Intervals in Dyck Paths and the Wreath Conjecture

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Abstract

Let $\iota_k(m,l)$ denote the total number of intervals of length m across all Dyck paths of semilength k such that each interval contains precisely l falls. We give the formula for $\iota_k(m,l)$ and show that $\iota_k(k,l) = \binom{k}{l}^2$. Motivated by this, we propose two stronger variants of the wreath conjecture due to Baranyai for $n = 2k+1$.

Mathematics Subject Classifications: 05A15, 05A19

1 Dyck paths and the main result

A *Dyck path of semilength k* (*Dyck k-path* for brevity) is a lattice path in $\mathbb{Z}^2$ that never goes below the x-axis, starts at $(0,0)$, ends at $(2k,0)$, and with each step of the form either $(1,1)$ – a *rise* –, or $(1,-1)$ – a *fall*. We will denote the set of all Dyck k-paths by $\mathcal{D}_k$. It is well known that $|\mathcal{D}_k| = C_k$, where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k-th *Catalan number*. The Catalan numbers appear in a great many combinatorial settings; the reader is referred to [9] for an extensive compilation. Dyck paths have also been widely studied; see the works by Deutsch [5] and by Blanco and Petersen [4] for a collection of statistics and other results about them.

For $D \in \mathcal{D}_k$ and non-negative integers $l \leqslant m \leqslant 2k$, we define $\iota_D(m,l)$ as the number of intervals of length m (that is, sequences of m consecutive steps) in D such that the interval contains precisely l falls. For a non-negative integer k we define $\iota_k(m,l)$ as $\sum_{D\in\mathcal{D}_k} \iota_D(m,l)$. Note that by considering reflections in the line $x = k$, we have $\iota_k(m,l) = \iota_k(m,m-l)$.

Our main result is a formula for $\iota_k(m,l)$. Note that in view of the paragraph above, we can restrict our attention to the case $2l \leqslant m$. We set $\binom{x}{y} = 0$ for integers $x \geqslant 0$ and $y < 0$.

Theorem 1. *Let l, m, k be three non-negative integers such that $2l \leqslant m \leqslant 2k$. Then*

$$\iota_k(m,l) = \sum_{d=0}^{k+l-m} \left[\binom{m}{l} - \binom{m}{l-d-1}\right]\left[\binom{2k-m+1}{k-m+l-d} - \binom{2k-m+1}{k-m+l-d-1}\right].$$

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A simple manipulation with the sum in Theorem 1 (see the beginning of the proof of Corollary 2 for a demonstration in the case $m = k$) provides an alternative formula for $\iota_k(m,l)$ given by

$$\iota_k(m,l) = \binom{m}{l}\binom{2k-m+1}{k-m+l} + \sum_{d'=0}^{l-1}\binom{m}{d'}\left[\binom{2k-m+1}{k+1-d'} - \binom{2k-m+1}{k-d'}\right].$$

The formula becomes particularly elegant in the special case $m = k$.

Corollary 2. *Let $l \leqslant k$ be two non-negative integers. Then $\iota_k(k,l) = \binom{k}{l}^2$.*

We remark that using an analogous technique as in the proof of Corollary 2 in Section 3, one can also obtain $\iota_k(k+1,l) = \binom{k}{l-1}\binom{k}{l}$ for $2l \leqslant k+1$. With a bit more effort, similar lines of reasoning eventually yield $\iota_k(k-1,l) = \binom{k+2}{l+1}\binom{k-1}{l} - \binom{k}{l+1}\binom{k-1}{l-1} - \binom{k}{l-1}\binom{k-1}{l}$ for $2l \leqslant k-1$.

2 The wreath conjecture

Kirkman's Schoolgirl problem from 1847 [7] gave rise to the question whether it is possible to partition the k-uniform complete hypergraph on n vertices into perfect matchings (that is, sets of hyperedges such that each vertex lies in exactly one of the hyperedges) whenever k divides n. The positive answer was confirmed by Baranyai in 1974 [3]. At the end of his paper, Baranyai posed a conjecture concerning a generalisation of his result.

This conjecture was originally stated in terms of 'staircase matrices'. Later, Katona [6] rephrased the conjecture in terms of 'wreaths'; it is this notion that we adapt here.

Let $k \leqslant n$ be two positive integers, and let $g = \gcd(n,k)$. We write $\mathbb{Z}_n$ for the set of integers modulo n and $\mathbb{Z}_n^{(k)}$ for the set of subsets of $\mathbb{Z}_n$ of size k. Given a permutation π of $\mathbb{Z}_n$ we define $\mathcal{F}_{n,k,\pi} \subset \mathbb{Z}_n^{(k)}$, the (n,k,π)-*wreath*, as

$$\{\{\pi((i-1)k+1), \pi((i-1)k+2), \ldots, \pi(ik)\} \mid i \in \mathbb{Z}_n\}.$$

It is easy to see that such a set has size $\frac{n}{g}$.

A set $\mathcal{F} \subset \mathbb{Z}_n^{(k)}$ is called an (n,k)-*wreath* if it is an (n,k,π)-wreath for some permutation π of $\mathbb{Z}_n$. The conjecture due to Baranyai [3] and Katona [6] (who nicknamed it *the wreath conjecture*) is as follows.

Conjecture 3 (The wreath conjecture)**.** For any positive integers $k \leqslant n$ there is a decomposition of $\mathbb{Z}_n^{(k)}$ into disjoint (n,k)-wreaths.

For example, for $n = 5$ and $k = 3$, the set $\mathbb{Z}_5^{(3)}$ can be decomposed into the following two $(5,3)$-wreaths: $\mathcal{F}_{5,3,\pi_1} = \{\{0,1,2\},\{1,2,3\},\{2,3,4\},\{3,4,0\},\{4,0,1\}\}$ and $\mathcal{F}_{5,3,\pi_2} = \{\{0,2,4\},\{2,4,1\},\{4,1,3\},\{1,3,0\},\{3,0,2\}\}$, where $\pi_1 = \mathrm{id}$ and $\pi_2 = \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3 \end{smallmatrix}\right)$.

We remark that for n and k coprime, this conjecture coincides with a later one due to Bailey and Stevens [2] concerning decompositions of complete k-uniform hypergraphs into tight Hamiltonian cycles. For more discussion of the case when n and k are not coprime, the reader is referred to a parallel article by the authors [8].

Let us consider the wreath conjecture for $n = 2k+1$. In this case we have $g = 1$, and hence, after rearranging, we can write $\mathcal{F}_{n,k,\pi} = \{\{\pi(i+1), \pi(i+2), \ldots, \pi(i+k)\} \mid i \in \mathbb{Z}_n\}$. Motivated by the fact that for $n = 2k+1$ the number of (n, k)-wreaths necessary to decompose $\mathbb{Z}_n^{(k)}$ coincides with the Catalan number C_k, we propose the following strengthening of Conjecture 3.

Conjecture 4. Let k be a positive integer. There exists a set $\Pi = \{\pi_1, \pi_2, \ldots, \pi_{C_k}\}$ of C_k permutations of $\mathbb{Z}_{2k+1}$ with each permutation fixing 0 and a bijection $\varphi : \Pi \to \mathcal{D}_k$ such that

- $\mathbb{Z}_{2k+1}^{(k)} = \bigcup_{i=1}^{C_k} \mathcal{F}_{2k+1,k,\pi_i}$, and
- for any i and j, the j-th step of $\varphi(\pi_i)$ is a rise if and only if $\pi_i(j) \in \{1, 2, \ldots, k\}$.

We verified the conjectures using a computer for $k \leqslant 4$. The bijections for $k \leqslant 3$ can be seen in Figure 1. This conjecture is also motivated by Corollary 2 in view of the following result.

Lemma 5. *The equality $\iota_k(k,l) = \binom{k}{l}^2$ from Corollary 2 is a necessary condition for Conjecture 4.*

Proof. If Conjecture 4 holds, then to each interval I of length k of a Dyck k-path D we can assign a set $\{\pi(s), \pi(s+1), \ldots, \pi(s+k-1)\}$, where I starts at the s-th step of D and $\pi = \varphi^{-1}(D)$.

By the first condition of Conjecture 4, this yields a bijection between intervals of length k of Dyck k-paths and sets in $\mathbb{Z}_{2k+1}^{(k)}$ not containing 0. The second condition then implies that intervals of length k with l falls are in bijection with sets in $\mathbb{Z}_{2k+1}^{(k)}$ without 0 and with l elements from $\{k+1, k+2, \ldots, 2k\}$. From the definition, there are $\iota_k(k,l)$ intervals of length k with l falls and there are $\binom{k}{l}^2$ sets in $\mathbb{Z}_{2k+1}^{(k)}$ without 0 and with l elements from $\{k+1, k+2, \ldots, 2k\}$. The result follows. $\square$

Our search through $k \leqslant 4$ suggests that an even stronger statement may be true. To state it, for a Dyck path $D \in \mathcal{D}_k$, we will denote by $D^{(R)}$ the Dyck path obtained by reflecting D in the line $x = k$.

Conjecture 6. Let k be a positive integer. There exists a set $\Pi = \{\pi_1, \pi_2, \ldots, \pi_{C_k}\}$ of C_k permutations of $\mathbb{Z}_{2k+1}$ with each permutation fixing 0 and a bijection $\varphi : \Pi \to \mathcal{D}_k$ such that

- $\mathbb{Z}_{2k+1}^{(k)} = \bigcup_{i=1}^{C_k} \mathcal{F}_{2k+1,k,\pi_i}$, and
- for any i and j, the j-th step of $\varphi(\pi_i)$ is a rise if and only if $\pi_i(j) \in \{1, 2, \ldots, k\}$, and
- for any Dyck k-path D and any $j \in \mathbb{Z}_{2k+1}$ we have $\varphi^{-1}(D)(j) + \varphi^{-1}(D^{(R)})(-j) = 0$.

The bijections from Figure 1 all satisfy the stronger Conjecture 6.

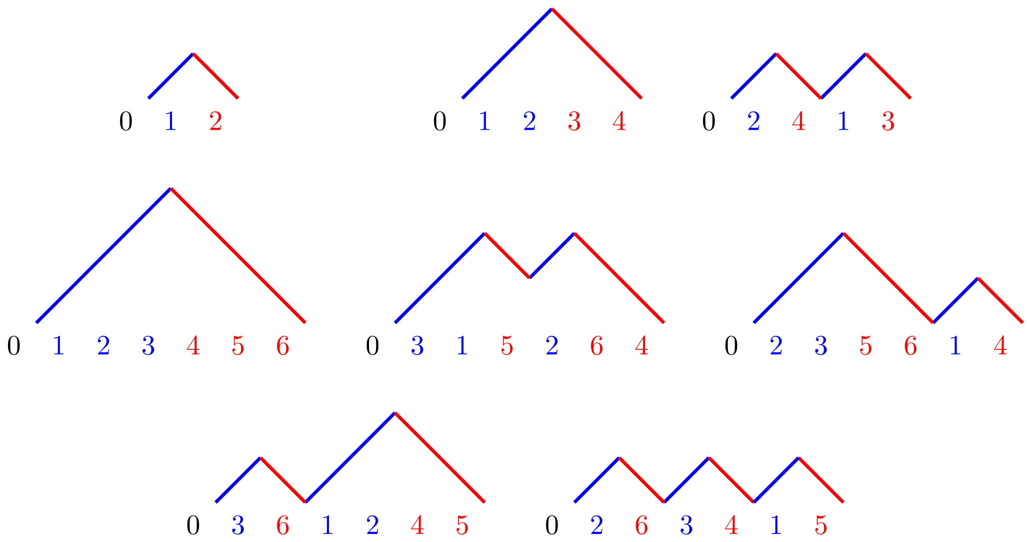

Figure 1: Examples of permutations which confirm Conjectures 4 and 6 for $k \leqslant 3$. Below each Dyck k-path the corresponding permutation π is written in the form $\pi(0), \pi(1), \ldots, \pi(2k)$. The second conditions of the conjectures require numbers $1, 2, \ldots, k$ to lie below rises and numbers $k+1, k+2, \ldots, 2k$ below falls.

3 Proofs of Theorem 1 and Corollary 2

To simplify the calculations in the proofs later, we introduce some additional notation. We call a lattice path a *NE upper path* if it never visits a point below the $y = x$ diagonal and each of its steps is either $(0, 1)$ – a *North* step –, or $(1, 0)$ – an *East* step. Instead of working with Dyck k-paths as defined in Section 1, we will work with NE upper paths from $(0, 0)$ to (k, k). To observe that there is a bijection between these two families of paths, consider a rotation of the plane by $\frac{\pi}{4}$ and an appropriate scaling. Under this bijection, rises translate to North steps and falls translate to East steps.

The proof uses the following well-known generalisation of Catalan numbers counting the number of NE upper paths between $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}^2$. This number, which we denote by $\mathrm{P}_{(x_1,y_1)}^{(x_2,y_2)}$, is nonzero if and only if $x_1 \leqslant x_2$, $y_1 \leqslant y_2$ and $x_i \leqslant y_i$ for $i \in \{1, 2\}$. Note that for $j \in \mathbb{Z}$ we have $\mathrm{P}_{(x_1,y_1)}^{(x_2,y_2)} = \mathrm{P}_{(x_1+j,y_1+j)}^{(x_2+j,y_2+j)}$.

Lemma 7. *Let $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}^2$ be such that $x_1 \leqslant x_2$, $y_1 \leqslant y_2$ and $x_i \leqslant y_i$ for $i \in \{1, 2\}$. Set $\delta = y_1 - x_1$, $\alpha = y_2 - y_1$ and $\beta = x_2 - x_1$. Then*

$$\mathrm{P}_{(x_1,y_1)}^{(x_2,y_2)} = \binom{\alpha+\beta}{\beta} - \binom{\alpha+\beta}{\beta-\delta-1}.$$

We give a proof of this lemma to keep the paper self-contained. The proof is analogous to André's reflection principle [1].

Proof. There are $\binom{\alpha+\beta}{\beta}$ paths from (x_1, y_1) to $(x_2, y_2) = (x_1+\beta, y_1+\alpha)$ consisting of North and East steps. We claim that the number of paths from (x_1, y_1) to (x_2, y_2) consisting of

North and East steps which visit a point below the diagonal $y = x$ is $\binom{\alpha+\beta}{\beta-\delta-1}$, which gives the result.

To show this claim, we find a bijection between the paths from (x_1, y_1) to (x_2, y_2) consisting of North and East steps which visit a point below the diagonal $y = x$ and the paths from (x_1, y_1) to $(x_1 + \alpha + \delta + 1, y_1 + \beta - \delta - 1)$ consisting of North and East steps.

Given a path W of the first type, consider the step after which the path visits a point below the diagonal $y = x$ for the first time. Let this be the s-th step. From the $(s+1)$-st step onward, we exchange North and East steps. The resulting path consists of $\beta - \delta - 1$ North and $\alpha + \delta + 1$ East steps, i.e., it is a path of the second type.

Now consider a path W' of the second type. Recall that $x_1 \leqslant y_1$, in other words, (x_1, y_1) is not below the diagonal $y = x$. We also have $y_1 + \beta - \delta - 1 = x_2 - 1 < y_2 + 1 = y_1 + \alpha + 1 = x_1 + \alpha + \delta + 1$, therefore $(x_1 + \alpha + \delta + 1, y_1 + \beta - \delta - 1)$ is below the diagonal $y = x$. Therefore, there exists a step of W' after which W' visits a point below the diagonal $y = x$ for the first time. Analogously to before, exchange all North and East steps after this step. The resulting path consists of α North and β East steps and visits a point below the diagonal $y = x$, therefore is of the first type.

The maps from the previous two paragraphs are inverses to each other, and so describe the desired bijection. See Figure 2 for an illustration.

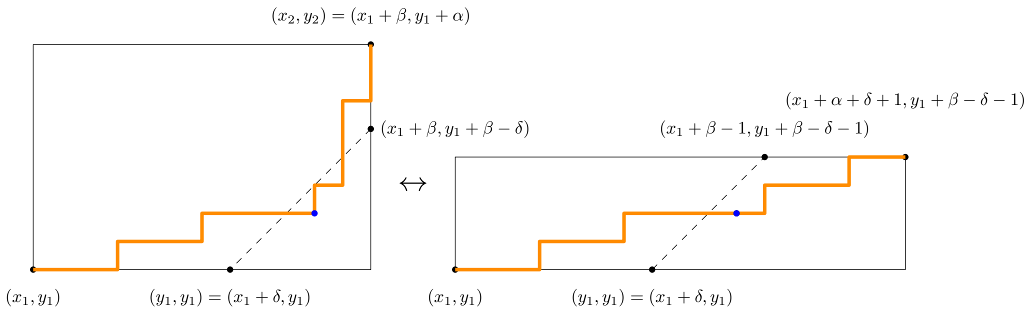

Figure 2: The bijection from Lemma 7 given by 'flipping' the section of the path after the highlighted point.

Hence we have $\mathrm{P}_{(x_1,y_1)}^{(x_2,y_2)} = \binom{\alpha+\beta}{\beta} - \binom{\alpha+\beta}{\beta-\delta-1}$, as claimed. □

Equipped with this lemma, we move on to the proof of the main theorem.

Proof of Theorem 1. We consider all possible 'starting points' $(i, i + d)$ of intervals of length m which contain exactly l East steps. To obtain $\iota_k(m, l)$, we sum over all such starting points the number of NE upper paths which visit the starting point and have exactly l East steps among the next m steps following this visit. Note that after these m steps, any such NE upper path visits $(i+l, i+d+m-l)$. See Figure 3 for an illustration.

As both the starting point and the corresponding endpoint lie in the square $[0, k]^2$, we have $i, d \geqslant 0$ as well as $i + l \leqslant k$ and $i + d + m - l \leqslant k$. Therefore i and d satisfy $0 \leqslant d \leqslant k + l - m$ and $0 \leqslant i \leqslant k + l - m - d$.

In the identities below, first we use $\mathrm{P}_{(x_1,y_1)}^{(x_2,y_2)} = \mathrm{P}_{(x_1+j,y_1+j)}^{(x_2+j,y_2+j)}$ and then simplify the sums:

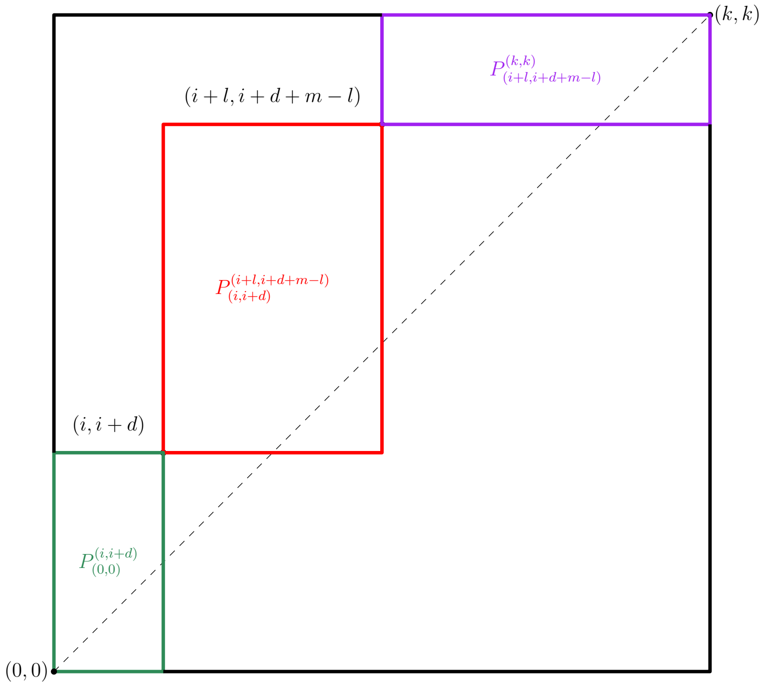

Figure 3: An illustration of NE upper paths whose interval of length m starting at $(i,i+d)$ contains exactly l East steps.

$$\begin{aligned}\iota_k(m,l) &= \sum_{d=0}^{k+l-m}\sum_{i=0}^{k+l-m-d} \mathrm{P}^{(i,i+d)}_{(0,0)}\mathrm{P}^{(i+l,i+d+m-l)}_{(i,i+d)}\mathrm{P}^{(k,k)}_{(i+l,i+d+m-l)}\\ &= \sum_{d=0}^{k+l-m}\sum_{i=0}^{k+l-m-d} \mathrm{P}^{(i,i+d)}_{(0,0)}\mathrm{P}^{(l,d+m-l)}_{(0,d)}\mathrm{P}^{(k,k)}_{(i+l,i+d+m-l)}\\ &= \sum_{d=0}^{k+l-m} \mathrm{P}^{(l,d+m-l)}_{(0,d)}\sum_{i=0}^{k+l-m-d} \mathrm{P}^{(i,i+d)}_{(0,0)}\mathrm{P}^{(k,k)}_{(i+l,i+d+m-l)}.\end{aligned}$$

Further simplification of this sum comes from the following claim.

Claim 8. *For any d such that $0 \leqslant d \leqslant k+l-m$ we have*

$$\sum_{i=0}^{k+l-m-d} \mathrm{P}^{(i,i+d)}_{(0,0)}\mathrm{P}^{(k,k)}_{(i+l,i+d+m-l)} = \binom{2k-m+1}{k-m+l-d} - \binom{2k-m+1}{k-m+l-d-1}. \tag{1}$$

Proof of Claim 8. Consider the pairs (W,i), where $i \in \{0,1,\ldots,k+l-m-d\}$ and W is a NE upper path from $(l-k, d+m-l-k)$ to $(0,d)$ which goes through the point $(-i,-i)$. Using $\mathrm{P}^{(x_2,y_2)}_{(x_1,y_1)} = \mathrm{P}^{(x_2+j,y_2+j)}_{(x_1+j,y_1+j)}$, observe that the number of distict pairs (W,i) is precisely the left-hand side of equation (1). See Figure 4 for an illustration.

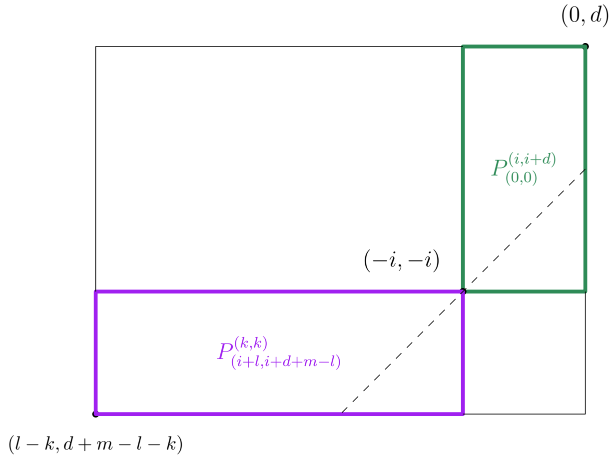

Figure 4: An illustration of the left-hand side of equation (1).

Next, we claim that the pairs (W, i) are in bijection with NE upper paths from $(l-k-1, d+m-l-k)$ to $(0, d)$ which visit the diagonal $y = x$ in at least one point. The bijection is given as follows.

Given a pair (W, i), consider the sequence of $k+l-m$ North and $k-l$ East steps corresponding to W. After the step at which W reaches $(-i,-i)$, add an additional East step. Denote by W' the path corresponding to the longer sequence of North and East steps starting at $(l-k-1, d+m-l-k)$. Observe that W' consists of $k+l-m$ North steps and $k-l+1$ East steps, never visits a point below the diagonal $y = x$ but visits a point on this diagonal. Moreover, the first visited point on this diagonal is $(-i,-i)$.

On the other hand, let W' be a path from $(l-k-1, d+m-l-k)$ to $(0, d)$ consisting of North and East steps that never goes below the diagonal $y = x$ but which visits this diagonal in at least one point. Then there is a first point at which W' visits this diagonal; let this point be $(-i,-i)$. Consider the sequence of $k+l-m$ North and $k-l+1$ East steps corresponding to W'. Remove the East step that leads to $(-i,-i)$ to obtain a shorter sequence of steps, and denote by W the corresponding path starting at $(l-k, d+m-l-k)$. Then (W, i) is a pair where $i \in \{0, 1, \ldots, k+l-m-d\}$ and W is a NE upper path from $(l-k, d+m-l-k)$ to $(0, d)$ which goes through the point $(-i,-i)$.

The maps from the previous two paragraphs are inverses to each other, and hence describe the desired bijection, see Figure 5 for an illustration.

We thus obtain that the left-hand side of equation (1) counts the number of NE upper paths from $(l-k-1, d+m-l-k)$ to $(0, d)$ which visit the diagonal $y = x$ in at least one point. Note that such paths are exactly the NE upper paths from $(l-k-1, d+m-l-k)$ to $(0, d)$ which visit a point below the diagonal $y = x+1$. The number of such paths is $\mathrm{P}_{(l-k-1,d+m-l-k)}^{(0,d)} - \mathrm{P}_{(l-k,d+m-l-k)}^{(1,d)}$.

If $d > 0$, we apply Lemma 7 twice to express this difference. If $d = 0$, we apply Lemma 7 to the first term and observe that $\mathrm{P}_{(l-k,m-l-k)}^{(1,0)}$, which equals 0 by its definition, can be written as $\binom{2k-m+1}{k-l+1} - \binom{2k-m+1}{(k-l+1)-(m-2l)-1}$. We get

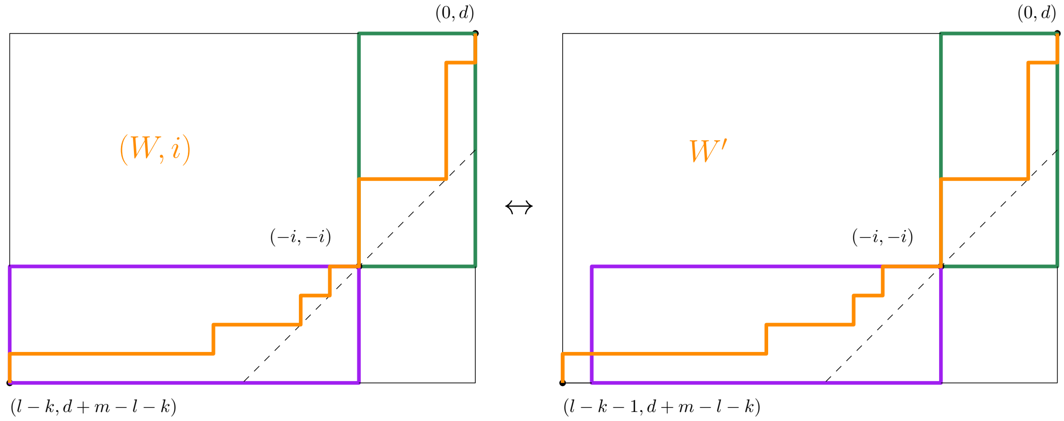

Figure 5: The bijection from Claim 8. The additional step in W' is the East step leading to $(-i, i)$.

$$\begin{aligned}
\sum_{i=0}^{k+l-m-d} \mathrm{P}_{(0,0)}^{(i,i+d)} \mathrm{P}_{(i+l,i+d+m-l)}^{(k,k)} &= \mathrm{P}_{(l-k-1,d+m-l-k)}^{(0,d)} - \mathrm{P}_{(l-k,d+m-l-k)}^{(1,d)} \\
&= \left[\binom{2k-m+1}{k-l+1} - \binom{2k-m+1}{(k-l+1)-(d+m-2l+1)-1}\right] \\
&\quad - \left[\binom{2k-m+1}{k-l+1} - \binom{2k-m+1}{(k-l+1)-(d+m-2l)-1}\right] \\
&= \binom{2k-m+1}{k-m+l-d} - \binom{2k-m+1}{k-m+l-d-1}.
\end{aligned}$$

This concludes the proof of Claim 8. $\square$

By applying Lemma 7 and Claim 8, we obtain

$$\begin{aligned}
\iota_k(m,l) &= \sum_{d=0}^{k+l-m} \mathrm{P}_{(0,d)}^{(l,d+m-l)} \sum_{i=0}^{k+l-m-d} \mathrm{P}_{(0,0)}^{(i,i+d)} \mathrm{P}_{(i+l,i+d+m-l)}^{(k,k)} \\
&= \sum_{d=0}^{k+l-m} \left[\binom{m}{l} - \binom{m}{l-d-1}\right]\left[\binom{2k-m+1}{k-m+l-d} - \binom{2k-m+1}{k-m+l-d-1}\right],
\end{aligned}$$

completing our proof of Theorem 1. $\square$

The formula from Theorem 1 simplifies considerably when $m = k$.

Proof of Corollary 2. Plugging $m = k$ into Theorem 1 we obtain

$$\begin{aligned}\iota_k(k,l) &= \sum_{d=0}^{l}\left[\binom{k}{l}-\binom{k}{l-d-1}\right]\left[\binom{k+1}{l-d}-\binom{k+1}{l-d-1}\right]\\ &= \binom{k}{l}\sum_{d=0}^{l}\left[\binom{k+1}{l-d}-\binom{k+1}{l-d-1}\right]\\ &\qquad -\sum_{d=0}^{l-1}\binom{k}{l-d-1}\left[\binom{k+1}{l-d}-\binom{k+1}{l-d-1}\right].\end{aligned}$$

Noting that the second factor in the first term is a telescoping sum, we find that

$$\binom{k}{l}\sum_{d=0}^{l}\left[\binom{k+1}{l-d}-\binom{k+1}{l-d-1}\right]=\binom{k}{l}\binom{k+1}{l}. \tag{2}$$

To simplify the second term, first we expand the bracket and substitute $d' = l - d - 1$ to get

$$\begin{aligned}&\sum_{d=0}^{l-1}\binom{k}{l-d-1}\left[\binom{k+1}{l-d}-\binom{k+1}{l-d-1}\right]=\\ &\qquad\sum_{d'=0}^{l-1}\binom{k}{d'}\binom{k+1}{k-d'}-\sum_{d'=0}^{l-1}\binom{k+1}{d'}\binom{k}{k-d'}.\end{aligned}$$

We now consider the task of choosing k objects out of $2k + 1$ ordered objects. The first sum counts the number of ways to do so in a way that at most $l - 1$ of the first k objects are picked. The second sum counts the number of ways to do so in a way that at most $l - 1$ of the first $k + 1$ objects are picked. Any choice of the second type is a choice of the first type. The only choices of the first type that are not of the second type are those which choose exactly $l - 1$ elements from the first k and also choose the $(k + 1)$-st element (which means $k - l$ elements are chosen from the last k elements).

We thus obtain

$$\sum_{d=0}^{l-1}\binom{k}{l-d-1}\left[\binom{k+1}{l-d}-\binom{k+1}{l-d-1}\right]=\binom{k}{l-1}\binom{k}{k-l}=\binom{k}{l}\binom{k}{l-1}. \tag{3}$$

Equations (2) and (3) imply

$$\iota_k(k,l)=\binom{k}{l}\binom{k+1}{l}-\binom{k}{l}\binom{k}{l-1}=\binom{k}{l}^2,$$

as claimed. ☐

4 Concluding remarks

The main open questions of interest are the conjectures from Section 2. As we have seen, Corollary 2 provides support for the newly introduced Conjecture 4 and Conjecture 6. The authors are also curious if the right-hand side of Theorem 1 could be simplified further in other cases than those with $m = k, k-1, k+1$ already considered.

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