A Characterization of Generalized Cospectrality of Rooted Graphs with Applications in Graph Reconstruction

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Abstract

Extending a classic result of Johnson and Newman, this paper provides a matrix characterization for two generalized cospectral graphs with a pair of generalized cospectral vertex-deleted subgraphs. As an application, we present a new condition for the reconstructibility of a graph. Namely, if a vertex-deleted subgraph $G - v$ of G is almost controllable, then the graph G is reconstructible if $G - v$ either has a nontrivial automorphism group, or is asymmetric with a specific property.

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1 Introduction

Let G be an n-vertex graph with adjacency matrix $A(G)$. The *spectrum* of G refers to the multiset of eigenvalues of $A(G)$. Two graphs G and H are *cospectral* if they share the same spectrum. It is known that if G and H are cospectral then there exists an orthogonal matrix Q such that $Q^{\mathrm{T}}A(G)Q = A(H)$.

We are interested in two kinds of enhancements of the ordinary cospectrality: rooted-cospectrality and generalized cospectrality. Let (G, u) be a rooted graph with u as the root vertex. We say two rooted graphs (G, u) and (H, v) are cospectral if (1) G and H are cospectral and (2) $G - u$ and $H - v$ are also cospectral. It turns out that the rooted-cospectrality of graphs can be characterized by specific orthogonal matrices, as described in the following theorem. Without loss of generality, we may assume that the root vertices are labeled as the last vertices in graphs.

Theorem 1 ([14, 18]). *Let G and H be two n-vertex graphs with vertex sets $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$. Then the following are equivalent:*

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(i) (G, u_n) *and* (H, v_n) *are cospectral.*

(ii) *There exists an orthogonal matrix of the form* $\begin{pmatrix} Q & O \\ O & 1 \end{pmatrix}$ *such that*

$$\begin{pmatrix} Q^{\mathrm{T}} & O \\ O & 1 \end{pmatrix} A(G) \begin{pmatrix} Q & O \\ O & 1 \end{pmatrix} = A(H).$$

We say that two graphs G and H are *generalized cospectral* if G and H are cospectral with cospectral complements. Similar to rooted-cospectrality, generalized cospectrality can be characterized by a special kind of orthogonal matrices. An orthogonal matrix is *regular* if the sum of each row is 1.

Theorem 2 ([12])**.** *Let G and H be two graphs. Then the following are equivalent:*
(i) *G and H are generalized cospectral.*
(ii) *There exists a regular orthogonal matrix Q such that $Q^{\mathrm{T}}A(G)Q = A(H)$.*

A primary goal of this paper is to unify the above two theorems. We say that two rooted graphs (G, u) and (H, v) are generalized cospectral if (1) G and H are generalized cospectral, and (2) $G - u$ and $H - v$ are also generalized cospectral. The first main result of this paper is the following theorem.

Theorem 3. *Let G and H be two n-vertex graphs with vertex sets $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$. Then the following are equivalent:*
(i) *(G, u_n) and (H, v_n) are generalized cospectral, i.e., four graphs $G, \overline{G}, G - u_n$ and $\overline{G - u_n}$ are cospectral with $H, \overline{H}, H - v_n$ and $\overline{H - v_n}$, respectively.*
(ii) *There exists a regular orthogonal matrix of the form* $\begin{pmatrix} Q & O \\ O & 1 \end{pmatrix}$ *such that*

$$\begin{pmatrix} Q^{\mathrm{T}} & O \\ O & 1 \end{pmatrix} A(G) \begin{pmatrix} Q & O \\ O & 1 \end{pmatrix} = A(H).$$

The proof of Theorem 3 will be given in Section 2. We remark that Theorem 3 was reported by Farrugia [3] in a different but essentially equivalent form, using the notion of *overgraphs*. However, the proof in [3] contains a gap, which the present work aims to fill; see Remark 10 in the next section. As an important application of Theorem 3, we obtain a new condition for the reconstructibility of graphs. To state the result, we recall some basic notions.

Given a graph G with vertex set $\{u_1, \ldots, u_n\}$, the *deck* of G, denoted by $\mathcal{D}(G)$, is the multiset of its vertex-deleted (unlabeled) subgraphs $G - u_i$ for $i = 1, \ldots, n$. A graph H is called a *reconstruction* of G if $\mathcal{D}(H) = \mathcal{D}(G)$. If every reconstruction of G is isomorphic to G, then G is said to be *reconstructible*. The Reconstruction Conjecture (also called Ulam Conjecture or Kelly-Ulam Conjecture) claims that every graph with at least three vertices is reconstructible. The conjecture has been proved for various graph classes, such as regular graphs, disconnected graphs, trees and outerplanar graphs. It has been verified by McKay [15] that all graphs with at most 13 vertices are reconstructible. In a probabilistic sense, Bollobás [1] showed that almost all graphs are reconstructible, i.e., the probability that a randomly chosen graph on n vertices is reconstructible approaches 1 as n tends to infinity.

A classic result of Tutte [19] states that the characteristic polynomial of a graph is reconstructible and, moreover, that graphs with irreducible characteristic polynomials are reconstructible. Tutte's result has received considerable attention, as it demonstrates that linear algebraic methods are powerful tools for tackling the Reconstruction Conjecture. Several authors have made efforts to rederive or extend Tutte's results, see, e.g., [4, 6, 7, 10, 11, 13, 22].

An eigenvalue λ of a graph G is called a *main eigenvalue* if the corresponding eigenspace is not orthogonal to the all-ones vector. An n-vertex graph G is called *controllable* [5] if it has n main eigenvalues, and *almost controllable* [23] if it has $n-1$ main eigenvalues. It is known that graphs with irreducible characteristic polynomials are controllable, but the converse does not hold. Tutte's result was further improved by Godsil and McKay [6], who showed that all graphs in a specific family—including all controllable and almost controllable graphs—are reconstructible.

Currently, it is unknown whether almost all graphs have irreducible characteristic polynomials. However, O'Rourke and Touri [16] proved that almost all graphs are controllable, improving upon the result of Tao and Vu [20], which states that almost all graphs have simple spectrum. Thus the result of Godsil and McKay [6] provides an alternative proof—using spectral graph theoretic methods—of the result of Bollobás [1] that almost all graphs are reconstructible.

As a natural extension of the Tutte's theorem and its subsequent improvement by Godsil and McKay, Hong [8] established the following reconstructibility criterion for a graph based on the properties of its deck.

Theorem 4 ([8])**.** *Let G be a graph with at least three vertices. If there exists a vertex-deleted subgraph that is controllable, then G is reconstructible.*

Inspired by the similarity between Hong's theorem and the theorem of Godsil and McKay, it is natural to ask whether the controllability assumption in Theorem 4 can be relaxed to almost controllability. As a direct application of Theorem 3, we provide an affirmative answer to this question under a mild restriction.

Let $\mathcal{H}_n$ denote the family of almost controllable graphs of order n. The family $\mathcal{H}_n$ can be naturally partitioned into two subsets $\mathcal{H}_n^s$ and $\mathcal{H}_n^a$, where $\mathcal{H}_n^s$ consists of graphs G in $\mathcal{H}_n$ that have a nontrivial automorphism, and $\mathcal{H}_n^a$ contains the asymmetric graphs. For any almost controllable graph $G \in \mathcal{H}_n$, we construct a unique rational regular orthogonal matrix $Q_0(G)$ distinct from the identity matrix (see Definition 11 in Section 3). We now state the second main result of this paper, which provides a new condition for reconstructibility of graphs.

Theorem 5. *Let $n \geqslant 3$ and G be an n-vertex graph with vertex set $\{u_1, \ldots, u_n\}$. Then G is reconstructible if there exists a vertex-deleted subgraph, say $G - u_n$, satisfying either of the following two conditions:*

(i) $G - u_n \in \mathcal{H}_{n-1}^s$,
(ii) $G - u_n \in \mathcal{H}_{n-1}^a$ *but* $Q_0 b \notin \{0,1\}^{n-1} \setminus \{b\}$, *where* $Q_0 = Q_0(G - u_n)$ *and* $b \in \{0,1\}^{n-1}$ *is the indicator vector of the neighborhood of* u_n *in* G.

In Section 2, we prove Theorem 3. Section 3 presents the proof of Theorem 5, which serves as a direct application of Theorem 3. We conclude with examples demonstrating the effectiveness of our results in graph reconstruction.

2 Proof of Theorem 3

Let G be a graph with vertex set $\{u_1, \ldots, u_n\}$. Let b be an n-dimensional $(0,1)$-vector. Following Farrugia [3], the *overgraph* $G + b$ is the graph whose adjacency matrix is

$$A(G+b) = \begin{pmatrix} A(G) & b \\ b^{\mathrm{T}} & 0 \end{pmatrix}.$$

Let A be an $n \times n$ real symmetric matrix and $\lambda_1, \ldots, \lambda_m$ be its distinct eigenvalues with multiplicities $k_1, \ldots, k_m$, respectively. Let P_i be any $n \times k_i$ matrix whose columns consist of an orthonormal basis of $\mathcal{E}_{\lambda_i}(A)$, the eigenspace of A corresponding to λ_i. Then A has the spectral decomposition

$$A = \lambda_1 P_1 P_1^{\mathrm{T}} + \cdots + \lambda_m P_m P_m^{\mathrm{T}}.$$

We note that $P_i P_i^{\mathrm{T}}$ is well defined although P_i is not unique. Indeed, if $\tilde{P}_i$ consists of another orthogonal basis of $\mathcal{E}_{\lambda_i}(A)$, then we must have $\tilde{P}_i = P_i Q$ for some orthogonal matrix Q, which clearly implies $\tilde{P}_i \tilde{P}_i^{\mathrm{T}} = P_i P_i^{\mathrm{T}}$.

For a graph G, we use $\chi(G; x)$ to denote the characteristic polynomial of G, i.e., $\chi(G; x) = \chi(A(G); x) = \det(xI - A(G))$. We use $e^{(n)}$ (or e) to denote the all-ones vector of dimension n.

Lemma 6 ([2])**.** *Let G be an n-vertex graph whose adjacency matrix A has spectral decomposition $A = \lambda_1 P_1 P_1^{\mathrm{T}} + \cdots + \lambda_m P_m P_m^{\mathrm{T}}$. Then*
(i) $\chi(G+b; x) = \chi(G; x) \left(x - \sum_{i=1}^m \frac{||P_i^{\mathrm{T}} b||^2}{x - \lambda_i} \right)$ *for any* $b \in \{0,1\}$.
(ii) $\chi(\overline{G}; x) = (-1)^n \chi(G; -x-1) \left(1 - \sum_{i=1}^m \frac{||P_i^{\mathrm{T}} e||^2}{x+1+\lambda_i} \right)$.

Suppose that $\chi(G; x)$ is known, then it is not difficult to see from Lemma 6 that knowledge of $\chi(G+e; x)$ is equivalent to knowledge of $\chi(\overline{G}; x)$. This yields an equivalent definition on generalized cospectrality:

Corollary 7. *Two graphs G and H are generalized cospectral if and only if two graphs G and $G+e$ are cospectral with H and $H+e$, respectively.*

We need a technical lemma.

Lemma 8. *Let $\lambda_1, \ldots, \lambda_m$ be m distinct real numbers. Let $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in \mathbb{R}$. If*

$$\left(1 + \sum_{i=1}^{m} \frac{a_i}{\lambda - \lambda_i}\right)^2 = \left(1 + \sum_{i=1}^{m} \frac{b_i}{\lambda - \lambda_i}\right)^2 \text{ for } \lambda \in \mathbb{R} \setminus \{\lambda_1, \ldots, \lambda_m\} \tag{1}$$

then $a_i = b_i$ for $i \in \{1, \ldots, m\}$.

Proof. By Eq. (1), we have

$$\left(1 + \sum_{i=1}^{m} \frac{a_i}{\lambda - \lambda_i}\right) = \pm \left(1 + \sum_{i=1}^{m} \frac{b_i}{\lambda - \lambda_i}\right).$$

We claim that the equality can hold with a negative sign for at most m different values of the variable λ. Indeed, suppose $\left(1 + \sum_{i=1}^{m} \frac{a_i}{\lambda - \lambda_i}\right) = -\left(1 + \sum_{i=1}^{m} \frac{b_i}{\lambda - \lambda_i}\right)$. By multiplying both sides by $\prod_{i=1}^{m}(\lambda - \lambda_i)$ and rearranging the terms, we obtain

$$2\prod_{i=1}^{m}(\lambda - \lambda_i) + \sum_{i=1}^{m}(a_i + b_i) \prod_{j=1, j\neq i}^{m} (\lambda - \lambda_j) = 0. \tag{2}$$

Noting that the left-hand side is a polynomial of degree m, we find that Eq. (2) has at most m roots. This proves the claim.

By the claim, we see that the equality $1 + \sum_{i=1}^{m} \frac{a_i}{\lambda - \lambda_i} = 1 + \sum_{i=1}^{m} \frac{b_i}{\lambda - \lambda_i}$, or equivalently,

$$\sum_{i=1}^{m} \frac{a_i - b_i}{\lambda - \lambda_i} = 0 \tag{3}$$

holds for all $\lambda \in \mathbb{R}$ except a finite number of values. Taking $\lambda \to \lambda_i$ in Eq. (3), we easily find that $a_i = b_i$. This completes the proof. $\square$

Proposition 9. *Let G and H be two n-vertex graphs and b, c be two vectors in $\{0, 1\}^n$. If (1) $G + b$ and $H + c$ are generalized cospectral and (2) G and H are also generalized cospectral, then there exists an orthogonal matrix Q such that $Q^{\mathrm{T}}A(G)Q = A(H)$, $Q^{\mathrm{T}}e = e$ and $Q^{\mathrm{T}}b = c$.*

Proof. Let $A = A(G)$ and $B = A(H)$. As A and B are cospectral, we may write the spectral decompositions of A and B as follows:

$$A = \sum_{i=1}^{m} \lambda_i P_i P_i^{\mathrm{T}} \quad \text{and} \quad B = \sum_{i=1}^{m} \lambda_i R_i R_i^{\mathrm{T}},$$

where each P_i and R_i consist of orthogonal bases of $\mathcal{E}_{\lambda_i}(A)$ and $\mathcal{E}_{\lambda_i}(B)$, respectively.
Claim: $||P_i^{\mathrm{T}}b|| = ||R_i^{\mathrm{T}}c||$, $||P_i^{\mathrm{T}}e|| = ||R_i^{\mathrm{T}}e||$ and $\langle P_i^{\mathrm{T}}b, P_i^{\mathrm{T}}e\rangle = \langle R_i^{\mathrm{T}}c, R_i^{\mathrm{T}}e\rangle$ for $i = 1, \ldots, m$.

The first two equalities of the claim are easy consequences of Lemma 6. Indeed, noting that $\chi(G+b;x)=\chi(H+c;x)$, it follows from Lemma 6 (i) that

$$\chi(G;x)\left(x-\sum_{i=1}^{m}\frac{||P_i^{\mathrm{T}}b||^2}{x-\lambda_i}\right)=\chi(H;x)\left(x-\sum_{i=1}^{m}\frac{||R_i^{\mathrm{T}}c||^2}{x-\lambda_i}\right).$$

As $\chi(G;x)=\chi(H;x)$, we must have

$$x-\sum_{i=1}^{m}\frac{||P_i^{\mathrm{T}}b||^2}{x-\lambda_i}=x-\sum_{i=1}^{m}\frac{||R_i^{\mathrm{T}}c||^2}{x-\lambda_i},$$

which clearly implies $||P_i^{\mathrm{T}}b||=||R_i^{\mathrm{T}}c||$. A similar argument shows that $||P_i^{\mathrm{T}}e||=||R_i^{\mathrm{T}}e||$. It remains to show the last equality $\langle P_i^{\mathrm{T}}b,P_i^{\mathrm{T}}e\rangle=\langle R_i^{\mathrm{T}}c,R_i^{\mathrm{T}}e\rangle$.

Let $\hat{G}=(G+b)+e^{(n+1)}$ and $\hat{H}=(H+c)+e^{(n+1)}$. By Corollary 7 and the first condition of this proposition, we see that $\hat{G}$ and $\hat{H}$ are cospectral. Note that the adjacency matrix of $\hat{G}$ is

$$\hat{A}=\begin{pmatrix}A & b & e\\ b^{\mathrm{T}} & 0 & 1\\ e^{\mathrm{T}} & 1 & 0\end{pmatrix}.$$

Direct calculation shows that

$$\begin{aligned}
\chi(\hat{G};x)&=\begin{vmatrix}\lambda I-A & -b & -e\\ -b^{\mathrm{T}} & \lambda & -1\\ -e^{\mathrm{T}} & -1 & \lambda\end{vmatrix}\\
&=\left|\begin{pmatrix}\lambda I-A & -b & -e\\ -b^{\mathrm{T}} & \lambda & -1\\ -e^{\mathrm{T}} & -1 & \lambda\end{pmatrix}\begin{pmatrix}I & (\lambda I-A)^{-1}b & (\lambda I-A)^{-1}e\\ O & 1 & 0\\ O & 0 & 1\end{pmatrix}\right|\\
&=\begin{vmatrix}\lambda I-A & O & O\\ -b^{\mathrm{T}} & \lambda-b^{\mathrm{T}}(\lambda I-A)^{-1}b & -1-b^{\mathrm{T}}(\lambda I-A)^{-1}e\\ -e^{\mathrm{T}} & -1-e^{\mathrm{T}}(\lambda I-A)^{-1}b & \lambda-e^{\mathrm{T}}(\lambda I-A)^{-1}e\end{vmatrix}\\
&=|\lambda I-A|\begin{vmatrix}\lambda-\sum\limits_{i=1}^{m}\frac{1}{\lambda-\lambda_i}b^{\mathrm{T}}P_iP_i^{\mathrm{T}}b & -\left(1+\sum\limits_{i=1}^{m}\frac{1}{\lambda-\lambda_i}b^{\mathrm{T}}P_iP_i^{\mathrm{T}}e\right)\\ -\left(1+\sum\limits_{i=1}^{m}\frac{1}{\lambda-\lambda_i}e^{\mathrm{T}}P_iP_i^{\mathrm{T}}b\right) & \lambda-\sum\limits_{i=1}^{m}\frac{1}{\lambda-\lambda_i}e^{\mathrm{T}}P_iP_i^{\mathrm{T}}e\end{vmatrix}\\
&=\chi(G;x)\begin{vmatrix}\lambda-\sum\limits_{i=1}^{m}\frac{||P_i^{\mathrm{T}}b||^2}{\lambda-\lambda_i} & -\left(1+\sum\limits_{i=1}^{m}\frac{\langle P_i^{\mathrm{T}}b,P_i^{\mathrm{T}}e\rangle}{\lambda-\lambda_i}\right)\\ -\left(1+\sum\limits_{i=1}^{m}\frac{\langle P_i^{\mathrm{T}}b,P_i^{\mathrm{T}}e\rangle}{\lambda-\lambda_i}\right) & \lambda-\sum\limits_{i=1}^{m}\frac{||P_i^{\mathrm{T}}e||^2}{\lambda-\lambda_i}\end{vmatrix}.
\end{aligned}\tag{4}$$

Similarly, we have

$$\chi(\hat{H};x)=\chi(H;x)\begin{vmatrix}\lambda-\sum\limits_{i=1}^{m}\frac{||R_i^{\mathrm{T}}c||^2}{\lambda-\lambda_i} & -\left(1+\sum\limits_{i=1}^{m}\frac{\langle R_i^{\mathrm{T}}c,R_i^{\mathrm{T}}e\rangle}{\lambda-\lambda_i}\right)\\ -\left(1+\sum\limits_{i=1}^{m}\frac{\langle R_i^{\mathrm{T}}c,R_i^{\mathrm{T}}e\rangle}{\lambda-\lambda_i}\right) & \lambda-\sum\limits_{i=1}^{m}\frac{||R_i^{\mathrm{T}}e||^2}{\lambda-\lambda_i}\end{vmatrix}.\tag{5}$$

As $\chi(\hat{G};x)=\chi(\hat{H};x)$, $\chi(G;x)=\chi(H;x)$, $||P_i^{\mathrm{T}}b||=||R_i^{\mathrm{T}}c||$ and $||P_i^{\mathrm{T}}e||=||R_i^{\mathrm{T}}e||$, combining Eqs. (4) and (5) leads to

$$\left(1+\sum_{i=1}^{m}\frac{\langle P_i^{\mathrm{T}}b,P_i^{\mathrm{T}}e\rangle}{\lambda-\lambda_i}\right)^2=\left(1+\sum_{i=1}^{m}\frac{\langle R_i^{\mathrm{T}}c,R_i^{\mathrm{T}}e\rangle}{\lambda-\lambda_i}\right)^2.$$

It follows from Lemma 8 that $\langle P_i^{\mathrm{T}}b,P_i^{\mathrm{T}}e\rangle=\langle R_i^{\mathrm{T}}c,R_i^{\mathrm{T}}e\rangle$ for $i=1,\ldots,m$. This completes the proof of the Claim.

For each $i\in\{1,\ldots,m\}$, let k_i be the multiplicity of the eigenvalue λ_i of A (or B). Note that $P_i^{\mathrm{T}}b,P_i^{\mathrm{T}}e,R_i^{\mathrm{T}}c,R_i^{\mathrm{T}}e\in\mathbb{R}^{k_i}$. By the Claim, we see that for each i there exists an orthogonal matrix Q_i of order k_i such that $Q_i(P_i^{\mathrm{T}}b)=R_i^{\mathrm{T}}c$ and $Q_i(P_i^{\mathrm{T}}e)=R_i^{\mathrm{T}}e$, see e.g. [9, Theorem 7.3.11]. Written in the form of block matrices, we have

$$\begin{pmatrix}Q_1P_1^{\mathrm{T}}\\ \vdots\\ Q_mP_m^{\mathrm{T}}\end{pmatrix}(b,e)=\begin{pmatrix}R_1^{\mathrm{T}}\\ \vdots\\ R_m^{\mathrm{T}}\end{pmatrix}(c,e). \tag{6}$$

Define

$$Q=(P_1Q_1^{\mathrm{T}},\ldots,P_mQ_m^{\mathrm{T}})\begin{pmatrix}R_1^{\mathrm{T}}\\ \vdots\\ R_m^{\mathrm{T}}\end{pmatrix}. \tag{7}$$

It is easy to see that both matrices on the right-hand side of Eq. (7) are orthogonal, implying Q is an orthogonal matrix. By Eq. (6), we have $Q^{\mathrm{T}}(b,e)=(c,e)$. Finally, noting that

$$(P_1Q_1^{\mathrm{T}},\ldots,P_mQ_m^{\mathrm{T}})^{\mathrm{T}}A(P_1Q_1^{\mathrm{T}},\ldots,P_mQ_m^{\mathrm{T}})=\mathrm{diag}(\lambda_1I_{k_1},\ldots,\lambda_mI_{k_m})$$

and

$$(R_1,\ldots,R_m)^{\mathrm{T}}B(R_1,\ldots,R_m)=\mathrm{diag}(\lambda_1I_{k_1},\ldots,\lambda_mI_{k_m}),$$

we obtain

$$\begin{aligned}Q^{\mathrm{T}}AQ &= (R_1,\ldots,R_m)(P_1Q_1^{\mathrm{T}},\ldots,P_mQ_m^{\mathrm{T}})^{\mathrm{T}}A(P_1Q_1^{\mathrm{T}},\ldots,P_mQ_m^{\mathrm{T}})\begin{pmatrix}R_1^{\mathrm{T}}\\ \vdots\\ R_m^{\mathrm{T}}\end{pmatrix}\\ &= (R_1,\ldots,R_m)\mathrm{diag}(\lambda_1I_{k_1},\ldots,\lambda_mI_{k_m})\begin{pmatrix}R_1^{\mathrm{T}}\\ \vdots\\ R_m^{\mathrm{T}}\end{pmatrix}\\ &= B.\end{aligned}$$

This completes the proof of this proposition. □

Remark 10. Proposition 9 was originally reported as Theorem 12 in [3]. In that paper, Farrugia essentially proved that there exist two orthogonal matrices Q_1 and Q_2 such that $Q_1^{\mathrm{T}}A(G)Q_1 = Q_2^{\mathrm{T}}A(G)Q_2 = A(H)$, $Q_1^{\mathrm{T}}e = e$ and $Q_2^{\mathrm{T}}b = c$. But his claim that the two matrices Q_1 and Q_2 can be chosen to be equal requires more justification. The key of the current proof is the newly established equality $\langle P_i^{\mathrm{T}}b, P_i^{\mathrm{T}}e\rangle = \langle R_i^{\mathrm{T}}c, R_i^{\mathrm{T}}e\rangle$, which guarantees the realizability of $Q_1 = Q_2$.

Proof of Theorem 3 The implication (ii) $\Longrightarrow$ (i) is straightforward; we only prove the other direction. Let $G_1 = G - u_n$ and $H_1 = H - v_n$. Then the adjacency matrices of G and H have the form

$$A(G) = \begin{pmatrix} A(G_1) & b \\ b^{\mathrm{T}} & 0 \end{pmatrix} \quad \text{and} \quad A(H) = \begin{pmatrix} A(H_1) & c \\ c^{\mathrm{T}} & 0 \end{pmatrix}.$$

Now the condition (i) can be restated as that both the pair $G_1 + b$, $H_1 + c$ and the pair G_1, H_1 are generalized cospectral graphs. Noting that G_1 and H_1 contain $(n-1)$ vertices, it follows from Proposition 9 that there exists an orthogonal matrix Q such that $Q^{\mathrm{T}}A(G_1)Q = A(H_1)$ $Q^{\mathrm{T}}e^{(n-1)} = e^{(n-1)}$ and $Q^{\mathrm{T}}b = c$. Direct calculation shows that

$$\begin{pmatrix} Q^{\mathrm{T}} & O \\ O & 1 \end{pmatrix} \begin{pmatrix} A(G_1) & b \\ b^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} Q & O \\ O & 1 \end{pmatrix} = \begin{pmatrix} Q^{\mathrm{T}}A(G_1)Q & Q^{\mathrm{T}}b \\ b^{\mathrm{T}}Q & 0 \end{pmatrix} = \begin{pmatrix} A(H_1) & c \\ c^{\mathrm{T}} & 0 \end{pmatrix}.$$

As Q is a regular orthogonal matrix, the block matrix $\begin{pmatrix} Q^{\mathrm{T}} & O \\ O & 1 \end{pmatrix}$ is also a regular orthogonal matrix. This completes the proof of Theorem 3.

3 Proof of Theorem 5

For an n-vertex graph G with adjacency matrix A, the *walk matrix* of G is

$$W(G) := [e, Ae, \ldots, A^{n-1}e].$$

It is known that the number of main eigenvalues of G equals the rank of $W(G)$. Recall that $\mathcal{H}_n$ denotes the set of all almost controllable graphs on n vertices. Clearly, if $G \in \mathcal{H}_n$ then $\operatorname{rank} W(G) = n - 1$. For a graph $G \in \mathcal{H}_n$, the following Householder matrix associated with G is crucial.

Definition 11 ([17])**.** For $G \in \mathcal{H}_n$, define

$$Q_0 = Q_0(G) = I_n - 2\frac{\xi\xi^{\mathrm{T}}}{\xi^{\mathrm{T}}\xi},$$

where ξ is a unique (up to the sign) nonzero integer vector $\xi = (a_1, a_2, \ldots, a_n)^{\mathrm{T}}$ satisfying $W^{\mathrm{T}}(G)\xi = 0$ and $\gcd(a_1, a_2, \ldots, a_n) = 1$.

Clearly, Q_0 is a symmetric orthogonal matrix. Moreover, from the equation $W^{\mathrm{T}}(G)\xi = 0$, we easily see that $\xi^{\mathrm{T}}e = 0$ and hence $Q_0 e = e$, i.e., the orthogonal matrix Q_0 is regular. Let $\mathrm{RO}(n)$ denote the group of regular orthogonal matrices of order n. Note that $\mathrm{RO}(n)$ contains the group of $n \times n$ permutation matrices as a subgroup. The importance of $Q_0(G)$ can be described in the following theorem.

Theorem 12 ([17, 23])**.** *For $G \in \mathcal{H}_n$, the solution set of the matrix equation $Q^{\mathrm{T}}A(G)Q = A(G)$ with variable $Q \in \mathrm{RO}(n)$ is exactly $\{I_n, Q_0(G)\}$.*

Let $G \in \mathcal{H}_n$. By Theorem 12, we see that $G \in \mathcal{H}_n^s$ if and only if $Q_0(G)$ is a permutation matrix. Now we can present a proof of Theorem 5.

Proof of Theorem 5 Let H be any reconstruction of G. It is known that G and H are generalized cospectral, see [19] or [7]. Let $A = A(G - u_n)$. Then the adjacency matrix of G can be written as

$$A(G) = \begin{pmatrix} A & b \\ b^{\mathrm{T}} & 0 \end{pmatrix}.$$

By relabeling vertices in H appropriately, we may assume the adjacency matrix of H has the form

$$A(H) = \begin{pmatrix} A & c \\ c^{\mathrm{T}} & 0 \end{pmatrix}.$$

It follows from Theorem 3 that there exists a regular orthogonal matrix of the form $\begin{pmatrix} Q & O \\ O & 1 \end{pmatrix}$ such that

$$\begin{pmatrix} Q^{\mathrm{T}} & O \\ O & 1 \end{pmatrix} \begin{pmatrix} A & b \\ b^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} Q & O \\ O & 1 \end{pmatrix} = \begin{pmatrix} A & c \\ c^{\mathrm{T}} & 0 \end{pmatrix}. \tag{8}$$

This means that $Q \in \mathrm{RO}(n-1)$, $Q^{\mathrm{T}}AQ = A$, and $Q^{\mathrm{T}}b = c$. Let $Q_0 = Q_0(G - u_n)$ be the Householder matrix as given in Definition 11. If $G - u_n \in \mathcal{H}_{n-1}^s$, then by Theorem 12 we have $Q \in \{I_{n-1}, Q_0\}$, where Q_0 is a permutation matrix. But this means G and H are isomorphic by Eq. (8). Now consider the other case $G - u_n \in \mathcal{H}_{n-1}^a$. We may assume $c \neq b$ since otherwise $H = G$ and we are done. Thus, $c \in \{0,1\}^{n-1} \setminus \{b\}$ and hence $Q_0 b \neq c$ by the condition of this theorem. Consequently, $Q \neq Q_0$ and hence we must have $Q = I_{n-1}$ by Theorem 12. This again forces $H = G$ by Eq. (8). Either case implies that H is isomorphic to G. Thus G is reconstructible, completing the proof of Theorem 5. $\square$

We present some examples to illustrate Theorem 5.

Example 1. Let n be an integer with $n \geqslant 4$ and $4 \nmid n$. Consider the family of graphs $\{D_n + b\colon\ b \in \{0,1\}^n\}$, where D_n is the Dynkin graph as shown in Fig. 1. It was proved that D_n ($4 \nmid n$) is almost controllable [21]. Clearly, $D_n \in \mathcal{H}_n^s$ as swapping vertices 1 and 2 is a nontrivial automorphism of D_n. Thus, by Theorem 5 (i), all graphs of the form $D_n + b$ are reconstructible.

Example 2. Let G be the graph as shown in Fig. 2. Let H be the vertex-deleted subgraph of G corresponding to the vertex 8. Direct calculation shows that $\operatorname{rank} W(H) =$

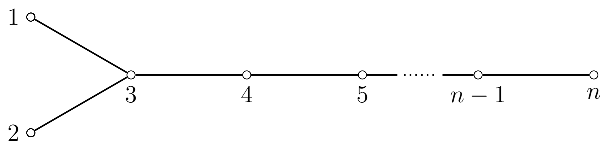

Figure 1: Graph D_n.

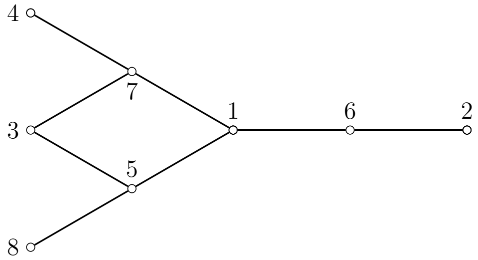

Figure 2: Graph G.

6 and hence H is almost controllable. Moreover, by Definition 11, we obtain $\xi = \pm(0,-1,1,0,-1,1,0)^{\mathrm{T}}$ and hence

$$Q_0 = I - 2\frac{\xi\xi^{\mathrm{T}}}{\xi^{\mathrm{T}}\xi} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

As Q_0 is not a permutation matrix, we see that $H \in \mathcal{H}_7^a$. Note that the indicator vector of the neighborhood of vertex 8 is $b = (0,0,0,0,1,0,0)^{\mathrm{T}}$. Thus $Q_0 b = \left(0,-\frac{1}{2},\frac{1}{2},0,\frac{1}{2},\frac{1}{2},0\right)^{\mathrm{T}} \notin \{0,1\}^7 \setminus \{b\}$ and hence G is reconstructible by Theorem 5 (ii).

Declaration of competing interest

There is no conflict of interest.

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