Stability of transversal Hamilton cycles and paths

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Abstract

Given graphs $G_1, \ldots, G_s$ all on a common vertex set and a graph H with $e(H) = s$, a copy of H is *transversal* if it contains one edge from each G_i. We establish a stability result for transversal Hamilton cycles: the minimum degree required to guarantee a transversal Hamilton cycle can be lowered as long as the graph collection $G_1, \ldots, G_n$ is far in edit distance from several extremal cases. We obtain an analogous result for Hamilton paths. The proof is a combination of our newly developed regularity-blow-up method for transversals, along with the absorption method.

Mathematics Subject Classifications: 05C35, 05C38

1 Introduction

Given graphs $G_1, \ldots, G_s$ on a common vertex set and a graph H with $e(H) = s$, a *transversal* copy of H is a copy of H containing exactly one edge from each of the graphs $G_1, \ldots, G_s$. We often think of each G_i having the colour i, and so a graph with at most one edge of each colour is called *rainbow*. Thus a transversal copy is also a rainbow copy. The following general question was formulated by Joos and Kim in [16].

Question 1. Let H be a graph with s edges and let $\boldsymbol{G} = (G_1, \ldots, G_s)$ be a collection of graphs on a common vertex set V. Which properties imposed on $\boldsymbol{G}$ yield a transversal copy of H?

Note that when $G_1 = \ldots = G_s = G \in \boldsymbol{G}$, a transversal copy of H inside $\boldsymbol{G}$ is equivalent to a subgraph H of G. So the question of Joos and Kim is a generalisation of the classical embedding problem. An important line of research for the classical problem has been to seek sufficient conditions for a graph to contain a copy of H, which is of particular interest as the problem is often computationally difficult. Question 1 is the transversal generalisation.

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1.1 General transversal embedding

There are transversal analogues of several classical embedding theorems. Mantel's well-known theorem from 1907 states that any n-vertex graph with more than $\lfloor n^2/4 \rfloor$ edges contains a triangle. Aharoni, DeVos, de la Maza, Montejano and Šámal [1] showed that in any collection of three graphs on the same vertex set, each one must have significantly more than $n^2/4$ edges to guarantee a transversal triangle. It is still a major open problem in this area to generalise their result to arbitrary K_r.

Dirac's theorem on Hamilton cycles from 1952 states that in an n-vertex graph G, minimum degree $\delta(G) \geqslant n/2$ suffices to guarantee a Hamilton cycle. Aharoni [1] conjectured that the transversal generalisation holds: given graphs $G_1, \ldots, G_n$ on a common vertex set of size n, if $\delta(G_i) \geqslant n/2$ for each $i \in [n]$, then there exists a transversal Hamilton cycle. This conjecture was asymptotically verified by Cheng, Wang and Zhao [11] and later fully proved by Joos and Kim [16], who used a short 'elementary' argument.

As in Dirac's theorem, to guarantee a transversal copy of a spanning graph, it is natural to impose a minimum degree condition on every graph in a collection. Given a graph H and a positive integer n, the *transversal minimum degree threshold* of H is the minimum d such that if every graph in a collection of $e(H)$ n-vertex graphs has minimum degree at least d, then the collection contains a transversal copy of H. If the transversal minimum degree threshold is, up to an additive $o(n)$ term, the same as the minimum degree threshold (for a single n-vertex graph) then we say that H is *colour-blind*. Note that the transversal minimum degree threshold is always at least as large as the minimum degree threshold for embedding in a single graph, which can be seen by taking a graph collection of identical copies of one graph.

Joos and Kim [16] showed that perfect matchings are colour-blind (in fact without any error term). Given a graph F with $v(F)|n$, an n-vertex F-factor H is a spanning graph consisting of vertex-disjoint copies of F. Cheng, Han, Wang and Wang [8] and independently, Montgomery, Müyesser and Pehova [21], showed that the K_k-factor is colour-blind. The authors of [21] further characterised those F for which F is colour-blind, and determined the correct transversal minimum degree threshold when F is not colour-blind. These results are approximate transversal versions of the Hajnal-Szemerédi theorem [14] and a theorem of Kühn and Osthus [19]. In [21] it was also shown that spanning trees with maximum degree $o(n/\log(n))$ are colour-blind, generalising a well-known result of Komlós, Sárközy and Szemerédi [17].

Gupta, Hamann, Müyesser, Parczyk and Sgueglia [13] showed that the $(k-1)$-th power C_n^{k-1} of a Hamilton cycle is colour-blind, as are certain hypergraph cycles. Chakraborti, Im, Kim and Liu [6] generalised many of the results in this section by proving a 'transversal bandwidth theorem' which gives an upper bound on the transversal minimum degree threshold for a large class of graphs, which is tight in many cases, and hence gives a rich class of graphs H which are colour-blind.

Other than the results of Joos and Kim on Hamilton cycles and perfect matchings [16], there are few results which determine the exact transversal minimum degree threshold. It would be interesting to improve all of the results in this section in this direction and determine whether a given H is 'exactly colour-blind'. We make further remarks on this at the end of the paper in Section 8.

1.2 Transversal embedding of Hamilton cycles

Following Joos and Kim's transversal generalisation of Dirac's theorem, there has been much progress on further questions related to the Hamiltonian case of Question 1 for graph collections on n vertices. Cheng, Wang and Zhao [11] also showed that minimum degree $(n+1)/2$ is sufficient to guarantee a rainbow cycle of every length $3, \dots, n-1$. Bradshaw [4] proved analogues of the results of Joos and Kim in bipartite graph collections. Chakraborti, Kim, Lee and Seo [7] studied the problem of finding transversal Hamilton cycles in tournaments (a tournament is a complete oriented graph). Li, Li and Li [20] and Zhang and van Dam [25] imposed Ore-type conditions rather than a minimum degree condition to find transversal Hamilton paths and cycles. Bradshaw, Halasz and Stacho [5] showed that minimum degree at least $n/2$ guarantees not just one transversal Hamilton cycles as per [16], but exponentially many.

Transversal Hamilton cycles have also been well studied in hypergraphs. Extending a result of Rödl, Ruciński and Szemerédi [22, 23], Cheng, Han, Wang, Wang and Yang [24] proved an asymptotically optimal bound on the minimum codegree guaranteeing a tight Hamilton cycle in a uniform hypergraph. The main result in [13] implies asymptotically optimal results for other types of Hamilton cycles in uniform hypergraphs, for various degree conditions. Answering a question left open here, the notable case of vertex degree for tight 3-uniform Hamilton cycles was obtained by Tang, Wang, Wang and Yan [24].

The problem of finding a Hamilton cycle with a given colouring in a graph collection (rather than a transversal Hamilton cycle which is *some* rainbow cycle) was studied by Bowtell, Morris, Pehova and Staden in [3]. It turns out that minimum degree $(\frac{1}{2}+o(1))n$ is enough to guarantee such a cycle, but at least $\lfloor n/2 \rfloor + 1$ is required. Ferber, Han and Mao [12] and Anastos and Chakraborti [2] studied *robustness* of transversal Hamiltonicity. In [2] the threshold at which a graph collection consisting of random subgraphs of graphs satisfying Dirac's condition contains a transversal Hamilton cycle is determined. This generalises a result of Krivelevich, Lee and Sudakov [18].

1.3 Stability for transversal Hamilton cycles and paths

In this paper, we are interested in the stability phenomenon for Hamilton paths and cycles, and will show that graph collections which do not contain a Hamilton cycle, but whose graphs all have minimum degree slightly smaller than that needed to guarantee one, in fact have a special structure.

We define two extremal graphs and of order n as follows:

- the n-vertex graph is the union of two disjoint cliques of size as equal as possible;
- the n-vertex graph is a complete bipartite graph whose two parts have size as equal as possible.

Since Dirac's theorem shows that a Hamilton cycle is guaranteed in any n-vertex graph G with minimum degree $\delta(G) \geqslant n/2$, graphs satisfying this minimum degree condition are called *Dirac graphs*. As mentioned in the previous section, Joos and Kim [16] showed that any collection of n Dirac graphs contains a transversal Hamilton cycle.

Dirac's theorem is tight: the graph does not contain a Hamilton cycle, and when n is even it has minimum degree $n/2 - 1 = \lceil n/2 \rceil - 1$. The graph does not contain a Hamilton cycle when n is odd, in which case it has minimum degree $(n-1)/2 = \lceil n/2 \rceil - 1$ and contains no Hamilton cycle. In fact, when n is odd, we can add arbitrary edges into the smaller part of without creating a Hamilton cycle. For any n, we can perturb either of these examples slightly, at the expense of slightly reducing the minimum degree, to get a non-Hamiltonian graph.

In fact, it is well-known that these are the only examples. Given $\kappa > 0$, we say that two graphs G, G' on the same vertex set of size n are *κ-close* if one can add and remove at most κn^2 edges of G to obtain G'.

Theorem 2 (Stability for Dirac's theorem, folklore)**.** *For all $\kappa > 0$, there exist $\mu > 0$ and n_0 such that the following holds for all integers $n \geqslant n_0$. Let G be an n-vertex graph with $\delta(G) \geqslant (\frac{1}{2} - \mu)n$. If G contains no Hamilton cycle, then either G is κ-close to with arbitrary edges added inside one part; or G is κ-close to .*

Given a graph collection $\boldsymbol{G}$, we write $\delta(\boldsymbol{G}) := \min_{G \in \boldsymbol{G}} \delta(G)$ for its *minimum degree*. Now suppose that a graph collection $\boldsymbol{G} = (G_1, \ldots, G_n)$ on a common vertex set of size n with $\delta(\boldsymbol{G}) \geqslant (\frac{1}{2} - \mu)n$ has no transversal Hamilton cycle (we say $\boldsymbol{G}$ is *non-Hamiltonian*). Of course, one can take $G_i = G$ where G is non-Hamiltonian; that is, we can take $\boldsymbol{G}$ to be a collection of identical copies of one of the graphs in Theorem 2. On the other hand, taking copies of and with, say, uniformly random partitions will give a graph collection which *is* Hamiltonian with high probability. However, one can combine these graphs to get further collections without a transversal Hamilton cycle.

Definition 3 ($\boldsymbol{H_a^b}$, half-split graph collection)**.** Given integers $a, b \geqslant 0$, let $\boldsymbol{H}_a^b$ be the graph collection on a common vertex set of size n obtained by taking a copies of and b copies of where they are defined on the same equitable partition $A \cup B$.

We say that a graph collection $\boldsymbol{J}$ on a common vertex set V of size n is *half-split* if there is $A \subseteq V$ with $|A| = \lfloor n/2 \rfloor + 1$ such that $J[A] = \emptyset$ and $J[A, V \setminus A]$ is complete for all $J \in \boldsymbol{J}$.

Note that $\delta(\boldsymbol{H}_a^b) = \lfloor n/2 \rfloor - \mathbb{1}(a > 0)$. Note that $\boldsymbol{H}_n^0$ is non-Hamiltonian since is, and when n is odd, $\boldsymbol{H}_0^n$ is also non-Hamiltonian since is. We claim that when b is odd and $a + b = n$, the collection $\boldsymbol{H}_a^b$ is non-Hamiltonian. Suppose not, and let C be a transversal Hamilton cycle and cyclically direct its edges. We say an edge of C is of 1-type if it comes from an copy and of 2-type if it comes from an copy. Note that the number of 2-type edges directed from A to B is equal to the number of 2-type edges directed from B to A. Hence, the total number of 2-type edges in C is even, which implies that b is even, a contradiction.

A half-split graph collection $\boldsymbol{J}$ on n vertices has $\delta(\boldsymbol{J}) = \lceil n/2 \rceil - 1$. Further, $\boldsymbol{J}$ is non-Hamiltonian since if it contained any Hamilton cycle C (with any colours), then between each pair of vertices in A on C there is at least one vertex of $V \setminus A$, so $|V \setminus A| \geqslant |A|$, a contradiction.

Given $\kappa > 0$, we say that two graph collections $\boldsymbol{G}, \boldsymbol{G'}$ on the same common vertex set of size n are *κ-close* if one can add and remove at most κn^3 edges of graphs in $\boldsymbol{G}$ to obtain $\boldsymbol{G'}$. Our first result shows that any non-Hamiltonian collection of n almost-Dirac graphs on n vertices is close to one of these two families.

Theorem 4. *For all $\kappa > 0$, there exist $\mu > 0$ and n_0 such that the following holds for all integers $n \geqslant n_0$. Let $\boldsymbol{G} = (G_1, \dots, G_n)$ be a graph collection on a common vertex set V of size n, with $\delta(\boldsymbol{G}) \geqslant (\frac{1}{2} - \mu)n$. If $\boldsymbol{G}$ contains no transversal Hamilton cycle, then either $\boldsymbol{G}$ is κ-close to $\boldsymbol{H}_a^b$ for some $a \in [n]$; or $\boldsymbol{G}$ is κ-close to a half-split graph collection.*

In contrast to the Hamilton cycle case, our second result demonstrates that half-split graph collections and $\boldsymbol{H}^0_{n-1}$ are the only extremal constructions (in an approximate sense) for transversal Hamilton paths. Clearly $\boldsymbol{H}^0_{n-1}$ does not contain a transversal Hamilton path, while if we increase the size of A by one when n is odd, a half-split collection of $n-1$ graphs does not contain a transversal Hamilton path either.

Theorem 5. *For all $\kappa > 0$, there exist $\mu > 0$ and n_0 such that the following holds for all integers $n \geqslant n_0$. Let $\boldsymbol{G} = (G_1, \dots, G_{n-1})$ be a graph collection on a common vertex set V of size n, with $\delta(\boldsymbol{G}) \geqslant (\frac{1}{2} - \mu)n$. If $\boldsymbol{G}$ does not contain a transversal Hamilton path, then $\boldsymbol{G}$ is either κ-close to $\boldsymbol{H}^0_{n-1}$ or $\boldsymbol{G}$ is κ-close to a half-split graph collection.*

We prove these theorems in a unified manner, combining our newly-developed regularity-blow-up method from [10] and the absorption method for transversals, which uses ideas from the papers [8, 9, 11] of the first author. We give a sketch of the proofs in Section 1.6.

Theorems 4 and 5 improve a result of Anastos and Chakraborti, [2, Theorem 4.4] proved in their work on robust Hamiltonicity mentioned earlier. They show that any graph collection $\boldsymbol{G}$ satisfying the hypotheses of Theorem 4 is close to some graph collection in a set $\mathcal{H}_{\mathrm{AC}}$, consisting of collections $\boldsymbol{H}^b_0 \cup \boldsymbol{J}_a$ where $\boldsymbol{J}_a$ is a half-split collection of $a = n - b$ graphs, and the common independent set A in $\boldsymbol{J}_a$ is one of the parts of the graphs in $\boldsymbol{H}^b_0$. Note that the fraction of $\mathcal{H}_{\mathrm{AC}}$ which are in fact Hamiltonian tends to 1 as n tends to infinity. On the other hand, Theorem 4 is best possible in the sense that none of the collections in its $\mathcal{H} = \mathcal{H}_4$ are Hamiltonian (and $\mathcal{H}_4$ is a very small subset of $\mathcal{H}_{\mathrm{AC}}$) so $\mathcal{H}_4$ is in some sense minimal.

The approach we take is very different to that of [2]. Indeed, for the 'stable case' of Theorem 4, we use the regularity-blow-up method and absorption method; while [2, Theorem 4.4], which is similar to this case, instead uses iterative absorption. Several new ideas (in particular those in Section 6) would be needed to deduce either Theorem 4 or 5 from [2, Theorem 4.4]. In fact, as needed for its application, [2, Theorem 4.4] actually guarantees that when a collection is far from any $\boldsymbol{H} \in \mathcal{H}_{\mathrm{AC}}$, not only is it Hamiltonian, there is a certain measure on the set of transversal Hamilton cycles which in particular implies there are many of them.

1.4 Notation and organisation

Notation. Let G be any graph. We denote its vertex set by $V(G)$ and its edge set by $E(G)$. We write $v(G) = |V(G)|$ and $e(G) = |E(G)|$ for their sizes. Given $v \in V(G)$, the *neighbourhood* $N_G(v)$ of v is the set of vertices that are incident to v and the *degree* of v is $d_G(v) := |N_G(v)|$. For any $U \subseteq V(G)$, let $G[U]$ be the induced graph of G on U, i.e., graph with vertex set U and those edges of G with both endpoints in U. Let $G - U := G[V(G) \setminus U]$ and $G - v := G - \{v\}$. For each vertex $v \in V(G)$ and subset $U \subseteq V(G)$, let $N_G(v, U) = N_G(v) \cap U$ and $d_G(v, U) = |N_G(v, U)|$. Let $E_G(X, Y)$ be the set of edges with one endpoint in X and another in Y (so edges in $G[X \cap Y]$ are only counted once) and let $e_G(X, Y) = |E_G(X, Y)|$. We write $E_G(X) := E_G(X, X) = E(G[X])$. We write $P = v_1 \dots v_t$ to denote a path of *length* t, and will sometimes write $v_1 P v_t$ or $v_1 P$ or $P v_t$ for P if we wish to emphase its endpoint(s).

Given any collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ of graphs on a common vertex set V, we call $\mathscr{C}$ the *colour set* of $\boldsymbol{G}$. For any two sets $X, Y \subseteq V$, let $E_{\boldsymbol{G}}(X, Y)$ be the multiset of edges of any colour with one endpoint in X and another in Y and $e_{\boldsymbol{G}}(X, Y) = |E_{\boldsymbol{G}}(X, Y)| = \sum_{G \in \boldsymbol{G}} e_G(X, Y)$. Given any edge-coloured graph H with $V(H) \subseteq V$ with edge-colouring $\sigma : E(H) \to \mathscr{C}$, we write $\mathrm{col}(H) := \bigcup_{e \in E(H)} \sigma(e)$ for the set of colours used on H.

We say a constant $x = a \pm b$ if we have $a - b \leqslant x \leqslant a + b$. For any two constants $\alpha, \beta \in (0, 1)$, we write $\alpha \ll \beta$ if there exists function $\alpha_0 = \alpha_0(\beta)$ such that the subsequent arguments hold for all $0 < \alpha \leqslant \alpha_0$. When we write multiple constants in a hierarchy, we mean that they are chosen from right to left. For any two integers $a \leqslant b$, let $[a, b] = \{a \leqslant x \leqslant b : x \in \mathbb{Z}\}$ and $[a] := [1, a]$. Given a set X and positive integer k, we write

- $\binom{X}{k}$ for the set of all k-subsets of X,
- X^k for the set of all k-tuples of elements of X,
- $(X)_k$ for the set of all k-tuples of *distinct* elements of X.

We use script letters e.g. $\mathscr{C}, \mathscr{A}$ to denote sets of colours and bold letters e.g. $\boldsymbol{G}, \boldsymbol{J}$ to denote graph collections.

Organisation. We conclude this section with a sketch of the proofs of Theorems 4 and 5. In Section 2, we introduce our regularity-blow-up method for transversals and state the definitions and tools we will need later. The remainder of the paper contains the proofs of our two main results. We prove them in a unified manner and all of the auxiliary results along the way concern the setting of Theorem 4, i.e. collections of n graphs and transversal Hamilton cycles (as opposed to paths). Section 3 introduces suitable notions of extremality and stability for graphs and graph collections; the proof will be split into a 'stable case' and an 'extremal case'. In Section 4, we prove some results about absorption that will be used in the proof of the stable case. Then, in Section 5 we show that there is always a transversal Hamilton cycle in this case. Section 6 deals with the extremal case. We combine the results of Sections 3–6 to give a unified proof of Theorems 4 and 5 in Section 7. In Section 8 with finish with some concluding remarks and a discussion of 'exact' results for transversal embedding.

1.5 Probabilistic tools

We use the following version of Chernoff's bound:

Lemma 6 (See e.g. Corollary 2.3 in [15])**.** *Let X be a random variable with binomial or hypergeometric distribution, and let $0 < \varepsilon < \frac{3}{2}$. Then*

$$\mathbb{P}[|X - \mathbb{E}(X)| \geqslant \varepsilon \mathbb{E}(X)] \leqslant 2e^{-\frac{\varepsilon^2}{3}\mathbb{E}(X)}.$$

1.6 Sketch of the proof of Theorems 4 and 5

We will define two kinds of stability for a graph family $\boldsymbol{G} = (G_1, \ldots, G_n)$ with minimum degree at least $(\frac{1}{2} - o(1))n$ (see Definition 18). We will say that

- $\boldsymbol{G}$ is *strongly stable* if $\boldsymbol{G}$ contains many graphs which are not close to containing either or , while
- $\boldsymbol{G}$ is *weakly stable* if almost all graphs in $\boldsymbol{G}$ are close to containing either or , but the vertex partitions associated with these subgraphs are not similar.

We say that $\boldsymbol{G}$ is *stable* if it is either strongly stable or weakly stable. Thus if $\boldsymbol{G}$ is not stable, most graphs in $\boldsymbol{G}$ are close to containing either or and their associated vertex partitions are almost the same. The first part of the proof is to show that if $\boldsymbol{G}$ is stable, then $\boldsymbol{G}$ contains a transversal Hamilton cycle. We then deal with the extremal case, where $\boldsymbol{G}$ is not stable.

The stable case

Step 1. *Build the absorbing cycle when $\boldsymbol{G}$ is stable.* (Section 4)
We build an 'absorbing cycle' C for $\boldsymbol{G}$ with the property that C is very small and there is a set $\mathcal{A}$ consisting of *some* colour-vertex pairs (c, v) and colour-vertex-vertex triples (c, u, v) such that whenever $\mathcal{A}_0 \subseteq \mathcal{A}$ is sufficiently small compared to $|C|$, C can *absorb* all of its elements (see Definition 25). This property implies

- we can add any $(c, v) \in \mathcal{A}_0$ to the cycle C, which means there is a new cycle C' with colour set $\mathrm{col}(C') = \mathrm{col}(C) \cup \{c\}$ and vertex set $V(C') = V(C) \cup \{v\}$; and,
- for any $(c, u, v) \in \mathcal{A}_0$, whenever P is a rainbow path with endpoints u, v, we can add P into C using only the new colour c.

In fact, when $\boldsymbol{G}$ is strongly stable, $\mathcal{A}$ contains *all* pairs and triples. But in general, we need to construct an additional auxiliary set to absorb those colours and vertices which C cannot. After C is constructed, we delete C and its colour set from $\boldsymbol{G}$.
Step 2. *Use the regularity-blow-up method for transversals to cover with long paths.*
We apply the regularity lemma for graph collections (Lemma 10) to $\boldsymbol{G}$ and thus get a reduced graph collection $\boldsymbol{R}$ that inherits the minimum degree condition of $\boldsymbol{G}$. By randomly partitioning its colour set, we obtain two families $\boldsymbol{R}^1$ and $\boldsymbol{R}^2$, each using about half of the colour clusters. Since $\boldsymbol{R}^1$ and $\boldsymbol{R}^2$ inherit the original degree condition, we are able to find two almost perfect rainbow matchings M_1, M_2 from $\boldsymbol{R}^1$ and $\boldsymbol{R}^2$ (see Lemma 20). The union of these matchings is 'locally' like a Hamilton cycle. We apply the transversal blow-up lemma (Theorem 13) to obtain almost spanning disjoint rainbow paths inside each edge of $M_1 \cup M_2$, which cover almost all the vertices outside C.

Step 3. *Connect the paths and cover remaining vertices via the absorbing cycle.*
The last step is to use the absorbing property of C as well as the auxiliary set to connect all the paths to a transversal Hamilton cycle.

The extremal case (Section 6)
The remaining case is when $\boldsymbol{G}$ is not stable and thus most graphs in $\boldsymbol{G}$ are close to containing either or and their vertex partitions are almost the same partition $V = A \cup B$. To prove Theorem 4, we need to show that if $\boldsymbol{G}$ is not close to a half-split graph collection or close to $\boldsymbol{H}_a^b$ for some $a \in [n]$ where $b = n - a$ is odd, then $\boldsymbol{G}$ contains a transversal Hamilton cycle. For this, we first find a short path that covers atypical vertices and colours, and in some cases, balances the two sides of the partition. Next, we find partitions $A = A^0 \cup A^{10} \cup A^{11}$ and $B = B^0 \cup B^{10} \cup B^{11}$ whose sizes depend on the exact number of graphs that are close to and to . The transversal blow-up lemma guarantees that there are spanning transversal paths in each of $\boldsymbol{G}[A^0, B^0]$, $\boldsymbol{G}[A^{10}, A^{11}]$, $\boldsymbol{G}[B^{01}, B^{11}]$ with disjoint colour sets and endpoints in any given large subsets. The structure of $\boldsymbol{G}$ allows us to find such paths which can be concatenated into a transversal Hamilton cycle, proving Theorem 4. The proof of Theorem 5 differs only at the end: we use the fact that given a collection $\boldsymbol{G}$ of $n-1$ graphs on a common vertex set of size n, the n-graph collection obtained from adding a complete graph to $\boldsymbol{G}$ has a Hamilton cycle if and only if $\boldsymbol{G}$ has a Hamilton path.

2 The regularity-blow-up method for transversals

In this section, we introduce the tools developed in our paper [10]. We first define (super)regularity for graph collections.

Definition 7 (Regularity and superregularity)**.** Suppose that $\boldsymbol{G}$ is a graph collection with colour set $\mathscr{C}$, where each G_c is bipartite with parts V_1, V_2. We say that

- $\boldsymbol{G}$ is (ε, d)*-regular* if whenever $V_i' \subseteq V_i$ with $|V_i'| \geqslant \varepsilon |V_i|$ for $i = 1, 2$ and $\mathscr{C}' \subseteq \mathscr{C}$ with $|\mathscr{C}'| \geqslant \varepsilon |\mathscr{C}|$, we have
$$\left| \frac{\sum_{c \in \mathscr{C}'} e_{G_c}(V_1', V_2')}{|\mathscr{C}'||V_1'||V_2'|} - \frac{\sum_{c \in \mathscr{C}} e_{G_c}(V_1, V_2)}{|\mathscr{C}||V_1||V_2|} \right| < \varepsilon$$
and $e_{\boldsymbol{G}}(V_1, V_2) \geqslant d|\mathscr{C}||V_1||V_2|$.
- $\boldsymbol{G}$ is (ε, d)*-superregular* if it is (ε, d)-regular and $\sum_{c \in \mathscr{C}} d_{G_c}(x) \geqslant d|\mathscr{C}||V_{3-i}|$ for all $x \in V_i$ where $i \in [2]$, and $e(G_c) \geqslant d|V_1||V_2|$ for all $c \in \mathscr{C}$.

Note that if every G_c with $c \in \mathscr{C}$ is the same, then $\boldsymbol{G}$ is (ε, d)-regular if and only if G_c is (ε, d)-regular; and $\boldsymbol{G}$ is (ε, d)-superregular if and only if G_c is (ε, d)-superregular.

Lemma 8 (Typical vertices and colours [10])**.** *Let $0 < \varepsilon \ll d \leqslant 1$, and let $\boldsymbol{G}$ be an (ε, d)-regular graph collection with colour set $\mathscr{C}$, where each G_c is bipartite with parts V_1, V_2. Then the following hold:*

(i) for every $i \in [2]$ and all but at most $\varepsilon |V_i|$ vertices $v \in V_i$ we have $\sum_{c \in \mathscr{C}} d_{G_c}(v) \geqslant (d - \varepsilon)|V_{3-i}||\mathscr{C}|$;

(ii) for all but at most $\varepsilon|\mathscr{C}|$ colours $c \in \mathscr{C}$ we have $e(G_c) \geqslant (d-\varepsilon)|V_1||V_2|$.

Lemma 9 (Slicing lemma [10]). *Let $0 < 1/n \ll \varepsilon \ll \alpha \ll d \leqslant 1$, and let $\boldsymbol{G}$ be a graph collection with colour set $\mathscr{C}$, where each G_c is bipartite with parts V_1, V_2 each of size at least n. Suppose that $\boldsymbol{G}$ is (ε, d)-regular. Let $V_i' \subseteq V_i$ with $|V_i'| \geqslant \alpha|V_i|$ for $i \in [2]$ and $\mathscr{C}' \subseteq \mathscr{C}$ with $|\mathscr{C}'| \geqslant \alpha|\mathscr{C}|$. Then $\boldsymbol{G}' := (G_c[V_1', V_2'] : c \in \mathscr{C}')$ is $(\varepsilon/\alpha, d/2)$-regular.*

We use the following regularity lemma for graph collections.

Lemma 10 (Regularity lemma for graph collections [10]). *For all integers $L_0 \geqslant 1$ and every $\varepsilon, \delta > 0$, there is an $n_0 = n_0(\varepsilon, \delta, L_0)$ such that for every $d \in [0,1)$ and every graph collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ on vertex set V of size $n \geqslant n_0$ with $\delta n \leqslant |\mathscr{C}| \leqslant n/\delta$, there exists a partition of V into $V_0, V_1, \ldots, V_L$, of $\mathscr{C}$ into $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_M$ and a spanning subgraph G_c' of G_c for each $c \in \mathscr{C}$ such that the following properties hold:*

(i) $L_0 \leqslant L, M \leqslant n_0$ and $|V_0| + |\mathscr{C}_0| \leqslant \varepsilon n$;

(ii) $|V_1| = \ldots = |V_L| = |\mathscr{C}_1| = \ldots = |\mathscr{C}_M| =: m$;

(iii) $\sum_{c\in\mathscr{C}} d_{G_c'}(v) > \sum_{c\in\mathscr{C}} d_{G_c}(v) - (3d/\delta^2 + \varepsilon)n^2$ for all $v \in V$ and $e(G_c') > e(G_c) - (3d/\delta^2 + \varepsilon)n^2$ for all $c \in \mathscr{C}$;

(iv) if, for $c \in \mathscr{C}$, the graph G_c' has an edge with both vertices in a single cluster V_i for some $i \in [L]$, then $c \in \mathscr{C}_0$;

(v) for all triples $(\{h,i\}, j) \in \binom{[L]}{2} \times [M]$, we have that either $G_c'[V_h, V_i] = \emptyset$ for all $c \in \mathscr{C}_j$, or $\boldsymbol{G}_{hi,j}' := (G_c'[V_h, V_i] : c \in \mathscr{C}_j)$ is (ε, d)-regular.

The sets V_i are called *vertex clusters* and the sets $\mathscr{C}_j$ are called *colour clusters*, while V_0 and $\mathscr{C}_0$ are the *exceptional* vertex and colour sets respectively.

Definition 11 (Reduced graph collection). Given a graph collection $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ on V and parameters $\varepsilon > 0, d \in [0,1)$ and $L_0 \geqslant 1$, the *reduced graph collection* $\boldsymbol{R} = \boldsymbol{R}(\varepsilon, d, L_0)$ of $\boldsymbol{G}$ is defined as follows. Apply Lemma 10 to $\boldsymbol{G}$ with parameters $\varepsilon, \delta, d, L_0$ to obtain $\boldsymbol{G}'$ and a partition $V_0, \ldots, V_L$ of V and $\mathscr{C}_0, \ldots, \mathscr{C}_M$ of $\mathscr{C}$ where V_0, $\mathscr{C}_0$ are the exceptional sets and $V_1, \ldots, V_L$ are the vertex clusters and $\mathscr{C}_1, \ldots, \mathscr{C}_M$ are the colour clusters. Then $\boldsymbol{R} = (R_1, \ldots, R_M)$ is a graph collection of M graphs each on the same vertex set $[L]$, where, for $(\{h,i\}, j) \in \binom{[L]}{2} \times [M]$, we have $hi \in R_j$ whenever $\boldsymbol{G}_{hi,j}'$ is (ε, d)-regular.

The next lemma states that clusters inherit a minimum degree bound in the reduced graph from G.

Lemma 12 (Degree inheritance [10]). *Suppose $L_0 \geqslant 1$ and $0 < 1/n \ll \varepsilon \leqslant d \ll \delta, \gamma, p \leqslant 1$. Let $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ be a graph collection on a vertex set V of size n with $\delta(G_c) \geqslant (p+\gamma)n$ for all $c \in \mathscr{C}$ and $\delta n \leqslant |\mathscr{C}| \leqslant n/\delta$. Let $\boldsymbol{R} = \boldsymbol{R}(\varepsilon, d, L_0)$ be the reduced graph collection of $\boldsymbol{G}$ on L vertices with M graphs. Then*

(i) for every $i \in [L]$ there are at least $(1 - d^{1/4})M$ colours $j \in [M]$ for which $d_{R_j}(i) \geqslant (p + \gamma/2)L$;

(ii) for every $j \in [M]$ there are at least $(1 - d^{1/4})L$ vertices $i \in [L]$ for which $d_{R_j}(i) \geqslant (p + \gamma/2)L$.

All of the above definitions and lemmas are merely convenient restatements of 'weak regularity' for 3-uniform hypergraphs, specialised to the transversal setting. However, the next lemma, our transversal blow-up lemma, which was the main result of [10], is a non-trivial tool which we will use in the proofs of Theorems 4 and 5.

An n-vertex graph H is *μ-separable* if there is $X \subseteq V(H)$ of size at most μn such that $H - X$ consists of components of size at most μn. This class of graphs includes many natural graphs including F-factors, 2-regular graphs and powers of Hamilton cycles. In this paper, we will only use it for Hamilton paths. Thus it suffices to state a simplified version of the main result of [10].

Theorem 13 (Transversal blow-up lemma [10])**.** *Let $0 < 1/n \ll \varepsilon, \mu, \alpha, \ll \nu, d, \delta, 1/\Delta \leqslant 1$. Let $\mathscr{C}$ be a set of at least δn colours and let $\boldsymbol{G} = (G_c : c \in \mathscr{C})$ be a collection of bipartite graphs with the same vertex partition V_1, V_2, where $n \leqslant |V_1| \leqslant |V_2| \leqslant n/\delta$, such that $\boldsymbol{G}$ is (ε, d)-superregular. Let H be a μ-separable bipartite graph with parts A_1, A_2 of sizes $|V_1|, |V_2|$ respectively, and $|\mathscr{C}|$ edges and maximum degree Δ. Suppose further that, for $i = 1, 2$, there is a set $U_i \subseteq A_i$ with $|U_i| \leqslant \alpha |A_i|$ and for each $x \in U_i$, a* target *set $T_x \subseteq V_i$ with $|T_x| \geqslant \nu |V_i|$. Then G contains a transversal copy of H such that for $i = 1, 2$, every $x \in U_i$ is embedded inside T_x.*

3 Extremality and stability

3.1 Extremal and stable graphs

In this section, we define extremality for a single graph. A graph G is not extremal if any two half-sized sets have many edges between them.

Definition 14 (nice, extremal)**.** Let G be a graph on a vertex set V of size n and let $\varepsilon > 0$. We say that

- G is *ε-nice* if for any two sets $A, B \subseteq V$ of size at least $(\frac{1}{2} - \varepsilon)n$, we have $e_G(A, B) \geqslant \varepsilon n^2$.
- G is *ε-extremal* if it is not ε^3-nice.

Note that whenever $\varepsilon' \geqslant \varepsilon > 0$, an ε'-nice graph is ε-nice and hence an ε-extremal graph is ε'-extremal.

The following is a version of a well-known fact about the structure of almost Dirac graphs which forms the basis of our extremal case distinction (for example, see [18]).

Lemma 15. *Suppose that $0 < 1/n \ll d \ll \varepsilon \leqslant 1$. Let G be a graph on a vertex set V of size n with $d_G(x) \geqslant (\frac{1}{2} - \varepsilon^3)n$ for all but at most dn vertices $x \in V$ which is ε-extremal. Then there is a* characteristic partition *(A, B, C) of G such that*

(i) A, B, C partition V;

(ii) $|A| = |B| = (\frac{1}{2} - \varepsilon)n$;

(iii) one of the following holds:

- *$d_G(a, A) \geqslant (\frac{1}{2} - 2\varepsilon)n$ for all $a \in A$ and $d_G(b, B) \geqslant (\frac{1}{2} - 2\varepsilon)n$ for all $b \in B$ and $e_G(A, B) \leqslant \varepsilon n^2$; here we say that G is $(\varepsilon, \substack{\bullet \\ \bullet})$*-extremal*;*

- $d_G(a, B) \geqslant (\frac{1}{2} - 2\varepsilon)n$ *for all* $a \in A$ *and* $d_G(b, A) \geqslant (\frac{1}{2} - 2\varepsilon)n$ *for all* $b \in B$*, and either* $e_G(A) \leqslant \varepsilon n^2$ *or* $e_G(B) \leqslant \varepsilon n^2$*; here we say that* G *is* $(\varepsilon, \;)$-extremal.

Proof. By adding the at most dn vertices with degree less than $(\frac{1}{2} - \varepsilon^3)n$ to C, it suffices to prove that if G has $\delta(G) \geqslant (\frac{1}{2} - \varepsilon^3)n$, then the conclusion holds with $\varepsilon/2$ in place of ε. Let $\mu := \varepsilon^3$. Since G is ε-extremal, there are $X, Y \subseteq V$ each of size at least $(\frac{1}{2} - \mu)n$ such that $e_G(X, Y) < \mu n^2$. Let $U := X \cap Y$ and $D := V \setminus (X \cup Y)$. We divide the proof into two cases based on the size of U.

Case 1. $|U| \geqslant 2\sqrt{\mu}n$. Let $U_0 := \{u \in U : d_G(u, D) \leqslant (\frac{1}{2} - 3\sqrt{\mu})n\}$. If $|U_0| > \sqrt{\mu}n$, then since $d_G(v, X \cup Y) \geqslant 2\sqrt{\mu}n$ for each $v \in U_0$, we have $e_G(X, Y) \geqslant e_G(U, X \cup Y) \geqslant \frac{1}{2} \cdot 2\sqrt{\mu}n \cdot \sqrt{\mu}n = \mu n^2$, a contradiction. Thus we have $|U_0| \leqslant \sqrt{\mu}n$ and hence $|U \setminus U_0| \geqslant \sqrt{\mu}n$. In particular, $U \setminus U_0 \neq \emptyset$ and thus $|D| \geqslant (\frac{1}{2} - 3\sqrt{\mu})n$, so $|X \cup Y| \leqslant (\frac{1}{2} + 3\sqrt{\mu})n$. We have

$$|U \setminus U_0| = |X| + |Y| - |X \cup Y| - |U_0| \geqslant 2(\tfrac{1}{2} - \mu)n - (\tfrac{1}{2} + 3\sqrt{\mu})n - \sqrt{\mu}n \geqslant (\tfrac{1}{2} - 5\sqrt{\mu})n.$$

Each vertex in $U \setminus U_0$ has at least $(\frac{1}{2} - 3\sqrt{\mu})n$ neighbours in $V \setminus U$. Since $d_G(v, D) \geqslant (\frac{1}{2} - 3\sqrt{\mu})n$ for all $v \in U \setminus U_0$, we have $e_G(U \setminus U_0, D) \geqslant (\frac{1}{2} - 5\sqrt{\mu})(\frac{1}{2} - 3\sqrt{\mu})n^2$ and hence $D_0 := \{x \in D : d_G(x, U \setminus U_0) \leqslant (\frac{1}{2} - 3\sqrt{\mu})n\}$ has size $|D_0| \leqslant 6\sqrt{\mu}n$. So $|D \setminus D_0| \geqslant (\frac{1}{2} - 9\sqrt{\mu})n$.

Now choose any $A \subseteq U \setminus U_0$ and $B \subseteq D \setminus D_0$ with $|A| = |B| = (\frac{1}{2} - \varepsilon/2)n$, and let $C := V \setminus (A \cup B)$. Then A and B are disjoint, and we have $d_G(a, B) \geqslant d_G(a, D \setminus D_0) - (\varepsilon/2 + 5\sqrt{\mu})n \geqslant (\frac{1}{2} - \varepsilon)n$ for all $a \in A$ and similarly for $d_G(b, A)$ for all $b \in B$. Finally, $e_G(A) \leqslant e_G(U) \leqslant e_G(X, Y) < \mu n^2 < \varepsilon n^2/2$. Thus G is $(\varepsilon/2, \;)$-extremal.

Case 2. $|U| < 2\sqrt{\mu}n$. This case is very similar so we only sketch the proof. It is easy to see that for all but most $2\sqrt{\mu}n$ vertices v in X we have $d_G(v, Y) \leqslant \sqrt{\mu}n$. Similarly, for all but most $2\sqrt{\mu}n$ vertices v in Y we have $d_G(v, X) \leqslant \sqrt{\mu}n$. We delete these exceptional vertices from X and from Y. Let $X_1 := X \setminus Y$ and $Y_1 := Y \setminus X$. Note that these sets are disjoint and each has size at least $(\frac{1}{2} - 5\sqrt{\mu})n$. Now it follows that for each vertex $v \in X_1$, we have $d_G(v, X_1) \geqslant (\frac{1}{2} - 8\sqrt{\mu})n$ and for each vertex $v \in Y_1$, we have $d_G(v, Y_1) \geqslant (\frac{1}{2} - 8\sqrt{\mu})n$. We choose $A \subseteq X_1$ and $B \subseteq Y_1$ of the correct sizes, and note that $e_G(A, B) \leqslant e_G(X, Y) \leqslant \mu n^2$. Thus G is $(\varepsilon/2, \;)$-extremal. $\square$

3.2 Stable graph collections

We now move on to consider stability for graph collections. Lemma 15 implies that given a positive integer n and $0 < 1/n \ll \varepsilon \leqslant 1$ and a graph collection $\boldsymbol{G} = (G_1, \ldots, G_n)$ on a common vertex set V of size n, whenever G_i is ε-extremal, we can fix a

$$\text{characteristic partition } (A_i, B_i, C_i) \text{ of } G_i.$$

We say that a vertex $v \in V$ is *i-good* if either G_i is not ε-extremal (that is, G_i is ε^3-nice) or G_i is ε-extremal and $v \in A_i \cup B_i$. Extremality of a graph collection depends on where graphs are in relation to one another, hence we make the following definitions.

Definition 16 (crossing, cross graph)**.** Let $0 < 1/n, \varepsilon, \delta < 1$ where $n \in \mathbb{N}$ and let $\boldsymbol{G} = (G_1, \ldots, G_n)$ be a graph collection on a common vertex set V of size n. Given $i, j \in [n]$ such that G_i and G_j are both ε-extremal, we say that they are δ*-crossing* if $|A_i \triangle A_j| \geqslant \delta n$ and $|A_i \triangle B_j| \geqslant \delta n$. We define the *cross graph* $C_{\boldsymbol{G}}^{\varepsilon,\delta}$ to be the graph with vertex set $[n]$ where i is adjacent to j if and only if G_i, G_j are both ε-extremal and δ-crossing.

Observation 17. *Suppose that* $0 < \varepsilon \leqslant \delta/8$. *If* G_i *and* G_j *are* δ*-crossing, then* $|X_i \cap Y_j| \geqslant \delta n/4$ *whenever* $X, Y \in \{A, B\}$.

Proof. Since $|A_i \triangle B_j| \geqslant \delta n$ and $|A_i| = |B_j|$ we get $|A_i \setminus B_j| \geqslant \delta n/2$. Thus $|A_i \cap A_j| = |A_i \setminus (B_j \cup C_j)| \geqslant \delta n/2 - 2\varepsilon n \geqslant \delta n/4$. The other assertions hold by symmetry. ☐

We can now define stability for graph collections.

Definition 18 (strongly stable, weakly stable, stable, nice)**.** Let $n \in \mathbb{N}$ and $0 < \mu, \gamma, \alpha, \varepsilon, \delta < 1$ be parameters. Suppose that $\boldsymbol{G} = (G_1, \ldots, G_n)$ is a graph collection on a common vertex set V of size n. We say that

- $\boldsymbol{G}$ is (γ, α)*-strongly stable* if G_i is α-nice for at least γn colours $i \in [n]$;
- $\boldsymbol{G}$ is (ε, δ)*-weakly stable* if $e(C_{\boldsymbol{G}}^{\varepsilon,\delta}) \geqslant \delta n^2$;
- $\boldsymbol{G}$ is $(\gamma, \alpha, \varepsilon, \delta)$*-stable* if it is either (γ, α)-strongly stable or (ε, δ)-weakly stable (or both).
- $\boldsymbol{G}$ is μ*-nice* if for every $A \subseteq V$ of size $\lfloor n/2 \rfloor$ we have $e_{\boldsymbol{G}}(A) > \mu n^3$ and $e_{\boldsymbol{G}}(A, V \setminus A) > \mu n^3$.

When the parameters are clear from the text, we simply say $\boldsymbol{G}$ is *strongly stable*, *weakly stable*, *stable* or *nice*.

Note that whenever $\gamma' \geqslant \gamma > 0$ and $\alpha' \geqslant \alpha > 0$, a (γ', α')-strongly stable collection is (γ, α)-strongly stable, and whenever $0 < \varepsilon' \leqslant \varepsilon$ and $\delta' \geqslant \delta > 0$, an (ε', δ')-weakly stable collection is (ε, δ)-weakly stable.

Lemma 19. *Suppose that* $0 < 1/n \ll \mu \ll \gamma, \alpha, \varepsilon, \delta < 1$. *Let* $\boldsymbol{G} = (G_1, \ldots, G_n)$ *be a graph collection on a common vertex set* V *of size* n. *If* $\boldsymbol{G}$ *is* $(\gamma, \alpha, \varepsilon, \delta)$*-stable, then* $\boldsymbol{G}$ *is* μ*-nice.*

Proof. Suppose that there is $A \subseteq V$ of size $\lfloor n/2 \rfloor$ such that $e_{\boldsymbol{G}}(A) \leqslant \mu n^3$. Therefore, for all but at most $\sqrt{\mu} n$ colours $i \in [n]$, we have $e_{G_i}(A) \leqslant \sqrt{\mu} n^2$, and hence G_i is not $\sqrt{\mu}$-nice. This implies that $\boldsymbol{G}$ is not $(1 - \sqrt{\mu}, \sqrt{\mu})$-strongly stable and hence not (γ, α)-strongly stable. Without loss of generality, suppose that for each $i \in [(1 - \sqrt{\mu})n]$, G_i is ε-extremal with characteristic partition (A_i, B_i, C_i) (as we fixed at the beginning of the subsection). Let $B := V \setminus A$. It follows that, after possibly relabelling A_i and B_i, we have $|A \triangle A_i| \leqslant \delta^2 n$ and $|B \triangle B_i| \leqslant \delta^2 n$ for all but at most $\delta^2 n$ colours $i \in [(1 - \sqrt{\mu})n]$, otherwise we have at least $\delta^4 n^3/2 > \mu n^3$ edges of $\boldsymbol{G}$ inside A by $\mu \ll \delta$, a contradiction. Delete such colours from $[(1 - \sqrt{\mu})n]$ and let $\mathscr{C}$ be the remaining colour set. Thus for every $i, j \in \mathscr{C}$, the graphs G_i and G_j are not δ^2-crossing. Therefore $e(C_{\boldsymbol{G}}^{\varepsilon,\delta}) \leqslant \sqrt{\mu} n^2 + \delta^2 n^2 < \delta n^2$ by $\mu \ll \delta$. Thus $\boldsymbol{G}$ is not (ε, δ)-weakly stable, a contradiction.

The proof of the other case is similar and we omit it. ☐

The next key lemma guarantees that a nice graph collection in which all vertices have degree at least $(\frac{1}{2}-\mu)n$ in *almost all* colours contains a transversal copy of a graph H which *locally* looks like a Hamilton cycle: almost all degrees equal two. We choose H to be a union of two almost perfect matchings. It is vital that here, 'almost all' means a $1-\theta$ proportion where $\theta \ll \mu$. If we were allowed to take, say, $\theta = 3\mu$, then the very first part of the argument of Joos and Kim on the transversal minimum degree threshold for matchings [16] would give a very short proof even without assuming the graph collection is stable. Instead, we need to analyse the structure of maximal matchings in a collection of almost Dirac graphs.

This lemma will be applied in a reduced graph, which is why we only assume a weaker degree condition, and then the transversal blow-up lemma will be applied to obtain a rainbow collection of long paths covering almost the whole of V.

Lemma 20. *Let $0 < 1/n \ll d \ll \theta \ll \mu \ll \gamma, \alpha, \varepsilon, \delta \ll 1$. Suppose that $\boldsymbol{G} = (G_1, \dots, G_n)$ is a graph collection defined on a common vertex set V of size n and for each vertex $v \in V$, we have $d_{G_i}(v) \geqslant (\frac{1}{2}-\mu)n$ for all but at most dn colours $i \in [n]$. If $\boldsymbol{G}$ is $(\gamma, \alpha, \varepsilon, \delta)$-stable, then $\boldsymbol{G}$ contains two edge-disjoint rainbow matchings M_1 and M_2 in $\boldsymbol{G}$ such that $e(M_\ell) \geqslant (\frac{1}{2}-\theta)n$ for $\ell = 1, 2$ and $\mathrm{col}(M_1) \cap \mathrm{col}(M_2) = \emptyset$.*

The lemma is an easy consequence of the next lemma which guarantees an almost spanning rainbow matching in a nice graph collection containing about $n/2$ graphs.

Lemma 21. *Let $0 < 1/n \ll d \ll \theta \ll \mu \ll 1$. Suppose that $\boldsymbol{G} = (G_1, \dots, G_{(\frac{1}{2}-\theta/4)n})$ is a graph collection defined on a common vertex set V of size n and for each vertex $v \in V$, we have $d_{G_i}(v) \geqslant (\frac{1}{2}-\mu)n$ for all but at most dn colours $i \in [n]$. If $\boldsymbol{G}$ is 100μ-nice, then $\boldsymbol{G}$ contains a rainbow matching M with $e(M) \geqslant (\frac{1}{2}-\theta)n$.*

Proof of Lemma 20 given Lemma 21. Let $\mathscr{C}_1$ be a random set obtained by selecting each colour in $[n]$ randomly and independently with probability $\frac{1}{2}$, and let $\mathscr{C}_2 := [n] \setminus \mathscr{C}_1$.

Claim 22. *With high probability, for $\ell = 1, 2$ we have $|\mathscr{C}_\ell| = n/2 \pm n^{3/4}$ and the graph collection $\boldsymbol{G}^\ell := (G_i : i \in \mathscr{C}_\ell)$ is μ-nice.*

Proof of Claim. The first statement follows from Chernoff's inequality (Lemma 6).

Suppose first that $\boldsymbol{G}$ is (γ, α)-strongly stable. Then $\mathscr{A} := \{i \in [n] : G_i \text{ is } \alpha\text{-nice}\}$ satisfies $|\mathscr{A}| \geqslant \gamma n$. By Chernoff's inequality, with high probability, we have $|\mathscr{A} \cap \mathscr{C}_\ell| \geqslant \gamma n/3$ and thus $\boldsymbol{G}^\ell$ is $(\gamma/2, \alpha)$-strongly stable for $\ell = 1, 2$.

Suppose instead that $\boldsymbol{G}$ is (ε, δ)-weakly stable. Let $\mathscr{A} := \{i \in [n] : G_i \text{ is } \varepsilon\text{-extremal}\}$. We have $e(C_{\boldsymbol{G}}^{\varepsilon,\delta}) \geqslant \delta n^2$, so in particular, $|\mathscr{A}| \geqslant \delta n$. Let $H := C_{\boldsymbol{G}}^{\varepsilon,\delta}$ and $\mathscr{B} := \{i \in \mathscr{A} : d_H(i, \mathscr{A}) \geqslant \delta n\}$. It follows that $|\mathscr{B}| \geqslant \delta n$. Let $\mathcal{B} := \{N_H(v, \mathscr{A}) : v \in \mathscr{B}\}$. By Chernoff's inequality, with high probability, we have $|\mathscr{B} \cap \mathscr{C}_\ell| \geqslant \delta n/3$ and $|N \cap \mathscr{C}_\ell| \geqslant \delta n/3$ for each $N \in \mathcal{B}$ and $\ell = 1, 2$. Thus with high probability, we have $e(H[\mathscr{C}_\ell]) \geqslant \delta^2 n^2/18$ and thus $\boldsymbol{G}^\ell$ is $(\varepsilon, \delta^2/18)$-weakly stable for $\ell = 1, 2$.

In both cases, Lemma 19 implies that $\boldsymbol{G}^\ell$ is μ-nice for $\ell = 1, 2$. $\square$

Fix such $\mathscr{C}_1$ and $\mathscr{C}_2$. Apply Lemma 21 to $\boldsymbol{G}^1$ to obtain a rainbow matching M_1 with $e(M_1) \geqslant (\frac{1}{2}-\theta)n$ and colours from $\mathscr{C}_1$. Remove any $xy \in E(M_1)$ from G_i^2 for each $i \in \mathscr{C}_2$. Every vertex degree in every graph reduces by at most one, so $\boldsymbol{G}^2$ is still 2μ-nice. Apply Lemma 21 to $\boldsymbol{G}^2$ to obtain a rainbow matching M_2 with $e(M_2) \geqslant (\frac{1}{2}-\theta)n$ and colours from $\mathscr{C}_2$. $\square$

Proof of Lemma 21. Let M be a maximal rainbow matching in $\boldsymbol{G}$ and suppose for a contradiction that $e(M) < (\frac{1}{2}-\theta)n$. Let $v_1, v_2 \notin V(M)$ and $c_1, c_2 \notin \mathrm{col}(M)$ be distinct such that $N_\ell := N_{G_{c_\ell}}(v_\ell)$ satisfies $|N_\ell| \geqslant (\frac{1}{2}-\mu)n$ for $\ell = 1, 2$. Let $U := V \setminus (V(M) \cup \{v_1, v_2\})$ and $\mathscr{C} := [(\frac{1}{2}-\theta/4)n] \setminus (\mathrm{col}(M) \cup \{c_1, c_2\})$. Given $x \in V(M)$, we write x^+ to denote the unique neighbour of x in M and given $A \subseteq V(M)$, we write $A^+ := \{x^+ : x \in A\}$. Given $A \subseteq V$, we write $\overline{A} := V \setminus A$.

Claim 23. *We have $N_\ell \subseteq V(M)$ and $(\frac{1}{2}-\mu)n \leqslant |N_\ell| \leqslant (\frac{1}{2}+3\mu)n$ for $\ell = 1, 2$ and exactly one of the following holds:*

- *$|\overline{N_1} \triangle N_2| \leqslant 6\mu n$, and writing $M_\ell := M[N_\ell]$ and $X_\ell := V(M_\ell)$ for $\ell = 1, 2$, we have $e(M_\ell) \geqslant (\frac{1}{4}-2\mu)n$ and $X_1 \cap X_2 = \emptyset$ and $e(G_c[X_1, X_2]) = 0$ for all $c \in \mathscr{C}$;*
- *$|N_1 \triangle N_2| \leqslant 6\mu n$, and there is $X \subseteq \overline{N_1 \cup N_2} \cap V(M)$ with $X^+ \subseteq N_1 \cap N_2$ such that $|\overline{N_1 \cup N_2} \setminus X|, |(N_1 \cap N_2) \setminus X^+| \leqslant 6\mu n$ and $|X| \geqslant (\frac{1}{2}-3\mu)n$ and $e(M[X^+]) = 0$ and $e(G_c[X]) = 0$ for all $c \in \mathscr{C}$.*

Proof of Claim. If there is $y \notin V(M)$ with $v_\ell y \in G_{c_\ell}$ for some $\ell = 1, 2$ then we can add $v_\ell y$ to M to get a larger matching. Thus $N_\ell \subseteq V(M)$ for $\ell = 1, 2$. Moreover, we have $N_\ell^+ \cap N_{3-\ell} = \emptyset$ since otherwise we get a 3-path $v_{3-\ell}ww^+v_\ell$ with $\mathrm{col}(v_{3-\ell}w) = c_{3-\ell}$, $\mathrm{col}(v_\ell w^+) = c_\ell$ and $ww^+ \in E(M)$. Replacing ww^+ by $v_{3-\ell}w$ and w^+v_ℓ gives a rainbow matching that is larger than M, a contradiction. Let $S_\ell := E(M[N_\ell])$ and $T_\ell := E(M[N_\ell, V - N_\ell])$ and write $s_\ell := |S_\ell|$ and $t_\ell := |T_\ell|$. Note that, since $N_\ell^+ \cap N_{3-\ell} = \emptyset$, we have

$$S_1 \cap S_2 = S_1 \cap T_2 = T_1 \cap S_2 = \emptyset. \tag{1}$$

Since $N_\ell \subseteq V(M)$ we have

$$2s_\ell + t_\ell = |N_\ell| \geqslant (\tfrac{1}{2}-\mu)n. \tag{2}$$

Case 1. $t_1 + t_2 \leqslant 2\mu n$. We will show that the first alternative holds. We have $x \in N_2$ only if $x^+ \notin N_1$ so $|N_1 \cap N_2| \leqslant t_1 + t_2 \leqslant 2\mu n$, and $|\overline{N_1 \cup N_2}| \leqslant n - 2(s_1+s_2) \leqslant 2\mu n + t_1 + t_2 \leqslant 4\mu n$. So $|\overline{N_1} \triangle N_2| \leqslant 6\mu n$. Also $|N_2| \leqslant |\overline{N_1}| + 2\mu n \leqslant (\frac{1}{2}+3\mu)n$ so by symmetry we have $(\frac{1}{2}-\mu)n \leqslant |N_\ell| \leqslant (\frac{1}{2}+3\mu)n$ for $\ell = 1, 2$. Next, $e(M_\ell) = s_\ell \geqslant \frac{1}{2}(|N_\ell| - t_\ell) \geqslant (\frac{1}{4}-2\mu)n$ by (2). We have $X_\ell = V(S_\ell)$ and since S_ℓ is a matching, $S_1 \cap S_2 = \emptyset$ implies that $X_1 \cap X_2 = \emptyset$.

Finally, suppose $E(G_c[X_1, X_2]) \neq \emptyset$ for some $c \in \mathscr{C}$. Then there are $w_\ell w_\ell^+ \in E(M_\ell)$ for $\ell = 1, 2$ with $w_1^+ w_2 \in G_c$. Thus there is a rainbow path $v_1 w_1 w_1^+ w_2 w_2^+ v_2$ using c_1, c_2, c and two colours from M. Hence we can replace $w_1 w_1^+$ and $w_2 w_2^+$ by $v_1 w_1, w_1^+ w_2, v_2 w_2^+$ to obtain a larger rainbow matching, a contradiction.

Case 2. $t_1 + t_2 \geqslant 2\mu n$. We will show that the second alternative holds. Now (1) implies that

$$\begin{aligned}|T_1 \cap T_2| &= t_1 + t_2 - |T_1 \cup T_2| \geqslant t_1 + t_2 - (e(M) - s_1 - s_2)\\ &= \frac{t_1+t_2}{2} + \frac{2s_1+t_1}{2} + \frac{2s_2+t_2}{2} - e(M)\\ &\overset{(2)}{\geqslant} \mu n + (\tfrac{1}{2}-\mu)n - (\tfrac{1}{2}-\theta)n \geqslant \theta n.\end{aligned}$$

Since $N_\ell^+ \cap N_{3-\ell} = \emptyset$, we can choose an edge $v_3v_3^+ \in T_1 \cap T_2$ where $v_3^+ \in N_1 \cap N_2$ and $v_3 \in \overline{N_1 \cup N_2}$. Choose any colour $c_3 \in \mathscr{C}$ such that $d_{G_{c_3}}(v_3) \geqslant (\frac{1}{2}-\mu)n$. Let $N_3 := N_{G_{c_3}}(v_3)$.

Suppose $N_3 \cap N_\ell^+ \neq \emptyset$ for $\ell \in [2]$. Thus there exists $ww^+ \in E(M)$ such that $\mathrm{col}(v_\ell w^+) = c_\ell$, $\mathrm{col}(v_3w) = c_3$ and hence $v_\ell w^+ w v_3 v_3^+ v_{3-\ell}$ is a rainbow path with $v_\ell w^+, wv_3, v_3^+v_{3-\ell}$ coloured by $c_\ell, c_3, c_{3-\ell}$. By replacing $v_3v_3^+, ww^+$ in M by $v_3^+v_{3-\ell}, v_\ell w^+, wv_3$, we get a larger rainbow matching, a contradiction. Thus $N_3 \cap N_\ell^+ = \emptyset$. Also, $N_3 \subseteq V(M)$: if not, there is $w \notin V(M)$ such that we can replace $v_3v_3^+$ in M with $v_3w, v_3^+v_1$ using colours c_3, c_1. We have shown that $N_3, N_1^+ \cup N_2^+$ are pairwise disjoint subsets of $V(M)$. By definition, we have the partition

$$\begin{aligned}&N_1^+ \cup N_2^+ = V(S_1) \cup V(S_2) \cup Y \quad \text{where}\\ &Y := \{x^+ : x \in N_1, xx^+ \in T_1\} \cup \{x^+ : x \in N_2, xx^+ \in T_2\}.\end{aligned}$$

Whenever $x \in Y$ we have $x^+ \notin Y$. So $|Y| \geqslant |T_1 \cup T_2|$ and hence $|V(M)| \geqslant |N_3| + 2s_1 + 2s_2 + |T_1 \cup T_2|$. Therefore

$$|T_1 \cap T_2| = t_1 + t_2 - |T_1 \cup T_2| \geqslant \sum_{\ell=1,2}(2s_\ell + t_\ell) + |N_3| - |V(M)| \overset{(2)}{\geqslant} 3(\tfrac{1}{2}-\mu)n - n \geqslant (\tfrac{1}{2}-3\mu)n.$$

Let $X := \{x \in \overline{N_1 \cup N_2} : xx^+ \in T_1 \cap T_2\}$. So $X \subseteq \overline{N_1 \cup N_2}$ and $X^+ \subseteq N_1 \cap N_2$ are disjoint, so $|X| = |X^+| \geqslant |T_1 \cap T_2|$. Thus also $|\overline{N_1 \cup N_2} \setminus X|, |(N_1 \cap N_2) \setminus X^+| \leqslant n - |X| - |X^+| \leqslant 6\mu n$. Since $N_1 \cup N_2$ and X are disjoint, we have $|N_\ell| \leqslant |N_1 \cup N_2| \leqslant (\frac{1}{2}+3\mu)n$ for $\ell = 1,2$. We have $|N_1 \triangle N_2| = |N_1 \cup N_2| - |N_1 \cap N_2| \leqslant n - |X^+| - |X| \leqslant 6\mu n$. The definition of X implies that $E(M[X^+]) = \emptyset$.

Finally, suppose $e(G_c[X]) \neq 0$ for some $c \in \mathscr{C}$. Then as before we obtain a contradiction by adding such an edge xy along with v_1x^+, v_2y^+ (of colours c, c_1, c_2) and removing xx^+, y^+y. This completes the proof of the claim. $\square$

By Claim 23, we have $N_\ell \subseteq V(M)$ and $(\frac{1}{2}-\mu)n \leqslant |N_\ell| \leqslant (\frac{1}{2}+3\mu)n$ for $\ell = 1,2$ and one of the following cases:

Case 1: $|\overline{N_1} \triangle N_2| \leqslant 6\mu n$, and writing $M_\ell := M[N_\ell]$ and $X_\ell := V(M_\ell)$ for $\ell = 1,2$, we have $e(M_\ell) \geqslant (\frac{1}{4}-2\mu)n$ and $X_1 \cap X_2 = \emptyset$ and $e(G_c[X_1, X_2]) = 0$ for all $c \in \mathscr{C}$.

It suffices to show that for all colours $c \in \mathrm{col}(M_1 \cup M_2)$, we have $e_{G_c}(X_1, X_2) \leqslant 13\mu n^2$. Indeed, there are at least $2(\frac{1}{4}-2\mu)n = (\frac{1}{2}-4\mu)n$ such colours so $e_{\boldsymbol{G}}(X_1, X_2) \leqslant 13\mu n^3 +$

$4\mu n^3 = 17\mu n^3$. Also X_1, X_2 are disjoint sets of size at least $(\frac{1}{2} - 4\mu)n$, so there is a set Y of size $n/2$ such that $e_{\boldsymbol{G}}(Y, \overline{Y}) \leqslant 30\mu n^3$, and therefore $\boldsymbol{G}$ is not 30μ-nice.

Now let $c \in \mathrm{col}(M_1 \cup M_2)$. Without loss of generality, there is $ww^+ \in E(M_1)$ such that $\mathrm{col}(ww^+) = c$. Choose a colour $c_3 \in \mathscr{C}$ so that $N_3 := N_{G_{c_3}}(w)$ satisfies $|N_3| \geqslant (\frac{1}{2} - \mu)n$. Since $w \in X_1$, our assumption that $e(G_{c_3}[X_1, X_2]) = 0$ implies that $N_3 \cap X_2 = \emptyset$. Now let $M' := M - ww^+ + v_1w^+$ be the rainbow matching with $\mathrm{col}(M') = (\mathrm{col}(M) \setminus \{c\}) \cup \{c_1\}$ and consider new vertex pair (w, v_2) and colour pair (c_3, c_2). We have $\{c, c_2, c_3\} \cap \mathrm{col}(M') = \emptyset$. The matching M' is maximal, otherwise the maximality of M is contradicted. So Claim 23 applies to M'.

Now, $N_3 \cap X_2 = \emptyset$ so $|N_3 \cap N_2| \leqslant |N_2| - |X_2| \leqslant (\frac{1}{2} + 3\mu)n - (\frac{1}{2} - 4\mu)n = 7\mu n$ and in particular the second alternative cannot hold. Thus $|\overline{N_3} \triangle N_2| \leqslant 6\mu n$, and writing $M'_\ell := M'[N_\ell]$ and $X'_\ell := V(M'_\ell)$ for $\ell = 2, 3$, we have $e(M'_\ell) \geqslant (\frac{1}{4} - 2\mu)n$ and $X'_2 \cap X'_3 = \emptyset$ and $E(G_c[X'_2, X'_3]) = \emptyset$. We have $|N_1 \triangle N_3| \leqslant |N_1 \triangle \overline{N_2}| + |\overline{N_2} \triangle N_3| \leqslant 12\mu n$. Since M, M' differ by two edges we therefore have $|X_1 \triangle X'_3| + |X_2 \triangle X'_2| \leqslant 13\mu n$. Thus $e_{G_c}(X_1, X_2) \leqslant e_{G_c}(X'_3, X'_2) + 13\mu n^2 = 13\mu n^2$, as required.

Case 2: $|N_1 \triangle N_2| \leqslant 6\mu n$, and there is $X \subseteq \overline{N_1 \cup N_2} \cap V(M)$ with $X^+ \subseteq N_1 \cap N_2$ such that $|\overline{N_1 \cup N_2} \setminus X|, |(N_1 \cap N_2) \setminus X^+| \leqslant 6\mu n$ such that $|X| \geqslant (\frac{1}{2} - 3\mu)n$ and $e(M[X^+]) = 0$ and $e(G_c[X]) = 0$ for all $c \in \mathscr{C}$.

It suffices to show that $e_{G_c}(X) \leqslant 24\mu n^2$ for all $c \in \mathrm{col}(M[X, X^+])$. Indeed, there are $|X| \geqslant (\frac{1}{2} - 3\mu)n$ such colours, so $e_{\boldsymbol{G}}(X) \leqslant 24\mu n^3 + 3\mu n^3 = 27\mu n^3$. Since $(\frac{1}{2} - 3\mu)n \leqslant |X| \leqslant n - |X^+| \leqslant (\frac{1}{2} + 3\mu)n$, there is a set Y of size $n/2$ with $e_{\boldsymbol{G}}(Y) \leqslant 40\mu n^3$. Thus $\boldsymbol{G}$ is not 40μ-nice.

Let $c \in \mathrm{col}(M[X, X^+])$. Let $w \in X$ be such that $\mathrm{col}(ww^+) = c$. Choose a colour c_3 from $\mathscr{C}$ such that $N_3 := N_{G_{c_3}}(w)$ satisfies $|N_3| \geqslant (\frac{1}{2} - \mu)n$. Since $w \in X$, our assumption that $e(G_{c_3}[X]) = 0$ implies that $N_3 \cap X = \emptyset$. Now let $M' := M - ww^+ + v_1w^+$ be the rainbow matching with $\mathrm{col}(M') = (\mathrm{col}(M \setminus \{c\}) \cup \{c_1\}$ and consider new vertex pair (w, v_2) and colour pair (c_3, c_2). So $\{c, c_2, c_3\} \cap \mathrm{col}(M') = \emptyset$. The matching M' is maximal, otherwise the maximality of M is contradicted. So Claim 23 applies to M'. We have $N_3 \cup N_2 \subseteq \overline{X}$ so $|N_3 \cap N_2| \geqslant |N_3| + |N_2| - (n - |X|) \geqslant 2(\frac{1}{2} - \mu)n - n + |X| > (\frac{1}{2} - 5\mu)n$, so in particular the first alternative cannot hold.

So $|N_3 \triangle N_2| \leqslant 6\mu n$, and there is $X' \subseteq \overline{N_3 \cup N_2}$ with $|\overline{N_3 \cup N_2} \setminus X'| \leqslant 6\mu n$ such that $|X'| \geqslant (\frac{1}{2} - 3\mu)n$ and $e(M[(X')^+]) = 0$ and $e(G_c[X']) = 0$. Now,

$$\begin{aligned}
|X' \triangle X| &\leqslant |X' \triangle \overline{N_3 \cup N_2}| + |\overline{N_3 \cup N_2} \triangle \overline{N_1 \cup N_2}| + |\overline{N_1 \cup N_2} \triangle X| \\
&\leqslant 12\mu n + |N_3 \triangle N_1| \\
&\leqslant 12\mu n + |N_3 \triangle N_2| + |N_2 \triangle N_1| \leqslant 24\mu n.
\end{aligned}$$

Thus $e_{G_c}(X) \leqslant 24\mu n^2$, as required.

This completes the proof of Lemma 21. $\square$

4 Absorbing

Recall that $(V)_k$ denotes the set of all k-tuples of distinct elements of V. A pair $H = (V, E)$ where V is a set and $E \subseteq (V)_k$ is called a *directed k-graph*, while if we allow E to be a multiset, it is a *directed multi-k-graph*. The elements of E are *edges*. Let $\boldsymbol{H} = (H_1, \dots, H_t)$ be a collection of directed k-graphs on the same vertex set V. A *matching* M is a directed k-graph where no $v \in V$ appears in more than one k-tuple in E, and it is *rainbow* if it uses at most one edge from each H_i with $1 \leqslant i \leqslant t$. We use the same notation for directed k-graphs as for graphs. The following lemma is the key tool that we will use to build an absorbing structure. Similar ideas have already been used in the papers [8, 9, 11] of the first author.

Lemma 24. *Let $k, C, n \in \mathbb{N}$ suppose that $0 < 1/n \ll \gamma \ll \varepsilon \ll 1/k, 1/C \leqslant 1$ and let $m \in [n^C]$ and $t := \gamma n$. Let $\boldsymbol{H} = (H_1, \dots, H_t)$ be a collection of directed k-graphs and let $\boldsymbol{Z} = (Z_1, \dots, Z_m)$ be a collection of directed multi-k-graphs all defined on a common vertex set V of size n. Suppose that $e(H_i) \geqslant \varepsilon n^k$ for all $i \in [t]$ and for each $j \in [m]$, we have $|E(Z_j) \cap E(H_i)| \geqslant \varepsilon n^k$ for at least εt indices $i \in [t]$. Then there is a rainbow matching M in $\boldsymbol{H}$ of size at least $(1-\varepsilon^2/4)t$ and $|E(Z_j) \cap E(M)| \geqslant \varepsilon^2 t/4$ for each $j \in [m]$.*

Proof. Let $W = \{e_1, \dots, e_t\}$ be obtained by independently picking e_i from $E(H_i)$, $i \in [t]$, uniformly at random. For each $i \in [t]$ and $j \in [m]$, let $\chi_{ij} = \mathbb{1}(e_i \in Z_j)$. By definition, we have $e(H_i) \leqslant n^k$ for each $i \in [t]$ and thus $\mathbb{E}(\chi_{ij}) = \mathbb{P}(e_i \in Z_j) = |E(Z_j) \cap E(H_i)|/|E(H_i)| \geqslant \varepsilon n^k/n^k = \varepsilon$ whenever $|E(Z_j) \cap E(H_i)| \geqslant \varepsilon n^k$. Therefore $\mathbb{E}(|W \cap E(H_j)|) = \mathbb{E}(\sum_{i \in [t]} \chi_{ij}) = \sum_{i \in [t]} \mathbb{E}(\chi_{ij}) \geqslant \varepsilon^2 t$ for each $j \in [m]$. Chernoff's inequality and a union bound imply that with probability at least $1 - e^{-\sqrt{n}}$, we have $|W \cap E(H_j)| \geqslant \varepsilon^2 t/2$ for all $j \in [m]$.

Let Y be the number of intersecting pairs of edges in W. So $Y \leqslant \sum_{ii' \in \binom{[t]}{2}} Y_{ii'}$ where $Y_{ii'} = \mathbb{1}(V(e_i) \cap V(e_{i'}) \neq \emptyset)$. Let $I_{ii'}$ be the number of intersecting pairs of edges $\{a_i, a_{i'}\}$ with $a_h \in E(H_h)$ for $h = i, i'$. So $|I_{ii'}| \leqslant kn^{2k-1}$. Hence, for every $ii' \in \binom{[t]}{2}$, we have

$$\mathbb{E}(Y_{ii'}) = \frac{|I_{ii'}|}{e(H_i)e(H_{i'})} \leqslant \frac{kn^{2k-1}}{\varepsilon^2 n^{2k}} = \frac{k}{\varepsilon^2 n} \quad \text{and so}$$
$$\mathbb{E}(Y) \leqslant \sum_{ii' \in \binom{[t]}{2}} \mathbb{E}(Y_{ii'}) \leqslant \binom{t}{2} \frac{k}{\varepsilon^2 n} \leqslant \frac{k}{2\varepsilon^2}\gamma^2 n \leqslant \frac{\varepsilon^2 t}{8}.$$

Markov's inequality implies that $\mathbb{P}(Y \leqslant \varepsilon^2 t/4) \geqslant 1/2$. Thus a union bound implies that there exists W with $|W| = t$, $|W \cap E(Z_j)| \geqslant \varepsilon^2 t/2$ for all $j \in [m]$ and there are at most $\varepsilon^2 t/4$ pairs of intersecting edges in W. The required M is obtained by deleting one edge from each intersecting pair of W. $\square$

Let $\boldsymbol{G} = (G_1, \dots, G_n)$ be a graph collection on a common vertex set V of size n. For every pair x, y of distinct vertices in V, we define $L(xy) := \{i \in [n] : xy \in E(G_i)\}$ to be the set of colours appearing on xy.

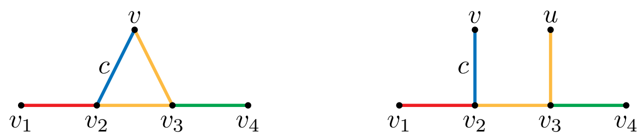

Figure 1: A c-absorbing path of (v, v) and a c-absorbing path of (v, u) for $v \neq u$.

Definition 25 (absorbing path, absorbing cycle)**.** Given any two not necessarily distinct vertices $u, v \in V$, a rainbow path $P = v_1v_2v_3v_4$ with $u, v \notin V(P)$ is called a *c-absorbing path of* (v, u) if $c \in L(v_2v)$ and $\mathrm{col}(v_2v_3) \in L(v_3u)$ (see Figure 1).

Given $\delta, \delta', \gamma, \gamma' > 0$, a rainbow cycle $C = v_1v_2 \ldots v_t$ is an *absorbing cycle* with parameters $(\delta, \delta', \gamma, \gamma')$ if $t \leqslant \gamma n$ and there exists a colour set $\mathscr{C}$ of size at least δn such that

(i) given any colour $c \in \mathscr{C}$ and any G_c-good vertex v, for all but at most $\delta' n$ vertices $u \in V$, there are at least $\gamma' n$ disjoint c-absorbing paths of (v, u) inside C;

(ii) given any colour $c \in \mathscr{C}$, for all but at most $\delta' n$ G_c-good vertices v, there are at least $\gamma' n$ disjoint c-absorbing paths of (v, v) inside C.

It is easy to see that if C is a rainbow cycle containing a c-absorbing path of (v, v) where $v \notin V(C)$ and $c \notin \mathrm{col}(C)$, then there exists a rainbow cycle C' such that $V(C') = V(C) \cup \{v\}$ and $\mathrm{col}(C') = \mathrm{col}(C) \cup \{c\}$ (insert v between v_2 and v_3). Similarly, if C is a rainbow cycle containing a c-absorbing path of (v, u) where u, v are the endpoints of a rainbow path P, $V(P) \cap V(C) = \emptyset$ and $\mathrm{col}(P), \{c\}, \mathrm{col}(C)$ are pairwise disjoint, then there exists a rainbow cycle C' such that $V(C') = V(C) \cup V(P)$ and $\mathrm{col}(C') = \mathrm{col}(C) \cup \mathrm{col}(P) \cup \{c\}$ (insert P between v_2 and v_3).

Using Lemma 24 it is quite straightforward to find an absorbing cycle when G is strongly stable, using the many nice graphs G_i guaranteed by strong stability.

Lemma 26. *Let $0 < 1/n \ll \lambda, \mu \ll \gamma, \alpha \ll 1$ and suppose that $\boldsymbol{G} = (G_1, \ldots, G_n)$ is a graph collection defined on a common vertex set V of size n with $\delta(\boldsymbol{G}) \geqslant (\frac{1}{2} - \mu)n$. If $\boldsymbol{G}$ is (γ, α)-strongly stable, then there exists an absorbing cycle with parameters $(1, 0, \lambda, \lambda^2)$.*

Proof. Without loss of generality, we assume that $G_1, \ldots, G_{\gamma n}$ are α-nice and $6|\lambda n$. We divide the colour set $[\lambda n/2]$ into consecutive sets $\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{\lambda n/2 - 2, \lambda n/2 - 1, \lambda n/2\}$ of three. For each $i \in I := [\frac{\lambda n}{6}]$, we define

$$\mathcal{F}_i := \{(v_1, v_2, v_3, v_4) \in (V)_4 : 3(i - 1) + \ell \in L(v_\ell v_{\ell+1}) \ \forall \ell \in [3]\}.$$

Our minimum degree condition implies that $e(\mathcal{F}_i) \geqslant n((\frac{1}{2} - \mu)n)^3 \geqslant n^4/9$. Now we fix a colour $c \in [n]$ and two not necessarily distinct vertices $v, v' \in V$ (which are not required to be G_c-good). We let $Z_i(c, vv')$ be the collection of $(v_1, v_2, v_3, v_4) \in \mathcal{F}_i$ for which $P = v_1v_2v_3v_4$ with the given colours is a c-absorbing path of (v, v'). Since the graph G_{3i+2} is α-nice, there are at least αn^2 ways to choose v_2v_3 with $v_2 \in N_{G_c}(v), v_3 \in N_{G_{3i+2}}(v')$ and $v_2v_3 \in E(G_{3i+2})$. When v_2, v_3 are fixed, there are at least $(\frac{1}{2} - \mu)^2 n^2$ ways to choose v_1 and v_4. Thus $|Z_i(c, vv')| \geqslant (\frac{1}{2} - \mu)^2 n^2 \alpha n^2 \geqslant \alpha n^4/5$.

We can apply Lemma 24 with $\boldsymbol{H} := (\mathcal{F}_i : i \in I)$, $\boldsymbol{Z} := (Z(c, vv') := \bigcup_{i \in I} Z_i(c, vv') : c \in [n], vv' \in V^2)$, where this is a multiset union, and parameters $|I| = t$ and $\alpha/5 = \varepsilon$, to obtain a rainbow matching M in $\boldsymbol{H}$ of size at least $(1 - \alpha^2/100)\lambda n/6$ such that $|E(Z(c, vv')) \cap E(M)| \geqslant \alpha^2 \lambda n/600$ for all $c \in [n]$ and $vv' \in V^2$. That is, there is $I' \subseteq I$ with $|I'| \geqslant (1-\alpha^2/100)\lambda n/6$ such that for each $i \in I'$ there is a rainbow path $P_i = v_1^i v_2^i v_3^i v_4^i$ with $3(i-1) + \ell \in L(v_\ell^i v_{\ell+1}^i)$ for all $\ell \in [3]$, and for every $c \in [n]$ and $vv' \in V^2$, there are at least $\alpha^2 \lambda n/600$ paths P_i which are c-absorbing of (v, v').

Relabel indices so that $I' = [s]$ where $s := (1 - \alpha^2/100)\lambda n/6$. We will connect the paths $P_1, \dots, P_s$ one by one into a rainbow cycle C. We first connect P_1 and P_2 into a single rainbow path $P_1 xy P_2$. For this, choose distinct unused colours $c_1, c_2 \in [\lambda n + 1, n]$ and $c_3 \in [\lambda n/2+1, \lambda n]$. Let $U := V \setminus V(\bigcup_{i \in [s]} V(P_i))$. Note that $d_{G_{c_1}}(v_4^1, U), d_{G_{c_2}}(v_1^2, U) \geqslant (\frac{1}{2} - \mu - \lambda)n \geqslant (\frac{1}{2} - \alpha)n$. Since $c_3 \in [\lambda n/2 + 1, \lambda n]$, the graph G_{c_3} is α-nice, so there are at least αn^2 c_3-coloured edges between $N_{G_{c_1}}(v_4^1, U)$ and $N_{G_{c_2}}(v_1^2, U)$. Thus we can choose a c_3-coloured edge xy between $N_{G_{c_1}}(v_4^1, U)$ and $N_{G_{c_2}}(v_1^2, U)$ such that $C = P_1 xy P_2$ is a rainbow path with colour set $\mathrm{col}(P_1) \cup \mathrm{col}(P_2) \cup \{c_1, c_2, c_3\}$.

We repeat the process above for each pair (P_i, P_{i+1}) for $i \in [s]$, where $P_{s+1} := P_1$. Each time we use two unused colours from $[\lambda n+1, n]$ and one unused colour from $[\lambda n/2+1, \lambda n]$. This is possible since $2s \leqslant \lambda n/3$. The number of used vertices at each step is at most $6s \leqslant \lambda n$ so we have the same bounds on degrees into the set of used vertices. By construction, C is an absorbing cycle with parameters $(1, 0, 6s/n, \alpha^2\lambda/600)$ and hence with parameters $(1, 0, \lambda, \lambda^2)$ since $\lambda \ll \alpha$. □

To complete this section, we find an absorbing cycle when $\boldsymbol{G}$ is weakly stable. This is much more involved than the strongly stable case. Again we use Lemma 24 to construct the cycle, using the many pairs of extremal graphs which are all highly crossing, as guaranteed by weak stability.

Lemma 27. *Let $0 < 1/n \ll \lambda, \mu, \varepsilon \ll \delta < 1$. Suppose that $\boldsymbol{G} = (G_1, \dots, G_n)$ is a graph collection defined on a common vertex set V where $|V| = n$ and $\delta(\boldsymbol{G}) \geqslant (\frac{1}{2} - \mu)n$. If $\boldsymbol{G}$ is (ε, δ)-weakly stable, then there exists an absorbing cycle C with parameters $(\delta/3, \sqrt{\varepsilon}, \lambda, \lambda^2)$.*

Proof. For each $j \leqslant n/3$, define the following family.

$$\mathcal{F}_j = \{(v_1, v_2, v_3, v_4) \in (V)_4 : 3(j-1) + \ell \in L(v_\ell v_{\ell+1}) \ \forall \ell \in [3]\}.$$

We first prove the following claim, which requires only the minimum degree condition on $\boldsymbol{G}$.

Claim 28. *Let $ij \in E(C_{\boldsymbol{G}}^{\varepsilon,\delta})$ where $j + 1 \leqslant n$ and let u be a G_j-good vertex and v be a G_i-good vertex. Let $Z_j(i, vu)$ be the collection of $(v_1, v_2, v_3, v_4) \in \mathcal{F}_{(j+1)/3}$ where $P = v_1 v_2 v_3 v_4$ is an i-absorbing path of (v, u). Then $|Z_j(i, vu)| \geqslant 2^{-7}\delta^2 n^4$.*

Proof of Claim. Fix any such i, j, u, v. By the definition of characteristic partition, we have $d_{G_i}(v, A_i) \geqslant (\frac{1}{2}-2\varepsilon)n$ or $d_{G_i}(v, B_i) \geqslant (\frac{1}{2}-2\varepsilon)n$. Without loss of generality, we assume that the former case holds since the latter case can be proved similarly. Since the graphs G_i and G_j are δ-crossing, we have $|A_i \cap A_j| \geqslant \delta n/4$ and $|A_i \cap B_j| \geqslant \delta n/4$ by Observation

17. Since $d_{G_i}(v, A_i) \geqslant (\frac{1}{2} - 2\varepsilon)n$, we have $d_{G_i}(v, A_j) \geqslant |A_i \cap A_j| - (|A_i| - d_{G_i}(v, A_i)) \geqslant \delta n/4 - \varepsilon n \geqslant \delta n/5$ by $\varepsilon \ll \delta$. Similarly, we have $d_{G_i}(v, B_j) \geqslant \delta n/5$. Since u is G_j-good, $u \in A_j \cup B_j$. We divide the proof into the following cases:

Case 1: $u \in A_j$ and G_j is $(\varepsilon, \substack{\bullet\\ \bullet})$-extremal. In this case, in the graph G_j, every vertex in A_j is adjacent to all but at most εn vertices in A_j. Thus $J := G_j[N_{G_i}(v, A_j)]$ is an almost complete graph of order at least $\delta n/5$ in the sense that each vertex in it is adjacent to all but at most εn vertices. Any choice of $x \in V(J)$, $y \in N_{G_j}(u) \cap N_J(x)$, $x' \in N_{G_{j-1}}(x)$ and $y' \in N_{G_{j+1}}(y)$ yields an i-absorbing path $x'xyy'$ of (v, u) in $\mathcal{F}_{(j+1)/3}$ (so its colours are, in order, $j-1, j, j+1$, and $i \in L(xv)$ and $j \in L(yu)$). The number of such paths is therefore at least $\delta n/5 \cdot (\delta n/5 - 2\varepsilon n) \cdot (\frac{1}{2} - \mu)^2 n^2 \geqslant 2^{-7}\delta^2 n^4$.

Case 2: $u \in A_j$ and G_j is $(\varepsilon, \substack{\circ\\ \circ})$-extremal. In this case, in the graph G_j, every vertex in A_j is adjacent to all but at most εn vertices in B_j and every vertex in B_j is adjacent to all but at most εn vertices in A_j. Thus $J := G_j[N_{G_i}(v, A_j), N_{G_j}(u, B_j)]$ is a bipartite graph with each part of order at least $\delta n/5$ which is almost complete in the sense that each vertex is adjacent to all but at most εn vertices from other side. Any choice of $x \in V(J) \cap A_j$, $y \in N_{G_j}(u) \cap N_J(x)$, $x' \in N_{G_{j-1}}(x)$ and $y' \in N_{G_{j+1}}(y)$ yields an i-absorbing path $x'xyy'$ of (v, u) in $\mathcal{F}_{(j+1)/3}$. The same lower bound from the previous case applies to the number of such choices.

The remaining cases when $u \in B_j$ are identical, so we omit them. $\square$

For any vertex v, recall Definition 16 and let

$$\mathscr{G}_v := \{i \in [n] : v \text{ is } G_i\text{-good}\}.$$

Since $e(C_{\boldsymbol{G}}^{\varepsilon,\delta}) \geqslant \delta n^2$, there is a subgraph H of $C_{\boldsymbol{G}}^{\varepsilon,\delta}$ such that $\delta(H) \geqslant \delta n$, so in particular, $|V(H)| > \delta n$. For each $i \in V(H)$, define the set $T_i := \{u \in V : |N_H(i) \backslash \mathscr{G}_u| \geqslant \delta n/2\}$. Note that since $|C_i| = 2\varepsilon n$ for each i where G_i is ε-extremal, we have $|T_i|\delta n/2 \leqslant \sum_{u \in V} |[n] \setminus \mathscr{G}_u| = \sum_{j \in [n]} |C_j| \leqslant 2\varepsilon n^2$ and thus $|T_i| \leqslant 4\varepsilon n/\delta \leqslant \sqrt{\varepsilon} n/2$ since $\varepsilon \ll \delta$. Let $\overline{T}_i := V \setminus T_i$. For each $i \in V(H)$ and $u \in \overline{T}_i$, we have $|\overline{T}_i| \geqslant (1 - \sqrt{\varepsilon}/2)n$ and $|\mathscr{G}_u \cap N_H(i)| \geqslant \delta n/2$.

Now we independently and randomly select vertices from $V(H)$ with probability $\kappa := \lambda/14$ to obtain a set $\mathscr{U}$ of colours. A Chernoff bound implies that, with high probability,

(i) $\kappa|V(H)|/2 \leqslant t := |\mathscr{U}| \leqslant 2\kappa|V(H)|$;

(ii) for every $i \in V(H)$ and $u \in \overline{T}_i$, we have $|\mathscr{G}_u \cap N_H(i, \mathscr{U})| \geqslant \delta\kappa n/4$.

Fix such a $\mathscr{U}$. By relabelling colours, we assume that $\mathscr{U} = \{3j - 1 : j \in [t]\}$. Let $\overline{\mathscr{U}} := V(H) \cap [3t + 1, n]$. So $\mathscr{U} \cap \overline{\mathscr{U}} = \emptyset$ and $|\overline{\mathscr{U}}| \geqslant \delta n/2$ and $\delta(H[\overline{\mathscr{U}}]) \geqslant \delta n/2$ since $\lambda \ll \delta$. For each colour $j \in [t]$, let I_j be the collection of $(i, vu) \in [n] \times V^2$ with $i \in \overline{\mathscr{U}}$, $u \in \overline{T}_i$, $j \in \mathscr{G}_u \cap N_H(i, \mathscr{U})$ and v a G_i-good vertex.

Given i and u, the number of choices of j is at least $\delta\kappa n/4$ by (ii). Since $i, j \in V(H)$ and u is G_j-good, Claim 28 implies that there are at least $2^{-7}\delta^2 n^4$ i-absorbing paths of (v, u) whose ordered vertex set is in $\mathcal{F}_{(j+1)/3}$. We can apply Lemma 24 with $\boldsymbol{H} := \{\mathcal{F}_j : j \in [t]\}$ and the directed multi-4-graph collection

$$\boldsymbol{Z} := (Z(i, vu) := \textstyle\bigcup_{j \in \mathscr{G}_u \cap N_H(i, \mathscr{U})} Z_j(i, vu) : i \in \overline{\mathscr{U}}, u \in \overline{T}_i, v \text{ is } G_i\text{-good})$$

and with $2^{-7}\delta^2$ playing the role of ε. Now, $e(\mathcal{F}_j) \geqslant n^4/9$ for all $j \in [t]$ and $\boldsymbol{H}$ contains $\kappa\delta n/2 \leqslant t \leqslant 2\kappa n$ graphs by (i), so $t/n \ll 2^{-7}\delta^2$. For each (i, vu) for which $Z(i, vu) \in \boldsymbol{Z}$ and every fixed $j \in \mathscr{G}_u \cap N_H(i, \mathscr{U})$, we have $(j+1)/3 \in [t]$ and $|E(Z(i, vu)) \cap E(\mathcal{F}_{(j+1)/3})| \geqslant 2^{-7}\delta^2 n^4$. The number of such j is at least $\delta\kappa n/4 \geqslant \delta t/8$ by (ii). Thus there is a rainbow matching M in $\boldsymbol{H}$ of size at least $(1 - 2^{-16}\delta^4)t$ such that $|E(Z(i, vu)) \cap E(M)| \geqslant 2^{-16}\delta^4 t$ for all $Z(i, vu) \in \boldsymbol{Z}$. That is, there is $I \subseteq [t]$ with $|I| \geqslant (1 - 2^{-16}\delta^4)t$ such that for each $j \in I$ there is a path $P_j = v_1^j v_2^j v_3^j v_4^j$ in $\mathcal{F}_{(j+1)/3}$ and for every $i \in \overline{\mathscr{U}}$, $u \in \overline{T}_i$ and v which is G_i-good, there are at least $2^{-16}\delta^4 t$ paths P_j which are i-absorbing of (v, u).

Relabel indices so that $I = [s]$ where $s := (1 - 2^{-16}\delta^4)t$. It remains to connect the $P_1, \ldots, P_s$ into an absorbing cycle C. Recall that no colour in $\overline{\mathscr{U}}$ appears on any P_i. We first connect P_1 and P_2 into a single rainbow path $P_1 v_4^1 xzy v_1^2 P_2$. For this, we first choose distinct colours $c_1, c_2, c_3, c_4 \in \overline{\mathscr{U}}$ where $c_1c_2 \in E(H)$. There are at least $|\overline{\mathscr{U}}| \geqslant \delta n/2$ choices for c_1, and given this at least $\delta(H[\overline{\mathscr{U}}]) \geqslant \delta n/2$ for c_2, and at least $|\overline{\mathscr{U}}| - 3 > \delta n/3$ choices for each of c_3, c_4. Next we choose unused vertices x, y such that $x \in N_{G_{c_3}}(v_4^1)$ is G_{c_1}-good and $y \in N_{G_{c_4}}(v_1^2)$ is G_{c_2}-good. There are at least $(\frac{1}{2} - \mu)n - 4e(M) - 2\varepsilon n$ choices for each of these. Finally, we choose $z \in N_{G_{c_1}}(x) \cap N_{G_{c_2}}(y)$. We claim that there are at least $\delta n/5$ choices for z.

To prove the claim, since $c_1c_2 \in E(H) \subseteq E(C_{\boldsymbol{G}}^{\varepsilon,\delta})$, both G_{c_1} and G_{c_2} are ε-extremal, so $x \in A_{c_1} \cup B_{c_1}$ and $y \in A_{c_2} \cup B_{c_2}$, and G_{c_1} and G_{c_2} are δ-crossing. By Observation 17, we have $|X_{c_1} \cap Y_{c_2}| \geqslant \delta n/4$ whenever $X, Y \in \{A, B\}$. By the definition of characteristic partition, there are $Z, W \in \{A, B\}$ such that $d_{G_{c_1}}(x, Z_{c_1}) \geqslant |Z_{c_1}| - \varepsilon n$ and $d_{G_{c_2}}(y, W_{c_2}) \geqslant |W_{c_2}| - \varepsilon n$. Thus $|N_{G_{c_1}}(x) \cap N_{G_{c_2}}(y)| \geqslant |N_{G_{c_1}}(x, Z_{c_1}) \cap N_{G_{c_2}}(y, W_{c_2})| \geqslant |Z_{c_1} \cap W_{c_2}| - 2\varepsilon n \geqslant \delta n/5$. This completes the proof of the claim.

Thus there are at least $\delta n/5$ choices for each of $c_1, c_2, c_3, c_4, x, y, z$ given any previous choices. So we can obtain the desired $P_1 xyz P_2$. We can further repeat the process for each pair (P_j, P_{j+1}) for $j \in [s]$ where $P_{s+1} := P_1$. Each time we use four unused colours in $\overline{\mathscr{U}}$ and three unused vertices, so for each choice of colour at most $4s \leqslant 4t \leqslant 8\kappa n < \delta n/10$ colours are forbidden and $3s < \delta n/10$ vertices are forbidden. Thus we obtain a cycle C of length $7s$ which contains every P_j with $j \in [s]$ as a segment. By construction, for every colour $i \in \overline{\mathscr{U}}$ and any G_i-good vertex v, for every vertex $u \in \overline{T}_i$, there are at least $2^{-16}\delta^4 t \geqslant 2^{-17}\kappa\delta^5 n$ disjoint i-absorbing paths of (v, u) inside C. Moreover, since at most $2\varepsilon n$ vertices are G_i-bad, the number of vertices in $\overline{T}_i$ which are G_i-good is at least $|\overline{T}_i| - 2\varepsilon n \geqslant (1 - \sqrt{\varepsilon}/2)n - 2\varepsilon n \geqslant (1 - \sqrt{\varepsilon})n$. Thus C is an absorbing cycle with parameters $(\delta/3, \sqrt{\varepsilon}, 14\kappa, 2^{-17}\kappa\delta^5)$ and hence with $(\delta/3, \sqrt{\varepsilon}, \lambda, \lambda^2)$. $\square$

5 The stable case

In this section we combine the results of the previous sections and use the regularity-blow-up method to find a transversal Hamilton cycle when $\boldsymbol{G}$ is stable. First, we show that the reduced graph inherits stability.

Lemma 29. *Suppose that $0 < 1/n \ll 1/L_0 \ll \varepsilon_0 \ll d \ll \mu, \alpha \ll \gamma, \varepsilon \ll \delta < 1$. Let $\boldsymbol{G} = (G_1, \ldots, G_n)$ be a graph collection on a common vertex set V of size n. Let*

$\boldsymbol{R} = \boldsymbol{R}(\varepsilon_0, 1, L_0)$ *be the reduced graph collection of* $\boldsymbol{G}$. *If* $\boldsymbol{G}$ *is* $(\gamma, \alpha, \varepsilon, \delta)$*-stable, then* $\boldsymbol{R}$ *is* $(\gamma/2, \alpha^2, \varepsilon, \delta/2)$*-stable.*

Proof. We may assume that $1/n \ll 1/n_0$ where $n_0 = n_0(\varepsilon, 1, L_0)$ is the constant from Lemma 10 (the regularity lemma for graph collections). Write $[L]$ for the common vertex set of $\boldsymbol{R}$ and $[M]$ for the set of colour clusters where $L_0 \leqslant L, M \leqslant n_0$. Thus there is a partition $V_0, \ldots, V_L$ of V and $\mathscr{C}_0, \ldots, \mathscr{C}_M$ of $[n]$ and a graph collection $\boldsymbol{G}'$ satisfying (i)–(v) of Lemma 10 and for $(\{h, i\}, j) \in \binom{[L]}{2} \times [M]$, we have $hi \in R_j$ whenever $\boldsymbol{G}'_{hi,j}$ is (ε_0, d)-regular, where $\boldsymbol{G}'_{hi,j} := (G'_c[V_h, V_i] : c \in \mathscr{C}_j)$. We have $Lm \leqslant n \leqslant Mm + \varepsilon_0 n$ and $Mm \leqslant n \leqslant Lm + \varepsilon_0 n$ so $|L - M| \leqslant \varepsilon_0 n/m \leqslant \varepsilon_0 L/(1 - \varepsilon_0)$. Thus we may assume that $M = L$ at the expense of assuming the slightly worse bound $|V_0| + |\mathscr{C}_0| \leqslant 3\varepsilon_0 n$. Given $X \subseteq V(R) = [L]$, we write $\widehat{X} := \bigcup_{j \in X} V_j$. So $|\widehat{X}| = m|X|$.

Claim 30. *Let* $\alpha' \gg d$. *Suppose that* $i \in [L]$ *and there are sets* $A, B \subseteq V(R) = [L]$ *of size at least* $(\frac{1}{2} - \alpha')L$ *such that* $e_{R_i}(A, B) \leqslant \alpha' L^2$. *Then, for all but at most* $2\sqrt{\alpha'}m$ *colours* $c \in \mathscr{C}_i$, *we have* $e_{G_c}(\widehat{A}, \widehat{B}) \leqslant \sqrt{\alpha'} n^2$.

Proof of Claim. Let t be the number of colours $c \in \mathscr{C}_i$ such that $e_{G_c}(\widehat{A}, \widehat{B}) > \sqrt{\widetilde{\alpha}} n^2$. Since $e(G_c) - e(G'_c) \leqslant (3d + \varepsilon_0)n^2$ for all $c \in [n]$, we have

$$\begin{aligned}\sqrt{\alpha'} n^2 t &\leqslant \sum_{c \in \mathscr{C}_i} e_{G_c}(\widehat{A}, \widehat{B}) \leqslant \sum_{c \in \mathscr{C}_i} (e_{G'_c}(\widehat{A}, \widehat{B}) + (3d + \varepsilon_0)n^2) \\ &\leqslant |\mathscr{C}_i| e_{R_i}(A, B) m^2 + |\mathscr{C}_i|(3d + \varepsilon_0)n^2 \leqslant \alpha' L^2 m^3 + 4dmn^2 \leqslant 2\alpha' mn^2.\end{aligned}$$

This implies $t \leqslant 2\sqrt{\alpha'} m$. ☐

Case 1: $\boldsymbol{G}$ is (γ, α)-strongly stable. We claim that $\boldsymbol{R}$ is $(\gamma/2, \alpha^2)$-strongly stable. Suppose for a contradiction that there exists a subset $I \subseteq [L]$ with $|I| \geqslant (1 - \gamma/2)L$ such that R_i is not α^2-nice for any $i \in I$. The claim applied with $\alpha' = \alpha^2$ implies that there are at least $(1 - 2\alpha)m$ colours $c \in \mathscr{C}_i$ such that G_i is not α-nice, since $|\widehat{A}|, |\widehat{B}| \geqslant (\frac{1}{2} - \alpha^2)Lm > (\frac{1}{2} - \alpha)n$. Thus the number of colours $c \in [n]$ for which G_c is not α-nice is at least $(1 - 2\alpha)m(1 - \gamma/2)L > (1 - \gamma)n$ since $\alpha \ll \gamma$. This contradicts the (γ, α)-strong stability of $\boldsymbol{G}$.

Case 2: $\boldsymbol{G}$ is (ε, δ)-weakly stable. Suppose that $\boldsymbol{R}$ is not $(\gamma/2, \alpha^2)$-strongly stable. It suffices to show that $\boldsymbol{R}$ is $(\varepsilon, \delta/2)$-weakly stable. There is $I \subseteq [L]$ with $|I| \geqslant (1 - \gamma/2)L$ such that for all $i \in I$, R_i is not α^2-nice. Thus R_i is $\alpha^{2/3}$-extremal, and hence ε-extremal. Lemma 12 implies that $d_{R_i}(j) \geqslant (\frac{1}{2} - 2\mu)L$ for all but at most $d^{1/4}L$ vertices $j \in [L]$. Lemma 15 now implies that R_i has a characteristic partition (A'_i, B'_i, C'_i). Furthermore, there are $Z, W \in \{A, B\}$ such that $e_{R_i}(Z'_i, W'_i) \leqslant \alpha^{2/3} L^2$ (and $Z \neq W$ if R_i is $(\alpha^{2/3}, \substack{\bullet\\\bullet})$-extremal, while $Z = W$ if R_i is $(\alpha^{2/3}, \substack{\circ\\\circ})$-extremal). We have $|\widehat{W'_i}| = |\widehat{Z'_i}| = |Z'_i|m = (\frac{1}{2} - \alpha^{2/3})Lm \geqslant (\frac{1}{2} - 2\alpha^{2/3})n$. By Claim 30 applied with (W'_i, Z'_i) and $\alpha' := \alpha^{1/3}$, there is $\mathscr{B}_i \subseteq \mathscr{C}_i$ with $|\mathscr{C}_i \setminus \mathscr{B}_i| \leqslant 2\alpha^{1/3}m$ such that for all $c \in \mathscr{B}_i$, we have $e_{G_c}(\widehat{W'_i}, \widehat{Z'_i}) \leqslant \alpha^{1/3} n^2$. It follows that every such G_c is $2\alpha^{1/3}$-extremal. Thus, recalling that (A_c, B_c, C_c) is the characteristic partition of G_c, there are $Z, Y, W \in \{A, B\}$ with $\{Z, Y\} = \{A, B\}$ such that $e_{G_c}(Z_c, W_c) \leqslant \alpha^{1/4}n^2$ and $e_{G_c}(Y_c, W_c) \geqslant (\frac{1}{4} - \alpha^{1/4})n^2$. But then we must have $|\widehat{Z_i'} \triangle Z_c|, |\widehat{Y_i'} \triangle Y_c| < \varepsilon n$ or $|\widehat{Z_i'} \triangle Y_c|, |\widehat{Y_i'} \triangle Z_c| < \varepsilon n$. So either $|\widehat{A_i'} \triangle A_c|, |\widehat{B_i'} \triangle B_c| < \varepsilon n$ or $|\widehat{A_i'} \triangle B_c|, |\widehat{B_i'} \triangle A_c| < \varepsilon n$. That is, for all $c \in \mathscr{B}_i$, the characteristic partition of G_c is almost the same as the union of clusters corresponding to the characteristic partition of R_i.

Suppose now that $c, c' \in [n]$ are such that G_c and $G_{c'}$ are ε-extremal and δ-crossing and $c \in \mathscr{B}_i$ and $c' \in \mathscr{B}_j$ for some $i, j \in [L]$. We must have $i \neq j$. Then, if $|\widehat{A_i'} \triangle A_c|, |\widehat{A_j'} \triangle A_{c'}| < \varepsilon n$, we have $|A_i' \triangle A_j'|m = |\widehat{A_i'} \triangle \widehat{A_j'}| \geqslant |A_c \triangle A_{c'}| - |\widehat{A_i'} \triangle A_c| - |\widehat{A_j'} \triangle A_{c'}| \geqslant \delta n - 2\varepsilon n \geqslant \delta n/2 \geqslant \delta mL/2$. The other cases are almost identical. Thus R_i and R_j are $\delta/2$-crossing. Since $\boldsymbol{G}$ is (ε, δ)-weakly stable, we have $e(C_{\boldsymbol{G}}^{\varepsilon,\delta}) \geqslant \delta n^2$. The number of pairs $i, j \in [L]$ with c, c' as above is therefore at least $\delta n^2/m^2 \geqslant \delta L^2/2$. Thus $e(C_{\boldsymbol{R}}^{\varepsilon,\delta/2}) \geqslant \delta L^2/2$, and hence $\boldsymbol{R}$ is $(\varepsilon, \delta/2)$-weakly stable. $\square$

Lemma 31. *Let $0 < 1/n \ll \mu \ll \alpha \ll \gamma, \varepsilon \ll \delta \ll 1$. Suppose that $\boldsymbol{G} = (G_1, \ldots, G_n)$ is a graph collection on a common vertex set V of size n and $\delta(\boldsymbol{G}) \geqslant (\frac{1}{2} - \mu)n$. If $\boldsymbol{G}$ is $(\gamma, \alpha, \varepsilon, \delta)$-stable, then $\boldsymbol{G}$ contains a transversal Hamilton cycle.*

Proof. Choose additional parameters $n_0, L_0, \varepsilon_0, d, \beta, \lambda$ where $n_0 = n_0(\varepsilon, 1, L_0)$ is obtained from Lemma 10 and so that

$$0 < 1/n \ll 1/n_0 \ll 1/L_0 \ll \varepsilon_0 \ll d \ll \mu \ll \beta \ll \lambda \ll \alpha \ll \gamma, \varepsilon \ll \delta \ll 1,$$

where the previous lemmas in this section hold with suitable parameters. By Lemmas 26 and 27, $\boldsymbol{G}$ has an absorbing cycle C with parameters $(\delta/3, \sqrt{\varepsilon}, \lambda, \lambda^2)$. For any colour c and two vertices $(x, y) \in V^2$, we say the triple (c, x, y) is *absorbable* if there are at least $\lambda^2 n$ disjoint c-absorbing paths of (x, y) inside C. Similarly, we say the pair (c, x) is *absorbable* if there are at least $\lambda^2 n$ disjoint c-absorbing paths of (x, x) inside C. Let $c, c' \in [n]$ be two colours and $x, y \in V$ be two vertices. We say (c, c', x, y) is *totally absorbable* if (c, x), (c', y) and (c, x, y) are all absorbable. By definition, C has length at most λn, and there exists a colour set $\mathscr{C} \subseteq [n]$ with $\mathscr{C} \cap \mathrm{col}(C) = \emptyset$ of size at least $\delta n/3$ such that

(i) given any colour $c \in \mathscr{C}$ and any G_c-good vertex v, the triple (c, v, u) is absorbable for all but at most $\sqrt{\varepsilon} n$ vertices u.

(ii) given any colour $c \in \mathscr{C}$, for all but at most $\sqrt{\varepsilon} n$ G_c-good vertices v, the pair (c, v) is absorbable.

Claim 32. *There is $(2 - 2^{-10})\beta n \leqslant r \leqslant 2\beta n$ and for each $i \in [r]$, disjoint vertex pairs $(v_i, v_i') \in (V)_2$ and disjoint colour pairs $(c_i, c_i') \in (\mathscr{C})_2$ such that the family $\mathcal{Q} := \{(c_i, c_i', v_i, v_i') : i \in [r]\}$ has the following properties:*

(a) (c_i, c_i', v_i, v_i') is totally absorbable for all $i \in [r]$;

(b) for every pair $(u_1, u_2) \in V^2$ and $c \in [n]$, there are at least $2^{-9}\beta n$ values $i \in [r]$ such that $c \in L(u_1 v_i)$ and $c_i' \in L(u_2 v_i')$.

Proof of Claim. For every $(b_1, b_2) \in (\mathscr{C})_2$, every $(u_1, u_2) \in V^2$ and $c \in \mathscr{C}$, let $S(b_1, b_2, u_1, u_2, c)$ be the multiset of pairs $(v_1, v_2) \in (V)_2$ such that $c \in L(u_1 v_1)$, $b_2 \in$

$L(u_2v_2)$ and (b_1, b_2, v_1, v_2) is totally absorbable. We will show that $|S(b_1, b_2, u_1, u_2, c)| \geqslant 2^{-4}n^2$.

For this, we first count the number of choices for v_1. Note that we have $d_{G_c}(u_1) \geqslant (\frac{1}{2} - \mu)n$ and the number of vertices v_1 such that v_1 is not G_{b_1}-good or (b_1, v_1) is not absorbable is at most $2\varepsilon n + \sqrt{\varepsilon}n$. Thus we have at least $n/4$ choices of v_1 such that $v_1 \in N_{G_c}(u_1)$, v_1 is G_{b_1}-good and (b_1, v_1) is absorbable. Now we fix v_1. Let $N_2 := \{x \in N_{G_{b_2}}(u_2) : x \text{ is } G_{b_2}\text{-good}\}$. So $|N_2| \geqslant (\frac{1}{2} - \mu)n - 2\varepsilon n$. By (i), for all but at most $\sqrt{\varepsilon}n$ vertices $v_2 \in V$, the pair (v_1, v_2) has at least $\lambda^2 n$ b_1-absorbing paths inside C. Thus, we can delete at most $\sqrt{\varepsilon}n + 1$ vertices from N_2 and obtain a set N_2' such that $|N_2'| \geqslant n/4$, and for each $v_2 \in N_2'$, $v_2 \neq v_1$ and (b_1, b_2, v_1, v_2) is totally absorbable. Therefore, the total number of choices for (v_1, v_2) is at least $2^{-4}n^2$.

Let $\{c_i, c_i' : i \in [2\beta n]\} \subseteq \mathscr{C}$ be a collection of distinct colours. Set

$$\boldsymbol{H} := (\mathcal{F}_i := \{(v_1, v_2) \in (V)_2 : (c_i, c_i', v_1, v_2) \text{ is totally absorbable}\} : i \in [2\beta n])$$

and define the collection $\boldsymbol{Z} := \{S(u_1, u_2, c) := \bigcup_{i \in [2\beta n]} S(c_i, c_i', u_1, u_2, c) : (u_1, u_2) \in V^2, c \in [n]\}$ of multi-2-graphs. For every $S = S(u_1, u_2, c) \in \boldsymbol{Z}$ and every index $i \in [2\beta n]$, we have $|E(S) \cap E(\mathcal{F}_i)| \geqslant 2^{-4}n^2$. Lemma 24 applied with $2\beta n, 2^{-4}$ playing the roles of t, ε, implies that there is a rainbow matching M in $\boldsymbol{H}$ of size at least $(2 - 2^{-10})\beta n$ (and at most $2\beta n$) and $|E(S) \cap E(M)| \geqslant 2^{-9}\beta n$ for all $S \in \boldsymbol{Z}$. $\square$

For the remainder of the proof, we will cover most of the unused vertices and colours by a small number of long paths. This step will use the regularity-blow-up method for transversals. Finally, we will use the absorbing cycle C and family $\mathcal{Q}$ to absorb the remaining colours and vertices and connect the long paths.

Let

$$V_{\text{abs}} := \bigcup_{i \in [r]} \{v_i, v_i'\}, \quad \mathscr{C}_{\text{abs}} := \bigcup_{i \in [r]} \{c_i, c_i'\}, \quad \mathscr{C}_{\text{rem}} := [n] \setminus (\text{col}(C) \cup \mathscr{C}_{\text{abs}}),$$
$$U := V \setminus (V(C) \cup V_{\text{abs}}) \quad \text{and} \quad \boldsymbol{J} = (J_i : i \in \mathscr{C}_{\text{rem}}) \text{ where } J_i := G_i[U].$$

So $|[n] \setminus \mathscr{C}_{\text{rem}}| = |V \setminus U| \leqslant \lambda n + 2\beta n \leqslant 2\lambda n$.

Since $\lambda \ll \gamma, \alpha, \varepsilon, \delta$, it is easy to see that since $\boldsymbol{G}$ is $(\gamma, \alpha, \varepsilon, \delta)$-stable, $\boldsymbol{J}$ is $(\gamma/2, \alpha/2, 2\varepsilon, \delta/2)$-stable, and $\delta(J_i) \geqslant (\frac{1}{2} - \mu - 2\lambda)n \geqslant (\frac{1}{2} - 3\lambda)n$ for $i \in \mathscr{C}_{\text{rem}}$. Apply Lemma 10 (the regularity lemma for graph collections) to $\boldsymbol{J}$ with parameters $(\varepsilon_0, 1, d, L_0)$. Let $\boldsymbol{R}$ be the reduced graph of $\boldsymbol{J}$. Write $[L]$ for the common vertex set of $\boldsymbol{R}$ where $L_0 \leqslant L \leqslant n_0$, and $[M]$ for the set of colour clusters. Thus there is a partition $V_0, \dots, V_L$ of V and $\mathscr{C}_0, \dots, \mathscr{C}_M$ of $[n]$ and a graph collection $\boldsymbol{J'}$ satisfying (i)–(v) of Lemma 10. Therefore, for $(\{h, i\}, j) \in \binom{[L]}{2} \times [M]$, we have $hi \in R_j$ whenever $\boldsymbol{J}'_{hi,j}$ is (ε_0, d)-regular, where $\boldsymbol{J}'_{hi,j} := (J_c'[V_h, V_i] : c \in \mathscr{C}_j)$. Recall that m is the order of each $|V_i|$ for $1 \leqslant i \leqslant L$ and each $|\mathscr{C}_j|$ for $1 \leqslant j \leqslant M$. We have $Lm \leqslant n \leqslant Mm + \varepsilon_0 n$ and $Mm \leqslant n \leqslant Lm + \varepsilon_0 n$ so $|L - M| \leqslant \varepsilon_0 n/m \leqslant \varepsilon_0 L/(1 - \varepsilon_0)$. Thus we may assume that $M = L$ at the expense of assuming the slightly worse bound $|V_0| + |\mathscr{C}_0| \leqslant 3\varepsilon_0 n$. Lemma 12 implies that for each vertex $i \in [L]$, there are at least $(1 - d^{1/4})L$ colours $j \in [L]$ such that $d_{R_j}(i) \geqslant (\frac{1}{2} - 4\lambda)L$.

By Lemma 29, $\boldsymbol{R}$ is $(\gamma/4, \alpha^2/4, 2\varepsilon, \delta/4)$-stable. Lemma 20 implies that there exist two edge-disjoint rainbow matchings M_1 and M_2 in $\boldsymbol{R}$ such that $e(M_\ell) \geqslant (\frac{1}{2} - \theta)L$ for $\ell = 1, 2$ and $\mathrm{col}(M_1) \cap \mathrm{col}(M_2) = \emptyset$.

Now for each vertex $i \in [L]$, let $V_i = V_i^1 \cup V_i^2$ be an arbitrary equipartition. Lemma 9(i) implies that whenever $hi \in E(R_j)$ where $j \in [L]$, we have that $\boldsymbol{J}^\ell_{hi,j} = (J'_c[V_h^\ell, V_i^\ell] : c \in \mathscr{C}_j)$ is $(2\varepsilon_0, d/2)$-regular for both $\ell = 1, 2$. By Lemma 8, for each $\ell = 1, 2$ and $hi \in E(M_\ell)$, we can remove at most $2\varepsilon_0 m$ vertices in V_h^ℓ and at most $2\varepsilon_0 m$ colours $c \in \mathscr{C}_j$, where j is the colour of hi in M_ℓ, so that the remaining graph collection is $(4\varepsilon_0, d/4)$-superregular.

Apply Theorem 13 (the transversal blow-up lemma) to obtain a rainbow path $P_{hi,j}$ with $2\min\{|V_h^\ell|, |V_i^\ell|\}$ vertices using colours from $\mathscr{C}_j$. (Note that we could have avoided using the blow-up lemma and instead used a tool for embedding an almost spanning structure inside a regular pair (as opposed to superregular), but it was convenient to follow the above approach.) The collection of $P_{hi,j}$ over $hi \in E(M_1) \cup E(M_2)$ (where hi has colour j in its matching) is vertex-disjoint, since the subcluster V_h^ℓ is used $d_{M_\ell}(h) \leqslant 1$ times, and rainbow since $M_1 \cup M_2$ is rainbow. The number of vertices not in any $P_{hi,j}$ is at most $4\varepsilon_0 mL + 3\varepsilon_0 n + 2(L - e(M_1) - e(M_2))m \leqslant 3\theta n$. Thus the number of colours not in any $P_{hi,j}$ is at most $3\theta n + e(M_1) + e(M_2) \leqslant 4\theta n$. We consider each vertex in U but not in any $P_{hi,j}$ to be a path (of length 0).

Now relabel so that the paths are $P_1, \dots, P_s$, so $s \leqslant e(M_1) + e(M_2) + 3\theta n \leqslant 3\theta n + L/2 \leqslant 4\theta n$, and let x_i, y_i be the startvertex and endvertex of P_i for each $i \in [s]$ (so $x_i = y_i$ if P_i has length 0). The paths $P_1, \dots, P_s$ are rainbow, pairwise vertex-disjoint, pairwise colour-disjoint and cover U. Thus the number of colours in $\mathscr{C}_{\mathrm{rem}}$ which are not used on any P_i is precisely s.

Do the following for each $i = 1, \dots, s$ in turn. Let a_i be an arbitrary unused colour, in $\mathscr{C}_{\mathrm{rem}}$. Choose an unused 4-tuple $Q_i = (c_{j_i}, c'_{j_i}, v_{j_i}, v'_{j_i}) \in \mathcal{Q}$ where $j_i \in [r]$, $a_i \in L(x_i v_{j_i})$ and $c'_{j_i} \in L(y_i v'_{j_i})$. This is possible since Claim 32 implies there are at least $2^{-9}\beta n$ choices for Q_i, of which at most $s \leqslant 4\theta n$ have been used. Now, $(c_{j_i}, c'_{j_i}, v_{j_i}, v'_{j_i})$ is totally absorbable, so there are at least $\lambda^2 n$ disjoint c_{j_i}-absorbing paths of (v_{j_i}, v'_{j_i}) inside C. So we can choose one of them, $S_i := x_1^i x_2^i x_3^i x_4^i$, whose vertices have not been previously chosen since $s/n \ll \lambda$, and whose colours are, in order, b_1^i, b_2^i, b_3^i.

At the end of this process, there remains $I \subseteq [r]$ such that the $(c_j, c'_j, v_j, v'_j) \in \mathcal{Q}$ with $j \in I$ are precisely the 4-tuples which were not chosen to be some Q_i. For each one, there are at least $\lambda^2 n$ disjoint c_j-absorbing paths of (v_j, v_j) and disjoint c'_j-absorbing paths of (v'_j, v'_j) inside C. So, since $\beta \ll \lambda$, we can choose one such path T_i, T'_i for each of (v_j, c_j) and (v'_j, c'_j), which are vertex-disjoint and whose vertices have not previously been chosen.

At the end of this process, we have a collection $\{S_i : i \in [s]\}, \{T_i : i \in I\}, \{T'_i : i \in I\}$ of vertex-disjoint paths in C where, for each $i \in [s]$, S_i is a c_{j_i}-absorbing path of (v_{j_i}, v'_{j_i}); for each $i \in I$, T_i is a c_i-absorbing path of (v_i, v_i) and T'_i is a c'_i-absorbing path of (v'_i, v'_i). For each $i \in [s]$, we replace S_i by $x_1^i x_2^i v_{j_i} x_i P_i y_i v'_{j_i} x_3^i x_4^i$ with colours b_1^i, c_{j_i}, a_i, followed by the colours inherited from P_i, followed by c'_{j_i}, b_2^i, b_3^i. That is, we have replaced S_i by a path with the same endpoints, vertices $V(S_i) \cup V(P_i) \cup \{v_{j_i}, v'_{j_i}\}$ and colours $\mathrm{col}(S_i) \cup \mathrm{col}(P_i) \cup \{c_{j_i}, c'_{j_i}, a_i\}$. For each $i \in I$, we replace $T_i = y_1^i y_2^i y_3^i y_4^i$ by $y_1^i y_2^i v_i y_3^i y_4^i$ where colours are inherited except $\mathrm{col}(y_2^i v_i) = c_i$ and $\mathrm{col}(v_i y_3^i) = \mathrm{col}(y_2^i y_3^i)$. We do a similar replacement of T_i', using new vertex v_i' and new colour c_i'. So we have replaced T_i by a path with the same endpoints, vertices $V(T_i) \cup \{v_i\}$ and colours $\mathrm{col}(T_i) \cup \{c_i\}$, and T_i' by a path with the same endpoints, vertices $V(T_i') \cup \{v_i'\}$ and colours $\mathrm{col}(T_i') \cup \{c_i'\}$. Thus we have obtained a cycle using vertices $V(C) \cup \bigcup_{i\in[s]} V(P_i) \cup \{v_i, v_i' : i \in [r]\} = [n]$ and colours $\mathrm{col}(C) \cup \bigcup_{i\in[s]} \mathrm{col}(P_i) \cup \{c_i, c_i' : i \in [r]\} = [n]$ where each colour is used at most once (and hence exactly once). That is, we have constructed a transversal Hamilton cycle. □

6 The extremal case

This section concerns the remaining case when $\boldsymbol{G}$ is not stable; so most graphs in the collection are close to containing one of the two extremal graphs $\substack{\bullet\\\bullet}$ or $\substack{\circ\\\circ}$, and moreover their characteristic partitions are similar. Throughout, we assume the following hypothesis:

(†) Suppose that
$$0 < 1/n \ll \mu \ll \varepsilon \ll \delta \ll \eta \ll 1.$$
Let $m = (1-3\delta)n$. Let $\boldsymbol{G} = (G_1, \dots, G_n)$ be a graph collection on a common vertex set V of size n and $\delta(\boldsymbol{G}) \geqslant (\frac{1}{2} - \mu)n$. Suppose that for each $i \in [m]$, G_i is ε-extremal with characteristic partition (A_i, B_i, C_i) and $|A_1 \triangle A_i|, |B_1 \triangle B_i| \leqslant \delta n$.

We also use the following notation:

$$\begin{aligned}
\mathscr{C}(H) &:= \{i \in [m] : G_i \text{ is } (\varepsilon, H)\text{-extremal}\},\\
\boldsymbol{G}(H) &:= (G_i : i \in \mathscr{C}(H)) \quad \text{for } H \in \{\substack{\bullet\\\bullet}, \substack{\circ\\\circ}\},\\
V_{\mathrm{bad}} &:= C_1 \cup V_{\mathrm{bad}}^A \cup V_{\mathrm{bad}}^B \quad \text{where}\\
&\qquad V_{\mathrm{bad}}^Z := \{x \in Z_1 : x \notin Z_i \text{ for at least } \sqrt{\delta} n \text{ colours } i \in [m]\} \quad \text{for all } Z \in \{A, B\},\\
\mathscr{C}_{\mathrm{bad}} &:= [m+1, n].
\end{aligned}$$

Now, $\sqrt{\delta} n |V_{\mathrm{bad}}^Z| \leqslant \sum_{i\in[m]} |Z_1 \setminus Z_i| \leqslant m\delta n$ for each $Z \in \{A, B\}$, so

$$|V_{\mathrm{bad}}| \leqslant 2\varepsilon n + 2\sqrt{\delta} m \leqslant 3\sqrt{\delta} n.$$

It is a consequence of Lemma 15 that $\mathscr{C}(\substack{\bullet\\\bullet}) \cap \mathscr{C}(\substack{\circ\\\circ}) = \emptyset$. The first result in this section is an application of the transversal blow-up lemma for embedding rainbow Hamilton paths inside very dense bipartite graph collections. We note that this application is more for convenience than necessity.

Lemma 33. *Suppose that (†) holds. Let $W, Z \in \{A, B\}$. Let $W^* \subseteq W_1 \setminus V_{\mathrm{bad}}$ and $Z^* \subseteq Z_1 \setminus V_{\mathrm{bad}}$ where $|W^*|, |Z^*| \geqslant \eta n$ and $W^* \cap Z^* = \emptyset$ and $|W^*| - |Z^*| =: t \in \{0, 1\}$. Let $T := Z$ if $t = 0$ and $T := W$ if $t = 1$. Let $\mathscr{C} \subseteq [n]$ satisfy $|\mathscr{C}| = |W^*| + |Z^*| - 1$, where $\mathscr{C} \subseteq \mathscr{C}(\substack{\bullet\\\bullet})$ if $W = Z$ and $\mathscr{C} \subseteq \mathscr{C}(\substack{\circ\\\circ})$ if $W \neq Z$. Let $W^- \subseteq W^*$ and $T^+ \subseteq T^*$ with $|W^-|, |T^+| \geqslant \eta n$. Then there is a rainbow Hamilton path in $\{G_i[W^*, Z^*] : i \in \mathscr{C}\}$ starting in W^- and ending in T^+.*

Proof. Choose a new constant ξ with $\eta \ll \xi \ll 1$. We will show that $\{G_i[W^*, Z^*] : i \in \mathscr{C}\}$ is $(\xi, 1-\xi)$-superregular.

Suppose that $W \neq Z$, so $\{W^*, Z^*\} = \{A^*, B^*\}$ and $\mathscr{C} \subseteq \mathscr{C}(8)$. For every $i \in \mathscr{C}$, we have $e_{G_i}(W_i, Z_i) \geqslant |W_i||Z_i| - \varepsilon n^2$ and hence $e_{G_i}(W^*, Z^*) > (1-\xi)|W^*||Z^*|$. Now let $x \in W^*$ and $\mathscr{C}' \subseteq \mathscr{C}$ with $|\mathscr{C}'| \geqslant \xi|\mathscr{C}|$ and $Z' \subseteq Z^*$ with $|Z'| \geqslant \xi|Z^*|$. Then $x \notin V_{\mathrm{bad}}$, so $x \in W_i$ for all but at most $\sqrt{\delta}n$ colours in $i \in [m]$. For each such i we have $d_{G_i}(x, Z_i) \geqslant (\frac{1}{2} - 2\varepsilon)n = |Z_i| - \varepsilon n$ by Lemma 15. Thus $d_{G_i}(x, Z') \geqslant |Z'| - \varepsilon n - |Z_1 \triangle Z_i| \geqslant |Z'| - 2\delta n$. Thus $\sum_{i \in \mathscr{C}'} d_{G_i}(x, Z') \geqslant (|\mathscr{C}'| - 2\sqrt{\delta}n)(|Z'| - 2\delta n) > (1-\xi)|\mathscr{C}'||Z'|$. An analogous statement holds for vertices in Z^*. Thus $\sum_{i \in \mathscr{C}} d_{G_i}(x, Z^*) \geqslant (1-\xi)|\mathscr{C}||Z^*|$ for all $x \in W^*$ and $\sum_{i \in \mathscr{C}} d_{G_i}(y, W^*) \geqslant (1-\xi)|\mathscr{C}||W^*|$ for all $y \in Z^*$ and $(1-\xi)|\mathscr{C}'||W'||Z'| \leqslant \sum_{i \in \mathscr{C}'} e_{G_i}(W', Z') \leqslant |\mathscr{C}'||W'||Z'|$. Thus $(G_i[W^*, Z^*] : i \in \mathscr{C})$ is $(\xi, 1-\xi)$-superregular.

Suppose instead that $W = Z$, so $\mathscr{C} \subseteq \mathscr{C}(\bullet\bullet)$. A very similar argument shows that, since every $G_i[Z^*]$ and hence $G_i[W^*, Z^*]$ is almost complete, the same conclusion holds here.

Let $s := |W^*| + |Z^*|$. Since $|W^*| - |Z^*| = t \in \{0, 1\}$, there is an s-vertex path P_s which is bipartite with parts A_W, A_Z of size $|W^*|, |Z^*|$ respectively, with $|\mathscr{C}| = |W^*| + |Z^*| - 1$ edges and whose first vertex lies in A_W and whose last lies in A_T. Thus we can apply Theorem 13 (the transversal blow-up lemma) with target sets W^-, T^+ for the first and last vertex of P_s respectively to obtain the required rainbow Hamilton path. $\square$

The next lemma shows that whenever there are many $(\varepsilon, 8)$-extremal graphs, we can find a short rainbow path which covers bad vertices and colours.

Lemma 34. *Suppose that (†) holds, and $|\mathscr{C}(8)| \geqslant \eta n$. Given any $\mathscr{F} \subseteq [n]$ with $|\mathscr{F}| \leqslant 1$, there is a rainbow path P in $\boldsymbol{G}$ with endpoints x, y such that the following holds:*

(i) $V_{\mathrm{bad}} \subseteq V(P)$ *and* $\mathscr{C}_{\mathrm{bad}} \setminus \mathscr{F} \subseteq \mathrm{col}(P)$*;*

(ii) $|V(P)| \leqslant 19\sqrt{\delta}n$*;*

(iii) $x, y \notin V_{\mathrm{bad}}$ *and there are distinct* $c, c' \in \mathscr{C}(8) \setminus \mathrm{col}(P)$ *such that* $x \in A_c \cap A_1$ *and* $y \in B_{c'} \cap B_1$*;*

(iv) $\mathscr{F} \cap (\mathrm{col}(P) \cup \{c, c'\}) = \emptyset$.

Proof. Since $|V_{\mathrm{bad}}| \leqslant 3\sqrt{\delta}n$ and $|\mathscr{C}_{\mathrm{bad}}| \leqslant 3\delta n$, by adding vertices to V_{bad} if necessary we may assume that $|\mathscr{C}_{\mathrm{bad}}| < r := |V_{\mathrm{bad}}| \leqslant 3\sqrt{\delta}n$. Let $A := A_1 \setminus V_{\mathrm{bad}}$ and $B := B_1 \setminus V_{\mathrm{bad}}$.

We will find a rainbow family $\mathcal{P} = \{y_i x_i y'_i : i \in [r]\}$ of vertex-disjoint 3-vertex paths, with colour set $\{c_i, c'_i : i \in [r]\} \supseteq \mathscr{C}_{\mathrm{bad}}$ where $V_{\mathrm{bad}} = \{x_1, \ldots, x_r\}$ and $\{y'_i, y'_i : i \in [r]\} \subseteq A \cup B$. For this, let $x \in V_{\mathrm{bad}}$ and let $c, c' \in [m]$ be distinct colours. Since $d_{G_c}(x) \geqslant (\frac{1}{2} - \mu)n$, there are $Z, Z' \in \{A, B\}$ such that $|N_{G_c}(x) \cap Z| \geqslant n/8$ and $|N_{G_{c'}}(x) \cap Z'| \geqslant n/8$. Choose $y \in N_{G_c}(x) \cap Z$ and $y' \in N_{G_{c'}}(x) \cap Z'$. This gives the path yxy' using the given colours c, c'. We find such paths for every $x \in V_{\mathrm{bad}}$, each time using unused vertices, which is possible since there are at least $n/8$ choices for each vertex and only $2r \leqslant 6\sqrt{\delta}n$ are used in total, and using all of the $3\delta n$ colours of $\mathscr{C}_{\mathrm{bad}}$ among the $2r$ colours used in total.

Next, we connect the paths of $\mathcal{P}$ into a single short rainbow path P. We start with two arbitrary paths P_1, P_2 in $\mathcal{P}$, with endpoints y_1, y'_1 and y_2, y'_2 respectively. Note that each of these endpoints is not in V_{bad} and is therefore G_i-good for at least $|\mathscr{C}(8)| - \sqrt{\delta}n - 2r \geqslant \eta n/2$

unused colours $i \in \mathscr{C}(\circ\circ)$. Thus we can choose distinct unused colours $j_1, j_2 \in \mathscr{C}(\circ\circ)$ such that y_1' is G_{j_1}-good and y_2 is G_{j_2}-good. Without loss of generality, we have the following two cases:

Case 1: $y_1' \in A_{G_{j_1}}$ and $y_2 \in A_{G_{j_2}}$. Choose a common neighbour $y \in N_{G_{j_1}}(y_1') \cap N_{G_{j_2}}(y_2) \cap B$ that avoids P_1, P_2. There are at least $n/3$ choices for y since y_1' is missing at most εn neighbours in B_{j_1} in the graph G_{j_1} and $|B_{j_1} \setminus B| \leqslant |V_{\text{bad}}| + |B_{j_1} \triangle B_1| \leqslant 4\sqrt{\delta} n$, and similarly for y_2 and G_{j_2}. Let $P_{12} := P_1 y_1' y y_2 P_2$ with colour set $\mathrm{col}(P) = \mathrm{col}(P_1) \cup \{j_1, j_2\} \cup \mathrm{col}(P_2)$.

Case 2: $y_1' \in A_{G_{j_1}}$ and $y_2 \in B_{G_{j_2}}$. Choose an unused colour $j_3 \in \mathscr{C}(\circ\circ)$, for which there are at least $\eta n/2$ choices. Choose a vertex $y \in N_{G_{j_1}}(y_1') \cap B_{G_{j_3}}$ that avoids P_1, P_2; there are at least $n/3$ choices. Choose a common neighbour $z \in N_{G_{j_3}}(y) \cap N_{G_{j_2}}(y_2) \cap A$ that is distinct from y_1' and avoids P_1, P_2; there are at least $n/3$ choices. Let $P_{12} := P_1 y_1' y z y_2 P_2$ with colour set $\mathrm{col}(P) = \mathrm{col}(P_1) \cup \{j_1, j_2, j_3\} \cup \mathrm{col}(P_2)$.

By considering P_{12}, P_3 and so on, we can find a rainbow path $P_{12\ldots r}$ that includes every path in $\mathcal{P}$. For this, we need to argue that we can always choose unused vertices and colours; this is indeed true since in the first step there are at least $n/3$ choices for any vertex and at least $\eta n/2$ choices for any colour, and at most $3r < \eta n/4$ vertices and colours are used in total. We have $|V(P_{12\ldots r})| \leqslant 6r$.

Using very similar arguments, we can extend the path by at most one vertex and colour in $\mathscr{C}(\circ\circ)$ so that its endpoints are in A and B. Since they do not lie in V_{bad}, there are many choices of unused colour for c, c' to satisfy (iii). The final path has length at most $6r + 1 \leqslant 19\sqrt{\delta} n$. To achieve (iv), we simply remove $\mathscr{F}$ from $\mathscr{C}_{\text{bad}}$ and choose not to use $\mathscr{F}$ at any step, which does not affect any of the above estimates. $\square$

The next lemma shows that some additional assumptions guarantee a short rainbow path that not only covers bad vertices and colours, but also balances A_1 and B_1. These assumptions are that almost every graph is $(\varepsilon, \circ\circ)$-extremal, and moreover both parts A_1, B_1 contain many internal edges from these graphs.

Lemma 35. *Suppose that* (†) *holds, and*

$$|\mathscr{C}(\bullet\bullet)| \leqslant \eta n \quad \textit{and} \quad e_{\boldsymbol{G}(\circ\circ)}(A_1) \geqslant 30\eta n^3 \quad \textit{and} \quad e_{\boldsymbol{G}(\circ\circ)}(B_1) \geqslant 30\eta n^3.$$

Then $\boldsymbol{G}$ contains a rainbow path P with endpoints x, y such that the following holds:

(i) $V_{\text{bad}} \subseteq V(P)$ *and* $\mathscr{C}_{\text{bad}} \cup \mathscr{C}(\bullet\bullet) \subseteq \mathrm{col}(P)$;

(ii) $|V(P)| \leqslant 4\eta n$;

(iii) $|A_1 \setminus V(P)| = |B_1 \setminus V(P)|$;

(iv) $x, y \notin V_{\text{bad}}$ *and there are distinct* $c, c' \in \mathscr{C}(\circ\circ) \setminus \mathrm{col}(P)$ *such that* $x \in A_c \cap A_1$ *and* $y \in B_{c'} \cap B_1$.

Proof. We have that $|\mathscr{C}(\circ\circ)| = m - |\mathscr{C}(\bullet\bullet)| \geqslant (1-2\eta)n$. Thus Lemma 34 applied with $\mathscr{F} := \emptyset$ implies that $\boldsymbol{G}$ contains a rainbow path $\widetilde{P}$ with endpoints $\widetilde{x}, \widetilde{y}$ such that $V_{\text{bad}} \subseteq V(\widetilde{P})$, $\mathscr{C}_{\text{bad}} \subseteq \mathrm{col}(\widetilde{P})$, $|V(\widetilde{P})| \leqslant 19\sqrt{\delta} n$, and $\widetilde{x}, \widetilde{y} \notin V_{\text{bad}}$ and $\widetilde{x} \in A_1$ and $\widetilde{y} \in B_1 \cap B_c$ for some $c \in \mathscr{C}(\circ\circ) \setminus \mathrm{col}(\widetilde{P})$.

Without loss of generality, suppose that $|A_1 \setminus V(\widetilde{P})| - |B_1 \setminus V(\widetilde{P})| := \widetilde{t} \geqslant 0$. Next we greedily extend $\widetilde{P}$ to a path whose colour set contains $\mathscr{C}(\substack{\bullet\\\bullet})$. Write $\{a_1, \ldots, a_r\}$ for the collection of unused colours in $\mathscr{C}(\substack{\bullet\\\bullet})$ and let a_0, a_{r+1} be distinct unused colours in $\mathscr{C}(\substack{\circ\\\circ})$, for which $\widetilde{x} \in A_{a_0}$, of which there are at least $\eta n/2$ choices because $\widetilde{x} \notin V_{\text{bad}}$. Choose an unused vertex $y_0 \in N_{G_{a_0}}(\widetilde{x}, B_1 \cap B_{a_1})$. There are at least $n/3$ choices since $|B_{a_0} \triangle (B_1 \cap B_{a_1})| \leqslant 2\delta n$ by (†) and $d_{G_{a_0}}(\widetilde{x}, B_{a_0}) \geqslant (\frac{1}{2} - 2\varepsilon)n = |B_{a_0}| - \varepsilon n$ because $a_0 \in \mathscr{C}(\substack{\circ\\\circ})$ and $\widetilde{x} \in A_{a_0}$. Now, given $i \in [r]$ and $y_{i-1} \in B_{a_i}$, we can choose an unused vertex $y_i \in N_{G_{a_i}}(y, B_1 \cap B_{a_{i+1}})$; again there are at least $n/3$ choices since $\{a_i, a_{i+1}\} \in [m]$ and (†) implies $|B_{a_i} \triangle (B_1 \cap B_{a_{i+1}})| \leqslant 2\delta n$, and $a_i \in \mathscr{C}(\substack{\bullet\\\bullet})$ implies $d_{G_{a_i}}(y, B_{a_i}) \geqslant (\frac{1}{2} - 2\varepsilon)n = |B_{a_i}| - \varepsilon n$. Thus we can obtain the rainbow path $\widetilde{x}y_0y_1 \ldots y_r$ using colours $a_0, a_1, \ldots, a_r$, and with $y_r \in B_1 \cap B_{a_{r+1}}$.

Let $P_1 = \widetilde{y}\widetilde{P}\widetilde{x}y_0y_1y_1 \ldots y_r$ be the final rainbow path obtained by concatenation with $\widetilde{P}$ and let $A := A_1 \setminus V(P_1)$ and $B := B_1 \setminus V(P_1)$. It satisfies (i) and has $|V(P_1)| \leqslant |V(\widetilde{P})| + |\mathscr{C}(\substack{\bullet\\\bullet})| + 1 \leqslant 2\eta n$, and also $||A| - |B|| \leqslant 2\eta n$. It has endpoints $\widetilde{y} \in B_1 \cap B_c$ and $\widetilde{z} := y_r \in B_b$ where $b := a_{r+1} \in \mathscr{C}(\substack{\circ\\\circ})$. Also, $|A_1 \setminus V(P_1)| - |B_1 \setminus V(P_1)| := t \geqslant \widetilde{t} + 1 \geqslant 1$. After removing used colours, we have $\sum_{i \in \mathscr{C}(\substack{\circ\\\circ})} e_{G_i}(A) \geqslant 20\eta n^3$ and $\sum_{i \in \mathscr{C}(\substack{\circ\\\circ})} e_{G_i}(B) \geqslant 20\eta n^3$ since we consumed at most $2\eta n$ vertices from $A_1 \cup B_1$ and at most $2\eta n$ colours from $\mathscr{C}(\substack{\circ\\\circ})$ in building P_1. For each vertex pair $uv \in \binom{V}{2}$, define

$$c(uv) := \{i \in \mathscr{C}(\substack{\circ\\\circ}) : uv \in E(G_i)\} \quad \text{and} \quad D := \{uv : u, v \in A \text{ and } c(uv) \geqslant 10\eta n\}.$$

Then we have $20\eta n^3 \leqslant |D|n + n^2 \cdot 10\eta n$ and thus $|D| \geqslant 10\eta n^2$. It follows that D contains a subgraph D' with minimum degree at least $10\eta n$.

Let $z \in N_{G_b}(\widetilde{z}, V(D'))$ be an unused vertex. Such a z exists since $b \in \mathscr{C}(\substack{\circ\\\circ})$ and $\widetilde{z} \in B_b$ imply that $d_{G_b}(\widetilde{z}, A_b) \geqslant (\frac{1}{2} - 2\varepsilon)n$ and $|V(D') \cap A_b| \geqslant |V(D')| - |A \triangle A_b| \geqslant 10\eta n - 2\eta n - \delta n \geqslant 7\eta n$ which is larger than the $2\eta n$ vertices used so far. Greedily construct a path of unused vertices inside D' starting at z, ending at some $w \in B_{c'} \cap B_1$ where $c' \in \mathscr{C}(\substack{\circ\\\circ})$ is unused, and consisting of t vertices. Greedily assign unused colours from the lists guaranteed by the definition of D. The path P obtained by concatenating this rainbow path with P_1 has endpoints $\widetilde{y}, w$ and has all of the required properties. $\square$

The final lemma of this section combines the previous ones to find a transversal Hamilton cycle in three cases. The proof proceeds by using Lemma 34 or 35 to find a short rainbow path covering bad vertices and colours, and then covering the remaining vertices with three long paths guaranteed by Lemma 33.

Lemma 36. *Suppose that (†) holds, along with one of the following:*

$$\begin{aligned} \textit{either (i)} \quad & |\mathscr{C}(\substack{\bullet\\\bullet})| < \eta n \quad \textit{and} \quad \min\left\{e_{\boldsymbol{G}(\substack{\circ\\\circ})}(A_1), e_{\boldsymbol{G}(\substack{\circ\\\circ})}(B_1)\right\} \geqslant 30\eta n^3; \\ \textit{or (ii)} \quad & |\mathscr{C}(\substack{\bullet\\\bullet})| \geqslant \eta n \quad \textit{and} \quad \max\left\{e_{\boldsymbol{G}(\substack{\circ\\\circ})}(A_1), e_{\boldsymbol{G}(\substack{\circ\\\circ})}(B_1)\right\} \geqslant 30\eta n^3; \\ \textit{or (iii)} \quad & |\mathscr{C}(\substack{\bullet\\\bullet})| \geqslant \eta n \quad \textit{and} \quad |\mathscr{C}(\substack{\circ\\\circ})| \geqslant \eta n \quad \textit{and} \quad G_n \cong K_n. \end{aligned}$$

Then $\boldsymbol{G}$ contains a transversal Hamilton cycle.

Proof. Suppose that (i) holds. By Lemma 35, there is a rainbow path P with endpoints $x \in A_c$ and $y \in B_{c'}$ where $c, c' \in \mathscr{C}(\substack{\circ\\\circ}) \setminus \mathrm{col}(P)$ are distinct, $\mathscr{A} := [n] \setminus (\mathrm{col}(P) \cup \{c, c'\}) \subseteq \mathscr{C}(\substack{\circ\\\circ})$ and $|A| = |B| \geqslant (\frac{1}{2} - 4\eta)n$ where $Z := Z_1 \setminus V(P)$ for $Z \in \{A, B\}$. So

$$|\mathscr{A}| = n - |\mathrm{col}(P)| - 2 = |A| + |B| + |V(P)| - |\mathrm{col}(P)| - 2 = |A| + |B| - 1.$$

Let $A^- := N_{G_{c'}}(y, A)$ and $B^+ := N_{G_c}(x, B)$. By Lemma 33 there is a rainbow path P' with colour set $\mathscr{A}$ starting at some $x' \in A^-$ and ending at some $y' \in B^+$. Concatenating this with P using the connecting edges yx' of colour c' and $y'x$ of colour c, we obtain a transversal Hamilton cycle, proving Case (i).

Thus it remains to consider Cases (ii) and (iii). In Case (ii), set $\mathscr{F} := \emptyset$ and in Case (iii), set $\mathscr{F} := \{n\}$. In both cases, we have $|\mathscr{C}(\substack{\circ\\\circ})| \geqslant \eta n$. Indeed, in Case (ii), each colour has at most n^2 edges and we have either $e_{\boldsymbol{G}(\substack{\circ\\\circ})}(A_1) \geqslant 30\eta n^3$ or $e_{\boldsymbol{G}(\substack{\circ\\\circ})}(B_1) \geqslant 30\eta n^3$, so this implies that $|\mathscr{C}(\substack{\circ\\\circ})| \geqslant 30\eta n^3/n^2 > \eta n$. Thus we can apply Lemma 34 with $\mathscr{F}$ to obtain a rainbow path P with endpoints $x, y \notin V_{\mathrm{bad}}$ such that $V_{\mathrm{bad}} \subseteq V(P)$ and $\mathscr{C}_{\mathrm{bad}} \setminus \mathscr{F} \subseteq \mathrm{col}(P)$ and $|V(P)| \leqslant 19\sqrt{\delta}n$, and $\mathscr{F} \cap (\mathrm{col}(P) \cup \{c, c'\} = \emptyset$, and $x \in A_1 \cap A_c$ and $y \in B_1 \cap B_{c'}$ for some colours $c, c' \in \mathscr{C}(\substack{\circ\\\circ})$. Let $Z := Z_1 \setminus V(P)$ for $Z \in \{A, B\}$.

In the next step, we proceed differently in each case. In Case (ii), by symmetry, we assume that $e_{G(\substack{\circ\\\circ})}(A_1) \geqslant 30\eta n^3$. Let $D := \{uv : u, v \in A \text{ and } c(uv) \geqslant 10\eta n\}$ as defined in the proof of Lemma 35 where $c(uv) := \{i \in \mathscr{C}(\substack{\circ\\\circ}) : uv \in E(G_i)\}$. There is a subgraph D' of D with minimum degree at least $10\eta n$. In Case (iii), let D' be an arbitrary clique of size $10\eta n$ inside A.

We resume a unified approach for both cases. Let $c_1, c_2 \in \mathscr{C}(\substack{\circ\\\circ})$ be distinct unused colours. Let $\mathscr{A}(H) := \mathscr{C}(H) \setminus (\mathrm{col}(P) \cup \{c, c', c_1, c_2\})$ for $H \in \{\substack{\bullet\\\bullet}, \substack{\circ\\\circ}\}$. We have

$$|A| + |B| - 3 = n - |V(P)| - 3 = n - |\mathrm{col}(P)| - 4 = |\mathscr{A}(\substack{\bullet\\\bullet})| + |\mathscr{A}(\substack{\circ\\\circ})|$$

and $||A| - |B|| \leqslant 5\delta n$. Choose partitions $A = A^0 \cup A^1$ and $A^1 = A^{10} \cup A^{11}$, and $B = B^0 \cup B^1$ and $B^1 = B^{10} \cup B^{11}$ such that

$$0 \leqslant |A^0| - |B^0| \leqslant 1, \quad |\mathscr{A}(\substack{\circ\\\circ})| = |A^0| + |B^0| - 1 \quad \text{and} \quad \delta(D'[V(D') \cap A^0]) \geqslant \eta^2 n,$$

and a partition $\mathscr{A}(\substack{\bullet\\\bullet}) = \mathscr{C}_A \cup \mathscr{C}_B$ with

$$0 \leqslant |A^{10}| - |A^{11}|, |B^{10}| - |B^{11}| \leqslant 1, \quad |\mathscr{C}_A| = |A^{10}| + |A^{11}| - 1 \quad \text{and} \quad |\mathscr{C}_B| = |B^{10}| + |B^{11}| - 1.$$

The only non-trivial part of this is the assertion about D' which holds since $|A^0| \geqslant |\mathscr{C}(\substack{\circ\\\circ})|/3 \geqslant \eta n/3$, and we could take e.g. a random partition and appeal to a Chernoff bound. Let

$$\boldsymbol{G}^0 = (G_i[A^0, B^0] : i \in \mathscr{A}(\substack{\circ\\\circ})) \quad \text{and} \quad \boldsymbol{G}^Z = (G_i[Z^{10}, Z^{11}] : i \in \mathscr{C}_Z) \quad \text{for } Z \in \{A, B\}.$$

Lemma 33 with $W = Z$ implies that there is a rainbow path P_A inside $\boldsymbol{G}^A$ with colour set $\mathscr{C}_A$ and a rainbow path P_B inside $\boldsymbol{G}^B$ with colour set $\mathscr{C}_B$ such that P_A starts at some $y_A \in N_{G_{c'}}(y) \cap A^{10}$ and ends at some $y'_A \in A_{c_1} \cap A^1$, and P_B starts at some $y_B \in N_{G_c}(x) \cap B^{10}$ and

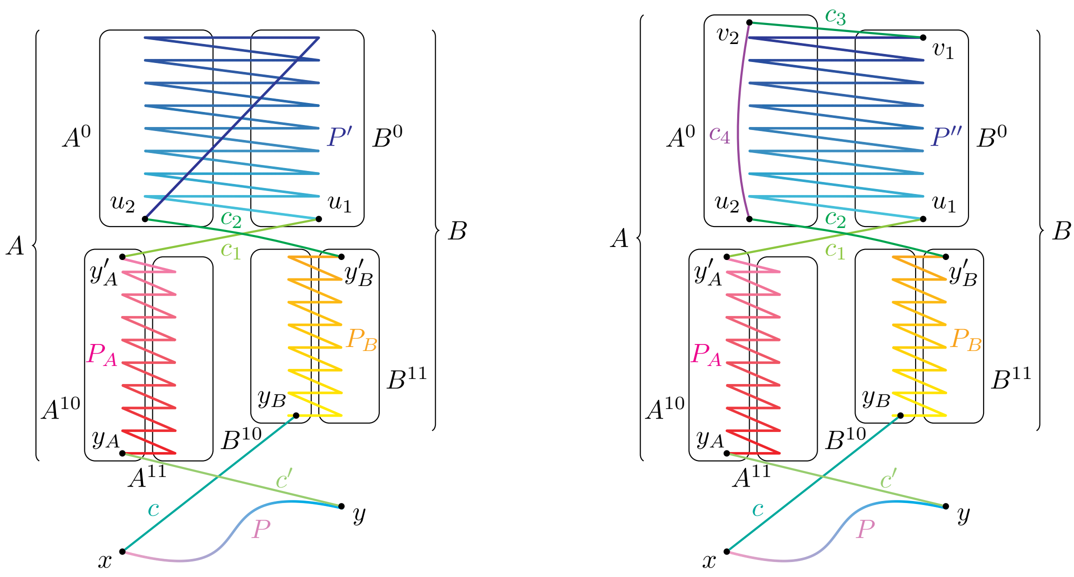

Figure 2: Finding a transversal Hamilton cycle in Lemma 36(ii) and (iii). Here, $|A^{10}| - |A^{11}| = 1$ and $|B^{10}| - |B^{11}| = 0$. On the left, $|A^0| - |B^0| = 0$, while on the right, $|A^0| - |B^0| = 1$.

ends at some $y'_B \in B_{c_2} \cap B^1$. (For example, to find P_A, we take $(W^*, Z^*) = (A^{10}, A^{11})$.) To complete the proof, it suffices to find a rainbow path P' in $\boldsymbol{G}^0$ with colour set $\mathscr{A}(\binom{\circ}{\circ})$, with startpoint $u_1 \in B^0$ where $c_1 \in L(u_1 y'_A)$, and endpoint $u_2 \in A^0$ where $c_2 \in L(u_2 y'_B)$. Then $xPyy_A P_A y'_A u_1 P' u_2 y'_B P_B y_B x$ with colours $\mathrm{col}(P), c', \mathrm{col}(P_A), c_1, \mathrm{col}(P'), c_2, \mathrm{col}(P_B), c$ is a transversal Hamilton cycle in $\boldsymbol{G}$.

Suppose first that $|\mathscr{A}(\binom{\circ}{\circ})|$ is odd. Then $|A^0| = |B^0|$ and Lemma 33 with $(W^*, Z^*) = (B^0, A^0)$, $t = 0$, $W^- = N_{G_{c_1}}(y'_A) \cap B^0$ and $T^+ = N_{G_{c_2}}(y'_B) \cap A^0$ immediately implies the existence of the required path between $u_1 \in W^-$ and $u_2 \in T^+$ (see the left-hand side of Figure 2).

Suppose secondly that $|\mathscr{A}(\binom{\circ}{\circ})|$ is even. So $|A^0| = |B^0| + 1$. Choose a colour $c_3 \in \mathscr{A}(\binom{\circ}{\circ})$ and $v_2 \in A_{c_3} \cap A^0$ which is a vertex of D'. Then choose $u_2 \in N_{G_{c_2}}(y'_B) \cap A^0$ which is a neighbour of v_2 in D'. These choices are both possible since there are at least $\eta^2 n$ such neighbours in D' of which at most $2\delta n$ are forbidden due to the G_{c_2} neighbourhood condition. In Case (ii), assign an unused colour $c_4 \in \mathscr{A}(\binom{\circ}{\circ})$ to $u_2 v_2$ using the large guaranteed colour lists. In Case (iii), assign the colour $c_4 := n$ to $u_2 v_2$. In both cases, $u_2 v_2 \in G_{c_4}$.

Now let $\boldsymbol{G}^{0'} = (G_i[A^0 \setminus \{u_2, v_2\}, B^0] : i \in \mathscr{A}(\binom{\circ}{\circ}) \setminus \{c_3, c_4\})$. By Lemma 33 with $(W^*, Z^*) = (B^0, A^0 \setminus \{u_2, v_2\})$ and $t = 1$, there is a rainbow path P'' inside $\boldsymbol{G}^{0'}$ with startpoint $u_1 \in N_{G_{c_1}}(y'_A) \cap B^0$ and endpoint $v_1 \in N_{G_{c_3}}(v_2) \cap B^0$ using colour set $\mathscr{A}(\binom{\circ}{\circ}) \setminus \{c_3, c_4\}$. We set $P' := u_1 P'' v_1 v_2 u_2$, using $\mathrm{col}(P''), c_3, c_4$ in that order (see the right-hand side of Figure 2). This completes the proof. $\square$

7 Proofs of Theorems 4 and 5

Finally, we can put all the ingredients together to prove our two main theorems on stability for transversal Hamilton cycles and paths.

Proof of Theorems 4 and 5. Let $\kappa > 0$ and assume that $\kappa < 1$ or the theorems are both vacuous. Choose additional parameters $n, \mu, \alpha, \gamma, \varepsilon, \delta, \eta$ such that $n \in \mathbb{N}$ and

$$0 < 1/n \ll \mu, \alpha \ll \gamma, \varepsilon \ll \delta \ll \eta \ll \kappa < 1$$

such that the conclusions of Lemma 15 and Lemmas 31–36 hold. Let $n' \in \{n-1, n\}$ and let $\boldsymbol{G} = (G_1, \dots, G_{n'})$ be a graph collection on a common vertex set V of size n with $\delta(\boldsymbol{G}) \geqslant (\frac{1}{2} - \mu)n$.

If $n' = n-1$, let $G_n := K_n$ and $\boldsymbol{J} := (G_1, \dots, G_n)$. If $n' = n$, let $\boldsymbol{J} := \boldsymbol{G}$. Suppose that $\boldsymbol{J}$ does not contain a transversal Hamilton cycle. Then Lemma 31 implies that $\boldsymbol{J}$ is not $(\gamma, \alpha, \varepsilon, \delta)$-stable. Thus, without loss of generality, for every colour $i \in [(1-\gamma)n]$, the graph G_i is not α-nice and hence is ε-extremal with characteristic partition (A_i, B_i, C_i), and $e(C_{\boldsymbol{J}}^{\varepsilon,\delta}) < \delta n^2$. We may further assume that 1 is the colour of minimum degree in the cross graph $C_{\boldsymbol{J}}^{\varepsilon,\delta}$, and $[(1-\gamma)n] \setminus N_{C_{\boldsymbol{J}}^{\varepsilon,\delta}}(1) = [m]$. Let $\mathscr{C}_{\mathrm{bad}} := [m+1, n]$ be the set of excluded colours and thus $|\mathscr{C}_{\mathrm{bad}}| \leqslant \gamma n + 2\delta n \leqslant 3\delta n$. It follows from Lemma 15 that $e_{\boldsymbol{J}(\overset{\bullet}{\bullet})}(A_1, B_1) \leqslant \varepsilon n^3$. For every $i \in [m]$, G_i and G_1 are not δ-crossing and hence we can swap the labels of A_i and B_i if necessary to get that $|A_1 \triangle A_i|, |B_1 \triangle B_i| \leqslant \delta n$. That is, $\boldsymbol{J}$ satisfies (†). We use the same notation defined after (†).

Suppose that one of the following hold.

(i) $|\mathscr{C}(\overset{\bullet}{\bullet})| \leqslant \eta n$ and either $e_{\boldsymbol{J}(\overset{\circ}{\circ})}(A_1) \leqslant 30\eta n^3$ or $e_{\boldsymbol{J}(\overset{\circ}{\circ})}(B_1) \leqslant 30\eta n^3$; or

(ii) $|\mathscr{C}(\overset{\bullet}{\bullet})| \geqslant \eta n$ and $e_{\boldsymbol{J}(\overset{\circ}{\circ})}(A_1) + e_{\boldsymbol{J}(\overset{\circ}{\circ})}(B_1) \leqslant 60\eta n^3$; or

(iii) $G_n = K_n$ and $|\mathscr{C}(\overset{\circ}{\circ})| < \eta n \leqslant |\mathscr{C}(\overset{\bullet}{\bullet})|$.

We claim that in these cases, $\boldsymbol{J}$ is κ-close to, respectively,

(i) a half-split graph collection;

(ii) $\boldsymbol{H}_a^b$ where $a = |\mathscr{C}(\overset{\bullet}{\bullet})|$ and $b = |\mathscr{C}(\overset{\circ}{\circ})| \pm 1$ is odd;

(iii) $\boldsymbol{H}_n^0$.

Indeed, for (i), we can remove at most $30\eta n^3$ edges so that some $Z_1 \in \{A_1, B_1\}$ has no $\boldsymbol{J}(\overset{\circ}{\circ})$-edges, and delete all edges in graphs in $\boldsymbol{J}(\overset{\bullet}{\bullet}) \cup \mathscr{C}_{\mathrm{bad}}$ by removing at most $\eta n^3 + 3\delta n^3$ edges. Finally, edit at most $2\delta n^3$ edges to increase $|Z_1|$ by less than εn so it has size $\lfloor n/2 \rfloor + 1$ and make $(Z_1, \overline{Z_1})$ complete in every graph in $\boldsymbol{J}(\overset{\circ}{\circ})$. The resulting graph collection is half-split and in total we have made $31\eta n^3 + 5\delta n^3 < \kappa n^3$ edits. For (ii), we can edit at most $3\delta n^3 + \varepsilon n^3 + 60\eta n^3$ edges so that for every $i \in \mathscr{C}_{\mathrm{bad}}$ the graph G_i becomes a copy of $\overset{\bullet}{\bullet}$ whose parts contain A_1, B_1; $e_{G_i}(A_1, B_1) = 0$ for all $G_i \in \boldsymbol{J}(\overset{\bullet}{\bullet})$; and $e_{\boldsymbol{J}(\overset{\circ}{\circ})}(A_1) + e_{\boldsymbol{J}(\overset{\circ}{\circ})}(B_1) = 0$. A further at most δn^3 edits will make $\boldsymbol{J}$ isomorphic to $\boldsymbol{H}_a^b$ where $a = |\mathscr{C}(\overset{\bullet}{\bullet})|$ and $b = |\mathscr{C}(\overset{\circ}{\circ})| \pm 1$ is odd. Thus we make at most $61\eta n^3 < \kappa\eta n^3$ edits in total. For (iii), we can make at most $2\eta n^3 + 2\delta n^3 < \kappa n^3$ edits to make $\boldsymbol{J}$ isomorphic to $\boldsymbol{H}_n^0$. This proves the claim.

Now we prove Theorem 4. Here, $\boldsymbol{J} = \boldsymbol{G}$. Lemma 36 implies that, if $|\mathscr{C}(\substack{\bullet \\ \bullet})| < \eta n$, then (i) holds, while if $|\mathscr{C}(\substack{\bullet \\ \bullet})| \geqslant \eta n$, then (ii) holds. This completes the proof of the theorem.

Finally, we prove Theorem 5. Here, $n' = n - 1$ and $\boldsymbol{J} = \boldsymbol{G} \cup \{K_n\}$, so $\boldsymbol{G}$ does not have a transversal Hamilton path if and only if $\boldsymbol{J}$ does not have a transversal Hamilton cycle. Lemma 36 implies that, if $|\mathscr{C}(\substack{\bullet \\ \bullet})| < \eta n$, then (i) holds, while if $|\mathscr{C}(\substack{\bullet \\ \bullet})| \geqslant \eta n$, then (iii) holds (and (ii) also holds in this case). This completes the proof of the theorem. □

8 Concluding remarks

In this paper, we proved Theorem 4 that any collection of n almost-Dirac graphs on the same large set of n vertices either has a transversal Hamilton cycle, or is close to one of several types of collection which do not contain a transversal Hamilton cycle. We proved Theorem 5, an analogue for Hamilton paths, characterising collections of $n-1$ almost-Dirac graphs without transversal Hamilton paths.

We proved these theorems in a unified manner, combining the regularity-blow-up method and absorption method for transversals. We believe that our method can be utilised to characterise stability for other spanning transversal embedding problems. It provides some hope for proving exact results: i.e. determining the best possible transversal minimum degree threshold. Indeed, while the transversal minimum degree thresholds for Hamilton cycles and perfect matchings are known by the results of Joos and Kim [16], exact results are commonly proved by first proving stability by showing that any graph without the required subgraph H and almost the conjectured minimum degree must have a specific structure. Then, an extremal analysis is conducted to show that there cannot be any imperfections in this structure. Given the literature on classical embedding, it seems unlikely that a short 'elementary' argument (i.e. without using any machinery) of the type used by Joos and Kim can be used to find exact thresholds for many of the natural graphs H in which we are interested. In particular, we think the ideas developed here may be useful in resolving the following conjecture, a transversal analogue of the Hanjal-Szemerédi theorem [14], which would indeed generalise this theorem.

Conjecture 37 (Transversal Hajnal-Szemerédi [8])**.** Let $k \geqslant 2$ be an integer and let n be a sufficiently large multiple of k. Let $\boldsymbol{G} = (G_1, G_2, \dots, G_{\frac{n}{k}\binom{k}{2}})$ be a graph collection on a common vertex set of size n. Suppose $\delta(\boldsymbol{G}) \geqslant (1 - \frac{1}{k})n$. Then $\boldsymbol{G}$ contains a transversal copy of a K_k-factor.

As mentioned, the case $k = 2$ of perfect matchings was proved by Joos and Kim in [16], and the asymptotic version of this conjecture was proved independently in [8] and [21].

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