Transitive cornerations in maps

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Abstract

A corner in a map is an edge-vertex-edge triple consisting of two distinct edges incident to the same vertex. A corneration is a set of corners that covers every arc of the map exactly once. Cornerations in a dart-transitive map generalize the notion of a cycle structure [6] in a symmetric graph. In this paper, we study the cornerations (and associated structures) that are preserved by a vertex-transitive group of automorphisms of the map.

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1 Introduction

Decompositions of complex structures into smaller and simpler substructures is a classical theme in mathematics. One instance of this phenomenon in graph theory is the problem of decomposing graphs into subgraphs of specific type and in particular, decompositions of graphs into cycles, where a number of difficult and deep results were proved and conjectured (consider, for example the famous Oberwolfach problem [5]).

Decomposition problems receive a somewhat different flavour when symmetry conditions are imposed. In graph theory, considerable attention has been given to the study of decomposition (in fact, factorisations) of graphs that are preserved under some vertex-transitive groups of automorphisms (see, for example, [10, 11, 12]).

In [6], a notion of *arc-transitive cycle decompositions* of graphs was introduced. In short, a cycle decomposition of a graph Γ is a partition of its edge-set $E(\Gamma)$ such that each part of the partition induces a connected 2-valent subgraph of Γ (in short, a *cycle* of Γ). Such a decomposition is *arc-transitive* provided that it is preserved by some group

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of automorphisms of Γ acting transitively on the *arc-set* (or the *dart-set*, as we call it in this paper), where an *arc* (or *dart*) is an unordered pair consisting of an edge and one of its end-vertices. (We may also think of an arc as a directed edge.) For the sake of brevity we shall call an arc-transitive cycle decomposition a *cycle structure*.

The investigation of cycle structures began in [6], where they were introduced as a means to understand an important class of tetravalent arc-transitive graphs, namely, those where the stabilizer of a vertex in the automorphism groups acts on the neighborhood as the dihedral group D_4.

The results proved there were used in constructions of *semisymmetric tetravalent graphs* [14] and played an instrumental role in the study of cubic vertex-transitive graphs [13]; the relationship with the cubic vertex-transitive graphs was described in [13] via a *split graph* construction, which takes cycle structures in tetravalent graph as an input and returns a cubic vertex-transitive graph.

In the present paper, we broaden the scope in several directions. While we keep the focus on finite graphs (and indeed tacitly assume the finiteness condition through the paper), we allow the graphs to have multiple edges (but no loops or semiedges). Further, we allow graphs to have higher valency. Next, we allow the parts of a cycle decomposition to consist of *circuits*, where, by a 'circuit', we mean the (undirected and unrooted) trail of a closed walk in a graph that passes through each of its edges at most once (but perhaps several times through the same vertex).

In fact, we shift our focus from 'circuits' and 'circuit decompositions' to 'corners' and 'cornerations'. Here, by a *corner* we mean an incident edge-vertex-edge triple, and by a *corneration*, a set of corners that covers every incident vertex-edge pair of the graph exactly once.

Surprisingly, this simple change in perspective simplifies many aspects of the investigation and constructions. Formal definitions will appear in Section 3. The connections between these cornerations and circuit decomposition will be explained in Section 4 and explicitly codified in Theorem 3. While corners and cornerations can be defined and studied in any graph, they become of particular interest when considered in the setting of (skeletons of) maps. In particular, a corner of a map divides the circular order of faces meeting its vertex into two set of consecutive parts. If the valency is q and the two parts have sizes j and $q - j$, with $j \leqslant q - j$, we call it a j-corner, and then draw conclusions about the corner or corneration to which it belongs relative to this j. If all corners of a corneration L are j-corners, then we say L is a j-corneration.

Throughout the paper, we will be interested in questions of symmetry. If L is a corneration in a map $\mathcal{M}$, let $\mathrm{Aut}(L)$ be the group of all symmetries of $\mathcal{M}$ which preserve L. If $\mathrm{Aut}(L)$ is transitive on L, we will call L *transitive*.

We begin to understand transitive cornerations by first examining their *local* behaviour, that is, how corners around a vertex are arranged (the *local corneration*) and how the stabilizer of that vertex acts on them (the *local action*). Our main results are Theorem 10, which says that a transitive corneration has one of three local actions, isomorphic to a $D_{\frac{q}{2}}, D_{\frac{q}{4}}$, or $C_{\frac{q}{2}}$, and Theorem 12, which says that the local corneration is equivalent to one of two standard ones, determined by the parity of j.

In Sections 7 and 8, we shift our focus to 1-cornerations (or *wedge-cornerations* as we also call them here), as in some sense every j-corneration can be viewed as a 1-corneration in a related j-hole map (see Section 3.1 for the definition of the j-hole operator and Remark 15 for further discussion). The main results about 1-cornerations are Theorem 13, which classifies the four ways that elements of a wedge corneration can appear in faces of the map, and Theorem 17, which uses the concept of symmetry type graphs to classify all transitive wedge cornerations of maps into 12 types.

Section 9 is devoted to the study of *symmetric cornerations* of maps, that is, cornerations whose symmetry groups act transitively on the darts of the map. These cornerations are of particular interest as in the simple maps (that is maps whose skeleta are simple graphs) the circuit decompositions that the cornerations induce are in fact arc-transitive cycle decompositions, as introduced in [6]. In Theorem 18, we prove that a map $\mathcal{M}$ admits a symmetric j-corneration if and only if j is odd and either $\mathcal{M}$ or the Petrie dual $\mathrm{P}(\mathcal{M})$ admit a group of symmetries that acts half-reflexibly on $\mathcal{M}$ (see Section 2.4 for the definition of the Petrie dual and Section 2.6 for the definition of half-reflexible actions).

Finally, in Section 10, we address one of our main motivations for this investigation, namely new constructions of vertex-transitive graphs of low valency. We do this by introducing several versions of the *split graph constructions*, which generalize the split graphs as defined in [13]. These constructions, in conjunction with existing censuses of highly symmetrical maps, will be used in the forthcoming work on enumerating and cataloguing vertex-transitive graphs.

2   Basic Definitions

2.1   Conventions

We will use exponential notation for permutations: if σ is an element of a permutation group G acting on a set X and $x \in X$, we write x^σ for the image of x under σ, and x^G for the orbit of x under G.

For a positive integer n let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ be the ring of integers modulo n with the convention that an arbitrary integer a can be seen as an element of $\mathbb{Z}_n$ by identifying it with the remainder $a \bmod n$.

In many situations it will be convenient to adopt the notation $|a|_n$, read, "the order of $a \bmod n$", to mean the additive order of the element a in $\mathbb{Z}_n$. It is equal to $\frac{n}{\gcd(n,a)}$.

2.2   Graphs

In this paper, we will use the word *graph* to indicate what is often called a *multigraph*. That is, we will allow graphs in this paper to have multiple (parallel) edges but no loops or semi-edges. Combinatorially, a graph will be a triple $\Gamma := (V, E, \partial)$ with V and E being disjoint sets of *vertices* and *edges*, respectively, and $\partial\colon E \to \binom{V}{2}$ being a function assigning a pair of distinct *end-vertices* to each edge. If Γ is a graph (V, E, ∂), then we let $\mathrm{V}(\Gamma) = V$ and $\mathrm{E}(\Gamma) = E$. Distinct edges that have the same ∂-images are said to be

parallel and graphs without pairs of parallel edges are said to be *simple*. We define the *valence* of a vertex v of Γ to be the number of edges e such that $v \in \partial(e)$.

It will sometimes be useful to consider a *topological form* Γ^T of Γ, which is obtained by taking for each edge of Γ, a space homeomorphic to a closed line segment, then labelling the endpoints of each segment with the end-vertices of the corresponding edge, and finally identifying all endpoints having the same label.

A permutation α of $V \cup E$ preserving each of the sets V and E and satisfying $\partial(e^\alpha) = \partial(e)^\alpha$ for every $e \in E$ is called an *automorphism* or *symmetry* of Γ. The set of all automorphisms of Γ forms a subgroup of the symmetric group $\mathrm{Sym}(V \cup E)$, called the *automorphism group of* Γ and denoted $\mathrm{Aut}(\Gamma)$. If a subgroup G of $\mathrm{Aut}(\Gamma)$ acts transitively on V or on E, then Γ is said to be G-vertex- or G-edge-transitive, respectively (with the prefix G omitted when $G = \mathrm{Aut}(\Gamma)$). Note that in a simple graph Γ, the action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{V}(\Gamma)$ is faithful and thus an automorphism group can be viewed as a permutation group on $\mathrm{V}(\Gamma)$.

A *dart* of a graph Γ is a pair $\{v, e\}$ with $e \in \mathrm{E}(\Gamma)$ and $v \in \partial(e)$. If a subgroup G of $\mathrm{Aut}(\Gamma)$ in its induced action on the set $\mathcal{D}(\Gamma)$ of all darts of Γ acts transitively, then Γ is said to be *G-dart-transitive* (or simply *dart-transitive* if we want to avoid the reference to G).

A *walk* of length k in a graph Γ is a sequence $[v_0, e_0, v_1, e_1, v_2, \ldots, e_{k-2}, v_{k-1}, e_{k-1}, v_k]$ with $v_i \in \mathrm{V}(\Gamma)$ and $e_i \in \mathrm{E}(\Gamma)$ such that $\partial(e_i) = \{v_i, v_{i+1}\}$ for all $i \in \{0, \ldots, k-1\}$. If $v_k = v_0$ then we say that the above walk is *closed*. A closed walk of length k is a *circuit* provided that it has k distinct edges and is a *cycle* provided that it has k distinct vertices.

A *circuit decomposition* in a graph is a partition of its edges such that each element consists of the set of edges of a circuit. A *circuit structure* is a circuit decomposition such that some dart-transitive group preserves the decomposition. Similar definitions hold for *cycle decomposition* and *cycle structure*

2.3 Maps on surfaces

A *surface* $\mathcal{S}$ is a connected compact 2-dimensional manifold (without boundary). A *map* $\mathcal{M}$ is an embedding of Γ^T for some graph Γ, called the *skeleton* of $\mathcal{M}$, into a surface $\mathcal{S}$ so that each connected component of $\mathcal{S} \backslash \Gamma^T$ is, topologically, a disk; these components are then called the *faces* of the map. Henceforward, we will not distinguish between the graph and its topological version. If the surface $\mathcal{S}$ is orientable, we call the map *orientable*.

The notions of vertices, valence, edges, darts, walks, cycles and circuits of a map are inherited from its skeleton. If Γ is bipartite, we say that $\mathcal{M}$ is vertex-bipartite. If there is a partition of the faces of map into two classes or 'colors' such that each face of a given color borders only those of the other, we say that the map is *face-bipartite*. Notice that the compactness of $\mathcal{S}$ implies that the map has a finite number of vertices, faces and edges.

2.4 Symmetries and operators on maps

In discussing any phenomenon concerning maps, it will be useful to have 'operators', that is, functions which relate one map to another, such as the classic *dual* operator. Another is the Petrie operator:

A *Petrie path* in a map is a circuit in $\mathcal{M}$ such that for any three consecutive edges e_1, e_2, e_3, the edges e_1 and e_2 lie in a face on one side of e_2, while e_2 and e_3 lie in the face on the other side of e_2. Removing the interior of each face, and spanning each Petrie path with a new membrane, gives a new map, $P(\mathcal{M})$. The faces of $P(\mathcal{M})$ are the Petrie paths of $\mathcal{M}$ and vice versa.

While the surface underlying $P(\mathcal{M})$ is usually not the same as the one underlying $\mathcal{M}$ itself, the skeletons are the same. If $\mathcal{M}$ is orientable then its Petrie paths must have even length. Of course, $P(P(\mathcal{M})) \cong \mathcal{M}$.

We will see a third operator, called a j-hole operator and denoted by H_j, in the next section.

A *symmetry* of a map $\mathcal{M}$ is a symmetry of its skeleton induced by a homeomorphism of the underlying surface. The group of all symmetries of $\mathcal{M}$ is denoted $\mathrm{Aut}(\mathcal{M})$. If $\mathrm{Aut}(\mathcal{M})$ acts transitively on the vertices of $\mathcal{M}$, we say that $\mathcal{M}$ is a vertex-transitive map. Further symmetry types of maps and groups acting on maps (such as reflexible, face reflexible and half reflexible) will be discussed in Section 8. For general background on symmetry and operators of maps, we refer the reader to [15].

2.5 Flags

We introduce the *barycentric subdivision* of a map: choose a point in the interior of each face to be its *center* and a point in the relative interior of each edge to be its *midpoint*. Draw dotted lines to connect each face-center with each incidence of the surrounding vertices and edge-midpoints. The original edges and these dotted lines divide the surface into triangles called *flags*. Let Ω be the set of all flags in the map.

Note that a flag f shares a side with three distinct flags (see Figure 1). One of them shares a face-center-to-edge-midpoint side with f; we will call this one $r_0 f$. One of the three shares a face-centre-to-vertex side with f; we will call this one $r_1 f$. The third shares a vertex-to-edge-mid-point side with f; this one is denoted $r_2 f$. In general, we shall call the flags f and $r_i f$ *i-adjacent* (note that this is well defined as $r_i(r_i f) = f$). Moreover, note that r_0, r_1 and r_2 can be viewed as involutory permutations on Ω and that $r_0 r_2 = r_2 r_0$. Finally, connectedness of the map implies that the group $\langle r_0, r_1, r_2 \rangle$ is transitive on Ω.

2.6 Face- and half-reflexible maps

Clearly, every symmetry of a map $\mathcal{M}$ induces a permutation on the set of flags Ω of $\mathcal{M}$. In fact, it can be seen that the induced action of $\mathrm{Aut}(\mathcal{M})$ is faithful and semiregular (that is, a symmetry fixing one flag fixes every flag of the map). In particular, a symmetry is determined by its action on a single flag.
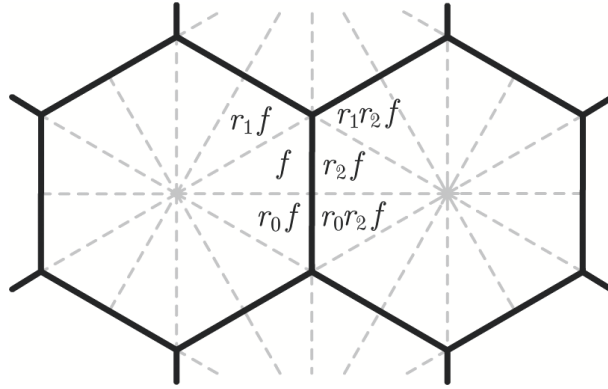
Figure 1: Flags in a trivalent map with hexagonal faces.

While there are many kinds of symmetry that a map may have, we wish to focus on just two. First, we call $\mathcal{M}$ *reflexible* provided that $\mathrm{Aut}(\mathcal{M})$ acts transitively on flags. Second, we will call a subgroup G of $\mathrm{Aut}(\mathcal{M})$ *half-reflexible* provided that it is not transitive on flags, but for each face F of $\mathcal{M}$, the stabilizer G_F is transitive on the flags of F. If $\mathrm{Aut}(\mathcal{M})$ is half-reflexible we call $\mathcal{M}$ *half-reflexible*. We will say a map $\mathcal{M}$ is *face-reflexible* if $\mathrm{Aut}(\mathcal{M})$ has a half-reflexible subgroup. From these definitions, the following facts should be clear:

Lemma 1. *For a face-reflexible map $\mathcal{M}$ and a half-reflexible subgroup G of $\mathrm{Aut}(\mathcal{M})$, each of the following holds:*

- *For any face F, all of its neighbors are in the same orbit under G and that orbit does not contain F.*

- *$\mathcal{M}$ is face-bipartite, and each partite set is an orbit under G.*

- *The index of G in $\mathrm{Aut}(\mathcal{M})$ is at most 2.*

- *If $G \neq \mathrm{Aut}(\mathcal{M})$, then $\mathcal{M}$ is reflexible.*

Lemma 2. *A reflexible map $\mathcal{M}$ is face-bipartite if and only if it is face-reflexible.*

Proof. From Lemma 1 we have that if $\mathcal{M}$ is face-reflexible, it is face-bipartite. To complete the proof of this lemma, let us assume that $\mathcal{M}$ is reflexible and face bipartite. Consider the faces in one class to be green, those of the other, red. Then every symmetry either sends red faces to red faces and greens to greens or it send each red face to a green and every green to a red. Let G be the subgroup of $\mathrm{Aut}(\mathcal{M})$ consisting of all symmetries of the first kind. Then because each face is monochromatic, for each face F, G_F acts transitively on the flags in that face. Thus G is half-reflexible and since $\mathcal{M}$ is reflexible, we see that $\mathcal{M}$ is face-reflexible. □

3 Corners in graphs and maps

A set $\{e_1, v, e_2\}$ where e_1 and e_2 are distinct edges of a graph Γ such that $v \in \partial_\Gamma(e_1) \cap \partial_\Gamma(e_2)$ is called a *corner* of Γ. If g is a symmetry of a graph Γ and $c = \{e_1, v, e_2\}$ is a corner, then clearly $c^g := \{e_1^g, v^g, e_2^g\}$ is also a corner of Γ. Thus, $\mathrm{Aut}(\Gamma)$ acts upon the set of all corners of Γ.

A *corner of a map* is defined to be a corner of its skeleton and a corner of a map $\mathcal{M}$ whose edges are consecutive in some face of $\mathcal{M}$ is called a *wedge* of $\mathcal{M}$. If v is a vertex of a map $\mathcal{M}$ of valence q, then a corner $\{e_1, v, e_2\}$ separates some j, $1 \leqslant j \leqslant \frac{q}{2}$, consecutive wedges at v from the remaining $q - j$ wedges, which are also consecutive. Such a corner is then called a *j-corner*. If q is even, then a $\frac{q}{2}$-corner is said to be *straight*. If c is a j-corner which is not straight, then the set of j consecutive wedges separated by c from the rest of the wedges at v is called the *interior* of c. Further, we say that the corner *subtends* j wedges. Figure 2 shows a 3-corner (in thick red edges) in a map of valence 10. The three wedges b, c, d are the interior of the corner. For a corner $\{e_1, v, e_2\}$, the two wedges containing the dart $\{e_1, v\}$ are *boundary wedges* for the corner, as are the two containing $\{v, e_2\}$; for example, in Figure 2 (left), wedges a and b are boundary wedges, as are d and e.
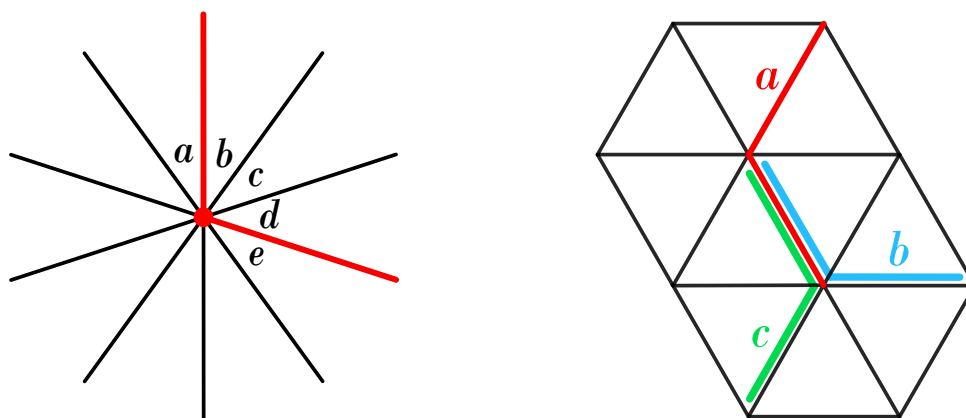

Figure 2: On the left, a 3-corner in thick red edges, in a map of valence 10. The wedges b, c and d are interior to $\{e_1, v, e_2\}$. On the right, three 2-corners in a 6-valent map. The 2-corner a (red) is convexly aligned with b (blue), and is aligned with inflexion with c (green).

If $c_1 = \{e_1, v, e_2\}$ and $c_2 = \{e_2, u, e_3\}$, $u \neq v$, are two j-corners sharing the edge e_2, we say that c_1 and c_2 are *convexly aligned* provided that the interior wedges of each containing e_2 are on the same side of e_2; they are *aligned with inflection* otherwise (see the right-hand side of Figure 2).

Given a closed walk $v_0, e_0, v_1, e_1, \ldots, v_{r-1}, e_{r-1}, v_r = v_0$ where every $\{e_{i-1}, v_i, e_i\}$, $i \in \mathbb{Z}_r$, is a j-corner, it might be that every pair of consecutive corners is convexly aligned, in which case, we call the sequence a *j-th order hole*. If, instead, every consecutive pair

is aligned with inflection, we say it is a *j-th order Petrie path*. Then j-th order holes generalize faces, in the sense that a face is a first-order hole; similarly, j-th order Petrie paths generalize Petrie paths.

3.1 j-th hole operator

Let us conclude the section by defining the j-th hole operator H_j on maps, first introduced in [15]. For any map $\mathcal{M}$, dissolve its faces, leaving only the skeleton of $\mathcal{M}$, and then span each circuit of the skeleton induced by a j-hole of $\mathcal{M}$ with a membrane.

If $d = \gcd(j, q)$ is equal to 1, then the result is a map on a surface, which we can call $H_j(\mathcal{M})$. If $d > 1$, then the resulting topological space is not a surface but rather a surface pinched at every vertex of the skeleton. We resolve the singularities by separating each vertex into d new vertices, each of valence $\frac{q}{d}$, one obtains a number of disjoint maps. Assign the name $H_j(\mathcal{M})$ to the set of all the component maps.

We could make this definition more rigorous by defining maps combinatorially in terms of set of flags Ω and the involutions r_0, r_1 and r_2 as described in Section 2.5. Then $H_j(\mathcal{M})$ is the set of maps corresponding to the orbits under the action of the group generated by r_0', r_1' and r_2' where $r_0' = r_0$, $r_1' = r_1(r_2 r_1)^{j-1}$ and $r_2' = r_2$.

For integers j, k, it should be clear that $H_j(H_k(\mathcal{M})) = H_{jk}(\mathcal{M})$. Thus if j and q are relatively prime and $\mathcal{N} = H_j(\mathcal{M})$, then there is an integer k such that $\mathcal{M} = H_k(\mathcal{N})$.

4 Cornerations

Note that each corner $\{e_0, v_0, e_1\}$ contains two darts, namely, $\{e_0, v_0\}$ and $\{v_0, e_1\}$. A *corneration* of a graph Γ is a set L of corners of Γ such that every dart of Γ is contained in exactly one corner of L; it follows that then every edge belongs to exactly two corners of L. A *corneration of a map* is then defined as a corneration of its skeleton.

If $\mathcal{C}$ is a circuit decomposition of a graph Γ, then a corner $\{e_0, v_0, e_1\}$ such that e_0, v_0, e_1 is a consecutive triple in some circuit of $\mathcal{C}$ is called a $\mathcal{C}$-*corner*. Clearly, the set $L(\mathcal{C})$ of all $\mathcal{C}$-corners is a corneration.

Suppose now that L is a corneration of a graph Γ. Given a dart $d_0 = \{e_0, v_0\}$, there is a unique edge e_1 such that $\{e_0, v_0, e_1\} \in L$. Denote this $\{e_0, v_0, e_1\}$ as $L(d_0)$. Now, v_0 is one endvertex of e_1; let v_1 be the other. We define $d_1 = \{e_1, v_1\}$ to be the *predecessor* to d_0. This predecessor to d_0 is unique and, similarly, each dart has a unique successor.

This allows to define d_{i+1} to be the predecessor of d_i for every $i \geqslant 0$. Let v_i and e_i be such that $d_i = \{v_i, e_i\}$. Since the number of darts is finite there exists a smallest integer $k > 0$ such that $d_k = d_0$. By uniqueness of successor and predecessor we see that the sequence $(v_0, e_0, \ldots, e_{k-1}, v_k)$ is a circuit in $\mathcal{M}$. Call it $O(d_0)$. Note that $O(d_0) = O(d)$ of every dart d contained in the circuit $O(d_0)$. Let $\mathrm{Cir}(L)$ be the set of all $O(d)$ such that d is a dart of Γ. We present, without proof, the following theorem, which follows easily from the above definitions.

Theorem 3. *If L is a corneration in the graph Γ, then $\mathrm{Cir}(L)$ is a circuit decomposition of Γ. Conversely, if X is a circuit decomposition of Γ, then $L(X)$ is a corneration L in Γ such that $X = \mathrm{Cir}(L)$.*

For a j-corneration L with $j < \frac{q}{2}$, define its *j-complement* L^c to be the set of all j-corners *not* in L. Note that the j-complement of a j-corneration is then also a j-corneration.

Remark 4. If L is a j-corneration for some $j < \frac{q}{2}$, consider a corner $c_0 = \{e_0, v, e_1\}$ of L. The other j-corner containing $\{v, e_1\}$ is some $c_1 = \{e_1, v, e_2\}$, which then must be in the j-complement of L. Proceeding inductively, we get a sequence of k corners $c_i = \{e_i, v, e_{i+1}\}$ for $i = 0, 1, 2, \ldots, k-1$, where $c_{k-1} = \{e_{k-1}, v, e_0\}$. In this sequence, $c_i \in L$ holds exactly when i is even. It follows that k is even.

This sequence may or may not include all of the j-corners at v. More precisely, the number k is $|j|_q$. Thus the sequence uses all j-corners at v if and only if j and q are relatively prime. In particular, if this happens, then j is odd (as the valence q is even in every graph admitting a corneration).

Remark 5. Let $\mathcal{N} = \mathrm{P}(\mathcal{M})$. If c is a j-corner of $\mathcal{M}$ then it is also a j-corner of $\mathcal{N}$, and has the same interior wedges and boundary wedges in $\mathcal{N}$ as in $\mathcal{M}$. If L is a corneration of $\mathcal{M}$ it is also a corneration of $\mathcal{N}$. Further, let $\mathcal{N}' = H_j(\mathcal{M})$. If c is a j-corner of $\mathcal{M}$ then it is also a wedge (1-corner) in $\mathcal{N}'$. If L is a j-corneration of $\mathcal{M}$ it is also a wedge-corneration of $\mathcal{N}'$. We note that if j is not coprime to q, then $\mathcal{N}'$ might be a union of several maps. In this case the definition of a corneration can be extended in a obvious way to $\mathcal{N}'$.

5 Symmetry in Cornerations

If L is a corneration of a graph Γ (or a map $\mathcal{M}$), let $\mathrm{Aut}(L)$ be the subgroup of $\mathrm{Aut}(\Gamma)$ (or of $\mathrm{Aut}(\mathcal{M})$) consisting of all symmetries of Γ (or of $\mathcal{M}$, respectively) which preserve L.

Let G be a subgroup of $\mathrm{Aut}(L)$ which is transitive on corners of L. Then we shall say that L is *G-transitive*. It might happen that G is transitive on the darts of Γ (or on $\mathcal{M}$), in which case, we call L *G-symmetric*. Since every G-transitive or G-symmetric corneration is also $\mathrm{Aut}(L)$-transitive or symmetric (respectively), we may omit G in the above definitions, and simply say that L is *transitive*, or *symmetric*, respectively. The first of the following two propositions is obvious while the second follows easily from Theorem 3.

Proposition 6. *Every transitive corneration of a map is a j-corneration for some j.*

Proposition 7. *If L is a corneration of a map $\mathcal{M}$, and $\mathcal{C} = \mathrm{Cir}(L)$, then L is symmetric if and only if $\mathcal{C}$ is a circuit structure. If, in addition, $\mathcal{M}$ is simple then $\mathcal{C}$ is a cycle structure.*

Remark 8. Following Remark 5, let $\mathcal{N}' = H_j(\mathcal{M})$. If L is a transitive corneration of $\mathcal{M}$, then it is a transitive corneration of $\mathcal{N}'$. However, if $\mathcal{N}'$ is disconnected, then a corneration L might be transitive (or even symmetric) on $\mathcal{N}'$ and still not be transitive on $\mathcal{M}$.

Lemma 9. *Let $\mathcal{M}$ be a map of valence q. If there exists a symmetric j-corneration of $\mathcal{M}$ with $j < \frac{q}{2}$, then j is odd.*

Proof. Suppose that L is a symmetric j-corneration of $\mathcal{M}$ and that $j < \frac{q}{2}$. Let $c = \{e_1, v, e_2\}$ be a corner of L and suppose, for a contradiction, that j is even. Then there exists an edge e_3 incident to v and interior to c that is "$\frac{j}{2}$ wedges apart" from e_1 and from e_2. That is, the corner $\{e_1, v, e_3\}$ (which is not in L) subtends $\frac{j}{2}$ wedges, as does the corner $\{e_2, v, e_3\}$. Since L is symmetric, there must be a $g \in \mathrm{Aut}(L)$ mapping the dart $\{e_1, v\}$ to $\{e_2, v\}$ and since it fixes c, it must map $\{e_2, v\}$ back to $\{e_1, v\}$. We see then that g acts like a reflection swapping e_1 and e_2 while fixing e_3 and v. Let e_4 be the unique edge such that $c' := \{e_3, v, e_4\}$ is a corner of L and observe that g must fix c' point-wise. It follows that g swaps every wedge interior to c' with an exterior wedge, since it swaps e_1 with e_2. This can only happen if the corner c' subtends exactly $\frac{q}{2}$ wedges, contradicting the assumption that every element of L is a j-corner with $j < \frac{q}{2}$. $\square$

The remainder of this paper is in two parts: first, in Sections 6–9, we discuss the group of a transitive corneration and discuss how it acts on vertices and on faces of the map; second, in Section 10, we introduce the generalizations of the split graph and consider the connectedness of the constructed graphs.

6 Local Actions and Local Cornerations

6.1 Local cornerations and local actions

In classifying actions of groups on a map one important step is to determine the ways in which the stabilizer of a vertex acts on its neighborhood. Loosely speaking, this is the *local action* of the group. In this section, we work to determine the possible local actions of a group which is transitive on a corneration.

First, notation: If L is a corneration of a map $\mathcal{M}$ and v is a vertex of $\mathcal{M}$, let L_v be the set of corners in L containing v and call this set the *local corneration* at v. Number the edges incident to v with $0, 1, 2, \ldots, q-1$ consecutively around v, and consider those labels to be in $\mathbb{Z}_q$.

There are, of course, q choices of where to start the numbering and two choices for which way to go around the circle. The $2q$ distinct numberings will be considered to be *equivalent*.

Let ρ be the permutation $i \to i+1$, and for each $f \in \mathbb{Z}_q$, let μ_f be the permutation $i \to f - i$. Then ρ acts as a rotation about v and each μ_f as a reflection. Those permutations generate a group isomorphic to the dihedral group isomorphic to D_q, and they may or may not extend to symmetries of $\mathcal{M}$.

Now consider a transitive j-corneration L of $\mathcal{M}$ for some $j < \frac{q}{2}$. Let $G = \mathrm{Aut}(L)$ and observe that since G is a group of symmetries of a map, the vertex-stabilizer G_v acts faithfully on the edges incident with v. Therefore, we may think of the vertex stabilizer G_v as a permutation group acting on the labels $0, 1, 2, \ldots, q-1$ of the edges incident to v. Then clearly $G_v \leqslant D_q$. It should now be clear that the following are true:

1. We can represent the j-corners at v by pairs $c_i = (i, i+j)$ for $i \in \mathbb{Z}_q$. There are q such pairs and half of them are in L_v.

2. G_v cannot contain ρ, because for any c_i in L_v, $c_i^{\rho^j}$ is in L_v, and shares the edge e_{i+j} with it.

3. If f is even, say $f = 2f'$, then G_v cannot contain μ_f because μ_f would fix the edge f' but not the other edge in its corner.

4. Because G_v is transitive on the elements of L_v, its order must be divisible by $\frac{q}{2}$ and thus, G_v has at most 2 orbits on the edges at v.

5. If G_v has two orbits on edges, then each corner c_i of L_v must contain one edge from each orbit.

There are only three subgroups of D_q that satisfy these requirements:

1. $D_{\frac{q}{2}}$, generated by μ_1 and μ_3, containing ρ^2; this subgroup is transitive. We say the local action is *half-dihedral* (HD, for short).

2. $C_{\frac{q}{2}}$, generated by ρ^2; it has two orbits of edges, those that are even and those that are odd. We say the local action is *half-cyclic* (HC, for short).

3. $D_{\frac{q}{4}}$, generated by μ_1 and μ_5, containing ρ^4. Here there are two orbits of edges; one consists of edges with numbers which are 0 or 1 mod 4, and one has those which are 2 or 3 mod 4. We say the local action is *quarter-dihedral*. (QD, for short)

Since $\mathrm{Aut}(L)$ is vertex-transitive by definition, the type of local action is independent of the choice of v. Then following theorem summarizes the discussion above.

Theorem 10. *If L is a transitive corneration in a map $\mathcal{M}$ of valence q, and $G = \mathrm{Aut}(L)$, then its local action is of type* HD, HC *or* QD. *As a consequence, the index of G in $\mathrm{Aut}(\mathcal{M})$ is 1, 2 or 4. Further, if L is symmetric, then the local action is of type* HD.

It will be convenient to define two standard local cornerations:

Definition 11. Using, as before, c_i to stand for the j-corner $(i, i+j)$ at vertex v in map $\mathcal{M}$, we define:

- For odd j and even q, the standard odd local corneration is $\{c_{2i} \mid i \in \{0, \ldots, \frac{q}{2} - 1\}\}$.

- For $j \equiv 2 \pmod{4}$ and q divisible by 4, the standard even local corneration is $\{c_i \mid i \bmod 4 \in \{0, 3\}\}$.

With these definitions established, the following theorem holds:

Theorem 12. *If L is a transitive j-corneration of a map $\mathcal{M}$ and v is a vertex of $\mathcal{M}$, then*

- j *is odd and* L_v *is equivalent to the standard odd local corneration,*

  or

- $j \equiv 2$ *(mod 4),* L_v *is equivalent to the standard even local corneration and the local action of* $\mathrm{Aut}(L)$ *is of type QD.*

Proof. Let $G = \mathrm{Aut}(L)$. We consider local cornerations having each of the groups mentioned in Theorem 10.

Suppose first that G_v is isomorphic to $C_{\frac{q}{2}}$. Then a corner in L_v must contain one even and one odd edge; thus j must be odd. Assume with no loss of generality, that $(0, j)$ is in L_v. Then L_v must contain and consist of all corners of the form $(2i, 2i + j)$ for $i \in \mathbb{Z}_q$. This is exactly the standard odd local corneration.

Now suppose that G is isomorphic to $D_{\frac{q}{2}}$. Then j must be odd, for otherwise ρ^j, which is in $\langle \rho^2 \rangle$, would send each $(i, i + j)$ in L_v to $(i + j, i + 2j)$ also in L_v, which would not satisfy the corneration definition. Renumbering the edges if necessary, we can assume that $(0, j) \in L_v$, and then because $\rho^2 \in G$, we have that L_v is equivalent to the standard odd local corneration.

Finally, suppose that G is isomorphic to $D_{\frac{q}{4}}$. In particular, suppose that G is generated by μ_1 and μ_5, and so contains ρ^4. Then, as noted above, one orbit of edges is the set of those which are 0 or 1 (mod 4), and the other, those that are 2 or 3 (mod 4). It follows that there are three orbits of wedges:

1. Those of the form $(4i, 4i + 1)$,

2. Those of the form $(4i + 2, 4i + 3)$,

3. Those of form $(4i + 1, 4i + 2)$ or $(4i - 1, 4i)$.

We can generalize this comment a little: For $j \equiv 1$ (mod 4), there are three orbits of j-corners:

1. Those of the form c_{4i},

2. Those of the form c_{4i+2},

3. Those of form c_{4i+1} or c_{4i-1}.

Only the third of these covers all edges at v, and so only this can be a local corneration, but it is equivalent to the standard odd local corneration, with which we are familiar.

On the other odd hand, for $j \equiv 3$ (mod 4), there are three orbits of j-corners:

1. Those of form c_{4i-1},

2. Those of the form c_{4i+1},

3. Those of the form c_{4i} or c_{4i+2}.

Only the third of these covers all edges at v, and so only this can be a local corneration. But it is identical to the standard odd local corneration, with which we are familiar.

Now, j cannot be divisible by 4, because edges which are equivalent mod 4 are in the same orbit. Thus the only remaining possibility for j is 2 (mod 4). Suppose, without any loss of generality, that $(0, j) \in L_v$; then L_v also contains $(0, j)^{\mu_{j-1}} = c_{-1}$ and thus L_v contains all pairs $(i, i + j)$ such that $i \equiv 0, 3$ (mod 4). Thus, corners in L_v come in consecutive pairs: $c_{4i-1} = (4i - 1, 4i - 1 + j)$ and $c_{4i} = (4i, 4i + j)$, while the corners $(4i + 1, 4i + 1 + j)$ and $(4i + 2, 4i + 2 + j)$ are not in L_v. Thus L_v is equivalent to the standard even local corneration. $\qquad\square$

7 Wedge Cornerations

In this section, we consider cornerations in which each corner is a 1-corner, also called a wedge. This restriction is not as severe as it might appear, as Remark 4 allows us to analyse a j-corneration of $\mathcal{M}$ by examining wedge cornerations (1-cornerations) of $H_j(\mathcal{M})$ where j is relatively prime to q. Note that each wedge *belongs* to a unique face in the sense that the two edges of the wedge appear as two consecutive edges on the boundary of a unique face.

Suppose that L is a corneration of a map $\mathcal{M}$ whose elements are wedges, and let $G = \mathrm{Aut}(L)$. For brevity, we refer to wedges in L as *in-wedges* and those not in L as *out-wedges*.

Given a face f of length p, here are some ways that in-wedges might appear around f:

- (A) Every wedge of f is an in-wedge.

- (B) p is even and every other wedge of f is an in-wedge.

- (C) p is divisible by 3 and every third wedge of f is an out-wedge.

- (D) p is divisible by 4 and pairs of in- and out-wedges alternate.

- (E) Every wedge of f is an out-wedge.

We say that f has *pattern* A, or B, for example. These possibilities are illustrated in Figure 3, where the in-wedges are indicated by dots. Theorem 13 will show that these are all the possible pattern.

Theorem 13. *Given that L is a transitive corneration of a map $\mathcal{M}$ whose elements are wedges there are four possibilities for the way that patterns of in-wedges appear in faces of $\mathcal{M}$:*

Type 1 *$\mathcal{M}$ is face-bipartite and all faces of one color have pattern A, while the rest have pattern E.*

Type 2 *Every face has pattern B.*

(a) Pattern A   (b) Pattern B   (c) Pattern C
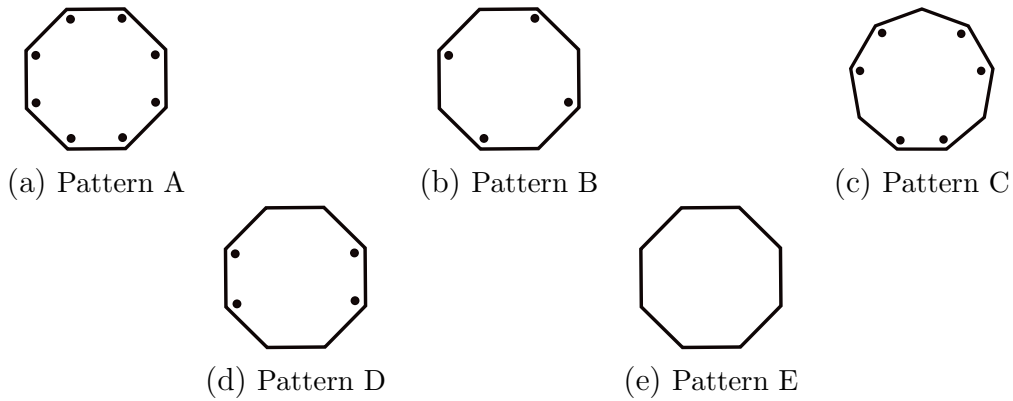
(d) Pattern D   (e) Pattern E

Figure 3: Patterns of corners in a face

Type 3 *Faces are of two patterns, C and E. Faces with pattern E border only those with pattern C, while a face with pattern C meets faces with patterns C and E.*

Type 4 *Every face has pattern D.*

Proof. Let $G = \mathrm{Aut}(L)$. The faces of $\mathcal{M}$ which contain in-wedges form an orbit W of G in its action on faces. Faces not in W must have pattern E. Let f be one element of W. Whatever pattern appears in f that same pattern in every face in W. Let W_f be the set of in-wedges in f.

We first note that W_f is an orbit of some corner of f under the stabilizer G_f in G, which is a subgroup of the dihedral group D_p.

If f has pattern A, then every edge of f is contained in two in-wedges in f itself. Hence, none of its neighboring faces contain an in-wedge containing an edge of f. In particular it cannot be of type A and thus it must be of type E. But faces neighboring faces of type E, in turn, have in-wedges and so must be in W which have pattern A. Thus, the map is face-bipartite, with one class of faces having pattern A, the other, pattern E.

If f has pattern B, due the fact that every dart is contained in exactly one in-wedge, every neighboring face contains an in-wedge and must thus be of type B as well. By connectivity all faces are of type B. We may henceforth assume that no face is of type A or B.

Let m be the length of a longest string S of consecutive in-wedges in f. Note that m cannot be greater than 2 if f is not of type A; otherwise, there would be an in-wedge adjacent to an out-wedge, and another in-wedge not adjacent to any out-wedge, contradicting that L is transitive.

Now, if $m = 1$, then no two consecutive wedges of f are in L. Since f is neither of type A or type B, there exists an edge of f that belongs to no in-wedges of f, implying that the neighboring face, f', of f containing this edge, has two consecutive in-wedges. This contradicts the fact that f and f' are in the same orbit of faces. Therefore $m = 2$.

We then see there is an integer $k > 1$ and integers $n_1, n_2, \ldots n_k$ such that the pattern of wedges around f consists of a string S_1 of two in-wedges, followed by a string T_1 of n_1

out-wedges, followed by a string S_2 of two in-wedges, followed by a string T_2 of n_2 out-wedges, and so on. Since L is transitive, there is a symmetry mapping the first in-wedge of S_1 to the first in-wedge of S_2. Such a symmetry cannot be a reflection, for it would map the last wedge in S_1 to the last wedge of T_1. Thus, it has to be a rotation and it must map each S_i to S_{i+1} and each T_i to T_{i+1}. It follows that for some n, each n_i is equal to n. Because half of all wedges in the map are in L, n cannot be more than m. So the remaining possibilities are

1. $m = 2$ and $n = 1$, giving pattern C for face f. Consider an edge e of f belonging to two in-wedges. In the face f' across e from f, the wedges containing e must be out-wedges. Thus f' is not in W, and so has pattern E.

2. $m = 2$ and $n = 2$. Then in each face of W, pairs of in-wedges alternate with pairs of out-wedges: pattern D. As before, the count shows that W must be the set of all faces.

This concludes the proof. $\square$

Remark 14. Some information about the circuit structure $\mathrm{Cir}(L)$ and local action of $\mathrm{Aut}(L)$ can be deduced in each of the four possible cases in the statement of Theorem 13

1. If L is of Type 1, then $\mathcal{M}$ is face-bipartite and the circuits in the circuit structure $\mathcal{C}(L)$ are the faces of one color in $\mathcal{M}$. The local action is half-dihedral or half-cyclic. Further, L forms a corneration of Type 2 in $\mathrm{P}(\mathcal{M})$.

2. If L is of Type 2, then G must be transitive on faces. The elements of $\mathcal{C}(L)$ are Petrie paths. The local action is half-dihedral or half-cyclic. Further L forms a corneration of Type 1 in $\mathrm{P}(\mathcal{M})$.

3. If L is of Type 3, then q must be divisible by 4, and G must not be transitive on faces. The orbit W contains at least $\frac{3}{4}$ of the faces. The local action of G is quarter dihedral. Here, L forms a corneration of Type 3 in $\mathrm{P}(\mathcal{M})$, as well.

4. If L is of Type 4, then G must be transitive on faces. The local action of G is half cyclic, and then L forms a corneration of Type 4 in $\mathrm{P}(\mathcal{M})$.

Remark 15. The previous sections have given us an understanding of transitive j-cornerations for which j and q are relatively prime, as these are equivalent to wedge cornerations in $H_j(\mathcal{M})$. From Remark 5, we see that even if $\gcd(q, j) > 1$, a j-corneration L of $\mathcal{M}$ is a wedge corneration of $H_j(\mathcal{M})$; it must then exhibit one of the four patterns from Theorem 13.

8 Symmetry-type graphs of cornerations

The topic of this section is the idea of a *symmetry-type graph*. Informally, a symmetry-type graph of a group G acting on a map $\mathcal{M}$ is a quotient of the flag graph of $\mathcal{M}$ with respect to G. The symmetry-type graph was first discussed in [3], under the name *Delaney symbol*. A definition closer to the one we wish to present is found in [7], where more (and earlier) references to this notion may be found. In order to present this more rigourous definition, we need to introduce some preliminary ideas.

8.1 Pre-graphs and diagrams

We begin with the idea of a *pregraph*. A pregraph is a quadruple $(V, D, \mathrm{beg}, \mathrm{inv})$ where V and D are two disjoint non-empty sets and where $\mathrm{beg}\colon D \to V$ and $\mathrm{inv}\colon D \to D$ are two mappings such that $\mathrm{inv}(\mathrm{inv}(x)) = x$ for every $x \in D$. Elements of V and D are then called vertices and darts, respectively. An unordered pair $\{x, \mathrm{inv}(x)\}$, $x \in D$, is then called an edge of the pregraph. The set of edges is denoted E. If $x = \mathrm{inv}(x)$, then the edge $\{x, \mathrm{inv}(x)\}$ is said to be a *semiedge*. Similarly, if $\mathrm{beg}(x) = \mathrm{beg}(\mathrm{inv}(x))$ but $x \neq \mathrm{inv}(x)$, then the edge $\{x, \mathrm{inv}(x)\}$ is a *loop*. All other edges are called *links*. A pregraph can be depicted by representing each vertex as a node (which might be a dot or a circle or a box). A link $\{x, \mathrm{inv}(x)\}$ is then represented as a line segment between the nodes representing $\mathrm{beg}(x)$ and $\mathrm{beg}(\mathrm{inv}(x))$, a loop $\{x, \mathrm{inv}(x)\}$ is drawn as a closed curve joining $\mathrm{beg}(x)$ to itself, while a semiedge $\{x\}$ is represented as a short line-segment emanating from $\mathrm{beg}(x)$. An isomorphism from a pregraph $(V, D, \mathrm{beg}, \mathrm{inv})$ to a pregraph $(V', D', \mathrm{beg}', \mathrm{inv}')$ is a bijection $\varphi\colon V \cup D \to V' \cup D'$, $\varphi(V) = V'$ and $\varphi(D) = D'$ such that $\varphi(\mathrm{beg}(x)) = \mathrm{beg}'(\varphi(x))$ and $\varphi(\mathrm{inv}(x)) = \mathrm{inv}'(\varphi(x))$ for every $x \in D$.

Note that the notion of a pregraph generalizes that of a graph. In particular, given a graph $\Gamma = (V, E, \partial)$, one can let $D = \{\{v, e\} : e \in E, v \in \partial(e)\}$ be the set of darts of Γ, let $\mathrm{beg}(\{v, e\}) = v$ and let $\mathrm{inv}(\{v, e\}) = \{u, e\}$ where $\partial(e) = \{v, u\}$. Then $(V, D, \mathrm{beg}, \mathrm{inv})$ is a pregraph with additional property that it has no loops or semiedges. Conversely, one can easily see that every pregraph without loops or semiedges can be obtained from a graph in such a way.

A mapping $c\colon E \to C$ from the edge-set E of a pregraph to a set C of colors is called an edge-coloring of a pregraph. Such an edge-coloring is *locally bijective* provided that the restriction of c to the set of edges incident to a vertex v is bijective onto C for every $v \in V$. In particular, existence of a locally bijective edge-coloring implies that the valence of every vertex is equal to the cardinality of $|C|$. A pair (Γ, c) where Γ is a pregraph and $c\colon E \to C$ is a locally-bijective edge-coloring of Γ is called an edge-colored pregraph (or, to be more precise, an edge-$|C|$-colored pregraph).

If (Γ, c) and (Γ', c') are two edge-colored pregraphs and $\varphi\colon \Gamma \to \Gamma'$ is a pregraph isomorphism such that $c(e) = c'(\varphi(e))$ for every edge e of Γ, then we call φ an isomorphism of edge-colored graphs.

Finally, for the purposes of this paper, we define a *vertex-shaped edge-3-colored pregraph* (or simply a *diagram*, for short) to be a edge-3-colored pregraph together with a labelling $V \to \{\mathrm{B}, \mathrm{O}\}$, where vertices assigned the label B will be called *boxes* and depicted as

squares, while those assigned the label O will be called *ovals* and will be depicted as such. Two diagrams are then isomorphic if they are isomorphic as edge-colored pregraphs via a vertex-label preserving isomorphism.

8.2 Symmetry-type graphs

The *flag (pre)graph* of a map $\mathcal{M}$ with the flag-set Ω is a edge-colored pregraph (Γ, c), where the vertex-set of Γ is Ω, and a dart (of color i) is an ordered pair $x = (f, r_i f)$ for some $i \in \{0, 1, 2\}$, where $\text{beg}(x) = f$ and $\text{inv}(x) = (r_i f, f)$. Let Δ_i denote the set of all darts of color i and let $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$. Note that the colors of x and $\text{inv}(x)$ are the same so we may define the color $c(e)$ of the edge $e = \{x, \text{inv}(x)\}$ to be their common color. An important observation is that the edges of colors 0 and 2 always induce closed walks of length dividing 4.

Assuming the above notation, we can now define the *symmetry-type graph* $\mathcal{M}/G$ of the map $\mathcal{M}$ with respect to a group of symmetries G as a pregraph $(\Omega/G, \Delta/G, \text{beg}', \text{inv}')$, where Ω/G and Δ/G are the orbits of the natural action of G on Ω and Δ, respectively, and where $\text{beg}'(x^G) = \text{beg}(x)^G$ and $\text{inv}'(x^G) = \text{inv}(x)^G$. That is, the symmetry type graph is the quotient (as defined in [9], for example) of the flag-graph of $\mathcal{M}$ with respect to the action of G. Finally, since the action of G on the flag-graph clearly preserves the colors of the edges, this allows us to define a color of an edge $e^G = \{x^G, \text{inv}'(x^G)\}$ to be that of the edge e in the flag-graph. Note that this coloring is locally-bijective and hence the symmetry type graph is an edge-3-colored pregraph. The symmetry type graph can be visualized by drawing as a pregraph with 0-, 1- and 2-edges depicted as dotted, single and double lines, as in Figure 4.

Furthermore, when given a wedge corneration L of a map $\mathcal{M}$, we may assign the label B (box) to every flag of $\mathcal{M}$ contained in an in-wedge of L, and the label O (oval) to every flag contained in an out-wedge. In this way, a symmetry type graph $\mathcal{M}/G$ with respect to some subgroup $G \leqslant \text{Aut}(L)$ is a diagram. Consider the sample diagram in Figure 4.
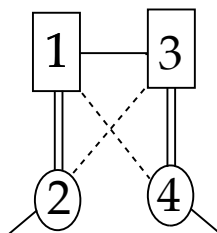

Figure 4: A symmetry-type graph (in fact, a diagram as defined in 8.1) of a map with respect to a group of symmetries acting on a wedge corneration. The 0-edges are depicted with dotted lines, 1-edges with single solid lines and 2-edges with double solid lines. Boxed vertices 1 and 3 correspond to orbits of in-wedges while the ovals 2 and 4 correspond to orbits of out-wedges.

Note how much about $\mathcal{M}$ we can deduce from $\mathcal{M}/G$:

- Any two consecutive flags in one face are either 0-adjacent or 1-adjacent (and a flag can be reached from any other flag in the same face by a sequence of i-adjacent flags, with $i \in \{0, 1\}$). Because the 0-edges and 1-edges of the symmetry-type graph join all four nodes in this diagram, every face contains a flag of each orbit. Thus the group acts transitively on faces.

- Looking at those edges in slightly more detail, we see that a sequence of 0-1-paths returns to the starting flag in four steps, we deduce that for each face, the group contains a rotation by four steps about that face.

- Because of the semi-edge at node 2, each flag of orbit 2 is 1-adjacent to another flag of the same orbit.

- As the flags around a vertex are joined by 1- and 2- adjacencies, and the 1- and 2-edges in the diagram connect all nodes, the group acts transitively on vertices as well. Thus G_v is QD.

- Two flags in a wedge are 1-adjacent. If they are in an in-wedge (rectangular nodes), one is in orbit 1, the other (if there is another) in orbit 3. Thus the group acts transitively on in-wedges.

- Of the two flags in an out-wedge (oval nodes), if one is in orbit 2, the other in orbit 2, as well. If one is in 4, so is the other. Thus the group has two orbits on out-wedges. It follows that the group must have a reflection at each such corner, and similarly, a reflection at each flag of orbit 4. Therefore faces have pattern B.

Remark 16. If $\mathcal{M}$ is a map with a G-transitive wedge-corneration, then the following holds for the symmetry type graph $\mathcal{M}/G$:

1. $\mathcal{M}/G$ is connected and has 2 or 4 nodes;

2. an edge of color 2 must connect nodes of different shapes (and thus there no semi-edges of color 2);

3. an edge of color 1 must connect nodes of the same type (and thus there are no full edges of color 1 if $\mathcal{M}/G$ has only two vertices, but only semi-edges);

4. two distinct rectangular nodes must be connected by an edge of color 1;

5. a walk of length 4 along the edges and semi-edges of $\mathcal{M}/G$ that alternates colors 0 and 2 must be closed.

There are 12 possible symmetry-type graphs for a transitive corneration, and these are shown in Figure 5. From these diagrams one can easily deduce information on the number of orbits on vertices, edges and faces of the corresponding quotient group, as well as the type of wedge corneration (see Theorem 13) and the local action. This is summarized in table 1

STG	V orbs	E orbs	F orbs	Pattern	local type
(a)	1	1	2	A, E	HD
(b)	1	1	1	A, E	HC
(c)	1	2	2	A, E	HC
(d)	1	1	2	A, E	QD
(e)	1	2	3	A, E	QD
(f)	1	1	1	B	HD
(g)	1	1	1	B	HC
(h)	1	2	2	B	HC
(i)	1	1	1	B	QD
(j)	1	2	2	B	QD
(k)	1	2	2	C, E	QD
(l)	1	2	1	D	HC

Table 1: Transitive corneration groups. Each line represents one of the 12 possible symmetry type graphs of transitive wedge cornerations, with the letters (a)–(l) in column STG corresponding to the 12 diagrams in Figure 5. The next three columns (V orbs, E orbs and F orbs) show the number of orbits on vertices, edges and faces, respectively of the transitive corneration group yielding the diagram. The column 'Pattern' shows the face patterns (as given in Figure 3) appearing in the corneration. The last column, labelled local type, shows the type of the local action as determined in Theorem 10.

Theorem 17. *A diagram is a symmetry type graph of a map $\mathcal{M}$ with respect to a group G acting transitively on a wedge corneration of $\mathcal{M}$ if and only if it is isomorphic to one of the diagrams in Figure 5 and this happens if and only if it satisfies the requirements on $\mathcal{M}/G$ in Remark 16.*

Proof. The readers may convince themselves that every diagram in Figure 5 satisfies the requirements of Remark 16 and that every such diagram appears in Figure 5. We then need to show only that each of these diagrams actually occurs as the symmetry-type graph of a transitive wedge corneration. In what follows, we will exhibit a map $\mathcal{M}$ admitting ten different transitive wedge cornerations, each realizing one of the diagrams (a)–(j) of Figure 5. This map $\mathcal{M}$ will be defined in term of the opposite operator, opp. Recall that if $\mathcal{N}$ is a map, then $\text{opp}(\mathcal{N})$ is defined as the dual of the Petrie of the dual of $\mathcal{N}$.

As was explained in [15], $\text{opp}(\mathcal{N})$ can also be obtained by separating the faces of $\mathcal{N}$ and glueing them again in such a way that if two faces F_1 and F_2 shared an edge with endvertices u and v then in $\text{opp}(\mathcal{N})$ the segment from u to v in F_1 gets identified with the segment from v to u in F_2. In that sense, we can say that $\mathcal{N}$ and $\text{opp}(\mathcal{N})$ have the same set of faces, and two faces are adjacent in $\mathcal{N}$ if and only if they are adjacent in $\text{opp}(\mathcal{N})$. In particular if $\mathcal{N}$ is face-bipartite then so is $\text{opp}(\mathcal{N})$.

Consider the graph with vertex-set $\mathbb{Z}_4 \times \mathbb{Z}_4$ with 'horizontal' edges of the form $(x, i) \sim (x, i+1)$ and 'vertical' edges of the form $(x, i) \sim (x+1, i)$. By glueing a disc along each
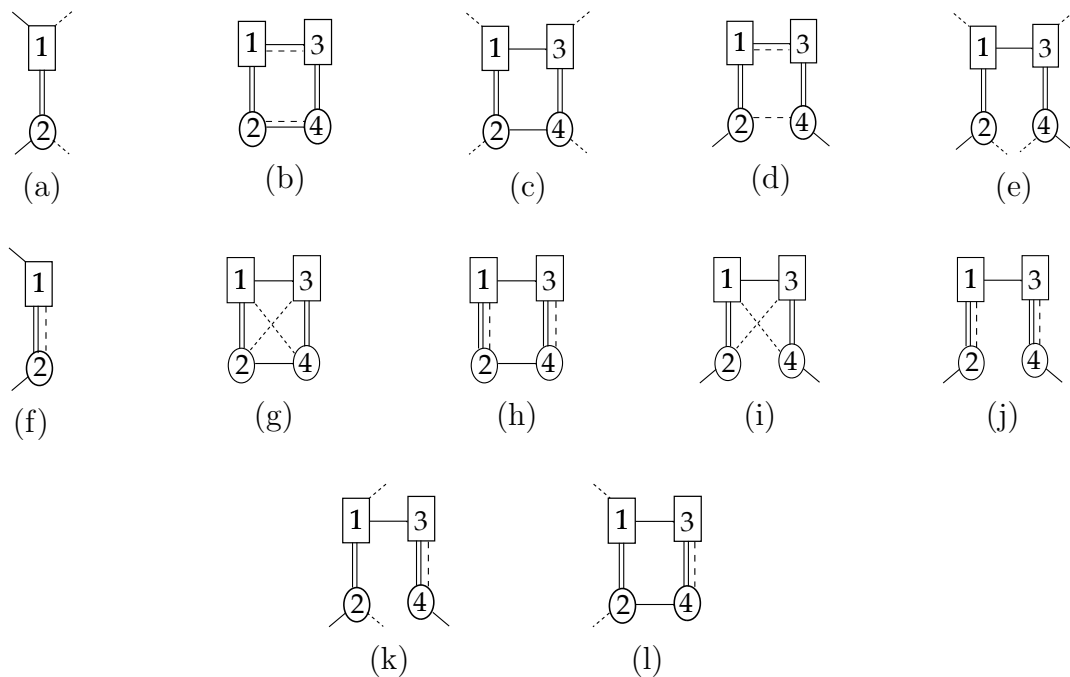
Figure 5: All symmetry-type graphs of transitive cornerations

4-cycle of alternating horizontal and vertical edges, one obtains a face-bipartite 4-valent map $\mathcal{N}$ on the torus with faces of length four and Petrie paths of length 8. This map is sometimes denoted as $\{4,4\}_{4,0}$ (see [2]) and is named RM[32, 3] in the census of rotary maps [8].

Now consider the opposite map $\mathcal{M} = \mathrm{opp}(\mathcal{N})$ (this map is named RM[32, 4] in [8] and it sits on an orientable surface of genus 5). It is an 8-valent face-bipartite map with Petrie paths of length 4. If we think of the faces as being colored red and green, and if we let L be the set of all wedges of green faces, this gives us a corneration of pattern A (in the green faces). Since $\mathrm{Aut}(\mathcal{M})$ acts transitively on the flags of $\mathcal{M}$, the group $G := \mathrm{Aut}(L)$ has index 2 in $\mathrm{Aut}(\mathcal{M})$ and acts transitively on flags in green faces as well as on flags in red faces, implying that L is a transitive corneration. In particular, G has two orbits on wedges and its symmetry-type graph is diagram (a) in Figure 5. Further, one can readily check with a computer package such as GAP [4] or Magma [1] (and using the census [8]) that G has four subgroups, each of index 2, each having 4 orbits of flags, and each are transitive on the wedges in L. Their symmetry-type graphs are diagrams (b)–(e) in Figure 5.

The map $\mathcal{M}$ is isomorphic to $\mathrm{P}(\mathcal{M})$. The corneration $\mathrm{P}(L)$ has pattern B and the operator P turns diagram (a) into (f), (b) into (g) and so on. Thus diagrams (a)–(j) have realizations on $\mathcal{M}$.

Diagram (k) has many realizations, but the easiest to see are in the geometric anti-prisms. As a map on the sphere, the n-antiprism is formed by two n-gonal faces joined by a band of $2n$ triangles. The corneration L_n contain none of the wedges in the n-gons, but does contain the two wedges of each triangle that are incident with the edge joining

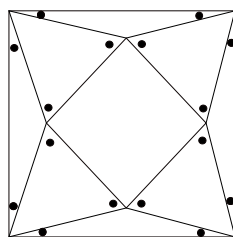the triangle to an n-gon. The case $n = 4$ is shown in Figure 6.

Figure 6: A corneration in the 4-antiprism

Diagram (l), also, has many realizations, and the easiest to see in this case are on the torus. Let the map $\mathcal{M}$ on the torus be formed from a $2m \times n$ rectangle of squares, with opposite sides identified in the usual way. The corneration L contains wedges incident with one of the two vertical edges in each square. The choice of which of those two edges to use is made consistently along each row of the grid but alternatingly along each column. The particular case with a 4×5 rectangle is shown in Figure 7. □
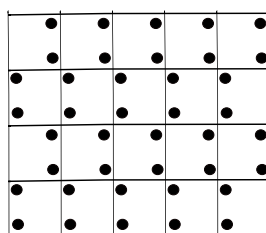

Figure 7: A corneration in a torus map

9 Symmetric Cornerations

For many applications, we are interested in the case in which $\mathrm{Aut}(L)$ acts transitively on the darts of the maps $\mathcal{M}$; these are the symmetric cornerations. We want to determine the maps in which these occur and how to find L (which we can assume is a wedge-corneration) in such a map. Recall that a dart can be seen as an orbit of flags under r_2. Therefore, if $\mathrm{Aut}(L)$ is arc-transitive, then the subgraph of the corresponding symmetry-type graph induced by edges of type 2 must have a single connected component (or, if you will, the edges of type 2 are a spanning tree in $\mathcal{M}/G$).

Scanning Figure 5, we see that only Diagrams (a) and (f) satisfy this condition, and thus any symmetric corneration must have one of those two as its symmetry-type graph.

Notice that in Diagram (a), all the flags of any one face (that is, an orbit under r_0 and r_1) are in one orbit, and therefore, the group must contain all reflections fixing that face. These are all the reflections whose axes pass through the center of that face.

Theorem 18. *Let $\mathcal{M}$ be a map of valence q and let $j < \frac{q}{2}$. Then, $\mathcal{M}$ admits a symmetric j-corneration if and only if j is odd and $\mathrm{Aut}(\mathcal{M})$ has a subgroup that acts half-reflexibly on $\mathcal{M}$ or $\mathrm{P}(\mathcal{M})$.*

Proof. First let L be a symmetric j-corneration of a map $\mathcal{M}$. Recall that this means that $\mathrm{Aut}(L)$ is transitive on the darts of $\mathcal{M}$, and that, by Lemma 9, j is odd. Then the local corneration is the standard odd local corneration, and so the stabilizer of a vertex must contain the reflections we called μ_i in Section 6 exactly when i is odd. These are reflections about each axis joining a vertex to a face-center.

For an edge $e = \{u, v\}$, the corners $L(e, u)$ and $L(e, v)$ either align convexly or with inflection, consistently throughout the map. Without loss of generality, suppose $\mathcal{M}$ (and not its Petrie) is the map for which those alignments are all convex. Then the element of $\mathrm{Aut}(L)$ which reverses an edge in $\mathcal{M}$ must be a reflection about the axis joining a face-center to an edge-midpoint. Thus $\mathrm{Aut}(L)$ acts half-reflexibly on $\mathcal{M}$, proving the first assertion of the theorem.

Conversely, suppose that $\mathcal{M}$ is a map, that $G \leqslant \mathrm{Aut}(\mathcal{M})$ is a group acting half-reflexibly on $\mathcal{M}$, and that j is odd. Then G has two orbits on flags, and we may think of those orbits as colors, red and green. Then for any one face, its flags are all of the same color, while any two adjacent faces have opposite colors.

Wedges are orbits under r_1, and so each wedge is a single color. Because j is odd, the interior boundary wedges of any j-corner are the same color. Define L_R to be the set of all j-corners of $\mathcal{M}$ whose interior boundary wedges are red and similarly define L_G to be the set of all j-corners of $\mathcal{M}$ whose interior boundary wedges are green. Then L_R and L_G are two cornerations each preserved by G, and thus are symmetric. $\square$

Corollary 19. *Let $\mathcal{M}$ be a reflexible map of valence q and $j < \frac{q}{2}$ a positive integer. Then $\mathcal{M}$ admits a symmetric j-corneration if and only if j is odd and at least one of $\mathcal{M}$ and $\mathrm{P}(\mathcal{M})$ is face-bipartite.*

Proof. This follows directly from Theorem 18 and Lemma 2. $\square$

10 Split graphs

In this section, we first define $\mathbb{S}(L, K)$ for a corneration L and a set K of corners. This is a the generalization of the Split Graph of a cycle decomposition. We give examples $\mathcal{A}(L), \mathcal{B}(L), \mathcal{C}(L), \mathcal{D}(L)$ of the $\mathbb{S}$ construction and indicate when these are vertex-transitive. We then discuss which of the graphs are locally connected (and thus connected), and the valence of each.

Definition 20. Let L be a corneration of Γ, and let K be any collection of corners in Γ disjoint from L. Let the split graph $\mathbb{S}(L, K)$ be the simple graph whose vertices are the elements of L and whose edges are of the two following kinds:

1. *'Old'* edges of $\mathbb{S}(L, K)$ are all $\{c_1, c_2\}$, where c_1 and c_2 are corners in L such that $c_1 \cap c_2 = \{e\}$ for some edge e of Γ.

2. *'New'* edges of $\mathbb{S}(L, K)$ are all pairs of the form $\{L(\{e_1, v\}), L(\{e_2, v\})\}$, where $\{e_1, v, e_2\} \in K$.

Proposition 21. *If L is a transitive corneration of a map $\mathcal{M}$ and a set K of corners disjoint from L is invariant under a subgroup G of $\mathrm{Aut}(L)$ acting transitively on L, then $\mathbb{S}(L, K)$ is a vertex-transitive graph.*

Proof. It should be clear that $H = \mathrm{Aut}(L)$ is a subgroup of $G = \mathrm{Aut}(\Gamma)$, where the latter is considered a group of permutations on the vertices and edges of Γ. Because H acts on L, it acts on the vertices and old edges of $\mathbb{S}(L, K)$. Because H acts on K, it acts on new edges of $\mathbb{S}(L, K)$ as well.

Because the action on L is transitive, the action of H on $\mathbb{S}(L, K)$ is vertex-transitive, as required. $\qquad\square$

Let L be a corneration in a graph Γ and let K be a set of corners in Γ disjoint from L. Then the split graph $\mathbb{S}(L, K)$ is called *locally connected* provided that for every $v \in \mathrm{V}(\Gamma)$ the subgraph of $\mathbb{S}(L, K)$ induced by all the corners in L containing v is connected. Note that if the split graph $\mathbb{S}(L, K)$ is locally connected, then it is connected, but the converse does not hold: it can happen that $\mathbb{S}(L, K)$ is connected even though it is not locally connected. The smallest example appears to come from 3-cornerations on a reflexible map of type $\{3, 12\}_9$ on 162 edges and its Petrie. This map is item $[162, 56]$ in the on-line census of rotary maps on (the section on regular maps) [8] and its Petrie is item $[162, 69]$ there.

10.1 Four constructions

We present here four examples of graphs constructed using Definition 20, and we prove that each is vertex-transitive when L is transitive. Then in the following subsection, we will determine for each, its valence and criteria for it to be locally connected.

Definition 22. If L is a j-corneration of a map $\mathcal{M}$ for $j < \frac{q}{2}$, let $\mathcal{A}(L) = \mathbb{S}(L, K)$, where K is the j-complement of L in $\mathcal{M}$.

Definition 23. If L is a j-corneration of a map $\mathcal{M}$, $2 \leqslant j \leqslant \frac{q}{2}$, and K is the set of all wedges in $\mathcal{M}$, let $\mathcal{B}(L) = \mathbb{S}(L, K)$.

Definition 24. If L is a j-corneration of a map $\mathcal{M}$, $2 \leqslant j < \frac{q}{2}$, and K_i is the set containing every wedge in $\mathcal{M}$ which is an interior boundary wedge of some corner of L, let $\mathcal{C}(L) = \mathbb{S}(L, K_i)$.

Definition 25. If L is a j-corneration of a map $\mathcal{M}$, $2 \leqslant j < \frac{q}{2}$, and K_x is the set containing every wedge in $\mathcal{M}$ which is an exterior boundary wedge of some corner of L, let $\mathcal{D}(L) = \mathbb{S}(L, K_x)$.

Theorem 26. *If L is a transitive corneration, then each of the graphs $\mathcal{A}(L)$, $\mathcal{B}(L)$, $\mathcal{C}(L)$, $\mathcal{D}(L)$ is vertex-transitive.*

Proof. In each case, the group $\mathrm{Aut}(L)$ preserves the corresponding set K, and so Proposition 21 applies. $\qquad\square$

10.2 Local Connectivity and valence of Split Graphs

Local connectivity is decided in the local corneration, and we use and enlarge on the notation from Section 6:

- L_v is the collection of edges and corners containing the vertex v.

- Edges are numbered $0, 1, 2, \ldots, q - 2, q - 1$ in order consecutively around v.

- A corner is just a pair $\{a, b\}$ of distinct elements $a, b \in \mathbb{Z}_q$.

- If L is a j-corneration, c_i stands for the corner $\{i, i + j\}$.

We first deal with the corneration of all straight corners. Here, the construction $\mathcal{A}$ does not apply because $L^c = \emptyset$. Similarly, neither $\mathcal{C}$ nor $\mathcal{D}$ applies because the interior of straight corners is not defined.

Theorem 27. *Let $\mathcal{M}$ be a dart-transitive map, and let L be the set of straight corners. Then L is transitive and $\mathcal{B}(L)$ is vertex-transitive and locally connected. Its valence is 4 unless $q = 4$, in which case the valence is 3.*

Proof. The straight corners at v are all $c_i = \{i, i + \frac{q}{2}\}$. The wedges sharing an edge with the straight corner c_0 are $\{0, 1\}, \{\frac{q}{2} - 1, \frac{q}{2}\}, \{\frac{q}{2}, \frac{q}{2} + 1\}, \{q - 1, 0\}$. The first and third of these join c_0 to c_1 and the second and fourth join c_0 to to c_{-1} Similarly, each c_i is joined to c_{i-1} and c_{i+1}, and so $\mathcal{B}(L)$ is locally connected. It is of valence 4 unless $c_1 = c_{-1}$, which happens if and only if $q = 4$. $\square$

Since we have dealt with straight cornerations in Theorem 27 (and the preceding comments), we can now assume that $j < \frac{q}{2}$ for any transitive j-corneration, so that all four of $\mathcal{A}(L), \mathcal{B}(L), \mathcal{C}(L), \mathcal{D}(L)$ are defined.

Of these, we now consider the graph $\mathcal{A}(L)$:

Theorem 28. *If L is a transitive corneration of width j, then $\mathcal{A}(L)$ is locally connected if and only if $\gcd(q, j) = 1$. Its valence is 4, unless $j = \frac{q}{4}$, in which case the valence is 3.*

Proof. We see from Remark 4 that $\mathcal{A}(L)$ is locally connected exactly when $\gcd(q, j) = 1$. Assuming, as we may, that $c_0 \in L$, and letting K be the j-complement of L, we have that $c_0 \in L, c_j \in K, c_{2j} \in L, \ldots$. Thus, corner c_0 is connected to c_{2j} and to c_{-2j}. Then c_0 is connected to two elements of L (and so has valence 4 in $\mathcal{A}(L)$) unless $2j = -2j$. This last can happen only if $4j = 0$, and then, the valence of v is 3. $\square$

We will present results about the remaining graphs, $\mathcal{B}(L), \mathcal{C}(L), \mathcal{D}(L)$ together, as the wedge-set in the construction of $\mathcal{B}(L)$ is the union of those for $\mathcal{C}(L)$ and $\mathcal{D}(L)$. But we will separate the presentation into two cases: the local corneration is the standard odd local corneration and the local corneration is the standard even local corneration.

Theorem 29. *If L is a transitive corneration of odd width $3 \leqslant j < \frac{q}{2}$, then:*

1. $\mathcal{B}(L)$ is locally connected,

2. $\mathcal{C}(L)$ is locally connected exactly when $\gcd(q, j - 1) = 2$,

3. $\mathcal{D}(L)$ is locally connected exactly when $\gcd(q, j + 1) = 2$.

Furthermore:

(5) $\mathcal{B}(L)$ has valence 6 unless q is divisible by 4 and $j = \frac{q}{2} - 1$, in which case it has valence 5,

(6) $\mathcal{C}(L)$ and $\mathcal{D}(L)$ each have valence 4 unless q is divisible by 4 and $j = \frac{q}{2} - 1$, in which case they have valence 3.

Proof. Because j is odd, we know that the local corneration is equivalent to the standard odd local corneration. Assuming, as we may, that $c_0 \in L_v$, we have that $L_v = \{c_{2i} | i \in \mathbb{Z}_{\frac{q}{2}}\}$. The wedge-set for $\mathcal{C}$ is $\{(2i, 2i + 1) | i \in \mathbb{Z}_{\frac{q}{2}}\}$, while that for $\mathcal{D}$ is $\{(2i - 1, 2i) | i \in \mathbb{Z}_{\frac{q}{2}}\}$.

Thus, in $\mathcal{C}$, $(0, 1)$ connects c_0 to c_{1-j} and $(j - 1, j)$ connects c_0 to c_{j-1}. In general, c_i is connected to c_{i+j-1}, and to c_{i-j+1}. Thus, each corner is connected by new edges to two other corners of L, giving it valence 4. Further, a connected component in the local corneration consists of all $c_{i(j-1)}$, and this is all of L_x if and only if $\gcd(q, j - 1) = 2$, and so the graph is locally connected if $\gcd(q, j - 1) = 2$.

In $\mathcal{D}$, $(-1, 0)$ connects c_0 to c_{-1-j}, and $(j, j + 1)$ connects c_0 to c_{j+1}, giving two new edges to c_0 unless $-1 - j = j + 1$. Thus, the valence of $\mathcal{D}(L)$ is 4 and that of $\mathcal{B}(L)$ is 6, unless $j = \frac{q}{2} - 1$. In that exceptional case, the valences are 3 and 5, respectively. Finally, since any c_i is connected to $c_{i \pm (j+1)}$, $\mathcal{D}(L)$ is locally connected if and only if $\gcd(q, j + 1) = 2$. $\square$

Our final effort in this section is to determine valence and connectivity for graphs $\mathcal{B}(L), \mathcal{C}(L), \mathcal{D}(L)$ when j is even.

Theorem 30. *If L is a transitive corneration of even width j, $2 \leqslant j < \frac{q}{2}$, then*

1. $\mathcal{B}(L)$ is locally connected,

2. $\mathcal{C}(L)$ is locally connected exactly when $\gcd(q, j - 2) = 4$,

3. $\mathcal{D}(L)$ is locally connected exactly when $\gcd(q, j + 2) = 4$.

Furthermore:

(4) $\mathcal{B}(L)$ has valence 6, unless $j = 2$, in which case it has valence 5,

(5) $\mathcal{C}(L)$ and $\mathcal{D}(L)$ each have valence 4 unless unless $j = 2$, in which case it has valence 3.

Proof. Recall from the proof of Theorem 12, that j must be 2 mod 4, and the local corneration L_v must be equivalent to the standard even local corneration. We can assume, then, that L_v is $\{c_i \mid i \bmod 4 \in \{0, 3\}\}$. Then the wedge-set for $\mathcal{C}(L)$ is all $(i, i+1)$ with i mod 4 in $\{0, 1, 3\}$, the wedge-set for $\mathcal{D}(L)$ is that for $\{1, 2, 3\}$, and, of course, the wedge-set for $\mathcal{B}(L)$ is the union of those, that is, all wedges. For $\mathcal{C}$, the wedge set includes $(-1, 0), (0, 1)$ and $(j-1, j)$, which connect c_0 to c_{-1}, c_{1-j}, c_{-1} respectively. Likewise, the wedge set includes $(-1, 0), (j-2, j-1)$ and $(j-1, j)$, which connect c_{-1} to c_0, c_{j-2}, c_0 respectively. Thus, each vertex has two new edges, giving it a valence of 4, unless $j-2 = 0$, where the valence is 3.

Further, each corner $(4i, 4i+j)$ is paired with $(4i-1, 4i+j-1)$, and the corresponding vertices are joined by a new edge. Moreover, one corner from each pair is joined to a corner from a pair that us $j-2$ places forward or backward around v; for instance, in the pair $\{c_{-1}, c_0\}$, c_{-1} is joined to c_{1-j}, of the pair $\{c_{1-j}, c_{2-j}\}$, while c_0 is joined to c_{j-2} of the pair $\{c_{j-3}, c_{j-2}\}$. Thus, $\mathcal{C}(L)$ is locally connected if and only if $\gcd(q, j-2) = 4$.

Similar reasoning supports the conclusions about $\mathcal{D}$ and $\mathcal{B}$. $\qquad\square$

10.3 Cubic splits

One of our motivating examples is the construction of a cubic split graph from cycle structure. To honor that heritage, we gather here the cases in which the constructed split graph has valence 3:

Corollary 31. *Suppose that L is a transitive j-corneration for $1 \leqslant j \leqslant \frac{q}{2}$. Then:*

- *by Theorem 28, $\mathcal{A}(L)$ has valence 3 if and only if $j = \frac{q}{2}$.*

- *by Theorem 27, $\mathcal{B}(L)$ has valence 3 if and only if $j = \frac{q}{4}$.*

- *by Theorem 30, $\mathcal{C}(L)$ has valence 3 if and only if $j = 2$.*

- *by Theorem 29, $\mathcal{D}(L)$ has valence 3 if and only if q is divisible by 4 and $j = \frac{q}{2} - 1$.*

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