# Structure of Tight $(k, 0)$-Stable Graphs

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**Abstract**

We say that a graph $G$ is $(k, \ell)$-stable if removing $k$ vertices from it reduces its independence number by at most $\ell$. We say that $G$ is tight $(k, \ell)$-stable if it is $(k, \ell)$-stable and its independence number equals $\lfloor \frac{n-k+1}{2} \rfloor + \ell$, the maximum possible, where $n$ is the vertex number of $G$. Answering a question of Dong and Wu, we show that every tight $(2, 0)$-stable graph with odd vertex number must be an odd cycle. Moreover, we show that for all $k \geqslant 3$, every tight $(k, 0)$-stable graph has at most $k + 6$ vertices.

**Mathematics Subject Classifications:** 05C69

## 1 Introduction

The resilience of graph properties is a fundamental question in graph theory. Motivated by studies on the Erdős–Rogers function, Dong and Wu [3] investigated the resilience of graph independence number with respect to removing vertices. For a graph $G = (V, E)$ and vertices $v_1, \dots, v_k \in V$, we let $G \setminus \{v_1, \dots, v_k\}$ denote the induced subgraph of $G$ on $V \setminus \{v_1, \dots, v_k\}$. For integers $k > \ell \geqslant 0$, we say that $G$ is $(k, \ell)$-*stable* if for any $k$ vertices $v_1, \dots, v_k \in V$, we have $\alpha(G \setminus \{v_1, \dots, v_k\}) \geqslant \alpha(G) - \ell$.

A result in [3] states that any $(k, \ell)$-stable graph $G$ on $n$ vertices satisfies

$$\alpha(G) \leqslant \lfloor \frac{n-k+1}{2} \rfloor + \ell.$$

A $(k, \ell)$-stable graph for which equality holds (i.e., $\alpha(G) = \lfloor \frac{n-k+1}{2} \rfloor + \ell$) is called *tight* $(k, \ell)$-*stable*. When $\ell = 0$, it is easily seen that every $K_{k+1}$ is tight $(k, 0)$-stable. When $\ell > 0$, one can always obtain a tight $(k, \ell)$-stable graph by taking the disjoint union of a tight $(k-1, \ell-1)$-stable graph with an isolated vertex. Starting with $K_{k-\ell+1}$, this yields a tight $(k, \ell)$-stable graph on $k + 1$ vertices for any $k > \ell \geqslant 0$.

Suppose $n$ is any integer greater than $k$. Does an $n$-vertex $(k, \ell)$-stable graph always exist? The answer is "yes" when $(k, \ell) = (1, 0)$. Indeed, when $n$ is even, any $n$-vertex

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balanced bipartite graph that contains a perfect matching is tight (1,0)-stable. When $n$ is odd, one can always take an $(n-1)$-vertex tight $(1,0)$-stable graph and add an extra vertex adjacent to all its vertices, to form an $n$-vertex tight (1,0)-stable graph. In general, by taking the disjoint union of a tight (1,0)-stable graph with $\ell$ isolated vertices, we immediately obtain examples of tight $(\ell+1,\ell)$-stable graphs for arbitrary $\ell$ on any number of vertices $n > \ell+1$.

Going one step further, we consider the case where $(k,\ell) = (2,0)$. When $n$ is even, one can find multiple examples of $n$-vertex tight $(2,0)$-stable graphs, such as a vertex-disjoint union of two odd cycles, or an even subdivision of $K_4$. (We say that $G'$ is an *even subdivision* of $G$ if we can obtain $G'$ from $G$ by iteratively performing the following two-step operation: (i) take edge $uv \in E$ and two new vertices $u', v' \notin V$; (ii) replace $V$ by $V \cup \{u', v'\}$ and $E$ by $(E \setminus \{uv\}) \cup \{uu', u'v', v'v\}$.) However, when $n$ is odd, the only example known so far is the $n$-cycle $C_n$.

Dong and Wu [3, Section 8] asked whether $C_n$ is the only $n$-vertex tight $(2,0)$-stable graph when $n$ is odd. We answer this question in the affirmative. Moreover, we will show the following structural properties on tight $(k,0)$-stable graphs for $k=1$ and $k=2$.

**Theorem 1.** *Let $G$ be a tight $(k,0)$-stable graph on $n$ vertices.*

*(a) If $k=1$ and $n$ is even, then $G$ contains a perfect matching.*

*(b) If $k=1$ and $n$ is odd, then $G$ has a spanning subgraph that is a vertex-disjoint union of an odd cycle and a (possibly empty) matching.*

*(c) If $k=2$ and $n$ is odd, then $G$ is an odd cycle.*

*(d) If $k=2$ and $n$ is even, then $G$ has a spanning subgraph that is either a vertex-disjoint union of two odd cycles, or an even subdivision of $K_4$.*

*Remark* 2. The authors of [3] conjectured that every tight (2,0)-stable graph is Hamiltonian. Here we point out that this is not the case, as every even subdivision of $K_4$ is tight (2,0)-stable, but not every such subdivision is Hamiltonian.

Note that for every $k = 1,2$ and $n > k$, one can always find a tight $(k,0)$-stable graph with $n$ vertices. We show that this becomes different for $k \geqslant 3$. In particular, a tight $(3,0)$-stable graph has at most 9 vertices.

**Theorem 3.** *Let $G$ be a tight $(3,0)$-stable graph. Then $G$ has a spanning subgraph that is among the five graphs $K_4$, $K_5$, $H_7$, $T_9$, $H_9$ as in Figure 1.*

Observe that if $G$ is tight $(k,0)$-stable, then for any vertex $v \in V$, $G \setminus \{v\}$ is tight $(k-1,0)$-stable. Thus, Theorem 3 immediately gives the following.

**Corollary 4.** *For all $k \geqslant 3$, any tight $(k,0)$-stable graph has at most $k+6$ vertices.*

We will prove Theorem 1(a)–(c) in Section 2. Our proof of Theorem 1(d) and Theorem 3 utilizes properties of *$\alpha$-critical graphs*, which are graphs whose independence number is affected by any *edge* removal. In Section 3, we introduce the notion of $\alpha$-criticality and prove these parts. Finally, in Section 4, we discuss some remaining questions about $(k,\ell)$-stable graphs.

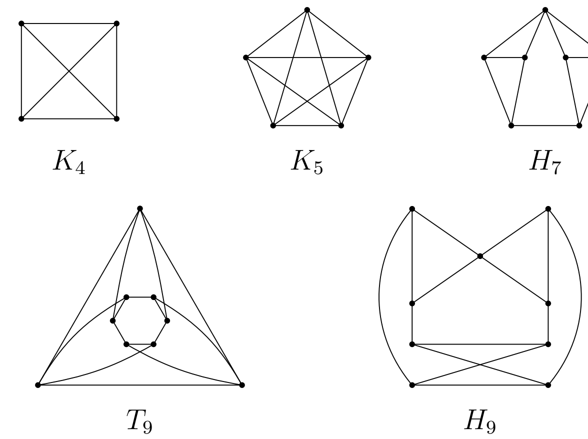

Figure 1: The five graphs listed in Theorem 3

## 2 Proof of Theorem 1(a)–(c)

We start by stating the following fact on (1,0)-stable graphs that was proved in [3]. For completeness, we also include a proof of this result.

**Lemma 5.** *Let $G = (V, E)$ be a $(1, 0)$-stable graph, and let $A \subseteq V$ be a maximum independent set in $G$. Then there exists a matching of size $|A|$ between $A$ and $V \setminus A$.*

*Proof.* It suffices to show that Hall's condition holds from $A$ to $V \setminus A$. For every subset $S \subseteq A$, we use $N(S)$ to denote the neighborhood of $S$ in $G$, which is a subset of $V \setminus A$ as $A$ is independent. By contradiction, suppose there exists $Z \subseteq A$ such that $Z \neq \emptyset$ and $|N(Z)| < |Z|$. Without loss of generality, we may assume that $Z$ is minimal. Take any $z \in Z$. Since $\alpha(G) = \alpha(G \setminus \{z\})$, there exists another maximum independent set $A'$ in $G$ that does not contain $z$.

Let $X_1 = Z \cap A'$ and $X_2 = Z \setminus A'$. Define $U = (A' \setminus N(X_2)) \cup X_2$. Since $A'$ is independent, $X_2 \subseteq Z \subseteq A$ is independent and $A' \setminus N(X_2)$ does not contain any neighbors of $X_2$, we know that $U = (A' \setminus N(X_2)) \cup X_2$ is independent.

Note that $Z$ is a disjoint union of $X_1$ and $X_2$. Since $X_1 \subseteq A'$ and $A'$ is independent, for every $a \in A'$, we cannot have $a \in N(X_1)$, so $a \notin N(X_2)$ if and only if $a \notin N(Z) \setminus N(X_1)$. Hence we have $U = (A' \setminus (N(Z) \setminus N(X_1))) \cup X_2$. Since $X_2$ and $A'$ are disjoint, this gives

$$\begin{aligned} |U| &= |A' \setminus (N(Z) \setminus N(X_1))| + |X_2| \\ &\geqslant |A'| - |N(Z) \setminus N(X_1)| + |X_2| \\ &= |A'| - |N(Z)| + |N(X_1)| + |X_2|. \end{aligned}$$

Since $z \in Z \setminus A'$, we have $X_1 \subsetneq Z$, so $|N(X_1)| \geqslant |X_1|$ by minimality of $Z$. Thus, we have

$$|U| \geqslant |A'| - |N(Z)| + |X_1| + |X_2| = |A'| - |N(Z)| + |Z| > |A'|,$$

contradicting the fact that $A'$ is a maximum independent set in $G$. $\square$

With Theorem 5, we can go ahead and prove Theorem 1(a)–(c).

*Proof of Theorem 1(a).* Suppose $G = (V, E)$ is a tight $(1, 0)$-stable graph on $n$ vertices, with $n$ even. Then $\alpha(G) = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. Let $A \subseteq V$ be a maximum independent set, so that $|A| = |V \setminus A| = \frac{n}{2}$. By Theorem 5, there exists a matching from $A$ to $V \setminus A$, which is a perfect matching in $G$. □

*Proof of Theorem 1(b).* Suppose $G = (V, E)$ is a tight $(1, 0)$-stable graph on $n$ vertices, with $n$ odd. Then $\alpha(G) = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. Let $m = \alpha(G) = \frac{n-1}{2}$ and $A \subseteq V$ be a maximum independent set, so that $|A| = m$ and $|V \setminus A| = m + 1$. By Theorem 5, there exists a matching of size $m$ from $A$ to $V \setminus A$. Label $V = \{a_1, \dots, a_m, b_1, \dots, b_m, c\}$ such that $A = \{a_1, \dots, a_m\}$, and $a_i b_i \in E$ for every $i \in [m]$. Let $B = \{b_1, \dots, b_m\}$.

For every $b_i \in B$, we say that $b_i$ has property $(*)$ if there exists $\{i_1, \dots, i_s\} \subseteq [m]$ such that $i_s = i$, and $c$–$a_{i_1}$–$b_{i_1}$–$\dots$–$a_{i_s}$–$b_{i_s}$ is a path in $G$. Consider the partition $[m] = X \cup Y$ given by

$$X = \{i \in [m] : b_i \text{ has property } (*)\},$$
$$Y = \{i \in [m] : b_i \text{ does not have property } (*)\}.$$

Observe that there is no edge between $\{a_i : i \in Y\}$ and $\{b_i : i \in X\} \cup \{c\}$. If for some $i_0 \in Y$, $a_{i_0}$ is adjacent to some vertex in $\{b_i : i \in X\} \cup \{c\}$, then $b_{i_0}$ would have property $(*)$, contradicting the definition of $Y$. Since $\{a_i : i \in Y\} \subseteq A$ is independent, to avoid the independent set $\{a_i : i \in Y\} \cup \{b_i : i \in X\} \cup \{c\}$ (as it has size $m+1$), there must be some edge within $\{b_i : i \in X\} \cup \{c\}$. One can then check that no matter where this edge occurs, it will create a spanning subgraph of $G$ that is a vertex-disjoint union of one odd cycle and a (possibly empty) matching (see Figure 2 for an illustration). □

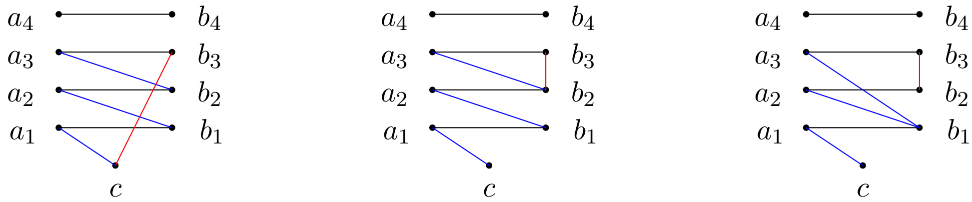


Figure 2: Some canonical cases in Theorem 1(b)

*Proof of Theorem 1(c).* Suppose $G = (V, E)$ is a tight $(2, 0)$-stable graph on $n$ vertices, with $n$ odd. Then $\alpha(G) = \lfloor \frac{n-1}{2} \rfloor = \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$. In particular, $G$ is tight (1,0)-stable. By Theorem 1(b), $G$ has a spanning subgraph $H$ that is a vertex-disjoint union of one odd cycle and a (possibly empty) matching.

We first verify that $H = C_n$. By contradiction, suppose $H$ is a vertex-disjoint union of a nonempty matching $e_1, \dots, e_s$ and an odd cycle $C_{n-2s}$, with $s \geqslant 1$. Then every independent set of size $\frac{n-1}{2}$ in $G$ must include one vertex from each of $e_1, \dots, e_s$. Thus, removing the two vertices in $e_1$ will destroy all maximum independent sets in $G$, contradicting the fact that $G$ is (2,0)-stable. Hence $H = C_n$. If $G \neq H$, i.e., $G$ is an odd cycle plus some extra chords, then we can find another spanning subgraph $H'$ of $G$ that is a vertex-disjoint union of a smaller odd cycle and a nonempty matching, which by the argument above implies that $G$ is not (2,0)-stable. Thus, we must have $G = H = C_n$. $\square$

## 3 Proof of Theorem 1(d) and Theorem 3

In this section, we prove Theorem 1(d) and Theorem 3. As mentioned earlier, we first introduce the concept of $\alpha$-critical graphs.

**Definition 6.** For graph $G = (V, E)$ and edge $uv \in E$, we let $G - uv$ denote the graph whose vertex set is $V$ and edge set is $E \setminus \{uv\}$.

**Definition 7.** We say that $G = (V, E)$ is $\alpha$-critical if for every edge $uv \in E$, we have $\alpha(G - uv) > \alpha(G)$.

We will utilize the following results, by Andrásfai [2] and Surányi [6], on connected $\alpha$-critical graphs with independence number close to $n/2$. A survey on these results can be found at [5, Chapter 18]. A proof of Theorem 9 can also be found at [7].

**Theorem 8** (Andrásfai)**.** *Let $G = (V, E)$ be a connected $\alpha$-critical graph with $|V| = 2\alpha(G) + 2$. Then $G$ is an even subdivision of $K_4$.*

**Theorem 9** (Surányi)**.** *Let $G = (V, E)$ be a connected $\alpha$-critical graph with $|V| = 2\alpha(G)+3$ and minimum degree at least 3. Then $G$ must be one of $K_5$, $H_7$, $H_9$ or $T_9$ as in Theorem 3.*

*Proof of Theorem 1(d).* Suppose $G = (V, E)$ is a tight $(2, 0)$-stable graph on $n$ vertices, with $n$ even. By a greedy removal of edges, we can obtain a spanning subgraph $G'$ of $G$, such that $\alpha(G') = \alpha(G) = n/2 - 1$ and $G'$ is $\alpha$-critical. Since $G$ is (2,0)-stable, so is the subgraph $G'$.

Let $G_1, \dots, G_t$ be the connected components of $G'$. For every $i \in [t]$, since $G_i$ is (2,0)-stable, we have $\alpha(G_i) \leqslant \left\lfloor \frac{|V(G_i)|-1}{2} \right\rfloor \leqslant \frac{|V(G_i)|-1}{2}$. This gives

$$\frac{n}{2} - 1 = \alpha(G') = \alpha(G_1) + \cdots + \alpha(G_t) \leqslant \frac{1}{2}(|V(G_1)| + \cdots + |V(G_t)| - t) = \frac{n-t}{2},$$

so $t \leqslant 2$.

If $t = 1$, then $G'$ is a connected $\alpha$-critical graph with $|V(G')| = 2\alpha(G') + 2$. By Theorem 8, $G'$ is an even subdivision of $K_4$.

If $t = 2$, then $G'$ is a vertex-disjoint union of $G_1$ and $G_2$, each of which has $\alpha(G_i) = \frac{|V(G_i)|-1}{2}$. This implies that both $G_1$ and $G_2$ are tight (2,0)-stable graphs with odd vertex number. By Theorem 1(c), $G'$ is a vertex-disjoint union of two odd cycles. $\square$

We now move on to prove Theorem 3.

*Proof of Theorem 3.* Suppose $G = (V, E)$ is a tight (3,0)-stable graph on $n$ vertices.

If $n$ is even, then $\alpha(G) = \left\lfloor \frac{n-2}{2} \right\rfloor = \frac{n}{2} - 1$. Fix any vertex $v \in V$. Since $G$ is (3,0)-stable, $G \setminus \{v\}$ is (2,0)-stable. Moreover, since $\alpha(G \setminus \{v\}) = \frac{n}{2} - 1 = \left\lfloor \frac{(n-1)-1}{2} \right\rfloor$, we know that $G \setminus \{v\}$ is tight (2,0)-stable. By Theorem 1(c), $G \setminus \{v\}$ is an odd cycle. We therefore know that $G \setminus \{v\}$ is an odd cycle for every $v \in V$. The only graph that has this property is the 4-vertex complete graph $K_4$.

If $n$ is odd, then $\alpha(G) = \left\lfloor \frac{n-2}{2} \right\rfloor = \frac{n-3}{2}$. Again, by a greedy removal of edges, we can obtain a spanning subgraph $G'$ of $G$, such that $\alpha(G') = \alpha(G) = \frac{n-3}{2}$ and $G'$ is $\alpha$-critical. Since $G$ is (3,0)-stable, so is the subgraph $G'$. Let $G_1, \ldots, G_t$ be the connected components of $G'$. For every $i \in [t]$, since $G_i$ is (3,0)-stable, we have $\alpha(G_i) \leqslant \left\lfloor \frac{|V(G_i)|-2}{2} \right\rfloor \leqslant \frac{|V(G_i)|-2}{2}$. This gives

$$\frac{n-3}{2} = \alpha(G') = \alpha(G_1) + \cdots + \alpha(G_t) \leqslant \frac{1}{2}(|V(G_1)| + \cdots + |V(G_t)| - 2t) = \frac{n-2t}{2},$$

so $t = 1$. Therefore, $G'$ is a connected $\alpha$-critical graph with $|V(G')| = 2\alpha(G') + 3$.

We further note that $G'$ has minimum degree at least 3. By contradiction, if $G'$ has a vertex $v$ of degree $\leqslant 2$, then by removing $v$ and its neighbors from $G'$, we are able to remove at most 3 vertices from $G'$ and reduce its independence number, which means that $G'$ is not (3,0)-stable. Hence $G'$ is a connected $\alpha$-critical graph with $|V(G')| = 2\alpha(G') + 3$ and minimum degree at least 3. By Theorem 9, $G'$ must be one of $K_5$, $H_7$, $H_9$ or $T_9$. $\square$

## 4 Further questions

In this work, we investigated the structure of $n$-vertex $(k, 0)$-stable graphs with independence number $\alpha = \left\lfloor \frac{n-k+1}{2} \right\rfloor$. Our results show that such graphs can be arbitrarily large for $k = 1, 2$ but are very sharply bounded in size when $k \geqslant 3$. While the cliques $K_{k+1}$ and $K_{k+2}$ are tight $(k, 0)$-stable, we do not know any other natural infinite family of graphs $G_k$ that is tight $(k, 0)$-stable for each $k \geqslant 3$. Thus, we ask the following.

**Question 10.** Does there exist a positive integer $k_0$ such that, for all $k \geqslant k_0$, the only tight $(k, 0)$-stable graphs are $K_{k+1}$ and $K_{k+2}$?

Our proofs suggest that there might be some connections between tight $(k, \ell)$-stable graphs – whose independence number is resilient under vertex removal, and $\alpha$-critical graphs – whose independence number is susceptible to edge removal. It would be interesting to understand these connections further.

Recall that every $(k, 0)$-stable graph on $n$ vertices has independence number at most $\left\lfloor \frac{n-k+1}{2} \right\rfloor$; Corollary 4 implies that for a fixed $k$, this upper bound cannot be attained for sufficiently large $n$. In the opposite direction, Dong and Wu [3] constructed a sequence of $n$-vertex $(3, 0)$-stable graphs with independence number $n/2 - O(\sqrt{n})$. This was extended by Alon [1] who showed that for every $k > l \geqslant 0$, there exists a sequence of $n$-vertex $(k, \ell)$-stable graphs with independence number $n/2 - o(n)$. For $k = 3$, we ask whether we can improve the $O(\sqrt{n})$ gap to a constant.

**Question 11.** Does there exist $c > 0$ such that there is a sequence of $(3, 0)$-stable graphs $G$, with vertex number $n \to \infty$ and $\alpha(G) \geqslant |V(G)|/2 - c$?

*Remark* 12. We remark that Questions 10 and 11 were both resolved in the recent paper by Liu, Song, and Wang [4]. Question 10 was answered in the positive, while Question 11 was answered in the negative. The paper also made further progress in understanding tight $(k, \ell)$-stable graphs for general $\ell$.


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