, there would be an odd number of coordinates $i \in [n]$ such that $|p_i - q_i| \in \mathbb{Z} + 1/2$, which implies that $\|\mathbf{p} - \mathbf{q}\|_1$ is not an integer. In particular, $|\phi(\mathcal{P})| \leq 2^{n-1}$.

Secondly, we claim that there cannot be three points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{P}$ with $\phi(\mathbf{a}) = \phi(\mathbf{b}) = \phi(\mathbf{c})$. Indeed, otherwise $a_i - b_i$, $b_i - c_i$, and $c_i - a_i$ are integers for all $i \in [n]$, and so

$$\begin{aligned}\|\mathbf{a} - \mathbf{b}\|_1 &= \sum_{i=1}^n |a_i - b_i| \equiv \sum_{i=1}^n (a_i - b_i) \pmod{2}, \\ \|\mathbf{b} - \mathbf{c}\|_1 &= \sum_{i=1}^n |b_i - c_i| \equiv \sum_{i=1}^n (b_i - c_i) \pmod{2}, \\ \|\mathbf{a} - \mathbf{c}\|_1 &= \sum_{i=1}^n |a_i - c_i| \equiv \sum_{i=1}^n (a_i - c_i) \pmod{2}.\end{aligned}$$

Adding the first two expressions and comparing with the last one, we conclude that

$$\|\mathbf{a} - \mathbf{b}\|_1 + \|\mathbf{b} - \mathbf{c}\|_1 \equiv \|\mathbf{a} - \mathbf{c}\|_1 \pmod{2},$$

which is impossible, as the sum of two odd numbers is even. Thus, $|\mathcal{P}| \leq 2 \cdot |\phi(\mathcal{P})| \leq 2^n$. $\square$

Finally, we show that we may assume all points in ℓ_1 -odd-distance sets $\mathcal{P}$ have rational coordinates. Indeed, the pairwise distances between the points of $\mathcal{P}$ can be expressed by a system of linear equations on the coordinates of the points of $\mathcal{P}$. Solutions to this system of linear equations can be characterised by a set of free variables whose choice determines all other variables. If we now replace all coordinates corresponding to free variables by nearby rational numbers, this determines all remaining coordinates, and those coordinates will also be rational since the coefficients of the linear equations were rational. This gives a point set with rational coefficients, and if the order of the coordinates did not change, then the distances between the points remain the same. So, the point set will be an odd-distance set.

Lemma 5. *For any ℓ_1 -odd-distance set $\mathcal{P} \subseteq \mathbb{R}^n$, there exists an odd-distance set $\mathcal{Q} \subseteq \mathbb{Q}^n$ of the same size as $\mathcal{P}$.*

Proof. By iteratively translating coordinates as in the proof of Theorem 1, we may assume that $|p_i - q_i| \geq 2$ for all distinct points $\mathbf{p}, \mathbf{q} \in \mathcal{P}$ and all $i \in [n]$. Let $s_i^{\mathbf{p}, \mathbf{q}} := 1$ if $p_i > q_i$ and $s_i^{\mathbf{p}, \mathbf{q}} := -1$ otherwise. Then

$$\|\mathbf{p} - \mathbf{q}\|_1 = \sum_{i=1}^n s_i^{\mathbf{p}, \mathbf{q}} (p_i - q_i).$$

Write $d_{\mathbf{p}, \mathbf{q}} := \|\mathbf{p} - \mathbf{q}\|_1$ and introduce a variable $x_{\mathbf{p}, i}$ for all $\mathbf{p} \in \mathcal{P}$ and all $i \in [n]$. This means that the system of linear equations

$$\sum_{i=1}^n s_i^{\mathbf{p}, \mathbf{q}} (x_{\mathbf{p}, i} - x_{\mathbf{q}, i}) = d_{\mathbf{p}, \mathbf{q}}$$

for all distinct $\mathbf{p}, \mathbf{q} \in \mathcal{P}$ has a solution given by $x_{\mathbf{p},i} = p_i$ for all $\mathbf{p} \in \mathcal{P}$ and all $i \in [n]$.

Since all coefficients and all $d_{\mathbf{p},\mathbf{q}}$ are integers, the reduced row echelon form of this system of linear equations only has rational coefficients. Thus, in the reduced row echelon form, there exists a set of indices $\mathcal{F} \subseteq \mathcal{P} \times [n]$ such that x_f is a free variable for all $f \in \mathcal{F}$ while x_g is a dependent variable for all $g \notin \mathcal{F}$, with $x_g = a_g + \sum_{f \in \mathcal{F}} b_{f,g} x_f$ for some $a_g, b_{f,g} \in \mathbb{Q}$.

Let $C := \max_{f \in \mathcal{F}, g \notin \mathcal{F}} |b_{f,g}|$. For each $(\mathbf{p}, i) \in \mathcal{F}$, choose $y_{\mathbf{p},i} \in \mathbb{Q}$ so that $|y_{\mathbf{p},i} - p_i| < 1/(C|\mathcal{P}|n)$. For each $(\mathbf{p}, i) \notin \mathcal{F}$ let $y_{\mathbf{p},i} = a_{\mathbf{p},i} + \sum_{f \in \mathcal{F}} b_{f,(\mathbf{p},i)} y_f$, and note that $y_{\mathbf{p},i} \in \mathbb{Q}$ and

$$\begin{aligned} |y_{\mathbf{p},i} - p_i| &= \left| \left(a_{\mathbf{p},i} + \sum_{(\mathbf{q},j) \in \mathcal{F}} b_{(\mathbf{q},j),(\mathbf{p},i)} y_{\mathbf{q},j} \right) - \left(a_{\mathbf{p},i} + \sum_{(\mathbf{q},j) \in \mathcal{F}} b_{(\mathbf{q},j),(\mathbf{p},i)} q_j \right) \right| \\ &\leq \sum_{(\mathbf{q},j) \in \mathcal{F}} |b_{(\mathbf{q},j),(\mathbf{p},i)} (y_{\mathbf{q},j} - q_j)| < \frac{|\mathcal{F}|C}{C|\mathcal{P}|n} \leq 1, \end{aligned}$$

where the first equality used the fact that $x_{\mathbf{p},i} = p_i$ is a solution to the system of equations.

For each $\mathbf{p} \in \mathcal{P}$, let $\mathbf{p}' \in \mathbb{Q}^n$ be the point with $p'_i = y_{\mathbf{p},i}$. Consider two distinct points $\mathbf{p}, \mathbf{q} \in \mathcal{P}$. For each i , if $p_i < q_i$, then $p_i + 2 \leq q_i$ by assumption, and since $|p'_i - p_i| < 1$ and $|q'_i - q_i| < 1$, this implies that $p'_i < q'_i$. Similarly, if $p_i > q_i$, then $p'_i > q'_i$. Therefore,

$$\|\mathbf{p}' - \mathbf{q}'\|_1 = \sum_{i=1}^n s_i^{\mathbf{p},\mathbf{q}} (p'_i - q'_i) = d_{\mathbf{p},\mathbf{q}},$$

where the last equality uses the fact that $x_{\mathbf{p},i} = p'_i$ is a solution to the system of equations. In particular, the distance between $\mathbf{p}'$ and $\mathbf{q}'$ is odd, and so $\mathcal{Q} = \{\mathbf{p}' : \mathbf{p} \in \mathcal{P}\} \subseteq \mathbb{Q}^n$ is an odd-distance set of the same size as $\mathcal{P}$. $\square$

Proposition 3 is now an easy consequence since, if we scale all coordinates of an odd-distance set by a fixed odd integer, the resulting set is still an odd-distance set.

Proposition 3. *For any ℓ_1 -odd-distance set $\mathcal{P} \subseteq \mathbb{R}^n$, there exists an ℓ_1 -odd-distance set $\mathcal{Q} \subseteq (\mathbb{Z}[\frac{1}{2}])^n$ of the same size as $\mathcal{P}$.*

Proof. By Lemma 5, there exists an odd-distance set $\mathcal{Q} \subseteq \mathbb{Q}^n$ of the same size as $\mathcal{P}$. Let C be the least common multiple of all odd integers that divide the denominator of any coordinate of any point of $\mathcal{Q}$. Clearly, C is an odd integer, and $\mathcal{Q}' := \{C\mathbf{q} : \mathbf{q} \in \mathcal{Q}\} \subseteq (\mathbb{Z}[\frac{1}{2}])^n$. Moreover, for all distinct points $C\mathbf{p}, C\mathbf{q} \in \mathcal{Q}'$ we have $\|C\mathbf{p} - C\mathbf{q}\|_1 = C\|\mathbf{p} - \mathbf{q}\|_1$, which is an odd integer as a product of two odd integers, and so $\mathcal{Q}'$ is an odd-distance set. $\square$

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