A conjecture on the Nordhaus-Gaddum product type inequality for Laplacian eigenvalues of a graph

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Abstract

For a graph G of n vertices, let $\mu_1(G)$ be its largest Laplacian eigenvalue. It was conjectured by Ashraf et al. in [*Electron. J. Combin.* **21**(3):#P3.6 (2014)] that

$$\mu_1(G)\mu_1(\bar{G}) \leqslant n(n-1),$$

where $\bar{G}$ is the complement of G, and equality holds if and only if G or $\bar{G}$ is isomorphic to the join of an isolated vertex and a disconnected graph of order $n-1$.

They proved that this conjecture holds for bipartite graphs. In this paper, we completely confirm this conjecture. Furthermore, we propose a more general conjecture that for any graph G with n vertices and $k \leqslant \frac{3n}{4}$,

$$\mu_k(G)\mu_k(\bar{G}) \leqslant n(n-k),$$

and equality holds if and only if G or $\bar{G}$ is isomorphic to the join of K_k and a disconnected graph on $n-k$ vertices with at least $k+1$ connected components.

We also prove that it is true for $\frac{n}{2} \leqslant k \leqslant \frac{3n}{4}$, and for each $k \geqslant \frac{3n}{4}+1$, a counterexample is given.

Mathematics Subject Classifications: 05C50

1 Introduction

In this paper, let $G = (V(G), E(G))$ be an undirected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. We denote the complement graph of G by $\bar{G}$. Denote by $N_G(v)$ (or $N(v)$) the set of vertices adjacent to a vertex v and $d_G(v) = |N_G(v)|$, or $d(v) = |N(v)|$ for simplicity. The distance between vertices u and v of G is denoted by

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$d(u,v)$. The diameter $d(G)$ is the maximum distance between any two vertices of G. As usual, let K_n, $K_{r,s}$, and P_n denote the complete graph, the complete bipartite graph with parts of sizes r and s, and the path of order n, respectively. Especially, K_1 indicates an isolated vertex.

Let $A(G)$ be the adjacency matrix, and $D(G) = diag(d(v_1), \cdots, d(v_n))$ be the diagonal matrix of vertex degrees of G. The Laplacian matrix of G is defined at $L(G) = D(G) - A(G)$. It is well known that $L(G)$ is a symmetric positive semidefinite matrix, and so its eigenvalues can be arranged as follows:

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_{n-1}(G) \geqslant \mu_n(G) = 0.$$

The second smallest eigenvalue $\mu_{n-1}(G)$, also called after Fiedler's seminal paper [12], the algebraic connectivity, can be denoted as $\alpha(G)$ for simplicity. A unit eigenvector associated with $\alpha(G)$ is called a Fiedler vector, after the pioneering work of Fiedler in [13].

Nordhaus and Gaddum [19] considered lower and upper bounds on the sum and the product of chromatic number of a graph G and its complement $\bar{G}$. Since then, Nordhaus-Gaddum type results are bounds of the sum or the product of a parameter for a graph and its complement. The Nordhaus-Gaddum type inequalities of various graph parameters have attracted much attention (see [2, 14, 18, 20, 24]).

As for the Laplacian eigenvalues of any graph, Zhai et al. [23] (see also You and Liu [22]) posed the following Nordhaus-Gaddum sum type inequality conjecture.

Conjecture 1. For any graph G of order $n \geqslant 2$,

$$\mu_1(G) + \mu_1(\bar{G}) \leqslant 2n - 1 \text{ (or equivalently } \alpha(G) + \alpha(\bar{G}) \geqslant 1),$$

with equality if and only if G or $\bar{G}$ is isomorphic to the join of K_1 and a disconnected graph of order $n - 1$.

In the past twenty years, many scholars successively confirmed this conjecture for various families of graphs, such as

- trees, in 2008 by Fan et al. [10];
- unicyclic graphs, in 2009 by Bao et al. [3];
- bicyclic graphs, in 2010 by Fan et al. [11];
- tricyclic graphs, in 2009 by Chen et al. [6];
- cactus graphs, in 2010 by Liu [16];
- quasi-tree graphs, in 2011 by Xu et al. [21];
- graphs with diameter not equal to 3, in 2011 by Zhai et al. [23];
- bipartite graphs, in 2014 by Ashraf et al. [2];
- K_3-free graphs, in 2016 by Chen et al. [5].

Excitingly, in 2021, Einollahzadeh et al. [9] confirmed Conjecture 1 completely.

In 2014, Ashraf et al. [2] posed the following Nordhaus-Gaddum product type inequality for Laplacian eigenvalues, and confirmed it for bipartite graphs.

Conjecture 2. For any graph G with n vertices,

$$\mu_1(G)\mu_1(\bar{G}) \leqslant n(n-1),$$

and equality holds if and only if G or $\bar{G}$ is isomorphic to the join of K_1 and a disconnected graph of order $n-1$.

In this paper, we completely solve this conjecture, and propose a more general conjecture on the Nordhaus-Gaddum product type inequality for Laplacian eigenvalue.

2 Preliminaries

Lemma 3. *[1, p. 143] For any graph G with n vertices, $\mu_1(G) \leqslant n$, with equality if and only if $\bar{G}$ is disconnected.*

Lemma 4. *[7, p. 4] For any graph G, $\mu_i(G) = n - \mu_{n-i}(\bar{G})$ for $i = 1, \dots, n-1$.*

Lemma 5. *[8, Theorem 2.1] Let G be a non-empty graph with vertex set $\{v_1, v_2, \dots, v_n\}$. Then*

$$\mu_1(G) \leqslant \max_{v_iv_j \in E(G)} \left\{ d(v_i) + d(v_j) - |N(v_i) \cap N(v_j)| \right\}.$$

Lemma 6. *[17, Theorem 4.2] For any connected graph G with diameter $d \geqslant 1$, $\alpha(G) \geqslant \frac{4}{nd}$.*

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. The vector x can be considered as a function defined on $V(G)$, which maps each vertex v_i of G to the value x_i, i.e. $x(v_i) = x_i$.

The "effective resistance" between two vertices v_r and v_s in a graph G is denoted by $R^G_{r,s}$ and defined by:

$$\frac{1}{R^G_{r,s}} := \min \sum_{v_iv_j \in E(G)} (x_i - x_j)^2,$$

where the minimum runs over all $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, with $x_r - x_s = 1$. So for an arbitrary vector $x \in \mathbb{R}^n$ and $1 \leqslant r < s \leqslant n$, we have

$$\sum_{v_iv_j \in E(G)} (x_i - x_j)^2 \geqslant \frac{(x_r - x_s)^2}{R^G_{r,s}}. \tag{1}$$

Lemma 7. *[9, Lemma 1] Let G be a simple graph with $n \geqslant 2$ vertices. For every subgraph H of G which contains the vertices v_1 and v_2, then $R^G_{1,2} \leqslant R^H_{1,2}$.*

Lemma 8. *[9, p. 241-245] Let $G = (V(G), E(G))$ be a graph with order n, $v_1, v_2 \in V(G)$, and $d(v_1, v_2) = 3$. Let $s \geqslant 1$ be the maximum number of vertex-disjoint paths with length 3 between v_1 and v_2, and S be the union of vertices of these paths,*

$$l = \max_{v \in S_1, u \in S_2} \left\{ |N_G(v) \cap A|, |N_G(u) \cap B| \right\},$$

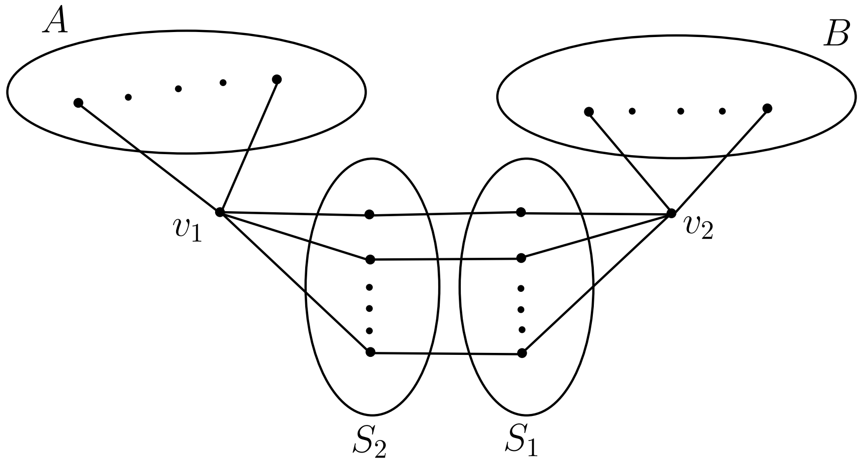

Figure 1: The subsets S_1, S_2, A, B of $V(G)$

where $S_1 = S \cap N_G(v_2)$, $S_2 = S \cap N_G(v_1)$, $A = N_G(v_1) \setminus S_2$, $B = N_G(v_2) \setminus S_1$ *(see Figure 1). Then*

$$\alpha(\bar{G}) \geqslant \frac{n}{n + 2 + 3s^2 + 6sl}.$$

Lemma 9. *[9, p. 240-241] Let* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$ *be Fiedler vectors of* G *and* $\bar{G}$*, respectively, and* $x_1 = \max_{1 \leqslant i \leqslant n} x_i$, $\quad x_2 = \min_{1 \leqslant i \leqslant n} x_i$*. If* $\max_{i<j} |x_i - x_j| \geqslant \max_{i<j} |y_i - y_j|$*, then*

$$\alpha(G) + \alpha(\bar{G}) \geqslant \frac{1}{(x_1 - x_2)^2}.$$

Lemma 10. *[9, Lemma 5] Suppose that* G *is a connected graph with* $n \geqslant 2$ *vertices. Let* $x = (x_1, \ldots, x_n)$ *be a Fiedler vector of* G*, and* $x_1 = \max_i x_i$, $\quad x_2 = \min_i x_i$*. If* $d(v_1, v_2) \leqslant 2$*, then* $\alpha(G) \geqslant 1$.

3 Proof of Conjecture 2

In this section, we give the proof of Conjecture 2.

Lemma 11. *If* G *or* $\bar{G}$ *is disconnected, then* $\mu_1(G)\mu_1(\bar{G}) \leqslant n(n-1)$*, and equality holds if and only if* G *or* $\bar{G}$ *is isomorphic to the join of* K_1 *and a disconnected graph of order* $n - 1$.

Proof. Without loss of generality, assume that $\bar{G}$ is disconnected. By Lemma 3, we have $\mu_1(G) = n$ and $\mu_1(\bar{G}) \leqslant n - 1$. The equality holds in the latter if and only if $\bar{G}$ has a connected component of order $n - 1$, say H, such that $\bar{H}$ is disconnected. So, G is isomorphic to the join of K_1 and $\bar{H}$. □

According to the above lemma, we can assume that both G and $\bar{G}$ are connected.

Lemma 12. *If both G and $\bar{G}$ are connected and $d(G) = 2$ or $d(\bar{G}) = 2$, then $\mu_1(G)\mu_1(\bar{G}) < n(n-1)$.*

Proof. Without loss of generality, assume that $d(G) = 2$. Then each pair of non-adjacent vertices v_i, v_j of G have common neighbors, that is, $|N_G(v_i) \cap N_G(v_j)| \geqslant 1$. This implies that each pair of adjacent vertices v_i, v_j of $\bar{G}$ have $|N_{\bar{G}}(v_i) \cup N_{\bar{G}}(v_j)| \leqslant n-1$. By Lemma 5, we have

$$\begin{aligned}\mu_1(\bar{G}) &\leqslant \max_{v_iv_j \in E(\bar{G})} \{d_{\bar{G}}(v_i) + d_{\bar{G}}(v_j) - |N_{\bar{G}}(v_i) \cap N_{\bar{G}}(v_j)|\} \\ &= \max_{v_iv_j \in E(\bar{G})} \{|N_{\bar{G}}(v_i) \cup N_{\bar{G}}(v_j)|\} \\ &\leqslant n-1. \end{aligned} \tag{2}$$

Besides, since $\bar{G}$ is connected, by Lemma 3, $\mu_1(G) < n$. Thus $\mu_1(G)\mu_1(\bar{G}) < n(n-1)$. □

Corollary 13. *For any graph with $d(G) = 2$, we have $\alpha(G) \geqslant 1$.*

Proof. For any graph with $d(G) = 2$, by Lemma 4 and Eq. (2), we have $\alpha(G) = n - \mu_1(\bar{G}) \geqslant 1$. □

Lemma 14. *For any connected graph G with order n and $d(G) \geqslant 4$, we have $\mu_1(G)\mu_1(\bar{G}) < n(n-1)$.*

Proof. It is well known that for all connected graphs with $d(G) \geqslant 4$, we have $d(\bar{G}) = 2$ [4]. By Lemma 12, we have $\mu_1(G)\mu_1(\bar{G}) < n(n-1)$. □

For any connected graph with $d(G) = 3$, we have $2 \leqslant d(\bar{G}) \leqslant 3$. By Lemma 12, we only need to discuss the case $d(G) = d(\bar{G}) = 3$.

Lemma 15. *For any graph G with order n and $d(G) = d(\bar{G}) = 3$, $\alpha(G) + \alpha(\bar{G}) > 2n - 2\sqrt{n(n-1)}$.*

Proof. Suppose that $\alpha = \alpha(G)$, $\bar{\alpha} = \alpha(\bar{G})$, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are Fiedler vectors of G and $\bar{G}$, respectively. Observe that y is also an eigenvector of $L(G)$ corresponding to $\mu_1(G)$. Thus, x and y are two orthonormal vectors in $\mathbb{R}^n$, both orthogonal to $e = (1, \ldots, 1)$, which is the eigenvector corresponding to $\mu_n(G) = 0$. Without loss of generality, we may assume that $\max_{i<j} |x_i - x_j| \geqslant \max_{i<j} |y_i - y_j|$ and

$$x_1 = \max_{1 \leqslant i \leqslant n} x_i, \quad x_2 = \min_{1 \leqslant i \leqslant n} x_i.$$

Note that since $\sum_i x_i = 0$ and x is nonzero, we have $x_2 < 0 < x_1$. In the following, we distinguish the following two cases:

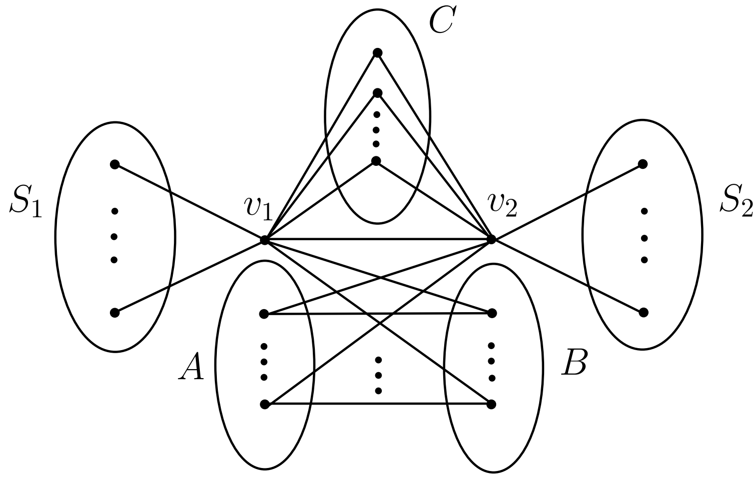

Figure 2: G_1: A spanning subgraph of $\bar{G}$

Case 1. $x_1 - x_2 < \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}}$. By Lemma 9, we have

$$\begin{aligned}\alpha(G) + \alpha(\bar{G}) &\geqslant \frac{1}{(x_1 - x_2)^2} \\ &> \frac{1}{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}} = 2n - 2\sqrt{n(n-1)}.\end{aligned}$$

Case 2. $x_1 - x_2 \geqslant \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}}$.

Owing to $d(G) = 3$, $d(v_1, v_2) \leqslant 3$. If $d(v_1, v_2) \leqslant 2$, by Lemma 10, we have $\alpha(G) \geqslant 1$. From $d(\bar{G}) = 3$ together with Lemma 6, we have $\alpha(\bar{G}) \geqslant \frac{4}{nd(\bar{G})} \geqslant \frac{4}{3n}$. Then

$$\alpha(G) + \alpha(\bar{G}) \geqslant 1 + \frac{4}{3n} > 2n - 2\sqrt{n(n-1)}.$$

Now we can suppose that $d(v_1, v_2) = 3$, and $s \geqslant 1$ is the maximum number of vertex-disjoint paths with length 3 between v_1 and v_2. Let the union of vertices of these paths be S, $S_1 = S \cap N_G(v_2)$, $S_2 = S \cap N_G(v_1)$, $A = N_G(v_1) \setminus S_2$, $B = N_G(v_2) \setminus S_1$ and $C = V(G) \setminus (A \cup B \cup S)$ (see Figure 1). It is easy to see that $|S_1| = |S_2| = s$, $\{A, B, C, S\}$ is a partition of $V(G)$. Then, G_1 is a spanning subgraph of $\bar{G}$, as shown in Figure 2.

Let $l = \max_{v \in S_1, u \in S_2} \{|N_G(v) \cap A|, |N_G(u) \cap B|\}$. Then, G contains a subgraph G_2, as shown in Figure 3. Recall that $x = (x_1, \ldots, x_n)$ is the Fiedler vector of G. By applying Eq. (1) together with the assumption, we have

$$\alpha = \sum_{v_i v_j \in E(G)} (x_i - x_j)^2 \geqslant \frac{(x_1 - x_2)^2}{R^G_{1,2}} \geqslant \frac{1}{R^G_{1,2}} \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}).$$

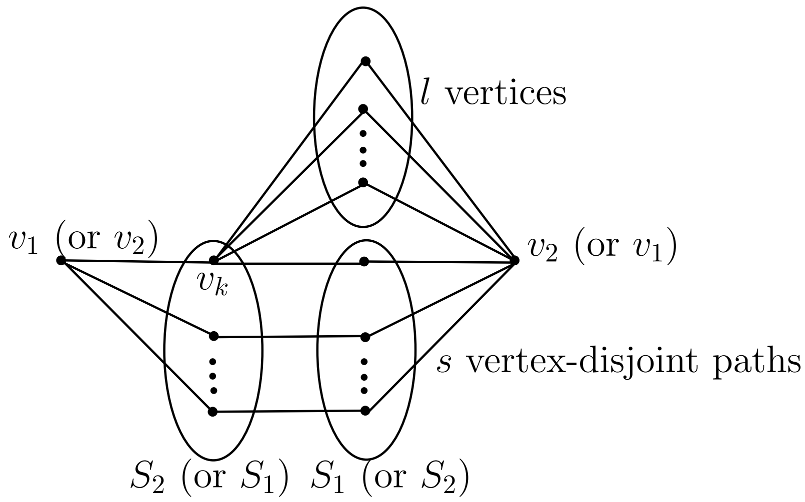

Figure 3: G_2: A subgraph of G

By Lemma 7, it has $\frac{1}{R^G_{1,2}} \geqslant \frac{1}{R^{G_2}_{1,2}}$. A computational result established in [9, p. 243] gives

$$\frac{1}{R^{G_2}_{1,2}} = \frac{s-1}{3} + \frac{l+1}{l+3}.$$

Thus, the lower bound for α in terms of s, l, and n is given by

$$\alpha \geqslant (\frac{s-1}{3} + \frac{l+1}{l+3}) \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{1}{n}}). \tag{3}$$

For $s \geqslant 4$ and $l \geqslant 0$, from Eq. (3), we have

$$\alpha \geqslant (1+\frac{1}{3}) \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{1}{n}}) > 2n - 2\sqrt{n(n-1)},\ n \geqslant 4.$$

So we only need to consider $s = 1, 2$ and 3.

By Lemma 8, we have

$$\bar{\alpha} \geqslant \frac{n}{n+2+3s^2+6sl}. \tag{4}$$

Using Eqs. (3) and (4), we obtain:

$$\alpha + \bar{\alpha} \geqslant (\frac{s-1}{3} + \frac{l+1}{l+3}) \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{1}{n}}) + \frac{n}{n+2+3s^2+6sl}. \tag{5}$$

The case $s = 1$. From Eq. (5),

$$\alpha + \bar{\alpha} \geqslant \frac{l+1}{l+3} \cdot \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}\right) + \frac{n}{n+5+6l} \triangleq f(n,l)$$

According to the definition of l, we know $0 \leqslant l \leqslant n-4$. Thus, for each $n \geqslant 16$, we have

$$\begin{aligned}\frac{\partial f}{\partial l} &= \frac{n + \sqrt{n(n-1)}}{n(l+3)^2} - \frac{6n}{(n+5+6l)^2} \\ &= \frac{(n + \sqrt{n(n-1)})(n+5+6l)^2 - 6n^2(l+3)^2}{n(l+3)^2(n+5+6l)^2} \triangleq \frac{g(n,l)}{n(l+3)^2(n+5+6l)^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial g}{\partial l} &= 12(n + \sqrt{n(n-1)})(n+5+6l) - 12n^2(l+3) \\ &= (-12n^2 + 72\sqrt{n(n-1)} + 72n)l - 24n^2 \\ &\quad + (12\sqrt{n(n-1)} + 60)n + 60\sqrt{n(n-1)}.\end{aligned}$$

Note that $-12n^2 + 72\sqrt{n(n-1)} + 72n < -12n^2 + 144n < 0$ and

$$\frac{\partial g}{\partial l} \leqslant -24n^2 + (12\sqrt{n(n-1)} + 60)n + 60\sqrt{n(n-1)} < -12n^2 + 120n < 0$$

for $n \geqslant 16$, so $g(n,l)$ is decreasing for $l \in [0, n-4]$, when $n \geqslant 16$.

Furthermore,

$$\begin{aligned}g(n,0) &= (n + \sqrt{n(n-1)})(n+5)^2 - 54n^2 > 0 \ (n \geqslant 16), \\ g(n, n-4) &= (n + \sqrt{n(n-1)})(7n-19)^2 - 6n^2(n-1)^2 < 0 \ (n \geqslant 16),\end{aligned}$$

thus $f(n,l)$ is increasing and then decreasing for $l \in [0, n-4]$. On the one hand, $f(n,0) = \frac{1}{3} \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + \frac{n}{n+5} > 2n - 2\sqrt{n(n-1)}$ $(n \geqslant 16)$. On the other hand, $f(n, n-4) = (1 - \frac{2}{n-1}) \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + \frac{n}{7n-19} > 2n - 2\sqrt{n(n-1)}(n \geqslant 16)$.

Thus, when $s = 1$ and $n \geqslant 16$, we have

$$\alpha + \bar{\alpha} > 2n - 2\sqrt{n(n-1)}.$$

The case $s = 2$. From Eq. (3), for $l \geqslant 5$,

$$\alpha \geqslant \left(\frac{1}{3} + \frac{3}{4}\right) \cdot \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}\right) > 2n - 2\sqrt{n(n-1)} \ (n \geqslant 7).$$

So, $\alpha + \bar{\alpha} > 2n - 2\sqrt{n(n-1)}$.

For each of cases $l = 0, 1, 2, 3, 4$, from Eq. (5), we have

$$\alpha + \bar{\alpha} \geqslant (\frac{1}{3} + \frac{l+1}{l+3}) \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + \frac{n}{n + 14 + 12l}.$$

If $l = 0$, then $n \geqslant 6$. $\alpha + \bar{\alpha} \geqslant \frac{2}{3} \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + \frac{n}{n+14} > 2n - 2\sqrt{n(n-1)}$ $(n \geqslant 9)$.

If $l = 1$, then $n \geqslant 7$. $\alpha + \bar{\alpha} \geqslant \frac{5}{6} \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + \frac{n}{n+26} > 2n - 2\sqrt{n(n-1)}$ $(n \geqslant 8)$.

If $l = 2$, then $n \geqslant 8$. $\alpha + \bar{\alpha} \geqslant \frac{14}{15} \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + \frac{n}{n+38} > 2n - 2\sqrt{n(n-1)}$ $(n \geqslant 8)$.

If $l = 3$, then $n \geqslant 9$. $\alpha + \bar{\alpha} \geqslant \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}} + \frac{n}{n+50} > 2n - 2\sqrt{n(n-1)}$ $(n \geqslant 9)$.

If $l = 4$, then $n \geqslant 10$. $\alpha + \bar{\alpha} \geqslant \frac{22}{21} \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + \frac{n}{n+62} > 2n - 2\sqrt{n(n-1)}$ $(n \geqslant 10)$.

The case $s = 3$. We have $n \geqslant 8$. From Eq. (3), if $l \geqslant 1$,

$$\alpha \geqslant (\frac{2}{3} + \frac{1}{2}) \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) > 2n - 2\sqrt{n(n-1)}.$$

So, $\alpha + \bar{\alpha} > 2n - 2\sqrt{n(n-1)}$.

For the case $l = 0$, from Eq. (5), we have

$$\alpha + \bar{\alpha} \geqslant \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}} + \frac{n}{n + 29} > 2n - 2\sqrt{n(n-1)} \ (n \geqslant 8).$$

The theorem for all remaining cases can be numerically verified as follows:

• $n = 4$. The only connected graph G with $d(G) = 3$ is P_4, then

$$\alpha + \bar{\alpha} = 2\alpha(P_4) = 2(2 - \sqrt{2}) > 8 - 4\sqrt{3}.$$

• $n = 5$. There are exactly three cases when $d(G) = 3$. In all these cases, the minimum value of $\alpha + \bar{\alpha}$ is at least 1.348, which exceeds $10 - 4\sqrt{5}$.

• $s = 1, 6 \leqslant n \leqslant 15$. The structure of graph G_1 is uniquely determined by three nonnegative integers $|A|$, $|B|$, and $|C|$, subject to the constraints $s = 1$ and $6 \leqslant |A| + |B| + |C| + 2s + 2 = n \leqslant 15$, where G_1 is shown in Figure 2. In this case, by numerical computations for all cases with $6 \leqslant n \leqslant 15$, we have $\alpha(G_1) > 0.415$.

Thus, by Eq. (3), for $l \geqslant 3$, we have

$$\alpha + \bar{\alpha} > \frac{2}{3} \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + 0.415 > 2n - 2\sqrt{n(n-1)}.$$

For the remaining cases $l = 0, 1, 2$, we have $|S_1| = |S_2| = s = 1$. By the definition of l, in $\bar{G}$, the vertex in $S_1 = \{u\}$ is not adjacent to at most l vertices in A, and similarly, the vertex in $S_2 = \{v\}$ is not adjacent to at most l vertices in B. Let G_3 be the graph obtained from G_1 by adding the edges in $\bar{G}$ that connect u to the $|A| - l$ vertices of A, as well as the edges that connect v to the $|B| - l$ vertices of B. Then G_3 is a spanning subgraph of $\bar{G}$.

Numerical computations for $6 \leqslant n \leqslant 15$ show that $\alpha(G_3) > 0.763$ $(l = 0)$, $\alpha(G_3) > 0.631$ $(l = 1)$, and $\alpha(G_3) > 0.485$ $(l = 2)$.

So from Eq. (3),

$$\alpha + \bar{\alpha} > \frac{1}{3} \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + 0.763 > 2n - 2\sqrt{n(n-1)} \ (l = 0).$$

$$\alpha + \bar{\alpha} > \frac{1}{2} \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + 0.631 > 2n - 2\sqrt{n(n-1)} \ (l = 1).$$

$$\alpha + \bar{\alpha} > \frac{3}{5} \cdot (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{n}}) + 0.485 > 2n - 2\sqrt{n(n-1)} \ (l = 2).$$

- $s = 2$, $l = 0, 6 \leqslant n \leqslant 8$ and $l = 1, n = 7$.

If $n = 6$, G_2 (see Figure 3) is a spanning subgraph of G. In fact, at this point G_2 is a cycle. So $\alpha \geqslant \alpha(G_1) = 1$. By Eq. (4) we have

$$\alpha + \bar{\alpha} \geqslant 1 + \frac{3}{10} > 12 - 2\sqrt{30}.$$

For $n = 7$, the graph G_1 is a spanning subgraph of $\bar{G}$, which implies that $\bar{\alpha} \geqslant \alpha(G_1)$. Among all possible configurations, $\alpha(G_1) > 0.398$.

Thus, by Eq. (3), we have

$$\alpha + \bar{\alpha} > \frac{2}{3} \cdot (\frac{1}{2} + \frac{\sqrt{42}}{14}) + 0.398 > 14 - 2\sqrt{42}.$$

If $n = 8$, we only need to consider $l = 0$.

Define G_4 as the graph obtained from G_1 by adding $|S_1| \times |A|$ edges in $\bar{G}$ between S_1 and A, as well as $|S_2| \times |B|$ edges between S_2 and B. Then G_4 is a spanning subgraph of $\bar{G}$. By directly computation for $n = 8$, $\alpha(G_4) > 0.627$.

Again by Eq. (3),

$$\alpha + \bar{\alpha} > \frac{2}{3} \cdot (\frac{1}{2} + \frac{\sqrt{14}}{8}) + 0.627 > 16 - 4\sqrt{14}.$$

Thus, we complete the proof. □

Theorem 16. *For any graph G with n vertices,*

$$\mu_1(G)\mu_1(\bar{G}) \leqslant n(n-1)$$

and equality holds if and only if G or $\bar{G}$ is isomorphic to the join of K_1 and a disconnected graph of order $n - 1$.

Proof. By Lemma 15, we know $\alpha(G) + \alpha(\bar{G}) > 2n - 2\sqrt{n(n-1)}$, for $d(G) = d(\bar{G}) = 3$. Then according to Lemma 4, we have $\mu_1(G) + \mu_1(\bar{G}) < 2\sqrt{n(n-1)}$. Then

$$\mu_1(G)\mu_1(\bar{G}) \leqslant \frac{(\mu_1(G) + \mu_1(\bar{G}))^2}{4} < \frac{(2\sqrt{n(n-1)})^2}{4} = n(n-1).$$

From Lemmas 11, 12, 14 and the above inequality, we complete the proof. □

4 Nordhaus-Gaddum product type inequality for $\mu_k(G)$

Lemma 17. *[15, Theorem 18] Let G be a graph with n vertices. If $k \leqslant \frac{n}{2}$, then $\mu_k(G) + \mu_k(\bar{G}) \geqslant n$.*

For a general integer k, we have the following theorem.

Theorem 18. *If $\frac{n}{2} \leqslant k \leqslant \frac{3n}{4}$, then $\mu_k(G)\mu_k(\bar{G}) \leqslant n(n-k)$.*

Proof. If $k \geqslant \frac{n}{2}$, by Lemma 17, then $\mu_{n-k}(G) + \mu_{n-k}(\bar{G}) \geqslant n$.
Combining with Lemma 4, we have

$$\mu_k(G) + \mu_k(\bar{G}) = n - \mu_{n-k}(\bar{G}) + n - \mu_{n-k}(G) \leqslant n.$$

Then

$$\mu_k(G)\mu_k(\bar{G}) \leqslant \frac{(\mu_k(G) + \mu_k(\bar{G}))^2}{4} \leqslant \frac{n^2}{4} \leqslant n(n-k) \ (k \leqslant \frac{3n}{4}). \quad \square$$

The following proposition give a counterexample for $\frac{3n}{4} + 1 \leqslant k \leqslant n-2$.

Proposition 19. *For any positive integer $n \geqslant 12$ and $\frac{3n}{4} + 1 \leqslant k \leqslant n-2$, there is a graph $H_n = K_1 \vee (K_{\lceil \frac{n-1}{2} \rceil} \cup K_{\lfloor \frac{n-1}{2} \rfloor})$, where $\vee$ is the join of two graphs, such that*

$$\mu_k(H_n)\mu_k(\overline{H_n}) > n(n-k).$$

Proof. It is easy to see that the Laplacian eigenvalues of $\overline{H_n} = K_1 \cup K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ are $n-1, \lceil \frac{n-1}{2} \rceil^{(\lfloor \frac{n-3}{2} \rfloor)}, \lfloor \frac{n-1}{2} \rfloor^{(\lceil \frac{n-3}{2} \rceil)}, 0^{(2)}$. By Lemma 4, the Laplacian eigenvalues of H_n are $n, \lceil \frac{n+1}{2} \rceil^{(\lceil \frac{n-3}{2} \rceil)}, \lfloor \frac{n+1}{2} \rfloor^{(\lfloor \frac{n-3}{2} \rfloor)}, 1, 0$. Thus, if $\frac{3n}{4} + 1 \leqslant k \leqslant n-2$,

$$\mu_k(H_n)\mu_k(\overline{H_n}) = \lfloor \frac{n-1}{2} \rfloor \cdot \lfloor \frac{n+1}{2} \rfloor.$$

If n is odd, then $\mu_k(H_n)\mu_k(\overline{H_n}) = \frac{1}{4}(n^2 - 1) > n(n-k)$.
If n is even, $\mu_k(H_n)\mu_k(\overline{H_n}) = \frac{n}{2}(\frac{n}{2} - 1) > n(n-k)$. $\square$

In the end, we propose the following conjecture.

Conjecture 20. For any graph G with n vertices and $k \leqslant \frac{3n}{4}$,

$$\mu_k(G)\mu_k(\bar{G}) \leqslant n(n-k),$$

and equality holds if and only if G or $\bar{G}$ is isomorphic to the join of K_k and a disconnected graph on $n-k$ vertices and has at least $k+1$ connected components.

Remark 1: We check that conjecture 3 is true for all graphs with at most 9 vertices.
Remark 2: In [15], the authors proposed the following Nordhaus-Gaddum sum type inequalities conjecture for the Laplacian eigenvalues of graphs.

Conjecture 21. Let G be a graph on n vertices and $\overline{G}$ be the complement of G. Then

$$\mu_k(G) + \mu_k(\overline{G}) \geqslant n - k,$$

for $k = 1, 2, \ldots, n-1$, with equality if and only if G or $\overline{G}$ is isomorphic to $K_{n-k} \vee H$, where H is a disconnected graph on k vertices and has at least $n-k+1$ connected components.

Note that Conjecture 4 is true for $k \leqslant \frac{n+1}{2}$ (see [15]). Thus if Conjecture 3 holds, then we have $\mu_{n-k}(G) + \mu_{n-k}(\overline{G}) \geqslant k + \frac{\mu_{n-k}(G)\mu_{n-k}(\overline{G})}{n}$ by Lemma 4, implies that Conjecture 4 holds.

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