Completing the Enumeration of Inversion Sequences Avoiding One or Two Patterns of Length 3

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Abstract

We present four constructions of inversion sequences, and use them to compute the enumeration sequences of 24 classes of pattern-avoiding inversion sequences. This completes the enumeration of inversion sequences avoiding one or two patterns of length 3. Some of our constructions are based on generating trees. Others involve pattern-avoiding words, which we also count using generating trees. To solve some of these cases, we introduce a generalization of inversion sequences, which we call *shifted* inversion sequences. Lastly, we briefly discuss the asymptotics of pattern-avoiding inversion sequences, focusing on their exponential or super-exponential behavior.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction

1.1 Basic definitions

Let $\mathbb{N}$ be the set of natural numbers, including 0. Given a natural number $n \in \mathbb{N}$, we call *integer sequences* of *size* n the elements of $\mathbb{N}^n$. We write the terms of an integer sequence $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n$. We denote by $\mathcal{I}_n$ the set of *inversion sequences* [19, 12] of size n, that is the set of sequences $\sigma \in \mathbb{N}^n$ such that $\sigma_i < i$ for all $i \in \{1, \dots, n\}$.

There is a simple bijection between $\mathcal{I}_n$ and the set of permutations of n elements, called the *Lehmer code*, which explains the name "inversion sequence". If π is a permutation of the set $\{1, \dots, n\}$, the inversion sequence $\sigma \in \mathcal{I}_n$ associated with π by the Lehmer code is defined by $\sigma_i = |\{j \ : \ \pi(j) > \pi(i) \text{ and } j < i\}|$ for all $i \in \{1, \dots, n\}$, i.e. σ_i counts the number of *inversions* of π whose second entry is at position i.

Given two integer sequences $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n$ and $\rho = (\rho_1, \dots, \rho_k) \in \mathbb{N}^k$, we say that σ *contains* the *pattern* ρ if there exists a subsequence of σ which is order-isomorphic

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to ρ. Such a subsequence is called an *occurrence* of ρ. In this work, we exclusively study patterns of length 3, which we denote $\rho_1\rho_2\rho_3$ instead of (ρ_1, ρ_2, ρ_3) for simplicity. For instance the sequence $(4, 3, 2, 5, 4)$ contains the pattern 021, since the subsequences $(3, 5, 4)$ and $(2, 5, 4)$ are both occurrences of 021. A sequence *avoids* a pattern ρ if it does not contain ρ, e.g. the inversion sequence $(0, 0, 2, 3, 2, 0, 1, 5)$ avoids the pattern 101. If P is a set of patterns, we denote by $\mathcal{I}_n(P)$ the set of inversion sequences of size n avoiding all patterns in P.

1.2 Context and summary of results

The study of pattern-avoiding inversion sequences (and many more types of sequences avoiding patterns) branched from pattern-avoiding permutations, a well-established field of research in enumerative combinatorics, see e.g. [15] or [31]. Pattern-avoiding inversion sequences were first introduced in [19] and [12], independently. Their study was continued in many articles, such as [22, 8, 3, 4, 35, 17, 5, 20, 27, 16], among others.

The enumeration of inversion sequences avoiding a single pattern of length 3 was already solved for all patterns except 010, see Table 1. A systematic study of inversion sequences avoiding pairs of patterns of length 3 was conducted by Yan and Lin in [35], which left open the enumeration of inversion sequences avoiding 32 of the 78 total pairs. Since then, eight cases were solved in [16], and one additional case was solved in [11] and [27] independently. Most of these cases were solved using bijections with other known combinatorial objects, or through generating trees.

Pattern ρ	$\lvert\mathcal{I}_n(\rho)\rvert$ for $n = 1, \ldots, 7$	Solved in	OEIS [24]
000	1, 2, 5, 16, 61, 272, 1385	[12]	A000111
001	1, 2, 4, 8, 16, 32, 64	[12]	A000079
010	**1, 2, 5, 15, 53, 215, 979**	**Theorem 56**	**A263779**
011	1, 2, 5, 15, 52, 203, 877	[12]	A000110
012	1, 2, 5, 13, 34, 89, 233	[12] and [19]	A001519
021	1, 2, 6, 22, 90, 394, 1806	[12] and [19]	A006318
100	1, 2, 6, 23, 106, 565, 3399	[16]	A263780
101 or 110	1, 2, 6, 23, 105, 549, 3207	[12]	A113227
102	1, 2, 6, 22, 89, 381, 1694	[19]	A200753
120	1, 2, 6, 23, 103, 515, 2803	[19]	A263778
201 or 210	1, 2, 6, 24, 118, 674, 4306	[19]	A263777

Table 1: Enumeration sequences of inversion sequences avoiding a single pattern of length 3.

In our work, we solve[1] the enumeration of inversion sequences avoiding 010, and the remaining 23 cases for pairs of patterns through four different constructions of pattern-avoiding inversion sequences, see Table 2.

Pattern pair P	$\lvert\mathcal{I}_n(P)\rvert$ for $n = 1, \ldots, 7$	Solved in	Performance	OEIS [24]
{000, 100}	1, 2, 5, 16, 60, 260, 1267	Theorem 2	990	A279564
{102, 201}	1, 2, 6, 22, 87, 354, 1465	Theorems 10,74	6100	A279566
{000, 102}	1, 2, 5, 14, 40, 121, 373	Theorem 17	5800	A374541
{102, 210}	1, 2, 6, 22, 87, 351, 1416	Theorems 20,76	5600	A374542
{000, 201} or {000, 210}	1, 2, 5, 16, 60, 257, 1218	Theorem 22	735	A374543
{100, 110}	1, 2, 6, 22, 93, 437, 2233	Theorem 25	820	A374544
{100, 101}	1, 2, 6, 22, 93, 439, 2267	Theorem 27	815	A374545
{110, 201}	1, 2, 6, 23, 103, 512, 2739	Theorem 29	825	A374546
{101, 210}	1, 2, 6, 23, 103, 513, 2763	Theorem 31	810	A374547
{011, 120}	1, 2, 5, 14, 42, 132, 431	Theorem 37	430	A374548
{100, 120}	1, 2, 6, 22, 92, 421, 2062	Theorem 40	350	A374549
{120, 201}	1, 2, 6, 23, 102, 498, 2607	Theorem 43	340	A374550
{110, 120}	1, 2, 6, 22, 92, 423, 2091	Theorem 46	330	A279570
{010, 120}	1, 2, 5, 15, 52, 201, 845	Theorem 48	330	A279559
{101, 120}	1, 2, 6, 22, 90, 397, 1859	Theorem 50	240	A374551
{000, 120}	1, 2, 5, 15, 50, 185, 737	Theorem 51	355	A374552
{000, 010}	1, 2, 4, 10, 29, 95, 345	Theorem 58	235	A279552
{010, 201} or {010, 210}	1, 2, 5, 15, 53, 214, 958	Theorem 61	185	A360052
{010, 110}	1, 2, 5, 15, 52, 201, 847	Theorem 64	145	A359191
{010, 102}	1, 2, 5, 15, 51, 186, 707	Theorem 67	265	A374553
{100, 102}	1, 2, 6, 21, 80, 318, 1305	Theorem 68	380	A374554

Table 2: Enumeration sequences of inversion sequences avoiding pairs of patterns studied in this article. Each section of the table corresponds to a different construction. The "performance" column indicates an approximation of the number of terms of each enumeration sequence we were able to compute in 1 minute, with C++ programs running on a personal computer (naive methods can compute around 20 terms at most).

[1] We consider the enumeration of P-avoiding inversion sequences "solved" once an algorithm is known to compute each number $\lvert\mathcal{I}_n(P)\rvert$ in polynomial time in n. Such an algorithm is sometimes called a "Wilfian formula" after Herbert Wilf's paper [34]. In our work, Wilfian formulas always take the form of a recurrence relation or an explicit expression.

Each construction is presented in a different section. The construction of Section 2 simply consists in inserting each entry of a sequence from left to right. In Section 3, we construct sequences by inserting their entries in increasing order of value (the order of insertion of entries having the same value varies according to the patterns considered). Section 4 is an improved and more complete version of a previous (unpublished) work [29]. It relies on a decomposition of inversion sequences around their first maximum; this means sequences are obtained by concatenating two smaller sequences. Section 5 introduces shifted inversion sequences and uses a decomposition around their first minimum, similar to that of Section 4. It is sometimes easier to construct pattern-avoiding inversion sequences by seeing them as a particular case of shifted inversion sequences.

In Section 6, we present conjectures about the algebraicity of the generating functions of several classes of inversion sequences avoiding pairs of patterns, and prove two of those conjectures. We conclude with a brief discussion on the asymptotic behavior of the number of pattern-avoiding inversion sequences in Section 7. In particular, we give sufficient conditions on a set of patterns P to show that the growth of the enumeration sequence of P-avoiding inversion sequences is bounded above by an exponential function, or to show that it is super-exponential.

1.3 Notation and preliminaries

Notation

For any integers $a, b \in \mathbb{Z}$, we denote the integer interval $[a, b] = \{k \in \mathbb{Z} \ : \ a \leqslant k \leqslant b\}$. We denote by $\delta_{a,b} = \begin{cases} 1 & \text{if} \quad a = b \\ 0 & \text{if} \quad a \neq b \end{cases}$ the Kronecker delta function. For all $n \in \mathbb{N}$, we denote by $C_n = \frac{1}{n+1}\binom{2n}{n}$ the Catalan numbers. We denote by ε the *empty sequence*, that is the only sequence of size 0.

Let $n, m \in \mathbb{N}$, and let $\sigma \in \mathbb{N}^n, \tau \in \mathbb{N}^m$ be two integer sequences. We define the *concatenation* of σ and τ as $\sigma \cdot \tau = (\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m) \in \mathbb{N}^{n+m}$. For all $k \in \mathbb{N}$, we denote by σ^k the concatenation of k copies of σ (in particular, $\sigma^0 = \varepsilon$).

For any $k \in \mathbb{N}$, we denote by $\sigma + k = (\sigma_i + k)_{i \in [1,n]}$ the sequence obtained by adding k to each term of σ.

Terminology

A *combinatorial class* is a set of objects $\mathcal{C}$ together with a size function $|\cdot| : \mathcal{C} \to \mathbb{N}$ such that there is a finite number of objects of size n for each $n \in \mathbb{N}$. For any set of patterns P, the set of all P-avoiding inversion sequences $\mathcal{I}(P) = \coprod_{n \geqslant 0} \mathcal{I}_n(P)$ is a combinatorial class.

If $\mathcal{C}$ is a combinatorial class, and $\mathcal{C}_n$ its subset of objects of size n for each $n \in \mathbb{N}$, we call $(|\mathcal{C}_n|)_{n \in \mathbb{N}}$ the *enumeration sequence* of $\mathcal{C}$ (here, the vertical bars are used to denote set cardinality). The (ordinary) *generating function* of the combinatorial class $\mathcal{C} = \coprod_{n \in \mathbb{N}} \mathcal{C}_n$ is the formal power series $\sum_{n \in \mathbb{N}} |\mathcal{C}_n| x^n$ in the indeterminate x.

Given two integer sequences $\sigma \in \mathbb{N}^n$ and $\tau \in \mathbb{N}^m$, we say that τ is a *factor* of σ if τ is a subsequence of consecutive terms of σ, i.e. if there exists two integers $a \leqslant b \in [1, n]$ such that $\tau = (\sigma_i)_{i \in [a,b]}$, or $\tau = \varepsilon$.

Statistics

For all $n \in \mathbb{N}$ and $\sigma \in \mathbb{N}^n$, let

- $\mathbf{Vals}(\sigma) = \{\sigma_i \ : \ i \in [1, n]\}$ be the *set of values* of σ,
- $\mathbf{min}(\sigma) = \mathbf{min}(\mathbf{Vals}(\sigma))$ be the *minimum* of σ, with the convention $\mathbf{min}(\varepsilon) = +\infty$,
- $\mathbf{max}(\sigma) = \mathbf{max}(\mathbf{Vals}(\sigma))$ be the *maximum* of σ, with the convention $\mathbf{max}(\varepsilon) = -1$,
- $\mathbf{dist}(\sigma) = |\mathbf{Vals}(\sigma)|$ be the number of *distinct values* of σ,
- $|\sigma| = n$ be the *size* or *length* of σ,
- $\mathbf{firstmax}(\sigma) = \mathbf{min}(i \in [1, n] \ : \ \sigma_i = \mathbf{max}(\sigma))$ be the position of the *first maximum* of σ, with the convention $\mathbf{firstmax}(\varepsilon) = 0$,
- $\mathbf{lastmax}(\sigma) = \mathbf{max}(i \in [1, n] \ : \ \sigma_i = \mathbf{max}(\sigma))$ be the position of the *last maximum* of σ, with the convention $\mathbf{lastmax}(\varepsilon) = 0$.

Words

For all $n, k \in \mathbb{N}$, we denote by $\mathcal{W}_{n,k} = [0, k-1]^n$ the set of words of length n over the alphabet $[0, k-1]$, and by $\overline{\mathcal{W}}_{n,k} = \{\omega \in \mathcal{W}_{n,k} \ : \ \mathbf{dist}(\omega) = k\}$ the subset of words in which each letter of the alphabet $[0, k-1]$ appears at least once[2]. For any set of patterns P, we denote by $\mathcal{W}_{n,k}(P)$ and $\overline{\mathcal{W}}_{n,k}(P)$ the subsets of P-avoiding words of $\mathcal{W}_{n,k}$ and $\overline{\mathcal{W}}_{n,k}$. The following proposition shows that it is essentially equivalent to solve the enumeration of $\mathcal{W}_{n,k}(P)$ or that of $\overline{\mathcal{W}}_{n,k}(P)$.

Proposition 1. *For any set of patterns P, for all $n, k \in \mathbb{N}$,*

$$|\mathcal{W}_{n,k}(P)| = \sum_{d=0}^{\mathbf{min}(n,k)} \binom{k}{d} |\overline{\mathcal{W}}_{n,d}(P)|.$$

Proof. Let $n, k \geqslant 0$, $d \in [0, \mathbf{min}(n, k)]$, and let $\mathfrak{W}_{n,k,d} = \{\omega \in \mathcal{W}_{n,k} \ : \ d = \mathbf{dist}(\omega)\}$ be the subset of $\mathcal{W}_{n,k}$ of words containing exactly d distinct letters. Let $F_{d,k}$ be the set of strictly increasing functions from $[0, d-1]$ to $[0, k-1]$. It is easy to see that the function $F_{d,k} \times \overline{\mathcal{W}}_{n,d} \to \mathfrak{W}_{n,k,d}$, $(\varphi, \omega) \mapsto (\varphi(\omega_i))_{i \in [1,n]}$ is a bijection. For any set of patterns P, restricting this function to the domain $F_{d,k} \times \overline{\mathcal{W}}_{n,d}(P)$ yields a bijection with the set of P-avoiding words in $\mathfrak{W}_{n,k,d}$ (the image of (φ, ω) is order-isomorphic to ω, so it contains the same patterns). This concludes the proof, since $\mathcal{W}_{n,k} = \coprod_{d=0}^{\mathbf{min}(n,k)} \mathfrak{W}_{n,k,d}$, and $|F_{d,k}| = \binom{k}{d}$. $\square$

[2] Equivalently, $\mathcal{W}_{n,k}$ is the set of maps $[1, n] \to [0, k-1]$, and $\overline{\mathcal{W}}_{n,k}$ is the subset of surjective maps. We remark that the set of surjective maps $[1, n] \to [1, k]$ is also known as the set of *Cayley permutations* [23] of length n and maximum k.

2 Generating trees growing on the right

2.1 Method

A *generating tree* is a rooted, labelled tree such that the label of each node determines its number of children, and their labels. Generating trees were first introduced in the context of pattern-avoiding permutations in [32, 33]. We call *combinatorial generating tree* a generating tree labelled by the objects of a combinatorial class and such that each object of size n labels exactly one node, at level n (with the convention that the root is at level 0). A combinatorial generating tree can be defined using a map, called *ECO operator* [7], which constructs each object of size $n+1$ from some object of size n. More generally, a generating tree can always be defined by a *succession rule*

$$\Omega = \begin{cases} a \\ \ell \rightsquigarrow \ell_1 \ldots \ell_{c(\ell)} \end{cases}$$

composed of an *axiom* a, which labels the root of the tree, and a *production* which associates to each label ℓ the labels of the children of any node labelled ℓ in the tree. In the example above, we denoted $c(\ell)$ the number of children of nodes labelled ℓ, and $\ell_1, \ldots, \ell_{c(\ell)}$ the labels[3] of those children. A simple combinatorial generating tree for the class of inversion sequences is defined by the succession rule

$$\Omega_{\text{inv}} = \begin{cases} \varepsilon \\ \sigma \rightsquigarrow \sigma \cdot i \quad \text{for} \quad i \in [0, |\sigma|]. \end{cases}$$

For any set of patterns P, a combinatorial generating tree for the class of P-avoiding inversion sequences $\mathcal{I}(P)$ can be obtained by restricting the above to only accept values of i such that $\sigma \cdot i$ avoids the patterns in P. This does define a generating tree: indeed, if $\sigma \cdot i$ avoids P, then σ must also avoid P. We call this tree the generating tree *growing on the right* for inversion sequences avoiding the patterns P. The present section is dedicated to such generating trees, which are one of the simplest and most common construction for pattern-avoiding inversion sequences. In all of Section 2, we call i a *forbidden* value for (σ, P) (or simply for σ, when P is implicit) if $\sigma \cdot i$ contains a pattern in P.

Most generating trees used in the literature to solve enumeration problems are not combinatorial generating trees, but instead use labels that are much simpler (typically, integers or tuples of integers). We define a generating tree to be *concise* if it does not contain two isomorphic subtrees rooted in nodes having different labels. Informally, a generating tree is concise if it involves the minimal number of labels required to describe the "shape" of the tree.

We say that two nodes of a tree $\mathcal{T}$ are *$\mathcal{T}$-equivalent* if they are roots of isomorphic subtrees of $\mathcal{T}$. In particular, two nodes of $\mathcal{T}$ which have the same label are always $\mathcal{T}$-equivalent: the label of a node determines its number of children and their labels, hence

[3] The same label may appear on several children of ℓ. Formally, the production maps each label to a multiset of labels.

a label also determines the entire subtree rooted in its node, by induction. Note that $\mathcal{T}$ is concise if and only if two $\mathcal{T}$-equivalent nodes always have the same label. We now explain how to go (in favorable cases) from a combinatorial generating tree to a concise generating tree.

Assume we have a combinatorial generating tree $\mathcal{T}$ for a combinatorial class $\mathcal{C}$ and a statistic $s : \mathcal{C} \to L$ for some set L, such that for each $\sigma \in \mathcal{C}$, the value $s(\sigma)$ determines the number of children of the node labelled σ in $\mathcal{T}$, and the value of s when applied to each child. Formally, this means there exists a function $f : L \to \mathbb{N}^L$ (where $\mathbb{N}^L$ is the set of multisets of elements in L) such that for all $\sigma \in \mathcal{C}$, $f(s(\sigma))$ is the finite multiset $s(\tau_1), \ldots, s(\tau_q)$, where $\tau_1, \ldots, \tau_q$ are the labels of the children of the node labelled σ in $\mathcal{T}$. Replacing each label σ by the value $s(\sigma)$ yields a generating tree $\mathcal{T}'$ which is isomorphic to $\mathcal{T}$. Notice that $\mathcal{T}'$ is defined by a succession rule which only involves the values of s (the axiom of this rule is the image under s of the label of the root of $\mathcal{T}$, and its production is the function we denoted f). On our previous example, we can use the statistic "size" to turn Ω_{inv} into a simpler succession rule

$$\Omega_{\text{factorial}} = \begin{cases} (0) \\ (n) \rightsquigarrow (n+1)^{n+1} \end{cases}$$

where the production means that each node labelled (n) has $n+1$ children, each labelled $(n+1)$. From this succession rule, we can easily see that there are $n!$ nodes at level n.

By keeping only the value $s(\sigma)$ rather than the "complete" object $\sigma \in \mathcal{C}$ as a label, we can retain a lower amount of information about objects, which is still sufficient to describe the tree (up to isomorphism). We say that two objects $\sigma, \tau \in \mathcal{C}$ such that $s(\sigma) = s(\tau)$ are *s-equivalent.* Each label $\ell \in L$ of $\mathcal{T}'$ corresponds to the *s-equivalence class* $s^{-1}(\ell) \subseteq \mathcal{C}$.

Since s-equivalent objects of $\mathcal{C}$ are always labels of $\mathcal{T}$-equivalent nodes, the coarsest s-equivalence relation is obtained by defining s to be the statistic which maps each object of $\mathcal{C}$ to the $\mathcal{T}$-equivalence class of the corresponding node. In particular, for any generating tree $\mathcal{T}$, there exists a concise generating tree $\mathcal{T}'$ isomorphic to $\mathcal{T}$, obtained by replacing the label of each node of $\mathcal{T}$ by its $\mathcal{T}$-equivalence class.

In practice, finding this tree $\mathcal{T}'$ requires some way of knowing whether two nodes are $\mathcal{T}$-equivalent, which is not always obvious. For slightly different definitions[4] of generating trees and equivalence, [16] presents an algorithm which can test whether two nodes are equivalent in finite time, for any combinatorial generating tree growing on the right for inversion sequences avoiding a finite set of patterns. This algorithm then labels each node of the tree by an inversion sequence (the minimal sequence in lexicographic order) which corresponds to a node in the same equivalence class.

Classically, generating trees are used to solve enumeration problems because they induce recurrence relations on the corresponding enumeration sequences (which may also

[4]In [16], generating trees are defined as plane trees, and two nodes are equivalent if they are roots of isomorphic plane subtrees. This relation relies on an order on the children of each node, and it is finer than the $\mathcal{T}$-equivalence we defined.

be turned into equations satisfied by their generating function). Here we formalize how a succession rule can be turned into such a recurrence relation. Assume we have a combinatorial generating tree $\mathcal{T}$ for a combinatorial class $\mathcal{C}$, a generating tree $\mathcal{T}'$ labelled by elements in some set L, and a function $s : \mathcal{C} \to L$ such that replacing each label σ of $\mathcal{T}$ by its image $s(\sigma)$ yields $\mathcal{T}'$. For all $(n, \ell) \in \mathbb{N} \times L$, let $\mathfrak{c}_{n,\ell} = |\{\sigma \in \mathcal{C} \ : \ |\sigma| = n, \ s(\sigma) = \ell\}|$, so that the number of objects of size n in $\mathcal{C}$ is $\sum_\ell \mathfrak{c}_{n,\ell}$. Let $f : L \to \mathbb{N}^L$ be the production of $\mathcal{T}'$. For all $n \geqslant 1$ and $\ell \in L$, we have

$$\mathfrak{c}_{n,\ell} = \sum_{k \in L} f(k)(\ell) \cdot \mathfrak{c}_{n-1,k},$$

where $f(k)(\ell)$ counts the multiplicity of the label ℓ among the children of a node labelled k in $\mathcal{T}'$.

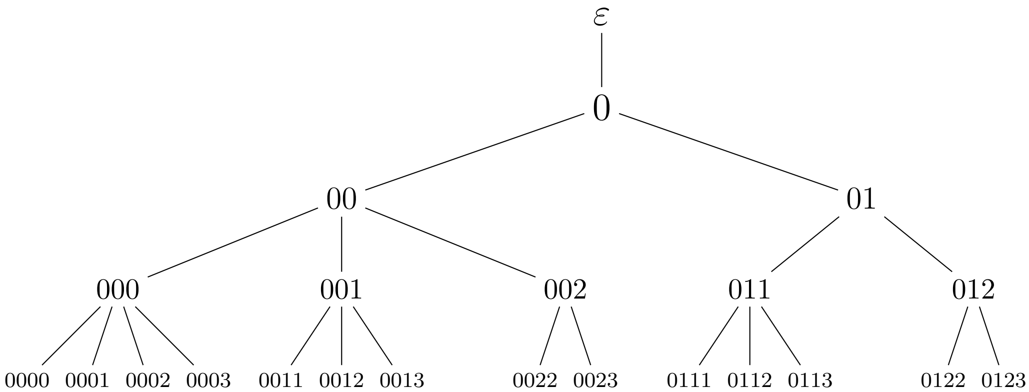

Figure 1: First five levels of the combinatorial generating tree growing on the right for $\mathcal{I}(10)$.

We end this introduction with a simple example. The following succession rule describes the generating tree growing on the right for inversion sequences avoiding the pattern 10 (i.e. nondecreasing inversion sequences), represented in Figure 1.

$$\Omega_{10} = \begin{cases} \varepsilon \\ \sigma \rightsquigarrow \sigma \cdot i \quad \text{for} \quad i \in [\mathbf{max}(\sigma), |\sigma|]. \end{cases}$$

For each $\sigma \in \mathcal{I}(10)$, let $s(\sigma)$ be the number of children of the node labelled σ in this tree. It can be shown that $s(\varepsilon) = 1$, and $s(\sigma) = 1 + |\sigma| - \mathbf{max}(\sigma)$ if σ is nonempty. By replacing each label σ by the value $s(\sigma)$, we obtain an isomorphic generating tree, represented in Figure 2, and described by the succession rule

$$\Omega_{\text{Cat}} = \begin{cases} (1) \\ (k) \rightsquigarrow (i) \quad \text{for} \quad i \in [2, k+1]. \end{cases}$$

It is known [32] that the succession rule Ω_{Cat} describes a tree in which the number of nodes at each level n is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. It is easy to see that

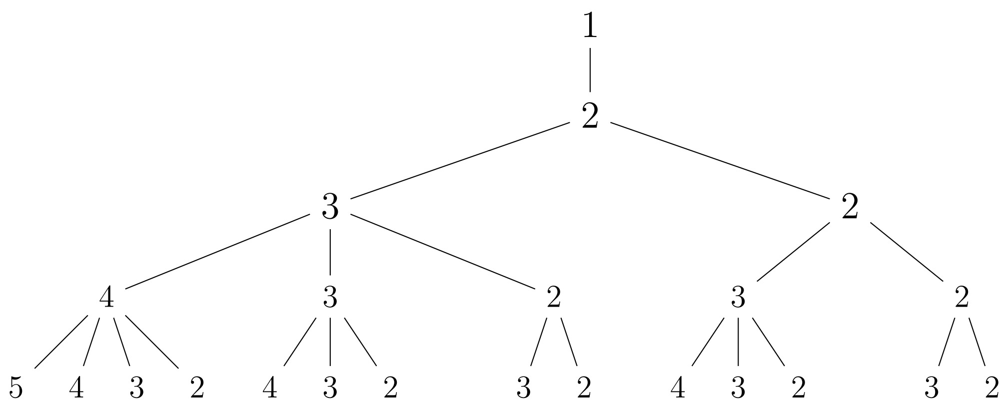

Figure 2: First five levels of the generating tree described by the succession rule Ω_{Cat}.

this succession rule describes a concise generating tree: the label of each node counts its number of children, so if two nodes are roots of isomorphic subtrees, they must have the same label.

From now on, we allow ourselves to represent a combinatorial generating tree and an isomorphic concise generating tree on the same figure, by placing two labels on each node.

2.2 The pair {000, 100}

Theorem 2. *The enumeration of inversion sequences avoiding the patterns 000 and 100 is given by the succession rule*

$$\Omega_{\{000,100\}} = \begin{cases} (1,0) \\ (a,b) \rightsquigarrow (a+1,b-1)^b \\ \qquad\qquad (a+1-j,b+j) \quad \textit{for} \quad j \in [1,a]. \end{cases}$$

Proof. For all $n \in \mathbb{N}$ and $\sigma \in \mathcal{I}_n(000,100)$, let

- $A(\sigma) = \{i > \mathbf{max}(\sigma) \ : \ \sigma \cdot i \in \mathcal{I}(000,100)\}$ be the set of values which can be inserted at the end of σ "strictly above" its maximum,
- $B(\sigma) = \{i \leqslant \mathbf{max}(\sigma) \ : \ \sigma \cdot i \in \mathcal{I}(000,100)\}$ be the set of values which can be inserted at the end of σ "weakly below" its maximum.

Inserting any value greater than $\mathbf{max}(\sigma)$ at the end of σ cannot create an occurrence of the patterns 000 or 100. As a result, $A(\sigma) = [\mathbf{max}(\sigma)+1, n]$. Let $a(\sigma) = |A(\sigma)|$, and $b(\sigma) = |B(\sigma)|$. In particular, $(a(\varepsilon), b(\varepsilon)) = (1,0)$.

We show that the combinatorial generating tree growing on the right for $\mathcal{I}(000,100)$ is isomorphic to the tree described by $\Omega_{\{000,100\}}$. This isomorphism relabels each node σ by $(a(\sigma), b(\sigma))$.

Let $n \in \mathbb{N}$, $\sigma \in \mathcal{I}_n(000,100)$, $i \in A(\sigma) \sqcup B(\sigma)$, and $\sigma' = \sigma \cdot i$.

- If $i \in B(\sigma)$, then $A(\sigma') = A(\sigma) \sqcup \{n+1\}$ and $B(\sigma') = B(\sigma) \setminus \{i\}$, since if $i < \mathbf{max}(\sigma)$, then i becomes forbidden to avoid 100, and if $i = \mathbf{max}(\sigma)$, then i becomes forbidden to avoid 000. In this case, σ' satisfies $(a(\sigma'), b(\sigma')) = (a(\sigma) + 1, b(\sigma) - 1)$.
- If $i \in A(\sigma)$, then no additional values are forbidden, and the values $[\mathbf{max}(\sigma) + 1, i]$ are now less than or equal to $\mathbf{max}(\sigma')$. Let $j = i - \mathbf{max}(\sigma)$. In this case, σ' satisfies $(a(\sigma'), b(\sigma')) = (a(\sigma) + 1 - j, b(\sigma) + j)$. As i ranges over $A(\sigma)$, j varies from 1 to $a(\sigma)$. □

2.3 The pair {102, 201}

In this section, we present a generating tree construction to enumerate $\mathcal{I}(102, 201)$. We later use this construction to find the generating function of $\mathcal{I}(102, 201)$ in Theorem 74.

A core idea in our construction of inversion sequences avoiding 102 and 201 is that we may restrict the "future" of combinatorial objects in a generating tree construction; we discuss this further in Remark 13. This idea comes from Pantone, who calls this a "commitment", see [27].

We begin by establishing some general properties of integer sequences avoiding 102 and 201. For any $\sigma \in \mathbb{N}^n$ such that $\mathbf{dist}(\sigma) \geqslant 2$, let $\mathbf{sec}(\sigma) = \mathbf{max}(\mathbf{Vals}(\sigma) \setminus \{\mathbf{max}(\sigma)\})$ be the second largest value of σ.

Proposition 3. *Let σ be an integer sequence of size n such that $\mathbf{dist}(\sigma) \geqslant 2$, let $m = \mathbf{max}(\sigma)$ and $s = \mathbf{sec}(\sigma)$. Then σ avoids 102 and 201 if and only if all three following conditions are satisfied:*

1. *$(\sigma_i)_{i \in [1, \mathbf{firstmax}(\sigma)]}$ is nondecreasing,*
2. *$(\sigma_i)_{i \in [\mathbf{lastmax}(\sigma), n]}$ is nonincreasing,*
3. *$\forall i \in [\mathbf{firstmax}(\sigma), \mathbf{lastmax}(\sigma)]$, $\sigma_i \in \{m, s\}$.*

Proof. First, we show that each condition is necessary:

1. If $i < j \in [1, \mathbf{firstmax}(\sigma)]$ satisfy $\sigma_i > \sigma_j$, then $\sigma_j < m$. In particular, we have $j \neq \mathbf{firstmax}(\sigma)$ hence (σ_i, σ_j, m) is an occurrence of 102.
2. If $i < j \in [\mathbf{lastmax}(\sigma), n]$ satisfy $\sigma_i < \sigma_j$, then $\sigma_i < m$. In particular, we have $i \neq \mathbf{lastmax}(\sigma)$ hence (m, σ_i, σ_j) is an occurrence of 201.
3. If $i \in [\mathbf{firstmax}(\sigma), \mathbf{lastmax}(\sigma)]$ satisfies $\sigma_i < s$, let j be an integer such that $\sigma_j = s$. If $j < i$, then (σ_j, σ_i, m) is an occurrence of 102. If $i < j$, then (m, σ_i, σ_j) is an occurrence of 201.

Assume, for the sake of contradiction, that σ satisfies all three conditions, and that $(\sigma_i, \sigma_j, \sigma_k)$ is an occurrence of 102 for some $i < j < k \in [1, n]$. Since $\sigma_i > \sigma_j$, condition 1 implies that $j > \mathbf{firstmax}(\sigma)$. Since $\sigma_j < \sigma_k$, condition 2 implies that $j < \mathbf{lastmax}(\sigma)$. Hence by condition 3, $\sigma_j \geqslant s$, which implies that $(\sigma_i, \sigma_j) = (m, s)$ because $\sigma_i > \sigma_j$. Finally $\sigma_k > \sigma_i = m$, a contradiction. The same reasoning holds for the pattern 201, by the reverse symmetry. □

Proposition 3 has two immediate corollaries.

Corollary 4. *Let σ be a $\{102, 201\}$-avoiding integer sequence such that $m = \mathbf{max}(\sigma)$, and $s = \mathbf{sec}(\sigma)$. Any occurrence of the pattern 101 in σ is a subsequence (m, s, m).*

Corollary 5. *An integer sequence avoids $\{101, 102, 201\}$ if and only if it is unimodal.*

We first study inversion sequences avoiding $\{101, 102, 201\}$, then those which avoid $\{102, 201\}$ and contain the pattern 101. The enumeration of inversion sequences avoiding 101, 102 and 201 was already solved and their generating function is given in [10][5]. We nevertheless describe a generating tree for this class, for the sake of completeness, and in order to better introduce the more difficult case of sequences containing 101.

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(10) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of nondecreasing inversion sequences of size n and maximum m for $n \geqslant 1$, and let $\mathfrak{A}_{0,0} = \{\varepsilon\}$ by convention. Let $\mathfrak{C}^{(1)}_{n,\ell} = \{\sigma \in \mathcal{I}_n(101, 102, 201) \ : \ \ell = \sigma_n < \mathbf{max}(\sigma)\}$ be the set of $\{101, 102, 201\}$-avoiding inversion sequences of size n, last value ℓ, and such that ℓ is not the maximum. Let $\mathfrak{A} = \coprod_{n,m\geqslant 0} \mathfrak{A}_{n,m}$, and $\mathfrak{C}^{(1)} = \coprod_{n>\ell\geqslant 0} \mathfrak{C}^{(1)}_{n,\ell}$. From Corollary 5, $\mathfrak{C}^{(1)}$ is the set of unimodal inversion sequences which decrease after reaching their maximum, and we have $\mathcal{I}(101, 102, 201) = \mathfrak{A} \sqcup \mathfrak{C}^{(1)}$. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, and $\mathfrak{c}^{(1)}_{n,\ell} = |\mathfrak{C}^{(1)}_{n,\ell}|$.

Lemma 6. *For all $n > m \geqslant 0$,*

$$\mathfrak{a}_{n,m} = \frac{\binom{n+m-1}{m}(n-m)}{n}.$$

For all $n \geqslant 3, \ell \geqslant 0$,

$$\mathfrak{c}^{(1)}_{n,\ell} = \sum_{i=\ell}^{n-2} \mathfrak{c}^{(1)}_{n-1,i} + \sum_{i=\ell+1}^{n-2} \mathfrak{a}_{n-1,i}.$$

Proof. Let $n \geqslant 1$, $m \in [0, n-1]$, and $\sigma \in \mathfrak{A}_{n,m}$. By definition of $\mathfrak{A}_{n,m}$, we have $\sigma_n = m$. Removing the last entry σ_n from σ yields a sequence in $\mathfrak{A}_{n-1,i}$ for some $i \in [0, m]$. Conversely, σ can be obtained by appending the value m to some sequence in $\mathfrak{A}_{n-1,i}$. Hence, there is a bijection between the sets $\mathfrak{A}_{n,m}$ and $\coprod_{i=0}^{m} \mathfrak{A}_{n-1,i}$. This yields the equation $\mathfrak{a}_{n,m} = \sum_{i=0}^{m} \mathfrak{a}_{n-1,i}$, from which we can easily prove that $\mathfrak{a}_{n,m} = \frac{\binom{n+m-1}{m}(n-m)}{n}$ for all $n > m \geqslant 0$. This corresponds to Catalan's Triangle (OEIS A009766).

Let $n \geqslant 3, \ell \geqslant 0$, $\sigma \in \mathfrak{C}^{(1)}_{n,\ell}$, and $i = \sigma_{n-1} \geqslant \ell$. Let $\sigma' = (\sigma_j)_{j\in[1,n-1]}$ be the sequence obtained by removing the last entry from σ. If σ' is nondecreasing, then $i = \mathbf{max}(\sigma) > \ell$, and $\sigma' \in \mathfrak{A}_{n-1,i}$. Otherwise, $\sigma' \in \mathfrak{C}^{(1)}_{n,\ell}$. As in the previous case, this is a bijection between $\mathfrak{C}^{(1)}_{n,\ell}$ and $\left(\coprod_{i=\ell}^{n-2} \mathfrak{C}^{(1)}_{n-1,i}\right) \sqcup \left(\coprod_{i=\ell+1}^{n-2} \mathfrak{A}_{n-1,i}\right)$. $\square$

[5] This article employs the same generating tree construction (that is, inserting each entry of a sequence from left to right). However, it does not present an explicit description of the labels in terms of statistics of inversion sequences, since the algorithmic approach from [16] was used to derive the succession rule.

Remark 7. The bijections from the proof of Lemma 6 can be turned into a succession rule, by labelling (n, m) each sequence in $\mathfrak{A}_{n,m}$, and labelling (ℓ) each sequence in $\mathfrak{C}^{(1)}_{n,\ell}$:

$$\Omega_{\{101,102,201\}} = \begin{cases} \quad (0,0) & & & \\ (n,m) & \rightsquigarrow & (n+1,i) & \text{for} \quad i \in [m,n] \\ & & (i) & \text{for} \quad i \in [0, m-1] \\ \\ \quad (\ell) & \rightsquigarrow & (i) & \text{for} \quad i \in [0,\ell]. \end{cases}$$

It remains to count inversion sequences in $\mathcal{I}(102, 201)$ which contain 101. By Proposition 3 and Corollary 4, an inversion sequence σ avoids $\{102, 201\}$ and contains 101 if and only if it can be split into three factors $\alpha \cdot \beta \cdot \gamma = \sigma$ such that:

1. α is a nondecreasing inversion sequence such that $\mathbf{max}(\alpha) < \mathbf{max}(\sigma)$,
2. β is a word over the alphabet $\{\mathbf{max}(\sigma), \mathbf{sec}(\sigma)\}$ which contains 101, and such that $\beta_1 = \mathbf{max}(\sigma)$,
3. γ is a (possibly empty) nonincreasing word over the alphabet $[0, \mathbf{sec}(\sigma) - 1]$.

We distinguish several sets of sequences according to how much of the first occurrence of the pattern 101 has already appeared. Let

- $\mathfrak{B}^{(1)}_{n,s} = \{\sigma \in \mathcal{I}_n(10) \ : \ s \in [\mathbf{sec}(\sigma), \mathbf{max}(\sigma) - 1]\}$, which can be seen as the sequences in which only the first 1 in a pattern 101 has yet appeared.
- $\mathfrak{B}^{(2)}_{n,s} = \{\sigma \cdot (s)^i \ : \ i \in [1, n-2],\ \sigma \in \mathfrak{B}^{(1)}_{n-i,s}\}$, which can be seen as the sequences in which only the first 10 in a pattern 101 has yet appeared.
- $\mathfrak{B}^{(3)}_{n,s} = \{\sigma \in \mathcal{I}_n(102, 201) \ : \ s = \mathbf{sec}(\sigma) \leqslant \sigma_n,\ \sigma \text{ contains } 101\}$ be the set of $\{102, 201\}$-avoiding inversion sequences of size n and second largest value s which contain 101, and for which the factor denoted γ earlier is empty.
- $\mathfrak{C}^{(2)}_{n,\ell} = \{\sigma \in \mathcal{I}_n(102, 201\} \ : \ \ell = \sigma_n < \mathbf{sec}(\sigma),\ \sigma \text{ contains } 101\}$ be the set of $\{102, 201)$-avoiding inversion sequences of size n and last value ℓ which contain 101, and for which the factor denoted γ earlier is nonempty.

Let $\mathfrak{B}^{(2)} = \coprod_{n,s\geqslant 0} \mathfrak{B}^{(2)}_{n,s}$, $\mathfrak{B}^{(3)} = \coprod_{n,s\geqslant 0} \mathfrak{B}^{(3)}_{n,s}$, and $\mathfrak{C}^{(2)} = \coprod_{n,\ell\geqslant 0} \mathfrak{C}^{(2)}_{n,\ell}$. In particular, $\{\sigma \in \mathcal{I}(102, 201) \ : \ \sigma \text{ contains } 101\} = \mathfrak{B}^{(3)} \sqcup \mathfrak{C}^{(2)}$.

Remark 8. The subsets of $\mathcal{I}(102, 201)$ we have defined are not disjoint. More precisely, the following holds.

1. The sets $\mathfrak{B}^{(1)}_{n,s}$ intersect for different values of s. For instance, the sequence $(0, 0, 2)$ belongs to both $\mathfrak{B}^{(1)}_{3,0}$ and $\mathfrak{B}^{(1)}_{3,1}$.
2. Each sequence in a set $\mathfrak{B}^{(1)}_{n,s}$ also belongs to a set $\mathfrak{A}_{n,m}$ for some $m > s$.

3. Each set $\mathfrak{B}^{(2)}_{n,s}$ is a subset of $\mathfrak{C}^{(1)}_{n,s}$.

Let $\mathfrak{b}^{(i)}_{n,s} = |\mathfrak{B}^{(i)}_{n,s}|$ for $i \in \{1,2,3\}$, and $\mathfrak{c}^{(2)}_{n,s} = |\mathfrak{C}^{(2)}_{n,s}|$.

Lemma 9. *For all* $0 \leqslant s \leqslant n-2$,

$$\mathfrak{b}^{(1)}_{n,s} = \mathfrak{b}^{(1)}_{n-1,s} + (n-1-s)\mathfrak{a}_{n,s}.$$

For all $0 \leqslant s \leqslant n-3$,

$$\mathfrak{b}^{(2)}_{n,s} = \mathfrak{b}^{(2)}_{n-1,s} + \mathfrak{b}^{(1)}_{n-1,s}.$$

For all $0 \leqslant s \leqslant n-4$,

$$\mathfrak{b}^{(3)}_{n,s} = 2\mathfrak{b}^{(3)}_{n-1,s} + \mathfrak{b}^{(2)}_{n-1,s}.$$

For all $0 \leqslant \ell \leqslant n-6$,

$$\mathfrak{c}^{(2)}_{n,\ell} = \sum_{i=\ell}^{n-5} \mathfrak{c}^{(2)}_{n-1,i} + \sum_{i=\ell+1}^{n-5} \mathfrak{b}^{(3)}_{n-1,i}.$$

Proof. We give a bijective proof of each identity by removing the last entry of an inversion sequence.

Let $n \geqslant 2, s \in [0, n-2]$, $\sigma \in \mathfrak{B}^{(1)}_{n,s}$. In particular, $\sigma_n = \mathbf{max}(\sigma)$. Let $\sigma' = (\sigma_j)_{j\in[1,n-1]}$ be the sequence obtained by removing the last entry from σ, and let $i = \sigma_{n-1}$. If $i = \sigma_n$, then $\sigma' \in \mathfrak{B}^{(1)}_{n-1,s}$. Otherwise, $i = \mathbf{sec}(\sigma) \leqslant s$, $\sigma' \in \mathfrak{A}_{n-1,i}$, and $\sigma_n \in [s+1, n-1]$. This yields a bijection between $\mathfrak{B}^{(1)}_{n,s}$ and $\mathfrak{B}^{(1)}_{n-1,s} \coprod \left([s+1, n-1] \times \coprod_{i=0}^{s} \mathfrak{A}_{n-1,i}\right)$. We know from the proof of Lemma 6 that $\coprod_{i=0}^{s} \mathfrak{A}_{n-1,i}$ is in bijection with $\mathfrak{A}_{n,s}$.

Let $n \geqslant 3, s \in [0, n-3]$, and $\sigma \in \mathfrak{B}^{(2)}_{n,s}$. In particular, $\sigma_n = s$, and $\sigma_{n-1} \in \{\mathbf{max}(\sigma), s\}$. Let $\sigma' = (\sigma_j)_{j\in[1,n-1]}$ be the sequence obtained by removing the last entry from σ, and let $i = \sigma_{n-1}$. If $i = s$, then $\sigma' \in \mathfrak{B}^{(2)}_{n-1,s}$. Otherwise $i = \mathbf{max}(\sigma)$, and $\sigma' \in \mathfrak{B}^{(1)}_{n-1,s}$.

Let $n \geqslant 4, s \in [0, n-4]$, and $\sigma \in \mathfrak{B}^{(3)}_{n,s}$. Let $\sigma' = (\sigma_j)_{j\in[1,n-1]}$ be the sequence obtained by removing the last entry from σ. If σ' contains 101, then $\sigma' \in \mathfrak{B}^{(3)}_{n-1,s}$, and $\sigma_n \in \{\mathbf{max}(\sigma), s\}$. Otherwise $\sigma' \in \mathfrak{B}^{(2)}_{n-1,s}$, and $\sigma_n = \mathbf{max}(\sigma)$.

Let $n \geqslant 6, \ell \in [0, n-6]$, and $\sigma \in \mathfrak{C}^{(2)}_{n,\ell}$. Let $\sigma' = (\sigma_j)_{j\in[1,n-1]}$ be the sequence obtained by removing the last entry from σ, and let $i = \sigma_{n-1} \geqslant \ell$. If $i < \mathbf{sec}(\sigma)$, then $\sigma' \in \mathfrak{C}^{(2)}_{n-1,i}$. Otherwise $i > \ell$, and $\sigma' \in \mathfrak{B}^{(3)}_{n-1,i}$. $\square$

Let $\mathfrak{C}_{n,\ell} = \mathfrak{C}^{(1)}_{n,\ell} \sqcup \mathfrak{C}^{(2)}_{n,\ell}$ and $\mathfrak{c}_{n,\ell} = |\mathfrak{C}_{n,\ell}| = \mathfrak{c}^{(1)}_{n,\ell} + \mathfrak{c}^{(2)}_{n,\ell}$. Let also $\mathfrak{C} = \mathfrak{C}^{(1)} \sqcup \mathfrak{C}^{(2)}$.

Theorem 10. *For all* $n \geqslant 1$,

$$|\mathcal{I}_n(102, 201)| = \left(\sum_{m=0}^{n-1} \mathfrak{a}_{n,m}\right) + \left(\sum_{s=0}^{n-4} \mathfrak{b}^{(3)}_{n,s}\right) + \left(\sum_{\ell=0}^{n-3} \mathfrak{c}_{n,\ell}\right).$$

Proof. Merging our results for inversion sequences which avoid or contain 101, we obtain

$$\mathcal{I}(102, 201) = \left(\mathfrak{A} \sqcup \mathfrak{C}^{(1)}\right) \sqcup \left(\mathfrak{B}^{(3)} \sqcup \mathfrak{C}^{(2)}\right) = \mathfrak{A} \sqcup \mathfrak{B}^{(3)} \sqcup \mathfrak{C}. \qquad \square$$

Remark 11. The equations for counting $\mathfrak{c}^{(1)}_{n,\ell}$ and $\mathfrak{c}^{(2)}_{n,\ell}$ from Lemmas 6 and 9 can be merged into a single recurrence relation for $\mathfrak{c}_{n,\ell}$. For all $0 \leqslant \ell \leqslant n-3$,

$$\mathfrak{c}_{n,\ell} = \mathfrak{c}_{n-1,\ell} + \sum_{i=\ell+1}^{n-2} \mathfrak{a}_{n-1,i} + \mathfrak{b}^{(3)}_{n-1,i} + \mathfrak{c}_{n-1,i}.$$

Remark 12. By labelling

- (a, n, m) each sequence of $\mathfrak{A}_{n,m}$,
- $(b^{(i)}, s)$ each sequence of $\mathfrak{B}^{(i)}_{n,s}$ for $i \in \{1, 2, 3\}$
- (c, ℓ) each sequence of $\mathfrak{C}_{n,\ell}$,

the recurrence relations from Lemmas 6, 9, and Remark 11 correspond to the succession rule

$$\Omega_{\{102,201\}} = \begin{cases} \quad (a,0,0) & & & \\ (a,n,m) & \rightsquigarrow & (a,n+1,i) & \text{for} \quad i \in [m,n] \\ & & (b^{(1)},i)^{n-i} & \text{for} \quad i \in [m,n-1] \\ & & (c,i) & \text{for} \quad i \in [0,m-1] \\ & & & \\ (b^{(1)},s) & \rightsquigarrow & (b^{(1)},s)\,(b^{(2)},s) & \\ (b^{(2)},s) & \rightsquigarrow & (b^{(2)},s)\,(b^{(3)},s) & \\ (b^{(3)},s) & \rightsquigarrow & (b^{(3)},s)^2 & \\ & & (c,i) & \text{for} \quad i \in [0,s-1] \\ & & & \\ (c,\ell) & \rightsquigarrow & (c,i) & \text{for} \quad i \in [0,\ell]. \end{cases}$$

Let $\mathcal{T}$ be the generating tree described by the succession rule $\Omega_{\{102,201\}}$. We now explain how to relate $\mathcal{T}$ with the definitions of Section 2.1. First, note that $\mathcal{T}$ is not isomorphic to a combinatorial generating tree for the class $\mathcal{I}(102, 201)$, because of the intersections mentioned in Remark 8. By construction, $\mathcal{T}$ is isomorphic to a combinatorial generating tree for the disjoint union $\mathcal{U}$ of the sets $\mathfrak{A}$, $\{\coprod_{n\geqslant 0} \mathfrak{B}^{(1)}_{n,s}\}_{s\geqslant 0}$, $\mathfrak{B}^{(2)}$, $\mathfrak{B}^{(3)}$ and $\mathfrak{C}$. We can view the combinatorial class $\mathcal{U}$ as a subset of $\mathbb{N} \times \mathcal{I}(102, 201)$:

$$\mathcal{U} \cong \big(\{0\} \times \big(\mathfrak{A} \sqcup \mathfrak{B}^{(3)} \sqcup \mathfrak{C}\big)\big) \sqcup \big(\{1\} \times \mathfrak{B}^{(2)}\big) \sqcup \left(\coprod_{s\geqslant 0} \left(\{s+2\} \times \coprod_{n\geqslant 0} \mathfrak{B}^{(1)}_{n,s}\right)\right),$$

and define the size of each object $(k, \sigma) \in \mathcal{U}$ as the size of σ. By the proof of Theorem 10, the objects of $\mathcal{U}$ of the form $(0, \sigma)$ are trivially in bijection with $\mathcal{I}(102, 201)$. The remaining objects of $\mathcal{U}$, of the form (k, σ) for $k \geqslant 1$, can be called *phantom objects* as they are ultimately not counted.

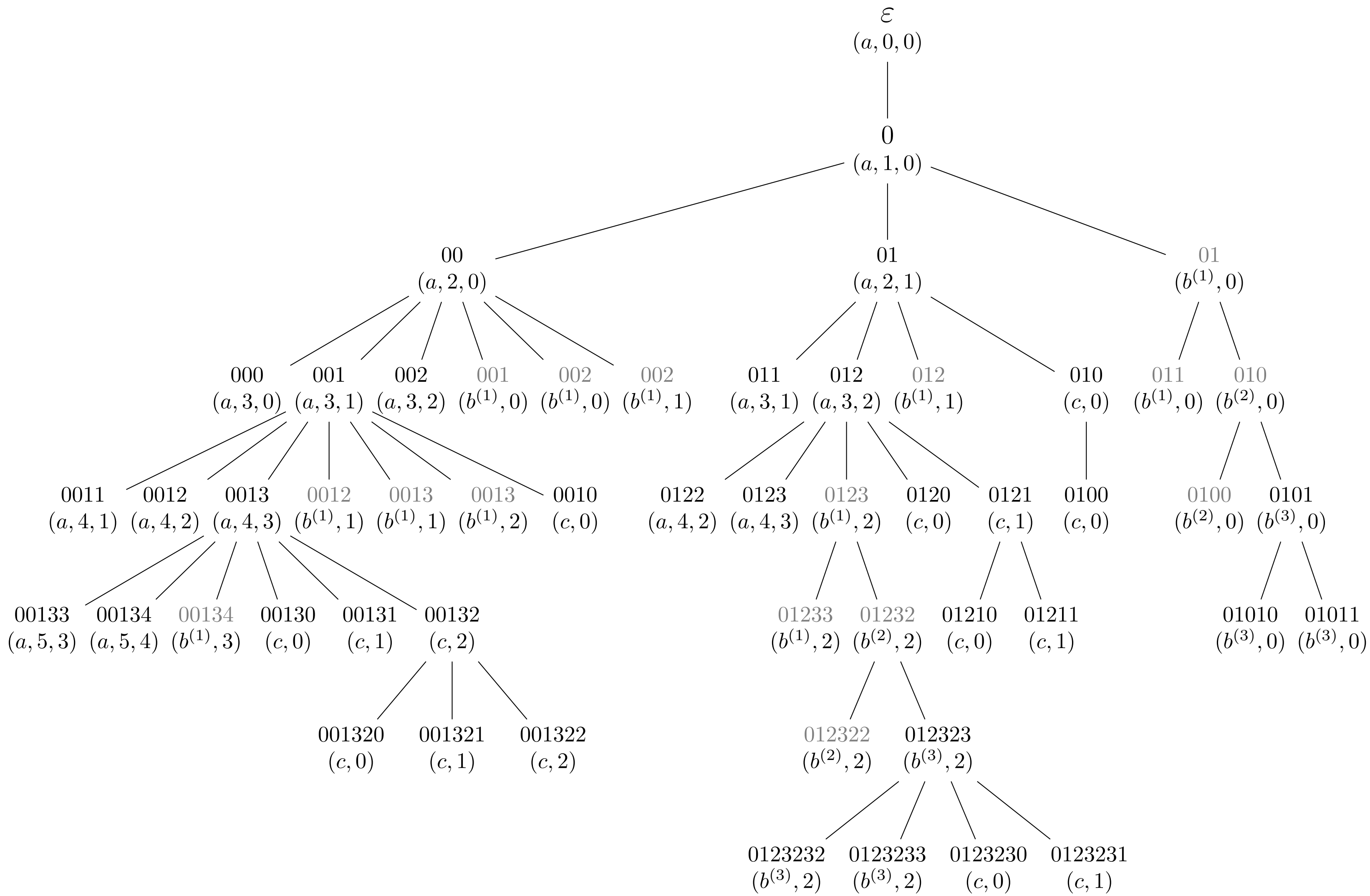

Figure 3: Part of the generating tree described by the succession rule $\Omega_{\{102,201\}}$. Phantom objects are colored in gray.

These phantom objects are nevertheless necessary in our construction, since removing all the nodes of $\mathcal{T}$ corresponding to phantom objects would disconnect[6] $\mathcal{T}$. The tree $\mathcal{T}$ is represented in Figure 3, on page 15.

Remark 13. The phantom objects can be understood as follows. In an object $(1,\sigma) \in \mathcal{U}$, $\sigma \in \mathcal{I}(101, 102, 201)$ is a prefix of an inversion sequence avoiding 102 and 201, whose maximum m and second maximum s form an occurrence (m, s) of the pattern 10 which must be completed into an occurrence (m, s, m) of 101 in the future. In particular, we may only insert the values m or s at the end of σ; otherwise, the subsequence (m, s, m) could never appear, by Corollary 4. Similarly, in an object $(s + 2, \sigma) \in \mathcal{U}$, $\sigma \in \mathcal{I}(10)$ is a prefix of an inversion sequence avoiding 102 and 201, that has maximum $m > s$ and which must contain an occurrence (m, s, m) of 101 in the future. In particular, we may only insert the values m or s at the end of σ; otherwise, the subsequence (m, s, m) could never appear, by Corollary 4.

This construction may appear more complicated than necessary. Indeed, we could simply consider the generating tree growing on the right for $\mathcal{I}(102, 201)$ instead, and this would not involve phantom objects. However, that would be less efficient, since this tree requires more labels. Specifically, the number of distinct labels appearing at level n in this tree would be quadratic in n (we would need to record both the values of the maximum and the second maximum of sequences in $\mathfrak{A}$), whereas our construction only involves a linear number of labels.

Remark 14. We could directly obtain an explicit expression for $|\mathcal{I}_n(102, 201)|$ from the observations of Proposition 3 and Corollaries 4 and 5, although using it for enumeration is not as efficient as our generating tree approach. The sequences of $\mathcal{I}_n(101, 102, 201)$ are counted by

$$1 + \sum_{m=1}^{n-1} \sum_{\ell=m+1}^{n} \frac{\binom{\ell+m-1}{m}(\ell - m)}{\ell} \cdot \binom{n - \ell + m - 1}{n - \ell},$$

as shown in [10]. The sequences of $\mathcal{I}_n(102, 201)$ which contain 101 are counted by

$$\sum_{f=2}^{n-2} \sum_{\ell=f+2}^{n} \sum_{s=0}^{f-2} (f - s - 1) \cdot \frac{\binom{f+s-1}{s}(f - s)}{f} \cdot (2^{\ell-f-1} - 1) \cdot \binom{n - \ell + s}{n - \ell}.$$

The summands count the number of sequences $\sigma \in \mathcal{I}_n(102, 201)$ such that $m = \mathbf{max}(\sigma)$, $s = \mathbf{sec}(\sigma)$, $f = \mathbf{firstmax}(\sigma)$, and $\ell = \mathbf{lastmax}(\sigma)$.

3 Removing the maximum

3.1 Method

Generally speaking, removing a term from an inversion sequence does not always create another inversion sequence. In fact, removing the i-th term σ_i from an inversion sequence

[6] More precisely, each node of $\mathcal{T}$ corresponding to an object in $\{0\} \times \mathfrak{B}^{(3)}$ or $\{0\} \times \mathfrak{C}^{(2)}$ has an ancestor which corresponds to a phantom object in $\{1\} \times \mathfrak{B}^{(2)}$.

$\sigma \in \mathcal{I}_n$ for some $i \in [1, n]$ yields an inversion sequence if and only if $\sigma_j < j - 1$ for all $j \in [i+1, n]$. In particular if $\sigma_i = \mathbf{max}(\sigma)$, then for all $j \in [i+1, n]$ we have $\sigma_j \leqslant \sigma_i < i \leqslant j - 1$, and the above condition is satisfied. Hence, removing a maximum from an inversion sequence always yields an inversion sequence.

Following this idea, we can decompose inversion sequences by removing their maxima one by one. For this decomposition to be unambiguous, we must choose the order of removal of the different occurrences of the maximum (this order is not always trivial). Observe that this decomposition defines a combinatorial generating tree $\mathcal{T}$ for inversion sequences. In that tree, the parent of each nonempty inversion sequence is obtained by removing its "next" maximum in the chosen order.

For any set of patterns P, a combinatorial generating tree for P-avoiding inversion sequences can be obtained by removing all nodes of $\mathcal{T}$ whose label contains a pattern in P. Indeed, this procedure does not disconnect $\mathcal{T}$ since the parent (in $\mathcal{T}$) of any nonempty P-avoiding inversion sequence also avoids P.

Let us quickly go over each pattern that appears in this section, and establish necessary and sufficient conditions for an integer sequence to avoid any occurrence of the given pattern which involves the maximum of the sequence.

- 000: The maximum does not appear more than twice.
- 100: All values appearing after the first maximum are distinct, or equal to the maximum.
- 101: All occurrences of the maximum are consecutive.
- 110: All maxima after the first one appear in a single factor, at the end of the sequence.
- 102: The subsequence to the left of the last maximum is nondecreasing, ignoring other occurrences of the maximum.
- 201: The subsequence to the right of the first maximum is nonincreasing, ignoring other occurrences of the maximum.
- 210: The subsequence to the right of the first maximum is nondecreasing, ignoring other occurrences of the maximum.

Since our generating tree construction inserts a maximum, the children of a node labelled σ in $\mathcal{T}'$ are the children of σ in $\mathcal{T}$ which satisfy the above conditions (for the patterns in P). Such facts will be used in proofs throughout this section without explicit reference.

3.2 The pair {000, 102}

In this subsection, we use a generalization of succession rules called *doubled succession rules*, introduced in [13], which allows a node at level n in a generating tree to have children at levels $n+1$ or $n+2$, denoted by $\overset{1}{\rightsquigarrow}$ and $\overset{2}{\rightsquigarrow}$ in the succession rule. We find a doubled succession rule describing a generating tree for $\mathcal{I}(000, 102)$ (up to isomorphism), and later use this rule to compute the generating function of $\mathcal{I}(000, 102)$ in the appendix.

For all $\sigma \in \mathcal{I}(000, 102)$, let $\mathbf{inc}(\sigma)$ be the size of the longest nondecreasing factor at the beginning of σ, and let $\mathbf{sites}(\sigma) = \mathbf{inc}(\sigma) - \mathbf{max}(\sigma)$. In particular, $\mathbf{sites}(\varepsilon) = 1$.

Remark 15. Let $\sigma \in \mathcal{I}(000, 102)\backslash\{\varepsilon\}$ be a nonempty inversion sequence. Since σ avoids the pattern 102, the factor $(\sigma_i)_{i\in[1,\mathbf{firstmax}(\sigma)]}$ is nondecreasing. In other words, the maximum of σ is reached before its first descent, and therefore $\sigma_{\mathbf{inc}(\sigma)} = \mathbf{max}(\sigma)$. This also implies that $\mathbf{inc}(\sigma) > \mathbf{max}(\sigma)$, hence $\mathbf{sites}(\sigma)$ is positive.

Given $n \geqslant 1, \sigma \in \mathbb{N}^n$, $p \in [1, n+1]$ and $v \in \mathbb{N}$, we denote by $[\sigma]_p^v$ the sequence obtained from σ by inserting the value v at position p, that is $[\sigma]_p^v = (\sigma_i)_{i\in[1,p-1]} \cdot v \cdot (\sigma_i)_{i\in[p,n]}$.

Lemma 16. *Let $n \geqslant 0$, $\sigma \in \mathcal{I}_n(000, 102)$, $m = \mathbf{max}(\sigma)$, $i \in [1, n+1]$, and $k > 0$. Let $\sigma' = [\sigma]_i^{m+k}$ be the sequence obtained from σ by inserting $m+k$ at position i. Then $\sigma' \in \mathcal{I}_{n+1}(000, 102)$ if and only if $i \in [m+k+1, \mathbf{inc}(\sigma)+1]$.*

Proof. Since σ does not contain the value $m+k$, inserting it cannot create an occurrence of the pattern 000. The value $m+k$ must be inserted in the interval $[m+k+1, n+1]$ in order to create an inversion sequence. By definition of $\mathbf{inc}(\sigma)$, $(\sigma_i)_{i\in[1,\mathbf{inc}(\sigma)]}$ is nondecreasing, hence inserting $m+k$ in any position $[m+k+1, \mathbf{inc}(\sigma)+1]$ does not create an occurrence of 102. On the contrary, inserting $m+k$ in any position $[\mathbf{inc}(\sigma)+2, n+1]$ creates an occurrence of 102, since $(\sigma_{\mathbf{inc}(\sigma)}, \sigma_{\mathbf{inc}(\sigma)+1})$ is an occurrence of 10 and $m+k > \sigma_{\mathbf{inc}(\sigma)}$. □

It follows from Lemma 16 that the statistic $\mathbf{sites}(\sigma)$ can be interpreted either as the number of values (denoted k earlier) greater than m which could be inserted in σ, or the number of positions (denoted i earlier) where a value greater than m can be inserted in σ, so that the resulting sequence is in $\mathcal{I}_{n+1}(000, 102)$.

Theorem 17. *The enumeration of inversion sequences avoiding the patterns 000 and 102 is given by the doubled succession rule*

$$\Omega_{\{000,102\}} = \begin{cases} (1) & \\ (s) \overset{1}{\rightsquigarrow} (j)^{s+1-j} & \textit{for} \quad j \in [1, s] \\ (s) \overset{2}{\rightsquigarrow} (j+1)^{s+1-j} (j)^{\binom{s+1-j}{2}} & \textit{for} \quad j \in [1, s]. \end{cases}$$

Proof. We consider the combinatorial generating tree (with jumps) for $\mathcal{I}(000, 102)$ such that the parent of any $\sigma \in \mathcal{I}(000, 102)\backslash\{\varepsilon\}$ is obtained from σ by removing all occurrences of $\mathbf{max}(\sigma)$. Due to the avoidance of 000, $\mathbf{max}(\sigma)$ appears either once or twice in σ, so the difference between the level of σ and that of its parent is either 1 or 2. We then replace each label σ by the value $\mathbf{sites}(\sigma)$ defined above. The axiom of the resulting succession rule is (1) since $\mathbf{sites}(\varepsilon) = 1$.

In the rest of this proof, let $n \geqslant 0$, $\sigma \in \mathcal{I}_n(000, 102)$ and $m = \mathbf{max}(\sigma)$.

Let $i, k \geqslant 1$ be two integers, and $\sigma' = [\sigma]_i^{m+k}$. From Lemma 16, $\sigma' \in \mathcal{I}_{n+1}(000, 102)$ if and only if $i \in [m+2, \mathbf{inc}(\sigma)+1]$ and $k \in [1, i-m-1]$. Let us now assume that is the case. By definition of σ', we have $\mathbf{inc}(\sigma') = i$ and $\mathbf{max}(\sigma') = m + k$. In particular, $\mathbf{sites}(\sigma') = i - (m + k) \in [1, \mathbf{sites}(\sigma)]$.

Let $j \in [1, \mathbf{sites}(\sigma)]$. Let us count how many choices of i and k satisfy the identity $\mathbf{sites}([\sigma]_i^{m+k}) = j$. The ordered pair (i, k) is a solution whenever $j = i - (m + k)$, i.e. whenever $k = i - m - j$. Since k is positive, there is one solution for each value of i such that $i > m + j$. The number of solutions (i, k) is the number of values i in the interval $[m + j + 1, \mathbf{inc}(\sigma) + 1]$, so there are $\mathbf{inc}(\sigma) + 1 - (m + j) = \mathbf{sites}(\sigma) + 1 - j$ solutions. This corresponds to $(s) \overset{1}{\rightsquigarrow} (j)^{s+1-j}$ in the succession rule. Now let us count in how many ways we may insert two occurrences of a value greater than m in σ. In the previous case, inserting a second $m + k$ adjacent to the first one yields a sequence $\sigma'' = [\sigma']_i^{m+k}$ which satisfies $\mathbf{sites}(\sigma'') = i + 1 - (m + k) = \mathbf{sites}(\sigma') + 1$. We obtain $\mathbf{sites}(\sigma) + 1 - j$ sequences of size $n + 2$ and label $j + 1$ for each $j \in [1, \mathbf{sites}(\sigma)]$. This corresponds to $(s) \overset{2}{\rightsquigarrow} (j + 1)^{s+1-j}$ in the succession rule.

In order to insert a value $m + k > m$ in two non adjacent positions in σ, we choose two positions $i_1 < i_2 \in [m + k + 1, \mathbf{inc}(\sigma) + 1]$ and define $\sigma'' = [[\sigma]_{i_2}^{m+k}]_{i_1}^{m+k}$, which satisfies $\mathbf{sites}(\sigma'') = i_1 - (m + k) \in [1, \mathbf{sites}(\sigma)]$. Let $j \in [1, \mathbf{sites}(\sigma)]$, and let us count how many choices of i_1, i_2 and k satisfy $\mathbf{sites}(\sigma'') = j$. The triple (i_1, i_2, k) is a solution whenever $j = i_1 - (m + k)$, i.e. whenever $k = i_1 - m - j$. Since k is positive, there is one solution for each (i_1, i_2) such that $i_1 > m + j$. The number of solutions is the number of ordered pairs (i_1, i_2) such that $i_1 < i_2$ and $i_1, i_2 \in [m + j + 1, \mathbf{inc}(\sigma) + 1]$, so there are $\binom{\mathbf{sites}(\sigma)+1-j}{2}$ solutions. This corresponds to $(s) \overset{2}{\rightsquigarrow} (j)^{\binom{s+1-j}{2}}$ in the succession rule. $\square$

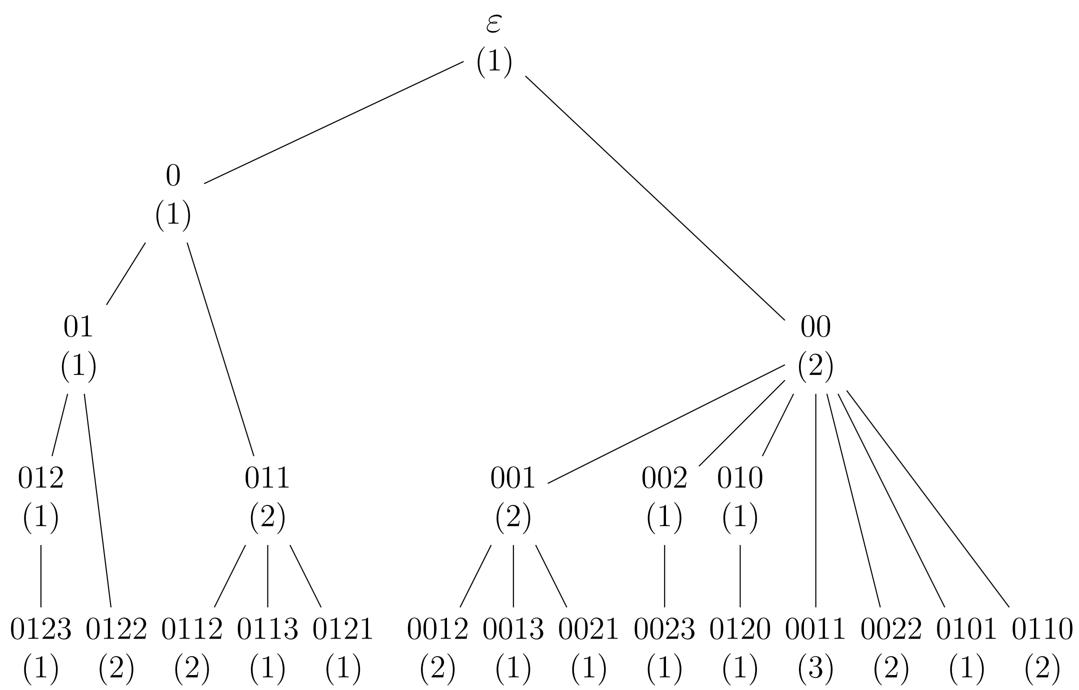

Figure 4: First five levels of the generating tree described by the succession rule $\Omega_{\{000,102\}}$.

The generating tree described by $\Omega_{\{000,102\}}$ is represented in Figure 4.

3.3 The pair {102, 210}

In this section, we present a generating tree for the class $\mathcal{I}(102, 210)$. We later use this construction to find the generating function of $\mathcal{I}(102, 210)$ in Theorem 76.

For any $n \geqslant 0$ and any $\sigma \in \mathbb{N}^n$, let $\mathbf{Des}(\sigma) = \{i \in [1, n-1] \ : \ \sigma_i > \sigma_{i+1}\}$ be the set of *descent tops* of σ.

Proposition 18. *Let σ be a $\{102, 210\}$-avoiding integer sequence. Then any descent top of σ is the position of a maximum of σ, i.e. if $d \in \mathbf{Des}(\sigma)$ then $\sigma_d = \mathbf{max}(\sigma)$.*

Proof. Assume, for the sake of contradiction, that $i \in \mathbf{Des}(\sigma)$ and $\sigma_i < \mathbf{max}(\sigma)$. Since $\mathbf{Des}(\sigma)$ is nonempty, $|\sigma| \geqslant 2$. Let $j \in [1, |\sigma|]$ be such that $\sigma_j = \mathbf{max}(\sigma)$. In particular, $j \notin \{i, i+1\}$. If $j < i$, then $(\sigma_j, \sigma_i, \sigma_{i+1})$ is an occurrence of 210. If $j > i+1$, then $(\sigma_i, \sigma_{i+1}, \sigma_j)$ is an occurrence of 102. $\square$

Proposition 19. *Let σ be a $\{102, 210\}$-avoiding integer sequence, and let $v > \mathbf{max}(\sigma)$.*

- *If σ is nondecreasing, then inserting the value v anywhere in σ does not create any occurrence of 102 or 210.*
- *If σ has exactly one descent $\sigma_d > \sigma_{d+1}$, then inserting the value v at some position p in σ yields a $\{102, 210\}$-avoiding sequence if and only if $p = d+1$.*
- *If σ has at least two descents, then inserting the value v anywhere in σ creates an occurrence of 102 or 210.*

Proof. We go case-by-case.

- Since v is greater than all values of σ, the value v could only take the role of the 2 in a pattern 102 or 210. Hence, in order for the insertion of the value v to create an occurrence of either pattern, σ must already contain the pattern 10.
- If d is the only descent top of σ, then all occurrences (σ_i, σ_j) of the pattern 10 in σ must satisfy $i \leqslant d$ and $j > d$. In particular, inserting v at position $d+1$ cannot create an occurrence of either pattern 102 or 210. If v is inserted at any other position, the resulting sequence still contains a descent top of value $\sigma_d < v$, so it contains 102 or 210 by Proposition 18.
- Inserting v anywhere in σ yields a sequence which still contains a descent top of value $\mathbf{max}(\sigma) < v$, and therefore contains 102 or 210 by Proposition 18. $\square$

It is now clear we want to partition $\mathcal{I}(102, 210)$ into three subsets. For all $n \geqslant 0$, let

- $\mathfrak{A}_n = \{\sigma \in \mathcal{I}_n(102, 210) \ : \ |\mathbf{Des}(\sigma)| = 0\} = \mathcal{I}_n(10)$ be the set of nondecreasing inversion sequences of size n,

- $\mathfrak{B}_n = \{\sigma \in \mathcal{I}_n(102, 210) \ : \ |\mathbf{Des}(\sigma)| = 1\}$ be the set of $\{102, 210\}$-avoiding inversion sequences of size n having exactly one descent,

- $\mathfrak{C}_n = \{\sigma \in \mathcal{I}_n(102, 210) \ : \ |\mathbf{Des}(\sigma)| \geqslant 2\}$ be the set of $\{102, 210\}$-avoiding inversion sequences of size n having at least two descents.

We define a statistic $\mathbf{bounce} : \mathcal{I}(102, 210) \to \mathbb{N}$, which behaves differently on each subset.

- For all $\sigma \in \mathfrak{A}_n$, let $\mathbf{bounce}(\sigma) = n - \sigma_n$, with the convention $\mathbf{bounce}(\varepsilon) = 0$.

- For all $\sigma \in \mathfrak{B}_n$, let $\mathbf{bounce}(\sigma) = d - \sigma_d$, where d is the only element of $\mathbf{Des}(\sigma)$.

- For all $\sigma \in \mathfrak{C}_n$, let $\mathbf{bounce}(\sigma) = \mathbf{firstmax}(\sigma) - \mathbf{max}(\sigma)$.

Note that when σ is nonempty, $\mathbf{bounce}(\sigma)$ counts the difference between some position of the maximum of σ and the value of this maximum; however, the exact position varies according to the number of descents of σ. In particular, $\mathbf{bounce}(\sigma) > 0$ if $\sigma \neq \varepsilon$.

Let $\mathfrak{A}_{n,k} = \{\sigma \in \mathfrak{A}_n \ : \ k = \mathbf{bounce}(\sigma)\}$, $\mathfrak{B}_{n,k} = \{\sigma \in \mathfrak{B}_n \ : \ k = \mathbf{bounce}(\sigma)\}$, and $\mathfrak{C}_{n,k} = \{\sigma \in \mathfrak{C}_n \ : \ k = \mathbf{bounce}(\sigma)\}$. We also consider some subsets of $\mathfrak{B}_{n,k}$ and $\mathfrak{C}_{n,k}$.

- Let $\mathfrak{B}^{(1)}_{n,k} = \{\sigma \in \mathfrak{B}_{n,k} \ : \ (\sigma_i)_{i \in [1,n] \setminus \mathbf{firstmax}(\sigma)} \in \mathfrak{A}_{n-1}\}$ be the subset of sequences of $\mathfrak{B}_{n,k}$ such that removing the first maximum yields a sequence in $\mathfrak{A}_{n-1}$. Let $\mathfrak{B}^{(2)}_{n,k} = \mathfrak{B}_{n,k} \setminus \mathfrak{B}^{(1)}_{n,k}$.

- Let $\mathfrak{C}^{(1)}_{n,k} = \{\sigma \in \mathfrak{C}_{n,k} \ : \ (\sigma_i)_{i \in [1,n] \setminus \mathbf{firstmax}(\sigma)} \in \mathfrak{B}_{n-1}\}$ be the subset of sequences of $\mathfrak{C}_{n,k}$ such that removing the first maximum yields a sequence in $\mathfrak{B}_{n-1}$. Let $\mathfrak{C}^{(2)}_{n,k} = \mathfrak{C}_{n,k} \setminus \mathfrak{C}^{(1)}_{n,k}$.

We give an example for each of the five cases.

- $\alpha = (0, 0, 1, 3, 3, 3, 4, 6, 6) \in \mathfrak{A}_{9,3}$, since $\mathbf{Des}(\alpha) = \emptyset$ and $\mathbf{bounce}(\alpha) = 9 - 6$.

- $\beta = (0, 0, 0, 1, 1, 4, 1, 3, 4) \in \mathfrak{B}^{(1)}_{9,2}$, since $\mathbf{Des}(\beta) = \{6\}$ and $\mathbf{bounce}(\beta) = 6 - 4$.

- $\beta' = (0, 0, 0, 1, 2, 2, 2, 0, 1) \in \mathfrak{B}^{(2)}_{9,5}$, since $\mathbf{Des}(\beta') = \{7\}$ and $\mathbf{bounce}(\beta') = 7 - 2$.

- $\gamma = (0, 0, 1, 1, 2, 4, 3, 4, 3) \in \mathfrak{C}^{(1)}_{9,2}$, since $\mathbf{Des}(\gamma) = \{6, 8\}$ and $\mathbf{bounce}(\gamma) = 6 - 4$.

- $\gamma' = (0, 1, 1, 3, 3, 1, 3, 2, 3) \in \mathfrak{C}^{(2)}_{9,1}$, since $\mathbf{Des}(\gamma') = \{5, 7\}$ and $\mathbf{bounce}(\gamma') = 4 - 3$.

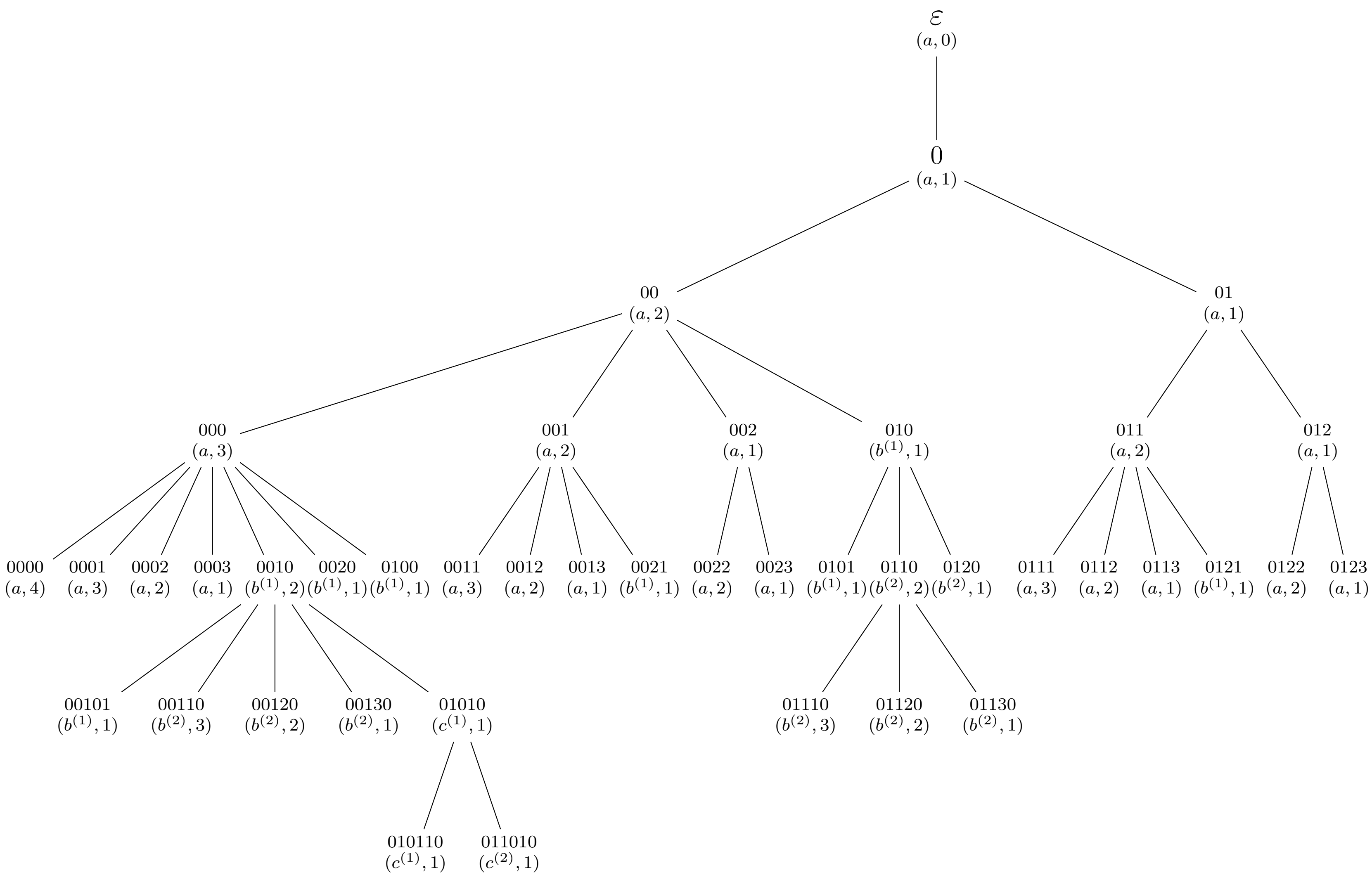

Figure 5: Part of the generating tree described by the succession rule $\Omega_{\{102,210\}}$.

Theorem 20. *The enumeration of inversion sequences avoiding the patterns 102 and 210 is given by the succession rule*

$$\Omega_{\{102,210\}} = \begin{cases} (a,0) & & \\ (a,k) \rightsquigarrow & (a,i) & \textit{for} \quad i \in [1,k+1] \\ & (b^{(1)},i)^{k-i} & \textit{for} \quad i \in [1,k-1] \\ (b^{(1)},k) \rightsquigarrow & (b^{(1)},k) & \\ & (b^{(2)},i) & \textit{for} \quad i \in [1,k+1] \\ & (c^{(1)},i) & \textit{for} \quad i \in [1,k-1] \\ (b^{(2)},k) \rightsquigarrow & (b^{(2)},i) & \textit{for} \quad i \in [1,k+1] \\ (c^{(1)},k) \rightsquigarrow & (c^{(1)},k) & \\ & (c^{(2)},i) & \textit{for} \quad i \in [1,k] \\ (c^{(2)},k) \rightsquigarrow & (c^{(2)},i) & \textit{for} \quad i \in [1,k]. \end{cases}$$

The generating tree described by $\Omega_{\{102,210\}}$ is represented in Figure 5, on page 22.

Proof. We describe a combinatorial generating tree for $\mathcal{I}(102,210)$ which "grows" inversion sequences by inserting entries in increasing order of value. We then label the sequences of $\mathcal{I}(102,210)$ according to which set they belong to: each sequence in a set $\mathfrak{A}_{n,k}$ is labelled (a,k), each sequence in a set $\mathfrak{B}^{(i)}_{n,k}$ is labelled $(b^{(i)},k)$, and each sequence in a set $\mathfrak{C}^{(i)}_{n,k}$ is labelled $(c^{(i)},k)$, for $i \in \{1,2\}$. In particular, the empty sequence ε is labelled $(a,0)$, which is the axiom of the succession rule.

The order of insertion of the different occurrences of each value is not simple. We first explain it informally, before describing the full construction in detail. Let $\sigma \in \mathcal{I}(102,210)$ be an inversion sequence of size $n \geqslant 1$ and maximum m. Let $\sigma' \in \mathcal{I}(102,210)$ be the subsequence obtained by removing all values m from σ. We generate σ from σ' by inserting entries of value m in the following order.

1. The rightmost descent top of σ (if σ has at least one descent).

2. Any entries of value m appearing in a single factor at the end of σ, from left to right.

3. The second rightmost descent top of σ (if σ has at least two descents).

4. Any entries appearing in a constant factor of value m which contains the rightmost descent top of σ (if σ has at least one descent), inserted from left to right, and to the right of the entry inserted at step 1.

5. Any remaining entries of value m, from right to left.

For instance if $\sigma = (0,0,0,1,2,2,6,6,2,3,6,4,6,6,6,4,4,6,6,6,4,6,6) \in \mathcal{I}_{23}(102,210)$, then the order of insertion of the 11 entries of maximal value 6 is shown on the bottom row of the example below.

$$\begin{array}{ccccccccccccccccccccccc}
0 & 0 & 0 & 1 & 2 & 2 & \color{red}{6} & \color{red}{6} & 2 & 3 & \color{red}{6} & 4 & \color{red}{6} & \color{red}{6} & \color{red}{6} & 4 & 4 & \color{red}{6} & \color{red}{6} & \color{red}{6} & 4 & \color{red}{6} & \color{red}{6} \\
 & & & & & & 11 & 10 & & & 9 & & 8 & 7 & 4 & & & 1 & 5 & 6 & & 2 & 3
\end{array}$$

This order of insertion may not be intuitive, but the resulting generating tree construction has some advantages compared to a simpler order:

- most importantly, it only requires one parameter (the value of the **bounce** statistic),
- it does not involve "phantom objects" as in Section 2.3,
- the resulting succession rule does not involve the size n of the sequences.

We now turn to a step-by-step description of the generating tree, which matches the succession rule $\Omega_{\{102,210\}}$. Let $\sigma \in \mathcal{I}(102,210)$, $n = |\sigma|$, $k = \mathbf{bounce}(\sigma)$, and $m = \mathbf{max}(\sigma)$.

- Suppose $\sigma \in \mathfrak{A}_{n,k}$. In particular, if σ is nonempty then $\mathbf{max}(\sigma) = \sigma_n = n-k$.
  - Inserting any value $v \in [n-k, n]$ at the end of σ yields a sequence in $\mathfrak{A}_{n+1,n+1-v}$, where $n+1-v \in [1, k+1]$. This production alone generates all nondecreasing inversion sequences.
  - If σ is nonempty, inserting any value $v \in [n-k+1, n-1]$ at any position $p \in [v+1, n]$ creates a sequence in $\mathfrak{B}^{(1)}_{n+1,p-v}$. For each $i = p - v \in [1, k-1]$, there are $k-i$ sequences in $\mathfrak{B}^{(1)}_{n+1,i}$ generated by σ (one sequence for each value $v \in [n-k+1, n-i]$, for $p = v+i$).
- Suppose $\sigma \in \mathfrak{B}^{(1)}_{n,k}$. In particular, $\mathbf{Des}(\sigma) = \{k+m\}$.
  - Inserting the value m at the end of σ yields a sequence in $\mathfrak{B}^{(1)}_{n+1,k}$.
  - Inserting any value $v \in [m, m+k]$ between the top and bottom of the only descent of σ (i.e. at position $k+m+1$) yields a sequence in $\mathfrak{B}^{(2)}_{n+1,k-v+m+1}$.
  - Inserting the value m at any position $p \in [m+1, k+m-1]$ yields a sequence in $\mathfrak{C}^{(1)}_{n+1,p-m}$.
- Suppose $\sigma \in \mathfrak{B}^{(2)}_{n,k}$. In particular, $\mathbf{Des}(\sigma) = \{k+m\}$. As before, inserting any value $v \in [m, m+k]$ between the top and bottom of the only descent of σ (i.e. at position $k+m+1$) yields a sequence in $\mathfrak{B}^{(2)}_{n+1,k-v+m+1}$.
- Suppose $\sigma \in \mathfrak{C}^{(1)}_{n,k}$. In particular, $\mathbf{firstmax}(\sigma) = k+m$.
  - Inserting the value m right-adjacent to the rightmost descent top of σ (i.e. at position $\mathbf{max}(\mathbf{Des}(\sigma))+1$) yields a sequence in $\mathfrak{C}^{(1)}_{n+1,k}$.

  - Inserting the value m at any position $p \in [m+1, \mathbf{firstmax}(\sigma)]$ yields a sequence in $\mathfrak{C}^{(2)}_{n+1,p-m}$.

- Suppose $\sigma \in \mathfrak{C}^{(2)}_{n,k}$. In particular, $\mathbf{firstmax}(\sigma) = k + m$. As before, inserting the value m at any position $p \in [m+1, \mathbf{firstmax}(\sigma)]$ yields a sequence in $\mathfrak{C}^{(2)}_{n+1,p-m}$.

Notice that inserting some value $v \geqslant m$ at any other position would either yield a sequence which already appears in this construction, or create an occurrence of a pattern 102 or 210. $\square$

3.4 The pairs {000, 201} or {000, 210}

A bijection between $\mathcal{I}(000, 201)$ and $\mathcal{I}(000, 210)$ was established in [35, Theorem 8.1]. In this section we enumerate $\mathcal{I}(000, 201)$, although the same construction can also be applied to $\mathcal{I}(000, 210)$ with a little more work, and results in the same equations. Let $\mathfrak{A}_{n,m,p} = \{\sigma \in \mathcal{I}_n(000, 201) \ : \ m = \mathbf{max}(\sigma), \ p = \mathbf{firstmax}(\sigma) = \mathbf{lastmax}(\sigma)\}$ be the set of $\{000, 201\}$-avoiding inversion sequences of size n, maximum m, and such that m appears only once and at position p. Let $\mathfrak{a}_{n,m,p} = |\mathfrak{A}_{n,m,p}|$. By definition, $\mathfrak{a}_{n,m,p} = 0$ if $n < p$ or $p \leqslant m$. When $m = 0$, $\mathfrak{a}_{n,0,p} = \delta_{n,1}\delta_{p,1}$. The numbers $(\mathfrak{a}_{n,m,p})_{n,m,p\in\mathbb{N}}$ can then be calculated using the following lemma.

Lemma 21. *For all* $1 \leqslant m < p \leqslant n$,

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{p} \mathfrak{a}_{n-1,s,q} + (p - q + \delta_{p,q} - \delta_{p,n})\mathfrak{a}_{n-2,s,q}$$

Proof. Let $m \geqslant 1, p > m, n \geqslant p$ be three integers, and let $\sigma \in \mathfrak{A}_{n,m,p}$. Let σ' be the sequence obtained by removing the value m from σ. Let $s = \mathbf{max}(\sigma')$ be the second largest value in σ, and $q = \mathbf{firstmax}(\sigma')$. Note that if $q < p$ then $\sigma_q = s$, and if $q \geqslant p$ then $\sigma_{q+1} = s$. In particular, if $q > p$ then the subsequence $(\sigma_p, \sigma_{p+1}, \sigma_{q+1})$ is an occurrence of the pattern 201. Hence q is less than or equal to p.

Since σ' avoids the pattern 000, its maximum s appears either once or twice.

- If s appears only once in σ', then $\sigma' \in \mathfrak{A}_{n-1,s,q}$. Note that inserting the value m at position p in any sequence in $\mathfrak{A}_{n-1,s,q}$ for $s < m$ and $q \leqslant p$ does not create an occurrence of 000 or 201, so this is a bijection.

- If s appears twice, then removing the second occurrence yields an inversion sequence $\sigma'' \in \mathfrak{A}_{n-2,s,q}$. The possible positions for the second s in σ' are:

  - Before p: $\begin{cases} [q+1, p-1] & \text{if} \quad q < p \\ \emptyset & \text{if} \quad q = p \end{cases}$, so there are $p - q - 1 + \delta_{p,q}$ choices.

  - After p: $\begin{cases} \{p+1\} & \text{if} \quad p < n \\ \emptyset & \text{if} \quad p = n \end{cases}$, so there are $1 - \delta_{p,n}$ choices.

This adds up to $p - q + \delta_{p,q} - \delta_{p,n}$ possible positions for the second s.

Summing over all possible values of m and p completes the proof. □

Theorem 22. *For all $n \geqslant 2$,*

$$|\mathcal{I}_n(000, 201)| = \sum_{m=0}^{n-1} \sum_{p=m+1}^{n} \mathfrak{a}_{n,m,p} + (n-p)\mathfrak{a}_{n-1,m,p}.$$

Proof. Let $n \geqslant 2$, $\sigma \in \mathcal{I}_n(000, 201)$, $m = \mathbf{max}(\sigma)$, and $p = \mathbf{firstmax}(\sigma)$. Since σ avoids 000, the maximum m appears either once or twice in σ.

- If m appears only once, then $\sigma \in \mathfrak{A}_{n,m,p}$.
- If m appears twice, then removing the second m yields a sequence $\sigma' \in \mathfrak{A}_{n-1,m,p}$. There are $n-p$ possible positions for the second m, hence $n-p$ different sequences σ correspond to the same σ'.

Summing over all possible values of m and p completes the proof. □

Remark 23. The pairs of patterns studied in the next four sections contain either 101 or 110. In a sequence that avoids 101 (resp. 110), all occurrences of the maximum after the leftmost one must appear in a single factor at the end of the sequence (resp. consecutive to the leftmost maximum). This allows us to restrict the enumeration to inversion sequences which contain only one occurrence of the maximum without loss of generality, since each repetition of that maximum can only be placed at a single position.

3.5 The pair {100, 110}

Let $\mathfrak{A}_{n,m,p} = \{\sigma \in \mathcal{I}_n(100, 110) \ : \ m = \mathbf{max}(\sigma),\ p = \mathbf{firstmax}(\sigma) = \mathbf{lastmax}(\sigma)\}$ be the set of $\{100, 110\}$-avoiding inversion sequences of size n, maximum m, and such that m appears only once and at position p. Let $\mathfrak{a}_{n,m,p} = |\mathfrak{A}_{n,m,p}|$. In particular, $\mathfrak{a}_{n,m,p} = 0$ if $n < p$ or $p \leqslant m$, and $\mathfrak{a}_{n,0,p} = \delta_{n,1}\delta_{p,1}$.

Lemma 24. *Let $1 \leqslant m < p \leqslant n$. If $p \in \{n-1, n\}$,*

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{n-1} \sum_{\ell=q}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

If $p < n-1$,

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} (n-p) \cdot \mathfrak{a}_{n-1,s,p} + \sum_{q=s+1}^{p-1} \mathfrak{a}_{n-1,s,q} + \mathfrak{a}_{n-2,s,q}.$$

Proof. Let $1 \leqslant m < p \leqslant n$, and $\sigma \in \mathfrak{A}_{n,m,p}$. Let $\sigma' = (\sigma_i)_{i\in[1,n]\setminus\{p\}}$ be the sequence obtained by removing the value m from σ. Let $s = \mathbf{max}(\sigma')$ and $q = \mathbf{firstmax}(\sigma')$. Let σ'' be the sequence obtained by removing all terms of value s except the first one from σ', so that $\sigma'' \in \mathfrak{A}_{\ell,s,q}$ for some $\ell \in [q, n-1]$.

Since σ avoids 100, all terms of σ after position p must have distinct values. In particular, s appears at most once after p. Since σ' avoids 110, every s after the first one must appear in a single factor at the end of σ'. At this point, there are two cases:

1. If $p \in \{n-1, n\}$, then the value s could appear any number of times without creating an occurrence of 100 in σ. In that case, σ'' can be any sequence in $\mathfrak{A}_{\ell,s,q}$ for any $\ell \in [q, n-1]$: starting from any sequence in $\mathfrak{A}_{\ell,s,q}$ for any $s < m$, $q > s$, and $\ell < n$, we can append $n-\ell-1$ terms of value s, then insert m at position $p \in \{n-1, n\}$ without creating an occurrence of 100 or 110, and this is a bijection. Thus, we obtain the first equation of the lemma.

2. If $p < n-1$:

   - If $q < p$:

     – If σ contains a single term of value s, then $\sigma'' = \sigma' \in \mathfrak{A}_{n-1,s,q}$.

     – If σ contains two terms of value s, then the second one is σ_n due to the avoidance of 110. In that case, $\sigma'' \in \mathfrak{A}_{n-2,s,q}$.

     In either case, this is a bijection for the same reason as before. Note that s cannot appear more than twice in σ, otherwise $(\sigma_p, \sigma_{n-1}, \sigma_n) = (m, s, s)$ would be an occurrence of 100.

   - If $q \geqslant p$, then s appears only once in σ in order to avoid 100. In that case, $\sigma'' = \sigma'$ can be any sequence in $\mathfrak{A}_{n-1,s,q}$ such that all values $(\sigma''_i)_{i\in[p,n-1]}$ are distinct. Moving the value s in σ'' from position q to position p yields a sequence $\tau \in \mathfrak{A}_{n-1,s,p}$. Conversely, starting from any sequence $\tau \in \mathfrak{A}_{n-1,s,p}$ and moving the value s from position p to position $q \in [p, n-1]$ yields a sequence $\sigma'' \in \mathfrak{A}_{n-1,s,q}$ such that all values $(\sigma''_i)_{i\in[p,n-1]}$ are distinct. Since there are $n-p$ possible values of q, each sequence $\tau \in \mathfrak{A}_{n-1,s,p}$ is obtained from $n-p$ different sequences σ''. □

Theorem 25. *For all $n \geqslant 2$,*

$$|\mathcal{I}_n(100, 110)| = \sum_{\ell=1}^{n} \sum_{m=0}^{\ell-1} \sum_{p=m+1}^{\ell} \mathfrak{a}_{\ell,m,p}.$$

Proof. Let $n \geqslant 2$, $\sigma \in \mathcal{I}_n(100, 110)$, $m = \mathbf{max}(\sigma)$, and $p = \mathbf{firstmax}(\sigma)$. Since σ avoids 110, all terms of value m after the first one σ_p are in a factor $(\sigma_i)_{i\in[\ell+1,n]}$ for some $\ell \in [p, n]$. Let σ' be the sequence obtained by removing all terms of value m from σ, except the first one σ_p. By definition, $\sigma' \in \mathfrak{A}_{\ell,m,p}$. Since we can rebuild σ by appending $n-\ell$ terms of value m to σ', this is a bijection between $\mathcal{I}_n(100, 110)$ and $\coprod_{0\leqslant m<p\leqslant \ell\leqslant n} \mathfrak{A}_{\ell,m,p}$. □

3.6 The pair {100, 101}

Let $\mathfrak{A}_{n,m,p} = \{\sigma \in \mathcal{I}_n(100, 101) \ : \ m = \mathbf{max}(\sigma),\ p = \mathbf{firstmax}(\sigma) = \mathbf{lastmax}(\sigma)\}$ be the set of $\{100, 101\}$-avoiding inversion sequences of size n, maximum m, and such that m appears only once and at position p. Let $\mathfrak{a}_{n,m,p} = |\mathfrak{A}_{n,m,p}|$. In particular, $\mathfrak{a}_{n,m,p} = 0$ if $n < p$ or $p \leqslant m$, and $\mathfrak{a}_{n,0,p} = \delta_{n,1}\delta_{p,1}$.

Lemma 26. *Let $1 \leqslant m < p \leqslant n$. If $p \in \{n-1, n\}$,*

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{n-1} \sum_{\ell=q}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

If $p < n-1$,

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} (n-p) \cdot \mathfrak{a}_{n-1,s,p} + \sum_{q=s+1}^{p-1} \sum_{\ell=n-p+q-1}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

Proof. This is quite similar to the proof of Lemma 24, except that now the terms of value s in σ' appear in a single factor starting at position q, instead of being split into σ'_q and a factor at the end of σ'. This makes a difference in only one case: if $p < n-1$ and $q < p$, then there may be more than two terms of value s. More precisely, the terms of value s in σ' are in a factor $(\sigma_i)_{i \in [q, q+k]}$ for some $k \in [0, p-q]$. In particular $\sigma'' = (\sigma'_i)_{i \in [1,n-1] \setminus [q+1,q+k]} \in \mathfrak{A}_{\ell,s,q}$, so $\ell = n-1-k \in [n-p+q-1, n-1]$. □

Theorem 27. *For all $n \geqslant 2$,*

$$|\mathcal{I}_n(100, 101)| = \sum_{\ell=1}^{n} \sum_{m=0}^{\ell-1} \sum_{p=m+1}^{\ell} \mathfrak{a}_{\ell,m,p}.$$

Proof. Very similar to the proof of Theorem 25, but now the factor of repetitions of the maximum is placed immediately after the first maximum, instead of being at the end of the sequence. □

3.7 The pair {110, 201}

Let $\mathfrak{A}_{n,m,p} = \{\sigma \in \mathcal{I}_n(110, 201) \ : \ m = \mathbf{max}(\sigma),\ p = \mathbf{firstmax}(\sigma) = \mathbf{lastmax}(\sigma)\}$ be the set of $\{110, 201\}$-avoiding inversion sequences of size n, maximum m, and such that m appears only once and at position p. Let $\mathfrak{a}_{n,m,p} = |\mathfrak{A}_{n,m,p}|$. In particular, $\mathfrak{a}_{n,m,p} = 0$ if $n < p$ or $p \leqslant m$, and $\mathfrak{a}_{n,0,p} = \delta_{n,1}\delta_{p,1}$.

Lemma 28. *Let $1 \leqslant m < p \leqslant n$. If $p \in \{n-1, n\}$,*

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{n-1} \sum_{\ell=q}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

If $p < n-1$,

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{p} \mathfrak{a}_{n-1,s,q} + \sum_{\ell=q}^{p-1+\delta_{p,q}} \mathfrak{a}_{\ell,s,q}.$$

Proof. Let $1 \leqslant m < p \leqslant n$, and $\sigma \in \mathfrak{A}_{n,m,p}$. Let $\sigma' = (\sigma_i)_{i\in[1,n]\setminus\{p\}}$ be the sequence obtained by removing the value m from σ. Let $s = \mathbf{max}(\sigma')$ and $q = \mathbf{firstmax}(\sigma')$. In particular $q \leqslant p$, otherwise $(\sigma_p, \sigma_{p+1}, \sigma_{q+1})$ would be an occurrence of 201. Let σ'' be the sequence obtained by removing all terms of value s except the first one from σ', so that $\sigma'' \in \mathfrak{A}_{\ell,s,q}$ for some $\ell \in [q, n-1]$.

- If $p \in \{n-1, n\}$, we obtain the same equation as in Lemma 24, for the same reason.
- If $p < n-1$,
  - If s appears only once in σ', then $\sigma'' = \sigma'$ can be any sequence in $\mathfrak{A}_{n-1,s,q}$
  - If s appears more than once, then every s after the first one must be in a single factor at the end of σ' to avoid 110. Since σ avoids 201, the factor $(\sigma_i)_{i\in[p+1,n]}$ is nonincreasing, which means every term in this factor has value s. The sequence σ'' obtained has length $\ell \in [q, p-1]$ if $q < p$, or $\ell = p$ if $q = p$. ☐

Theorem 29. *For all $n \geqslant 2$,*

$$|\mathcal{I}_n(110, 201)| = \sum_{\ell=1}^{n} \sum_{m=0}^{\ell-1} \sum_{p=m+1}^{\ell} \mathfrak{a}_{\ell,m,p}.$$

Proof. Identical to the proof of Theorem 25, since inserting a term of maximal value anywhere after the first maximum cannot create an occurrence of 201. ☐

3.8 The pair {101, 210}

Let $\mathfrak{A}_{n,m,p} = \{\sigma \in \mathcal{I}_n(101, 210) \ : \ m = \mathbf{max}(\sigma),\ p = \mathbf{firstmax}(\sigma) = \mathbf{lastmax}(\sigma)\}$ be the set of $\{101, 210\}$-avoiding inversion sequences of size n, maximum m, and such that m appears only once and at position p. Let $\mathfrak{a}_{n,m,p} = |\mathfrak{A}_{n,m,p}|$. In particular, $\mathfrak{a}_{n,m,p} = 0$ if $n < p$ or $p \leqslant m$, and $\mathfrak{a}_{n,0,p} = \delta_{n,1}\delta_{p,1}$.

Lemma 30. *Let $1 \leqslant m < p \leqslant n$. If $p \in \{n-1, n\}$,*

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \sum_{q=s+1}^{n-1} \sum_{\ell=q}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

If $p < n-1$,

$$\mathfrak{a}_{n,m,p} = \sum_{s=0}^{m-1} \left(\sum_{q=p}^{n-1} \mathfrak{a}_{q,s,p} \right) + \sum_{q=s+1}^{p-1} \mathfrak{a}_{q,s,q} + \sum_{\ell=n-p+q}^{n-1} \mathfrak{a}_{\ell,s,q}.$$

Proof. Let $1 \leqslant m < p \leqslant n$, and $\sigma \in \mathfrak{A}_{n,m,p}$. Let $\sigma' = (\sigma_i)_{i\in[1,n]\setminus\{p\}}$ be the sequence obtained by removing the value m from σ. Let $s = \mathbf{max}(\sigma')$ and $q = \mathbf{firstmax}(\sigma')$. Let σ'' be the sequence obtained by removing all terms of value s except the first one from σ', so that $\sigma'' \in \mathfrak{A}_{\ell,s,q}$ for some $\ell \in [q, n-1]$.

- If $p \in \{n-1, n\}$, we obtain the same equation as in Lemma 26, for the same reason.
- If $p < n-1$:
  - If $q \geqslant p$, then every term in the factor $(\sigma_i)_{i\in[q+1,n]}$ has value s, because σ avoids 210. In that case σ'' is a sequence in $\mathfrak{A}_{q,s,q}$ such that $(\sigma''_i)_{i\in[p,q]}$ is nondecreasing. The set of such sequences σ'' is in bijection with $\mathfrak{A}_{q,s,p}$ (to see that, simply move the value s from position q to position p).
  - If $q < p$, let $k+1$ be the number of terms of value s in σ. In particular, $k = |\sigma'| - |\sigma''| = n-1-\ell$. Note that the terms of value s in σ' are exactly $(\sigma'_i)_{i\in[q,q+k]}$.
    * If $k \geqslant p-q$, then $\sigma_{p+1} = s$, which implies that every term $(\sigma_i)_{i\in[q,n]\setminus\{p\}}$ has value s because σ avoids 210. In particular $k = n-1-q$, $\ell = q$, and we obtain a sequence $\sigma'' \in \mathfrak{A}_{q,s,q}$.
    * If $k < p-q$, then the value s does not appear after position p, and we obtain a sequence $\sigma'' \in \mathfrak{A}_{\ell,s,q}$ for $\ell = n-1-k \in [n-p+q, n-1]$. □

Theorem 31. *For all $n \geqslant 2$,*

$$|\mathcal{I}_n(101, 210)| = \sum_{\ell=1}^{n} \sum_{m=0}^{\ell-1} \sum_{p=m+1}^{\ell} \mathfrak{a}_{\ell,m,p}.$$

Proof. Identical to the proof of Theorem 27, since inserting a term of maximal value anywhere after the first maximum cannot create an occurrence of 210. □

3.9 A generating tree perspective on the previous constructions

The inversion sequences studied in Sections 3.4 to 3.8 could also be generated by inserting their values one by one, in increasing order (and inserting any repetitions of a value from left to right). This yields generating trees which differ only slightly from our construction. We chose not to present them that way, since it would require distinguishing several subsets of inversion sequences, as we did in Sections 2.3 and 3.3. In all five cases, the resulting succession rules would have two integer parameters, which always record the number of positions where a new maximum may be inserted, and the number of possible values of a new maximum. The first parameter is influenced by pattern-avoidance, and the second one is used to maintain the "subdiagonal" property characterizing inversion sequences.

Note that Sections 3.2 and 3.3 use such generating tree constructions, but only have one integer parameter.

- For the pair of patterns $\{000, 102\}$, the two parameters are one and the same, and correspond to the statistic we denoted **sites**, as explained in the paragraph just before Theorem 17.
- For the pair $\{102, 210\}$:

– a new maximum cannot be inserted in a sequence with several descents; it follows that such sequences only require one parameter to record the number of positions where repetitions of the current maximum could be inserted.

– in a sequence which has 0 or 1 descents, the positions where a new maximum may be inserted are trivial (for 0 descents any position is fine, and for 1 descent the only available position is between the top and the bottom of the descent), so only the number of possible values of a new maximum is required.

4 Splitting at the first maximum

4.1 Method

The method described in this section is a generalization of the construction used in [19] for 120-avoiding inversion sequences.

Let $\sigma \in \mathcal{I}_n$ be an inversion sequence of size $n \geqslant 1$, $m = \mathbf{max}(\sigma)$, and $p = \mathbf{firstmax}(\sigma)$. By definition, $p > m$. Let $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\beta = (\sigma_i)_{i\in[p,n]}$ be two factors of σ, which satisfy $\alpha{\cdot}\beta = \sigma$. The case $m = 0$ corresponds to constant inversion sequences, whose enumeration is trivial. When $m > 0$, we use this decomposition to count inversion sequences σ from the number of possible choices for α and β. Note that α is an inversion sequence of size $p - 1$ and maximum less than m, and β is a word of length $n - p + 1$ over the alphabet $[0, m]$ such that $\beta_1 = m$. More precisely, the following holds (where $\mathcal{W}_{n,k}$ is the set of words of length n over the alphabet $[0, k - 1]$).

Remark 32. For all $0 \leqslant m < p < n$, $\{\sigma \in \mathcal{I}_n \ : \ m = \mathbf{max}(\sigma), p = \mathbf{firstmax}(\sigma)\} = \{\alpha \cdot \beta \ : \ \alpha \in \mathcal{I}_{p-1}, \mathbf{max}(\alpha) < m, \beta \in \mathcal{W}_{n-p+1,m+1}, \beta_1 = m\}$.

Now we can express pattern avoidance on σ in terms of necessary and sufficient conditions on α and β.

Remark 33. Let τ be a pattern of length $k \geqslant 2$. Then σ avoids τ if and only if all three following conditions are satisfied:

1. α avoids τ,
2. β avoids τ,
3. There are no $i_1 < i_2 < \cdots < i_k \in [1, n]$ such that $i_1 < p$, $i_k \geqslant p$ and $(\sigma_{i_j})_{j\in[1,k]}$ is an occurrence of τ.

Condition 3 is difficult to work with in general, but it can be turned into simple conditions on α and β for certain patterns, such as 120 or 010.

Proposition 34. *If $\tau = 120$, condition 3 is equivalent to* $\mathbf{max}(\alpha) \leqslant \mathbf{min}(\beta)$.

Proof. If $\mathbf{max}(\alpha) > \mathbf{min}(\beta)$, then $(\mathbf{max}(\alpha), m, \mathbf{min}(\beta))$ is an occurrence of 120. Conversely, if there are three integers $i_1 < i_2 < i_3$ such that $i_1 < p$, $i_3 \geqslant p$ and $(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3})$ is an occurrence of 120, then $\mathbf{max}(\alpha) \geqslant \sigma_{i_1} > \sigma_{i_3} \geqslant \mathbf{min}(\beta)$. ☐

Proposition 35. *If $\tau = 010$, condition 3 is equivalent to* $\mathbf{Vals}(\alpha) \cap \mathbf{Vals}(\beta) = \emptyset$.

Proof. If $v \in \mathbf{Vals}(\alpha) \cap \mathbf{Vals}(\beta)$, then (v, m, v) is an occurrence of 010. Conversely, if there are three integers $i_1 < i_2 < i_3$ such that $i_1 < p$, $i_3 \geqslant p$ and $(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3})$ is an occurrence of 010, then $\sigma_{i_1} = \sigma_{i_3} \in \mathbf{Vals}(\alpha) \cap \mathbf{Vals}(\beta)$. $\square$

Sections 4.2 to 4.9 are focused on the pattern 120, while Sections 4.10 to 4.14 are focused on the pattern 010.

We always count the possible choices for α by recurrence, since α is an inversion sequence that is shorter than σ and must avoid the same patterns. On the other hand, the words β form a different family of objects which must be enumerated independently. This could be solved by applying the same decomposition around the first maximum to words, since Remark 32 has an analogue for words[7]. We use generating trees instead, since it results in simpler and more efficient recurrence formulas.

Depending on the pair of patterns studied, we consider different types of words in our decomposition. We always denote β the word defined as above, $\gamma = (\beta_i)_{i \in [2, n-p]}$ the same word without the first letter m (so that $\sigma = \alpha \cdot m \cdot \gamma$), and $\gamma' = (\beta_i)_{i \in [1, n-p], \beta_i \neq m}$ the word obtained by removing all letters m from β.

4.2 A useful symmetry

Given two integers $n, k \in \mathbb{N}$, a *composition* of n into k (possibly empty) *parts* is an integer sequence $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{N}^k$ such that $\sum_{i=1}^{k} \lambda_i = n$. Let $n, k \in \mathbb{N}$ and let λ be a composition of n into k parts. We denote by $\mathcal{W}_\lambda \subseteq \mathcal{W}_{n,k}$ the set of words in which each letter $\ell \in [0, k-1]$ appears exactly $\lambda_{\ell+1}$ times (i.e. permutations of the multiset $\{\ell^{\lambda_{\ell+1}} \ : \ \ell \in [0, k-1]\}$). As before, we denote by $\mathcal{W}_\lambda(P)$ the subset of words avoiding some set of patterns P. The following theorem is a result of [2] (up to symmetry).

Theorem 36. *Let $k \in \mathbb{N}$. The function $\mathbb{N}^k \to \mathbb{N}$, $(\lambda_1, \ldots, \lambda_k) \mapsto |\mathcal{W}_\lambda(120)|$ is symmetric.*

In other words, if $\lambda, \mu \in \mathbb{N}^k$ are two sequences such that there exists a permutation π of $[1, k]$ which satisfies $\mu_i = \lambda_{\pi(i)}$ for all $i \in [1, k]$, then the identity $|\mathcal{W}_\lambda(120)| = |\mathcal{W}_\mu(120)|$ holds. This result was generalized in [1] and [28] to show that the number $|\mathcal{W}_\lambda(120)|$ remains unchanged when the pattern 120 is replaced by any permutation of $\{0, 1, 2\}$.

4.3 The pair {011, 120}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(011, 120) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of $\{011, 120\}$-avoiding inversion sequences of size n and maximum m. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(120) \ : \ \omega_i = \omega_j \implies i = j\}$ be the set of 120-avoiding words of length n over the alphabet $[0, k-1]$ which do not contain any repeated letter. Let $\mathfrak{C}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(120) \ : \ \omega_i = \omega_j \neq 0 \implies i = j\}$ be

[7]More precisely, the set of words $\omega \in \mathcal{W}_{n,k}$ such that $m = \mathbf{max}(\omega)$ and $p = \mathbf{firstmax}(\omega)$ (note that p does not have to be greater than m now) is in bijection with the set of ordered pairs (α, β) such that $\alpha \in \mathcal{W}_{p-1,m}$ and $\beta \in \mathcal{W}_{n-p+1,m+1}$ satisfies $\beta_1 = m$. Remark 33 and Propositions 34 and 35 apply to these words as well.

the set of 120-avoiding words of length n over the alphabet $[0, k-1]$ which do not contain any nonzero repeated letter. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$, and $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$.

Theorem 37. *For all $0 < m < n$,*

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \mathfrak{c}_{n-p,m} + \sum_{j=1}^{m-1} \mathfrak{a}_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j-1}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, and let $j = \mathbf{max}(\alpha)$. In particular, $j < m$.

- If $j = 0$, then $\alpha = (0)^{p-1}$, and $\gamma \in \mathfrak{C}_{n-p,m}$ since $\sigma_1 = 0$ implies that any nonzero repeated letter creates an occurrence of the pattern 011.
- If $j \geqslant 1$, then $\alpha \in \mathfrak{A}_{p-1,j}$. By Proposition 34, γ is a 120-avoiding word of length $n-p$ over the alphabet $[j+1, m-1]$ (since repeating j or m would create an occurrence of 011) in which all letters are distinct (to avoid 011). Subtracting $j+1$ from every letter in γ yields a word in $\mathfrak{B}_{n-p,m-j-1}$. □

Observing that these two constructions are bijective concludes the proof.

Let us now enumerate the families of words $(\mathfrak{B}_{n,k})_{n,k\in\mathbb{N}}$ and $(\mathfrak{C}_{n,k})_{n,k\in\mathbb{N}}$.

Proposition 38. *For all $n, k \geqslant 0$,*

$$\mathfrak{b}_{n,k} = \frac{\binom{k}{n}\binom{2n}{n}}{n+1}.$$

Proof. There are $\binom{k}{n}$ possible choices for the set of n distinct letters in $[0, k-1]$ which appear in a word of $\mathfrak{B}_{n,k}$. Once this set of letters is decided, choosing their order amounts to choosing a 120-avoiding permutation of size n, counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. □

Lemma 39. *For all $n \geqslant 1, k \geqslant 2$,*

$$\mathfrak{c}_{n,k} = 2\mathfrak{c}_{n,k-1} + \mathfrak{c}_{n-1,k} - \mathfrak{c}_{n-1,k-1} - \frac{\binom{k-2}{n}\binom{2n}{n}}{n+1}.$$

Proof. Let $n \geqslant 0, k \geqslant 2$, and $\omega \in \mathfrak{C}_{n,k}$. We consider three different cases, based on the position of the letter 1 in ω.

- If ω does not contain the letter 1, then subtracting 1 from every nonzero letter in ω yields a word in $\mathfrak{C}_{n,k-1}$, and this is a bijection. As such, the words of $\mathfrak{C}_{n,k}$ which do not contain the letter 1 are counted by $\mathfrak{c}_{n,k-1}$.
- Otherwise, ω contains exactly one occurrence of the letter 1. This implies that $n \geqslant 1$ and $k \geqslant 2$.

– If there is any occurrence of the letter 0 appearing to the right of the letter 1 in ω, then every such occurrence must be in a single factor adjacent to the letter 1 (otherwise, ω would contain the pattern 120). Hence, removing the last letter 0 in ω can yield any word in $\mathfrak{C}_{n-1,k}$ which contains the letter 1, and this is a bijection. We know from the previous case that the words of $\mathfrak{C}_{n-1,k}$ which do not contain the letter 1 are counted by $\mathfrak{c}_{n-1,k-1}$, hence the number of possible words ω in this case is $\mathfrak{c}_{n-1,k} - \mathfrak{c}_{n-1,k-1}$.

– Otherwise, every letter 0 in ω appears to the left of the letter 1. Subtracting 1 from every nonzero letter in ω then yields a word in $\mathfrak{C}_{n,k-1}$ which contains the letter 0. Note this defines a bijection: its inverse takes any word in $\mathfrak{C}_{n,k-1}$ which contains the letter 0, adds 1 to every nonzero letter, and replaces the last letter 0 by the letter 1. Since the words of $\mathfrak{C}_{n,k-1}$ which do not contain the letter 0 are trivially in bijection with $\mathfrak{B}_{n,k-2}$, we find there are $\mathfrak{c}_{n,k-1} - \mathfrak{b}_{n,k-2}$ possible choices for the word ω in this case. ☐

We also give an explicit expression for $\mathfrak{c}_{n,k}$ in Lemma 42.

4.4 The pair {100, 120}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(100, 120) \,:\, m = \mathbf{max}(\sigma)\}$ be the set of $\{100, 120\}$-avoiding inversion sequences of size n and maximum m. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(120) \,:\, \omega_i = \omega_j \neq k-1 \implies i = j\}$ be the set of 120-avoiding words of length n over the alphabet $[0, k-1]$ which do not contain any non-maximal repeated letter. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 40. *For all* $0 < m < n$,

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j+1}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, and let $j = \mathbf{max}(\alpha)$. In particular, $\alpha \in \mathfrak{A}_{p-1,j}$. By Proposition 34, $\gamma \in [j, m]^{n-p}$. Since σ avoids 100, γ cannot contain any repeated value lower than m. Subtracting j from each letter of γ yields a word in $\mathfrak{B}_{n-p,m-j+1}$. ☐

As in Section 4.3, let $\mathfrak{C}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(120) \,:\, \omega_i = \omega_j \neq 0 \implies i = j\}$ be the set of 120-avoiding words of length n over the alphabet $[0, k-1]$ which do not contain any nonzero repeated letter, and let $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$. The number of words $\mathfrak{c}_{n,k}$ was counted in Lemma 39.

Lemma 41. *For all* $n, k \geqslant 0$, $\mathfrak{b}_{n,k} = \mathfrak{c}_{n,k}$.

Proof. Let $\mathfrak{B}_{n,k,r} = \{\omega \in \mathfrak{B}_{n,k} \,:\, r = |\{i \in [1, n] \,:\, \omega_i = k-1\}|\}$ be the subset of $\mathfrak{B}_{n,k}$ of words in which the largest letter $k-1$ appears exactly r times. Likewise, let

$\mathfrak{C}_{n,k,r} = \{\omega \in \mathfrak{C}_{n,k} \ : \ r = |\{i \in [1,n] \ : \ \omega_i = 0\}|\}$ be the subset of $\mathfrak{C}_{n,k}$ of words in which the letter 0 appears exactly r times. By definition, we have

$$\mathfrak{B}_{n,k} = \coprod_{r=0}^{n} \mathfrak{B}_{n,k,r} \quad \text{and} \quad \mathfrak{C}_{n,k} = \coprod_{r=0}^{n} \mathfrak{C}_{n,k,r}.$$

Let $\mathfrak{D}_{n,k} = \{\lambda \in \{0,1\}^k \ : \ n = \sum_{i=1}^{k} \lambda_i\}$ be the set of compositions of n into k parts of size 0 or 1. In the notation of Section 4.2, for all $r \geqslant 0$,

$$\mathfrak{B}_{n,k,r} = \coprod_{\lambda \in \mathfrak{D}_{n-r,k-1}} \mathcal{W}_{\lambda \cdot r}(120) \quad \text{and} \quad \mathfrak{C}_{n,k,r} = \coprod_{\lambda \in \mathfrak{D}_{n-r,k-1}} \mathcal{W}_{r \cdot \lambda}(120).$$

By Theorem 36, for all $r \geqslant 0$ and $\lambda \in \mathfrak{D}_{n-r,k-1}$, $|\mathcal{W}_{\lambda \cdot r}(120)| = |\mathcal{W}_{r \cdot \lambda}(120)|$. This implies that $|\mathfrak{B}_{n,k,r}| = |\mathfrak{C}_{n,k,r}|$ for all $n, k, r \geqslant 0$, hence $\mathfrak{b}_{n,k} = \mathfrak{c}_{n,k}$. □

We also provide an explicit expression of $\mathfrak{b}_{n,k}$.

Lemma 42. *For all $n \geqslant 0, k \geqslant 1$,*

$$\mathfrak{b}_{n,k} = \frac{1}{n+1} \sum_{d=0}^{\min(n,k)} \binom{k-1}{d} \binom{n+d}{n} (n-d+1).$$

Proof. Let $\mathfrak{E}_{n,k} = \{\omega \in \mathfrak{B}_{n,k+1} \ : \ [0,k-1] \subseteq \mathbf{Vals}(\omega)\}$ be the subset of words of $\mathfrak{B}_{n,k+1}$ which contain all letters $[0,k-1]$ (and may contain the letter k). In particular, for all $n \geqslant 0$, $\mathfrak{E}_{n,k}$ is empty if $k > n$, and $\mathfrak{E}_{n,0} = \{(0)^n\}$. Note that any word of $\mathfrak{E}_{n,k}$ contains exactly one occurrence of each letter in $[0,k-1]$, and $n-k$ occurrences of the letter k. Let $\mathfrak{e}_{n,k} = |\mathfrak{E}_{n,k}|$.

The same construction as in Proposition 1 gives the following identity, seeing that $\binom{k-1}{d}$ is the number of increasing functions $[0,d] \to [0,k-1]$ such that $d \mapsto k-1$. For all $n \geqslant 0, k \geqslant 1$,

$$\mathfrak{b}_{n,k} = \sum_{d=0}^{\min(n,k)} \binom{k-1}{d} \mathfrak{e}_{n,d}.$$

Let us now count the words of $\mathfrak{E}_{n,k}$. Let $1 \leqslant k \leqslant n$, and $\omega \in \mathfrak{E}_{n,k}$.

- If $\omega_n = k$, then removing ω_n yields a word in $\mathfrak{E}_{n-1,k}$ and this is a bijection.
- If $\omega_n < k$, then no letter k may appear after the letter $k-1$ in ω, otherwise the subsequence $(k-1, k, \omega_n)$ would be an occurrence of 120. Replacing every letter k in ω by $k-1$ yields a word $\omega' \in \mathfrak{E}_{n,k-1}$ (since each letter in $[0,k-2]$ still appears exactly once in ω'), and this is a bijection. Indeed, it can be reversed by taking any word $\omega' \in \mathfrak{E}_{n,k-1}$ and replacing every letter $k-1$ except the last one by the letter k (notice that $\omega' \in \mathfrak{E}_{n,k-1}$ must contain the letter $k-1$ since $k \leqslant n$).

Hence, $(\mathfrak{e}_{n,k})_{n,k\geqslant 0}$ satisfies the recurrence relation $\mathfrak{e}_{n,k} = \mathfrak{e}_{n-1,k} + \mathfrak{e}_{n,k-1}$ for all $1 \leqslant k \leqslant n$, with initial conditions $\mathfrak{e}_{n,0} = 1$ for all $n \geqslant 0$ and $\mathfrak{e}_{0,k} = 0$ for all $k \geqslant 1$. From this recurrence, we can easily prove by induction that for all $0 \leqslant k \leqslant n$,

$$\mathfrak{e}_{n,k} = \frac{\binom{n+k}{n}(n-k+1)}{n+1},$$

which shows that $(\mathfrak{e}_{n,k})_{n,k\geqslant 0}$ is Catalan's triangle (entry A009766 of the OEIS). ☐

4.5 The pair {120, 201}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(120, 201) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of $\{120, 201\}$-avoiding inversion sequences of size n and maximum m. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k+1}(120) \ : \ \omega_i < \omega_j \neq k \implies j < i\}$ be the set of 120-avoiding words of length n over the alphabet $[0, k]$ whose non-maximal letters appear in nonincreasing order. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 43. *For all $0 < m < n$,*

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, and let $j = \mathbf{max}(\alpha)$. In particular, $\alpha \in \mathfrak{A}_{p-1,j}$. By Proposition 34, $\gamma \in [j, m]^{n-p}$. Since σ avoids 201 and $\sigma_p = m$, all letters different from m in γ appear in nonincreasing order. By subtracting j from all letters in γ, we obtain a word in the set $\mathfrak{B}_{n-p,m-j}$. We easily observe that this construction is a bijection. ☐

Remark 44. For all $n \geqslant 0$ and $k \geqslant 1$, there are $\binom{k+n-1}{n}$ nonincreasing words of length n over the alphabet $[0, k-1]$. Indeed, the set of nonincreasing words of length n over the alphabet $[0, k-1]$ is in bijection with the set of compositions of n into k (possibly empty) parts. More precisely, for each composition of n into k parts λ, there is a single nonincreasing word in the set $\mathcal{W}_\lambda$ (using the notation from Section 4.2).

Lemma 45. *For all $n, k \geqslant 1$,*

$$\mathfrak{b}_{n,k} = \binom{k+n+1}{n} + k\left(2^n - 1 - \frac{n(n+1)}{2}\right) - n.$$

Proof. Let $n, k \geqslant 1$, and $\omega \in \mathfrak{B}_{n,k}$.

- If the letter k does not appear in ω, then ω can be any nonincreasing word of length n over the alphabet $[0, k-1]$. By Remark 44, there are $\binom{k+n-1}{n}$ such words.
- Otherwise, let $p = \mathbf{firstmax}(\omega)$ be the position of the first letter k in ω.
  - If $p = 1$, then $\omega_1 = k$ and $(\omega_i)_{i\in[2,n]}$ can be any word of $\mathfrak{B}_{n-1,k}$.

– If $p \geqslant 2$, let $\alpha = (\omega_i)_{i\in[1,p-1]}$ and $\gamma = (\omega_i)_{i\in[p+1,n]}$. In particular, α is a nonincreasing word of length $p-1$ over the alphabet $[0,k-1]$. By Remark 44, there are $\binom{k+p-2}{p-1}$ possible words α. Note that any letter $v \in \mathbf{Vals}(\gamma)$ such that $v \neq k$ must satisfy both $v \leqslant \mathbf{min}(\alpha)$ (by the nonincreasing property) and $v \geqslant \mathbf{max}(\alpha)$ (to avoid 120). We distinguish two cases:
  * If α is constant, then $\alpha = (v)^{p-1}$ for some $v \in [0,k-1]$, and $\gamma \in \{v,k\}^{n-p}$. In that case, there are k possible choices for α, and for each α there are 2^{n-p} possible choices for γ.
  * Otherwise, there are $\binom{k+p-2}{p-1} - k$ remaining choices for α. Since α is not constant, $\mathbf{min}(\alpha) < \mathbf{max}(\alpha)$, hence no letter in $[0,k-1]$ may appear in γ, so $\gamma = (k)^{n-p}$.

We obtain the following equation for all $n,k \geqslant 1$:

$$\begin{aligned}
\mathfrak{b}_{n,k} &= \binom{k+n-1}{n} + \mathfrak{b}_{n-1,k} + \sum_{p=2}^{n}\left(k\cdot 2^{n-p} + \binom{k+p-2}{p-1} - k\right)\\
&= \binom{k+n-1}{n} + \mathfrak{b}_{n-1,k} + k(2^{n-1}-1) + \binom{k+n-1}{n-1} - 1 - k(n-1)\\
&= \mathfrak{b}_{n-1,k} + \binom{k+n}{n} + k(2^{n-1}-n) - 1.
\end{aligned}$$

We conclude the proof with a telescoping sum. For all $n,k \geqslant 1$,

$$\begin{aligned}
\mathfrak{b}_{n,k} &= \mathfrak{b}_{0,k} + \sum_{i=1}^{n}(\mathfrak{b}_{i,k} - \mathfrak{b}_{i-1,k})\\
&= 1 + \sum_{i=1}^{n}\left(\binom{k+i}{i} + k(2^{i-1}-i) - 1\right)\\
&= \binom{k+n+1}{n} + k\left(2^n - 1 - \frac{n(n+1)}{2}\right) - n. \qquad \square
\end{aligned}$$

4.6 The pair {110, 120}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(110,120) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of $\{110,120\}$-avoiding inversion sequences of size n and maximum m. Let $\mathfrak{B}_{n,k} = \coprod_{\ell\in[0,n]} \mathcal{W}_{\ell,k}(110,120)$ be the set of $\{110,120\}$-avoiding words of length at most n over the alphabet $[0,k-1]$. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 46. *For all $0 < m < n$,*

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n}\sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j}\cdot\mathfrak{b}_{n-p,m-j}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, $j = \mathbf{max}(\alpha)$. In particular, $\alpha \in \mathfrak{A}_{p-1,j}$. By Proposition 34, $\gamma \in [j,m]^{n-p}$. Since σ avoids 110, if $\gamma_q = m$ for some $q \in [1, n-p]$ then $\gamma_i = m$ for all $i \in [q, n-p]$. In other words, all occurrences of m in γ (if any) must appear in a single factor at the end of γ. Let $k \in [0, n-p]$ be the number of occurrences of m in γ, and $\gamma' = (\gamma_i)_{i\in[1,n-p-k]}$ be the word obtained by removing all letters m from γ. By subtracting j from all letters in γ', we obtain a word in the set $\mathfrak{B}_{n-p,m-j}$. We easily observe that this construction is a bijection. □

Let $\mathfrak{C}_{n,k} = \overline{\mathcal{W}}_{n,k}(110, 120)$ be the set of $\{110, 120\}$-avoiding words of length n which contain all letters of the alphabet $[0, k-1]$. Let $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$. By definition of $\mathfrak{B}_{n,k}$ and Proposition 1, for all $n, k \geqslant 0$,

$$\mathfrak{b}_{n,k} = \sum_{\ell=0}^{n} \sum_{d=0}^{\mathbf{min}(\ell,k)} \binom{k}{d} \mathfrak{c}_{\ell,d}.$$

Lemma 47. *For all* $1 \leqslant k \leqslant n$,

$$\mathfrak{c}_{n,k} = \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1}.$$

Proof. For all $0 \leqslant k \leqslant n$ and $\omega \in \mathfrak{C}_{n,k}$, let $\mathbf{dec}(\omega)$ be the size of the longest strictly decreasing factor at the beginning of ω. In particular, if $n > 0$ then $\omega_{\mathbf{dec}(\omega)}$ is the first letter 0 in ω (otherwise the first 0 would appear after a weak increase, creating an occurrence of 110 or 120). Also note that $\mathbf{dec}(\omega) \leqslant k$ since all letters of a strictly decreasing factor are distinct. Let $\mathfrak{C}_{n,k,s} = \{\omega \in \mathfrak{C}_{n,k} \ :\ s = \mathbf{dec}(\omega)\}$, and $\mathfrak{c}_{n,k,s} = |\mathfrak{C}_{n,k,s}|$.

We consider the combinatorial generating tree for $\coprod_{n\geqslant k\geqslant 0} \mathfrak{C}_{n,k}$ obtained by inserting letters in decreasing order of value, and inserting repeats of a letter from right to left. This amounts to inserting the letter 0 before the first 0 in a word $\omega \in \mathfrak{C}_{n,k}$, or inserting the letter 0 in $\omega+1$. This indeed defines a generating tree since any nonempty word ω has a unique parent of size $|\omega| - 1$, obtained by removing the first 0 from ω, and subtracting 1 from every letter if the resulting word does not contain the letter 0.

For any word $\omega \in \mathfrak{C}_{n,k}$, the positions where the letter 0 may be inserted before the first 0 in ω are $[1, \mathbf{dec}(\omega)]$, and this clearly cannot create an occurrence of 110 or 120. The positions where the letter 0 may be inserted in $\omega + 1$ without creating an occurrence of 110 or 120 are $[1, \mathbf{dec}(\omega) + 1]$. This implies that each word of $\mathfrak{C}_{n,k,s}$ has one child in $\mathfrak{C}_{n+1,k,i}$ for all $i \in [1, s]$, and one child in $\mathfrak{C}_{n+1,k+1,i}$ for all $i \in [1, s+1]$. We obtain the following equation for all $2 \leqslant i \leqslant k \leqslant n$:

$$\mathfrak{c}_{n,k,i} = \sum_{s=i}^{n-1} \mathfrak{c}_{n-1,k,s} + \sum_{s=i-1}^{n-1} \mathfrak{c}_{n-1,k-1,s}.$$

In particular, $\mathfrak{c}_{n,k,i} - \mathfrak{c}_{n,k,i+1} = \mathfrak{c}_{n-1,k,i} + \mathfrak{c}_{n-1,k-1,i-1}$, so a simpler recurrence relation holds: for all $2 \leqslant s \leqslant k \leqslant n$,

$$\mathfrak{c}_{n,k,s} = \mathfrak{c}_{n,k,s+1} + \mathfrak{c}_{n-1,k,s} + \mathfrak{c}_{n-1,k-1,s-1}.$$

From this equation, we can easily prove by induction that for all $1 \leqslant s \leqslant k \leqslant n$,

$$\mathfrak{c}_{n,k,s} = \frac{s(n+k-s-1)!}{(n-k)!k!(k-s)!}.$$

Finally, the expression of $\mathfrak{c}_{n,k}$ is obtained by summing over s. For all $1 \leqslant k \leqslant n$,

$$\begin{aligned}
\mathfrak{c}_{n,k} &= \sum_{s=1}^{k} \mathfrak{c}_{n,k,s} \\
&= \sum_{s=1}^{k} \frac{s(n+k-s-1)!}{(n-k)!k!(k-s)!} \\
&= \frac{1}{k}\binom{n-1}{k-1} \sum_{s=1}^{k} s \binom{n+k-s-1}{n-1} \\
&= \frac{1}{k}\binom{n-1}{k-1} \sum_{i=1}^{k} \sum_{s=i}^{k} \binom{n+k-s-1}{n-1} \\
&= \frac{1}{k}\binom{n-1}{k-1} \sum_{i=1}^{k} \binom{n+k-i}{n} \\
&= \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1}.
\end{aligned}$$

□

4.7 The pair {010, 120}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(010, 120) \; : \; m = \mathbf{max}(\sigma)\}$ be the set of $\{010, 120\}$-avoiding inversion sequences of size n and maximum m. Let $\mathfrak{B}_{n,k} = \mathcal{W}_{n,k}(010, 120)$ be the set of {010, 120}-avoiding words of length n over the alphabet $[0, k-1]$. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 48. *For all* $0 < m < n$,

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j}.$$

Proof. It follows from Remarks 32 and 33 and Propositions 34 and 35 that for $0 < m < n$, $\mathfrak{A}_{n,m}$ is the set of inversion sequences $\alpha \cdot m \cdot \gamma$ such that:

- $\alpha \in \mathfrak{a}_{p-1,j}$ for some $p \in [m+1, n]$ and $j \in [0, m-1]$,
- $\gamma \in [0, m]^{n-p}$ avoids 010 and 120 and satisfies $\mathbf{min}(\gamma) > \mathbf{max}(\alpha) = j$.

In other words, γ is a {010, 120}-avoiding word of length $n-p$ over the alphabet $[j+1, m]$, and the set of such words is clearly in bijection with $\mathfrak{B}_{n-p,m-j}$. □

Let $\mathfrak{C}_{n,k} = \overline{\mathcal{W}}_{n,k}(010, 120)$ be the set of $\{010,\ 120\}$-avoiding words of length n which contain all letters of the alphabet $[0, k-1]$. Let $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$. By Proposition 1, for all $n, k \geqslant 0$,

$$\mathfrak{b}_{n,k} = \sum_{d=0}^{\mathbf{min}(n,k)} \binom{k}{d} \mathfrak{c}_{n,d}.$$

Lemma 49. *For all $n, k \geqslant 1$,*

$$\mathfrak{c}_{n,k} = \frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1}.$$

Proof. We consider the combinatorial generating tree for $\coprod_{n \geqslant k \geqslant 0} \mathfrak{C}_{n,k}$ obtained by inserting letters in increasing order of value, and inserting repeats of a letter from right to left. In other words, at each step of the construction, we either insert a letter k in a word $\omega \in \mathfrak{C}_{n,k}$, or insert a letter $k-1$ before the first $k-1$ in ω.

For all $1 \leqslant k \leqslant n$ and $\omega \in \mathfrak{C}_{n,k}$, let

$$\mathbf{Sites}(\omega) = \{q \in [1, \mathbf{firstmax}(\omega)] \ : \ \mathbf{max}((\omega_i)_{i \in [1,q-1]}) < \mathbf{min}((\omega_i)_{i \in [q,n]})\}$$

be the set of positions where the letter $k-1$ may be inserted before the first letter $k-1$ in ω without creating an occurrence of 010 or 120. The set of positions where the letter k may be inserted in ω without creating an occurrence of 010 or 120 is $\mathbf{Sites}(\omega) \sqcup \{n+1\}$. For all $1 \leqslant s \leqslant k \leqslant n$, let $\mathfrak{C}_{n,k,s} = \{\omega \in \mathfrak{C}_{n,k} \ : \ s = |\mathbf{Sites}(\omega)|\}$.

Let $1 \leqslant s \leqslant k \leqslant n$ and $\omega \in \mathfrak{C}_{n,k,s}$. Let $q_1 < q_2 < \cdots < q_s$ be the elements of $\mathbf{Sites}(\omega)$. Then for all $i \in [1, s]$, inserting the letter $k-1$ at position q_i yields a word of $\mathfrak{C}_{n+1,k,i}$, and inserting the letter k at position q_i yields a word of $\mathfrak{C}_{n+1,k+1,i}$. Additionally, inserting the letter k at position $n+1$ yields a word in $\mathfrak{C}_{n+1,k+1,s+1}$. Hence the following equation holds: for all $2 \leqslant i \leqslant k \leqslant n$,

$$\mathfrak{c}_{n,k,i} = \sum_{s=i}^{n-1} \mathfrak{c}_{n-1,k,s} + \sum_{s=i-1}^{n-1} \mathfrak{c}_{n-1,k-1,s}.$$

The same recurrence relation was found in the proof of Lemma 47, with the same initial condition $\mathfrak{c}_{1,1,1} = 1$. This exhibits a bijection between the sets of words $\overline{\mathcal{W}}_{n,k}(010, 120)$ and $\overline{\mathcal{W}}_{n,k}(110, 120)$ (which trivially extends to a bijection between $\mathcal{W}_{n,k}(010, 120)$ and $\mathcal{W}_{n,k}(110, 120)$), and concludes our proof. $\square$

4.8 The pair {101, 120}

Let $\mathfrak{A}_{n,m} = \{\sigma \in \mathcal{I}_n(101, 120) \ : \ m = \mathbf{max}(\sigma)\}$ be the set of $\{101, 120\}$-avoiding inversion sequences of size n and maximum m. Since the sequences in $\mathfrak{A}_{n,m}$ avoid 101, all occurrences of their maximum m are consecutive. Let $\mathfrak{A}'_{n,m} = \{\sigma \in \mathfrak{A}_{n,m} \ : \ \sigma_n = m\}$ be the subset of $\mathfrak{A}_{n,m}$ of sequences whose last value is their maximum. Let $\mathfrak{B}_{n,k} = \coprod_{\ell \in [0,n]} \mathcal{W}_{\ell,k}(101, 120)$ be the set of $\{101, 120\}$-avoiding words of length at most n over the alphabet $[0, k-1]$. Let $\mathfrak{a}_{n,m} = |\mathfrak{A}_{n,m}|$, $\mathfrak{a}'_{n,m} = |\mathfrak{A}'_{n,m}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 50. *For all* $0 < m < n$,

$$\mathfrak{a}_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}'_{p-1,j} \cdot \mathfrak{b}_{n-p,m-j} + (\mathfrak{a}_{p-1,j} - \mathfrak{a}'_{p-1,j}) \cdot \mathfrak{b}_{n-p,m-j-1},$$

$$\mathfrak{a}'_{n,m} = \sum_{p=m+1}^{n} \sum_{j=0}^{n-1} \mathfrak{a}_{p-1,j}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, $j = \mathbf{max}(\alpha)$. In particular, $\alpha \in \mathfrak{A}_{p-1,j}$ and $\gamma \in [j,m]^{n-p}$. Let $\ell = \mathbf{lastmax}(\sigma) \in [p,n]$ be the position of the last value m in σ. Since σ avoids 101, all occurrences of the maximum m are consecutive, hence their positions are $[p,\ell]$. Let $\gamma' = (\sigma_i)_{i\in[\ell+1,n]}$ be the word obtained by removing all letters m from γ.

- If $\alpha \in \mathfrak{A}'_{p-1,j}$, then γ' can be any $\{101, 120\}$-avoiding word of length $n-\ell \in [0, n-p]$ over the alphabet $[j, m-1]$. Subtracting j from all letters in γ' yields any word in the set $\mathfrak{B}_{n-p,m-j}$.
- If $\alpha \in \mathfrak{A}_{p-1,j} \backslash \mathfrak{A}'_{p-1,j}$, then $\sigma_{p-1} < j$. The subsequence (j, σ_{p-1}) is an occurrence of the pattern 10, which implies that the value j does not appear in γ' (otherwise σ would contain 101). In that case, γ' is a word over the alphabet $[j+1, m-1]$, and the number of possible words γ' is $\mathfrak{b}_{n-p,m-j-1}$.

Finally, if $\sigma \in \mathfrak{A}'_{n,m}$ then $\gamma = (m)^{n-p}$ and α can be any sequence in $\mathfrak{A}_{p-1,j}$. This yields the second equation. ☐

It remains to count the words of $\mathfrak{B}_{n,k}$. Let $F(x,y) = \sum_{\ell,k\geqslant 0} |\mathcal{W}_{\ell,k}(101,120)| x^\ell y^k$ be the ordinary generating function of $\{101, 120\}$-avoiding words. The enumeration of $\{101, 120\}$-avoiding words was solved in [18], with the following expression:

$$F(x,y) = \frac{(1-x)^2 - (1-2x)y - \sqrt{(1-x)^4 - 2(1-x)^2 y + (1-4x^2+4x^3)y^2}}{2xy(1-y)}.$$

The generating function of $\mathfrak{B}_{n,k}$ is

$$\begin{aligned} B(x,y) &= \sum_{n,k\geqslant 0} \mathfrak{b}_{n,k} x^n y^k \\ &= \sum_{n,k\geqslant 0} \sum_{\ell\in[0,n]} |\mathcal{W}_{\ell,k}(101,120)| x^n y^k \\ &= \sum_{\ell,k\geqslant 0} |\mathcal{W}_{\ell,k}(101,120)| \sum_{n\geqslant \ell} x^n y^k \\ &= \sum_{\ell,k\geqslant 0} |\mathcal{W}_{\ell,k}(101,120)| \frac{x^\ell y^k}{1-x} \\ &= \frac{F(x,y)}{1-x}. \end{aligned}$$

4.9 The pair {000, 120}

Given a sequence $\sigma \in \mathbb{N}^n$ and an integer $v \in \mathbb{N}$, let $\mathbf{occ}(\sigma, v) = |\{i \in [1,n] \ : \ \sigma_i = v\}|$ be the number of occurrences of the value v in σ.

Let $\mathfrak{A}_{n,m,r} = \{\sigma \in \mathcal{I}_n(000,120) \ : \ m = \mathbf{max}(\sigma),\ r = \mathbf{occ}(\sigma, m)\}$ be the set of $\{000,120\}$-avoiding inversion sequences of size n, maximum m, and r occurrences of m. Let $\mathfrak{B}_{n,k} = \mathcal{W}_{n,k}(000,120)$ be the set of $\{000,120\}$-avoiding words of length n over the alphabet $[0,k-1]$, and $\mathfrak{C}_{n,k} = \{\omega \in \mathfrak{B}_{n,k} \ : \ \mathbf{occ}(\omega, k-1) = 1\}$ be the subset of words of $\mathfrak{B}_{n,k}$ in which the largest letter $k-1$ appears exactly once. We also consider the subsets of $\mathfrak{B}_{n,k}$ and $\mathfrak{C}_{n,k}$ of words in which the letter 0 appears less than twice: let $\mathfrak{B}_{n,k,\cancel{00}} = \{\omega \in \mathfrak{B}_{n,k} \ : \ \mathbf{occ}(\omega, 0) < 2\}$ and $\mathfrak{C}_{n,k,\cancel{00}} = \{\omega \in \mathfrak{C}_{n,k} \ : \ \mathbf{occ}(\omega, 0) < 2\}$. Let $\mathfrak{a}_{n,m,r} = |\mathfrak{A}_{n,m,r}|$, $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$, $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$, $\mathfrak{b}_{n,k,\cancel{00}} = |\mathfrak{B}_{n,k,\cancel{00}}|$, and $\mathfrak{c}_{n,k,\cancel{00}} = |\mathfrak{C}_{n,k,\cancel{00}}|$.

Theorem 51. *For all* $0 < m < n$,

$$\mathfrak{a}_{n,m,1} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,1} \cdot \mathfrak{b}_{n-p,m-j,\cancel{00}} + \mathfrak{a}_{p-1,j,2} \cdot \mathfrak{b}_{n-p,m-j-1},$$

$$\mathfrak{a}_{n,m,2} = \sum_{p=m+1}^{n} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,1} \cdot \mathfrak{c}_{n-p,m-j+1,\cancel{00}} + \mathfrak{a}_{p-1,j,2} \cdot \mathfrak{c}_{n-p,m-j}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, $\gamma = (\sigma_i)_{i\in[p+1,n]}$, $j = \mathbf{max}(\alpha)$. In particular, $\alpha \in \mathfrak{A}_{p-1,j}$ and $\gamma \in [j,m]^{n-p}$ by Proposition 34.

- If $\sigma \in \mathfrak{A}_{n,m,1}$, then the letter m cannot appear in γ, so $\gamma \in [j, m-1]^{n-p}$.
  - If $\alpha \in \mathfrak{A}_{p-1,j,1}$ then the letter j appears at most once in γ (to avoid the pattern 000). By subtracting j from every letter in γ, we obtain a word in the set $\mathfrak{B}_{n-p,m-j,\cancel{00}}$.
  - If $\alpha \in \mathfrak{A}_{p-1,j,2}$ then the letter j cannot appear in γ. By subtracting $j+1$ from every letter in γ, we obtain a word in the set $\mathfrak{B}_{n-p,m-j-1}$.
- If $\sigma \in \mathfrak{A}_{n,m,2}$, then the letter m appears exactly once in γ.
  - If $\alpha \in \mathfrak{A}_{p-1,j,1}$ then subtracting j from every letter in γ yields a word in the set $\mathfrak{C}_{n-p,m-j+1,\cancel{00}}$.
  - If $\alpha \in \mathfrak{A}_{p-1,j,2}$ then subtracting $j+1$ from every letter in γ yields a word in the set $\mathfrak{C}_{n-p,m-j}$.

We easily observe that this construction is a bijection. ☐

For $a, b \in \{1,2\}$, let $\mathfrak{D}_{n,k,a,b} = \{\omega \in \overline{\mathcal{W}}_{n,k}(000,120) \ : \ a = \mathbf{occ}(\omega, 0),\ b = \mathbf{occ}(\omega, k-1)\}$ be the set of $\{000,120\}$-avoiding words of length n which contain every letter of the alphabet $[0,k-1]$ and in which the letter 0 appears a times and the letter $k-1$ appears b times. Let $\mathfrak{d}_{n,k,a,b} = |\mathfrak{D}_{n,k,a,b}|$.

Proposition 52. *For all $n, k \geqslant 1$,*

$$\mathfrak{b}_{n,k} = \sum_{d=1}^{\mathbf{min}(n,k)} \binom{k}{d} \sum_{a,b\in\{1,2\}} \mathfrak{d}_{n,d,a,b},$$

$$\mathfrak{c}_{n,k} = \sum_{d=1}^{\mathbf{min}(n,k)} \binom{k-1}{d-1} \sum_{a\in\{1,2\}} \mathfrak{d}_{n,d,a,1},$$

$$\mathfrak{b}_{n,k,\cancel{00}} = \sum_{d=1}^{\mathbf{min}(n,k)} \left(\binom{k-1}{d-1} \sum_{b\in\{1,2\}} \mathfrak{d}_{n,d,1,b} + \binom{k-1}{d} \sum_{a,b\in\{1,2\}} \mathfrak{d}_{n,d,a,b} \right),$$

$$\mathfrak{c}_{n,k,\cancel{00}} = \sum_{d=1}^{\mathbf{min}(n,k)} \left(\binom{k-2}{d-2} \mathfrak{d}_{n,d,1,1} + \binom{k-2}{d-1} \sum_{a\in\{1,2\}} \mathfrak{d}_{n,d,a,1} \right).$$

Proof. Let $F_{d,k}$ be the set of increasing functions from $[0, d-1]$ to $[0, k-1]$. The four identities above can be proven using the same construction as in the proof of Proposition 1, observing that

1. $\binom{k}{d} = |F_{d,k}|$,
2. $\binom{k-1}{d-1} = |\{\varphi \in F_{d,k} \ : \ \varphi(d-1) = k-1\}|$,
3. $\binom{k-1}{d-1} = |\{\varphi \in F_{d,k} \ : \ \varphi(0) = 0\}|$, and $\binom{k-1}{d} = |\{\varphi \in F_{d,k} \ : \ \varphi(0) > 0\}|$,
4. $\binom{k-2}{d-2} = |\{\varphi \in F_{d,k} \ : \ \varphi(0) = 0, \varphi(d-1) = k-1\}|$, and $\binom{k-2}{d-1} = |\{\varphi \in F_{d,k} \ : \ \varphi(0) > 0, \varphi(d-1) = k-1\}|$. □

Let $\mathfrak{E}_{n,k} = \{\lambda \in \{1,2\}^k \ : \ n = \sum_{i=1}^{k} \lambda_i\}$ be the set of compositions of n into k parts of size 1 or 2. Equivalently, $\mathfrak{E}_{n,k}$ is the set of words of length k over the alphabet $\{1,2\}$ which contain exactly $2k-n$ occurrences of the letter 1 and $n-k$ occurrences of the letter 2. Observe that $\mathfrak{E}_{n,k}$ is empty if $n < k$ or $n > 2k$.

Remark 53. In the notation of Section 4.2,

$$\mathfrak{D}_{n,k,a,b} = \coprod_{\substack{\lambda\in\mathfrak{E}_{n,k} \\ (\lambda_1,\lambda_k)=(a,b)}} \mathcal{W}_\lambda(120).$$

Let $\mathfrak{F}_{n,k} = \mathcal{W}_{(2)^{n-k}\cdot(1)^{2k-n}}(120)$ be the set of 120-avoiding words of length n over the alphabet $[0, k-1]$, which contain exactly 2 occurrences of each letter in $[0, n-k-1]$ and 1 occurrence of each letter in $[n-k, k-1]$, and let $\mathfrak{f}_{n,k} = |\mathfrak{F}_{n,k}|$.

Proposition 54. *For all $k \geqslant 2, n \in [k, 2k]$ and $a, b \in \{1,2\}$,*

$$\mathfrak{d}_{n,k,a,b} = \binom{k-2}{n-k+2-a-b} \mathfrak{f}_{n,k}$$

Proof. For all $k \geqslant 2, n \in [k, 2k]$ and $a, b \in \{1, 2\}$, the set of sequences $\lambda \in \mathfrak{E}_{n,k}$ such that $\lambda_1 = a$ and $\lambda_k = b$ is the set of sequences of the form $a \cdot \lambda' \cdot b$ where λ' is a word of length $k-2$ over the alphabet $\{1, 2\}$ which contains exactly $n-k+2-a-b$ occurrences of the letter 2. Hence, there are $\binom{k-2}{n-k+2-a-b}$ such sequences.

By Theorem 36, the number of words $|\mathcal{W}_\lambda(120)|$ is independent of the choice of $\lambda \in \mathfrak{E}_{n,k}$, and counted by $\mathfrak{f}_{n,k}$. Remark 53 concludes the proof. $\square$

Lemma 55. *The following recurrence relation holds for all $k \geqslant 1, n \in [k+1, 2k]$:*

$$\mathfrak{f}_{n,k} = \mathfrak{f}_{n,k+1} - \mathfrak{f}_{n-1,k},$$

with initial conditions $\mathfrak{f}_{n,n} = \frac{1}{n+1}\binom{2n}{n}$ for all $n \geqslant 0$.

Proof. Let $k \geqslant 2, n \in [k, 2k-2]$, and $\omega \in \mathfrak{F}_{n,k}$. In particular, ω contains a single letter $k-1$ and a single letter $k-2$.

- If the letter $k-2$ is on the left of $k-1$, then $k-1$ is the last letter of ω (because ω avoids the pattern 120). In that case, removing the letter $k-1$ yields a word in $\mathfrak{F}_{n-1,k-1}$, and this is a bijection.
- Otherwise, the letter $k-2$ is on the right of $k-1$. In that case, replacing the letter $k-1$ by $k-2$ yields a word in $\mathcal{W}_{(2)^{n-k}\cdot(1)^{2k-n-2}\cdot(2)}(120)$, and this is a bijection. By Theorem 36, this set is equinumerous with $\mathcal{W}_{(2)^{n-k+1}\cdot(1)^{2k-n-2}}(120) = \mathfrak{F}_{n,k-1}$.

This shows that for all $k \geqslant 2$ and $n \in [k, 2k-2]$, $\mathfrak{f}_{n,k} = \mathfrak{f}_{n,k-1} + \mathfrak{f}_{n-1,k-1}$, which is equivalent to the recurrence relation of the lemma.

For all $n \geqslant 0$, $\mathfrak{F}_{n,n}$ is trivially in bijection with the set of 120-avoiding permutations of size n, which is known to be counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. $\square$

The numbers $\mathfrak{f}_{n,k}$ can be found in entry A059346 of the OEIS.

4.10 The pattern 010

We denote by $\left[{n \atop k}\right]$ the unsigned Stirling numbers of the first kind, which count the number of permutations of size n with k cycles (among other combinatorial interpretations, cf. entry A132393 of the OEIS).

Let $\mathfrak{A}_{n,m,d} = \{\sigma \in \mathcal{I}_n(010) \ : \ m = \mathbf{max}(\sigma),\, d = \mathbf{dist}(\sigma)\}$ be the set of 010-avoiding inversion sequences of size n, maximum m, and having exactly d distinct values. Let also $\mathfrak{B}_{n,k} = \{\omega \in \overline{\mathcal{W}}_{n,k}(010) \ : \ \omega_1 = k-1\}$ be the set of 010-avoiding words of length n which contain all letters of the alphabet $[0, k-1]$ and begin with their largest letter. Let $\mathfrak{a}_{n,m,d} = |\mathfrak{A}_{n,m,d}|$ and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 56. *For all $2 \leqslant d \leqslant m+1 \leqslant n$,*

$$\mathfrak{a}_{n,m,d} = \sum_{i=0}^{d-1} \binom{m-i}{d-i-1} \sum_{p=m+1}^{n} \mathfrak{b}_{n-p+1,d-i} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,i}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m,d}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i\in[1,p-1]}$, and $\beta = (\sigma_i)_{i\in[p,n]}$.

Since σ avoids the pattern 010, α and β avoid 010, and $\mathbf{Vals}(\alpha) \cap \mathbf{Vals}(\beta) = \emptyset$ by Proposition 35. In particular, $\alpha \in \mathfrak{A}_{p-1,j,i}$ for some $j < m$ and $i < d$, and β is a 010-avoiding word of length $n-p+1$, which contains exactly $d-i$ distinct values chosen from the remaining $m+1-i$ (that is, all values in $[0,m]$ except for the i values in α), and such that $\beta_1 = m$. Since m is always in β, there are $\binom{m-i}{d-i-1}$ possible choices for the set $\mathbf{Vals}(\beta)$. Once the values of β are chosen, there are $\mathfrak{b}_{n-p+1,d-i}$ ways to arrange them into a word avoiding 010 and beginning with its largest letter. □

Lemma 57. *For all $n, k \geqslant 1$,*

$$\mathfrak{b}_{n,k} = \begin{bmatrix} n \\ n+1-k \end{bmatrix}.$$

Proof. Let $n \geqslant k \geqslant 2$, and $\omega \in \mathfrak{B}_{n,k}$. Since ω avoids the pattern 010, all letters 0 in ω are consecutive.

- If ω contains several letters 0, then removing one of them yields a word $\omega' \in \mathfrak{B}_{n-1,k}$, and this is clearly a bijection.
- If ω contains a single letter 0, then removing it and subtracting 1 from all other letters yields a word $\omega' \in \mathfrak{B}_{n-1,k-1}$. Since $\omega_1 = k-1 > 0$, there are $n-1$ possible positions for the letter 0 in ω, so exactly $n-1$ words $\omega \in \mathfrak{B}_{n,k}$ yield the same ω'.

Hence the following recurrence relation holds for all $n \geqslant k \geqslant 2$:

$$\mathfrak{b}_{n,k} = \mathfrak{b}_{n-1,k} + (n-1)\mathfrak{b}_{n-1,k-1}.$$

This recurrence relation is also satisfied by the Stirling numbers of the first kind $\begin{bmatrix} n \\ n+1-k \end{bmatrix}$, and we can easily verify that initial conditions also match. □

4.11 The pair of patterns {000, 010}

Let $\mathfrak{A}_{n,m,d} = \{\sigma \in \mathcal{I}_n(000, 010) \ : \ m = \mathbf{max}(\sigma),\ d = \mathbf{dist}(\sigma)\}$ be the set of $\{000, 010\}$-avoiding inversion sequences of size n, maximum m, and having d distinct values. Let $\mathfrak{B}_{n,k} = \{\omega \in \overline{\mathcal{W}}_{n,k}(000, 010) \ : \ \omega_1 = k-1\}$ be the set of $\{000, 010\}$-avoiding words of length n which contain all letters of the alphabet $[0, k-1]$ and begin with their largest letter. Let $\mathfrak{a}_{n,m,d} = |\mathfrak{A}_{n,m,d}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$. Since the sequences of $\mathfrak{A}_{n,m,d}$ and the words of $\mathfrak{B}_{n,k}$ avoid the pattern 000, we have $\mathfrak{a}_{n,m,d} = 0$ if $n > 2d$, and $\mathfrak{b}_{n,k} = 0$ if $n > 2k$.

Theorem 58. *For all $2 \leqslant d \leqslant m+1 \leqslant n$,*

$$\mathfrak{a}_{n,m,d} = \sum_{i=0}^{d-1} \binom{m-i}{d-i-1} \sum_{p=m+1}^{n} \mathfrak{b}_{n-p+1,d-i} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,i}.$$

Proof. Identical to the proof of Theorem 56: the pattern 000 could not spread over α and β, since the avoidance of 010 already implies that α and β do not share any values. □

Lemma 59. *For all $n, k \geqslant 2$,*

$$\mathfrak{b}_{n,k} = (n-1)\mathfrak{b}_{n-1,k-1} + (n-2)\mathfrak{b}_{n-2,k-1}.$$

Proof. Let $n \geqslant k \geqslant 2$, and $\omega \in \mathfrak{B}_{n,k}$. Since ω avoids 010, all letters 0 in ω are consecutive. Since ω avoids 000, ω has at most two letters 0.

- If ω contains a single letter 0, then removing it and subtracting 1 from all other letters yields a word $\omega' \in \mathfrak{B}_{n-1,k-1}$. Since $\omega_1 = k-1 > 0$, there are $n-1$ possible positions for the letter 0 in ω, so exactly $n-1$ words $\omega \in \mathfrak{B}_{n,k}$ yield the same ω'.

- If ω contains two letters 0, then removing them and subtracting 1 from all other letters yields a word $\omega' \in \mathfrak{B}_{n-2,k-1}$. There are $n-2$ possible positions for the two consecutive letters 0 in ω, so exactly $n-2$ words $\omega \in \mathfrak{B}_{n,k}$ yield the same ω'. □

4.12 Forbidden values

In the remainder of Section 4, if α is a sequence avoiding a set of patterns P, we say that a value $v \in \{0, \ldots, \mathbf{max}(\alpha)\}$ is *forbidden* by α and P if $\alpha \cdot mv$ contains a pattern in P when $m > \mathbf{max}(\alpha)$. We denote by $\mathbf{Forb}(\alpha, P)$ the set of values forbidden by α and P, or simply $\mathbf{Forb}(\alpha)$ when there is no ambiguity.

In our decomposition around the first maximum of an inversion sequence $\sigma = \alpha \cdot m \cdot \gamma$, if γ contains a value in $\mathbf{Forb}(\alpha, \tau)$, then σ contains τ. This means that in order for σ to avoid τ, γ must be a word over the alphabet $[0, m] \backslash \mathbf{Forb}(\alpha)$. This is in fact a weaker version of condition 3 from Remark 33, since it only ensures that σ does not contain any occurrence of τ whose last entry only is in γ.

Note that Propositions 34, 35 can be expressed in terms of forbidden values:

$$\mathbf{Forb}(\alpha, 120) = [0, \mathbf{max}(\alpha) - 1], \qquad \mathbf{Forb}(\alpha, 010) = \mathbf{Vals}(\alpha),$$

and these identities hold for any integer sequence α (not only for inversion sequences). In particular, if α is an inversion sequence such that $\mathbf{max}(\alpha) > 0$, then $|\mathbf{Forb}(\alpha, 010)| \geqslant 2$.

Earlier, we refined the enumeration of pattern-avoiding inversion sequences σ according to one or two parameters (in addition to their size). The first one was the maximum m of σ, and is required for our decomposition around the first maximum (so that the maximum of the left part α is less than m). For every pair of pattern which contained 120, the number of forbidden values was redundant with the maximum of σ, and therefore unnecessary. In the previous two cases (the patterns 010 and $\{000, 010\}$), the second parameter counting the number of distinct values d of σ was in fact the number of forbidden values $|\mathbf{Forb}(\sigma, 010)|$.

For each pair of patterns P which follows, there is no simple equivalent description of $|\mathbf{Forb}(\sigma, P)|$, and introducing this parameter allows us to solve the enumeration of P-avoiding inversion sequences.

4.13 The pairs {010, 201} and {010, 210}

A bijection between $\mathcal{I}(010, 201)$ and $\mathcal{I}(010, 210)$ was established in [35]. In this section we work with the pair of patterns $\{010, 210\}$, although our construction can also be applied to inversion sequences avoiding the pair $\{010, 201\}$, resulting in the same equations.

Remark 60. Let σ be a $\{010, 210\}$-avoiding integer sequence. Let q be the largest value of σ such that a larger value appears to its left, or $q = 0$ if there is no such value (i.e. if σ is nondecreasing). Then we have $\mathbf{Forb}(\sigma, 210) = [0, q-1]$. Recalling also that $\mathbf{Forb}(\sigma, 010) = \mathbf{Vals}(\sigma)$, we find $\mathbf{Forb}(\sigma, \{010, 210\}) = [0, q-1] \sqcup \{i \in \mathbf{Vals}(\sigma) \ : \ i \geqslant q\}$.

Let $\mathfrak{A}_{n,m,f} = \{\sigma \in \mathcal{I}_n(010, 210) \ : \ m = \mathbf{max}(\sigma),\ f = |\mathbf{Forb}(\sigma)|\}$ be the set of $\{010, 210\}$-avoiding inversion sequences of size n, maximum m, and having f forbidden values. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k+1}(010) \ : \ \omega_i < \omega_j < k \implies i < j \ \text{ and } \ k-1 \in \mathbf{Vals}(\omega)\}$ if $k > 0$, or $\mathfrak{B}_{n,k} = \{(0)^n\}$ if $k = 0$, be the set of 010-avoiding words ω of length n over the alphabet $[0, k]$ such that the subword ω' obtained by removing all letters k from ω is nondecreasing, and $k-1$ is the maximum of ω'. Let $\mathfrak{a}_{n,m,f} = |\mathfrak{A}_{n,m,f}|$, and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$.

Theorem 61. *For all* $2 \leqslant f \leqslant m+1 \leqslant n$*,*

$$\mathfrak{a}_{n,m,f} = \sum_{p=m+1}^{n} \sum_{i=0}^{f-1} \mathfrak{b}_{n-p,f-i-1} \sum_{j=0}^{m-1} \mathfrak{a}_{p-1,j,i}.$$

Proof. Let $\sigma \in \mathfrak{A}_{n,m,f}$, $p = \mathbf{firstmax}(\sigma)$, $\alpha = (\sigma_i)_{i \in [1,p-1]}$, and $\gamma = (\sigma_i)_{i \in [p+1,n]}$. Let γ' be the subsequence of γ obtained by removing all values m from γ. Since σ avoids the pattern 010, α and γ avoid 010, and $\mathbf{Vals}(\alpha) \cap \mathbf{Vals}(\gamma) = \emptyset$ by Proposition 35. Since σ avoids the pattern 210, α avoids 210, and γ' is nondecreasing.

The subsequence α is in $\mathfrak{A}_{p-1,j,i}$ for some $j < m$ and $i \leqslant f$. Notice that we actually have $i < f$ since m is a forbidden value for σ (because of the avoidance of 010), but not for α (since $m > \mathbf{max}(\alpha)$). The subsequence γ is a 010-avoiding word of length $n-p$ over the alphabet $\Sigma = [0, m] \backslash \mathbf{Forb}(\alpha)$ of size $m+1-i$, and such that γ' is nondecreasing.

The maximum of γ' is the largest letter of $m \cdot \gamma$ such that a larger letter appears to its left, and m is the only value in $m \cdot \gamma$ which is greater than $\mathbf{max}(\gamma')$. Hence, by Remark 60, $\mathbf{Forb}(\sigma) = \mathbf{Forb}(\alpha) \sqcup \{\ell \in \Sigma \ : \ \ell \leqslant \mathbf{max}(\gamma')\} \sqcup \{m\}$ (this still holds if γ' is empty, i.e. $\mathbf{max}(\gamma') = -1$). Since $|\{\ell \in \Sigma \ : \ \ell \leqslant \mathbf{max}(\gamma')\}| = |\mathbf{Forb}(\sigma)| - |\mathbf{Forb}(\alpha)| - |\{m\}| = f - i - 1$, replacing the letters $\{\ell \in \Sigma \ : \ \ell \leqslant \mathbf{max}(\gamma')\}$ by $[0, f-i-2]$ and the letter m by $f-i-1$ yields a bijection between the words γ and the words of $\mathfrak{B}_{n-p,f-i-1}$. $\square$

Let $\mathfrak{C}_{n,k}$ be the set of 010-avoiding words ω of length n over the alphabet $[0, k-1] \sqcup \{\infty\}$ (where ∞ is the largest letter) such that the subword ω' defined by removing all letters ∞ from ω is nondecreasing, and $k-1$ is the maximum of ω'. Clearly, taking any word $\omega \in \mathfrak{C}_{n,k}$ and replacing each letter ∞ in ω by the letter k yields a word in $\mathfrak{B}_{n,k}$, and this is a bijection. It is more convenient for the proof of the following lemma to count the words of $\mathfrak{C}_{n,k}$ rather than those of $\mathfrak{B}_{n,k}$.

Let $\mathfrak{C}^{(1)}_{n,k} = \{\omega \in \mathfrak{C}_{n,k} \mid \omega_n = \infty\}$, and $\mathfrak{C}^{(2)}_{n,k} = \{\omega \in \mathfrak{C}_{n,k} \mid \omega_n \neq \infty\}$, so that $\mathfrak{C}_{n,k} = \mathfrak{C}^{(1)}_{n,k} \sqcup \mathfrak{C}^{(2)}_{n,k}$. Let $\mathfrak{c}^{(1)}_{n,k} = |\mathfrak{C}^{(1)}_{n,k}|$, and $\mathfrak{c}^{(2)}_{n,k} = |\mathfrak{C}^{(2)}_{n,k}|$. In particular, $\mathfrak{b}_{n,k} = \mathfrak{c}^{(1)}_{n,k} + \mathfrak{c}^{(2)}_{n,k}$.

Lemma 62. *For all $n \geqslant 2, k \geqslant 1$,*

$$\mathfrak{b}_{n,k} = \mathfrak{b}_{n,k-1} + 2\mathfrak{b}_{n-1,k} - \mathfrak{b}_{n-2,k} - \mathfrak{b}_{n-1,k-1} + \mathfrak{b}_{n-2,k-1}.$$

Proof. Let $n \geqslant 1, k \geqslant 0$, and $\omega \in \mathfrak{C}_{n,k}$.

- If $\omega \in \mathfrak{C}^{(1)}_{n,k}$, then $\omega_n = \infty$. Removing ω_n yields a word in $\mathfrak{C}_{n-1,k}$.
- If $\omega \in \mathfrak{C}^{(2)}_{n,k}$, then by the nondecreasing property, $\omega_n = k - 1$.
  - If $\omega_{n-1} = k - 1$, then removing ω_n yields a word in $\mathfrak{C}^{(2)}_{n-1,k}$.
  - If $\omega_{n-1} = \infty$, then the avoidance of the pattern 010 ensures ω cannot contain another letter $k - 1$, therefore removing ω_n yields a word in $\mathfrak{C}^{(1)}_{n-1,i}$ for some $i < k$.
  - Otherwise, $\omega_{n-1} = i$ for some $i < k$, and removing ω_n yields a word in $\mathfrak{C}^{(2)}_{n-1,i}$.

The maps described above are all bijections, hence for all $n \geqslant 1, k \geqslant 0$,

$$\mathfrak{c}^{(1)}_{n,k} = \mathfrak{b}_{n-1,k},$$

and for all $n \geqslant 2, k \geqslant 0$,

$$\begin{aligned}
\mathfrak{c}^{(2)}_{n,k} &= \mathfrak{c}^{(2)}_{n-1,k} + \sum_{i=0}^{k-1} \mathfrak{c}^{(1)}_{n-1,i} + \mathfrak{c}^{(2)}_{n-1,i} \\
&= \mathfrak{b}_{n-1,k} - \mathfrak{c}^{(1)}_{n-1,k} + \sum_{i=0}^{k-1} \mathfrak{b}_{n-1,i} \\
&= \mathfrak{b}_{n-1,k} - \mathfrak{b}_{n-2,k} + \sum_{i=0}^{k-1} \mathfrak{b}_{n-1,i}.
\end{aligned}$$

By summing $\mathfrak{c}^{(1)}_{n,k}$ and $\mathfrak{c}^{(2)}_{n,k}$, we have for all $n \geqslant 2, k \geqslant 0$,

$$\mathfrak{b}_{n,k} = 2\mathfrak{b}_{n-1,k} - \mathfrak{b}_{n-2,k} + \sum_{i=0}^{k-1} \mathfrak{b}_{n-1,i}.$$

We conclude by telescoping the sum over i. For all $n \geqslant 2, k \geqslant 1$,

$$\begin{aligned}
\mathfrak{b}_{n,k} - \mathfrak{b}_{n,k-1} &= 2\mathfrak{b}_{n-1,k} - \mathfrak{b}_{n-2,k} + \sum_{i=0}^{k-1} \mathfrak{b}_{n-1,i} - (2\mathfrak{b}_{n-1,k-1} - \mathfrak{b}_{n-2,k-1} + \sum_{i=0}^{k-2} \mathfrak{b}_{n-1,i}) \\
&= 2\mathfrak{b}_{n-1,k} - \mathfrak{b}_{n-2,k} - \mathfrak{b}_{n-1,k-1} + \mathfrak{b}_{n-2,k-1}.
\end{aligned}$$

□

4.14 The pair $\{010, 110\}$

Remark 63. Let σ be a $\{010,110\}$-avoiding integer sequence. Let q be the largest repeated value in σ, or $q=0$ if there is no such value. Then $\mathbf{Forb}(\sigma,110)=[0,q-1]$. Recalling that $\mathbf{Forb}(\sigma,010)=\mathbf{Vals}(\sigma)$, we find $\mathbf{Forb}(\sigma,\{010,110\})=[0,q-1]\sqcup\{i\in\mathbf{Vals}(\sigma)\ :\ i\geqslant q\}$.

Let $\mathfrak{A}_{n,m,f}=\{\sigma\in\mathcal{I}_n(010,110)\ :\ m=\mathbf{max}(\sigma),\ f=|\mathbf{Forb}(\sigma)|\}$ be the set of $\{010,110\}$-avoiding inversion sequences of size n, maximum m, and having f forbidden values. Let $\mathfrak{B}_{n,k,f}=\{\omega\in\mathcal{W}_{n,k}(010,110)\ :\ f=|\mathbf{Forb}(\omega)|\}$ be the set of $\{010,110\}$-avoiding words of length n over the alphabet $[0,k-1]$ having f forbidden values. Let $\mathfrak{C}_{n,k}=\coprod_{\ell\in[0,n]}\mathcal{W}_{\ell,k}(010,110)$ be the set of $\{010,110\}$-avoiding words of length at most n over the alphabet $[0,k-1]$. Let $\mathfrak{a}_{n,m,f}=|\mathfrak{A}_{n,m,f}|$, $\mathfrak{b}_{n,k,f}=|\mathfrak{B}_{n,k,f}|$, and $\mathfrak{c}_{n,k}=|\mathfrak{C}_{n,k}|$.

Theorem 64. *For all $2\leqslant f\leqslant m+1\leqslant n$,*

$$\mathfrak{a}_{n,m,f}=\sum_{p=m+1}^{n}\sum_{i=0}^{f-1}\sum_{j=0}^{m-1}\mathfrak{a}_{p-1,j,i}\cdot\left(\mathfrak{b}_{n-p,m-i,f-i-1}+\delta_{f,m+1}\cdot\mathfrak{c}_{n-p-1,m-i}\right).$$

Proof. Let $\sigma\in\mathfrak{A}_{n,m,f}$, $p=\mathbf{firstmax}(\sigma)$, $\alpha=(\sigma_i)_{i\in[1,p-1]}$, and $\gamma=(\sigma_i)_{i\in[p+1,n]}$. Since σ avoids the pattern 010, α and γ avoid 010, and $\mathbf{Vals}(\alpha)\cap\mathbf{Vals}(\gamma)=\emptyset$ by Proposition 35. Since σ avoids the pattern 110, α and γ avoid 110, and all letters of γ are greater than any repeated letter of α.

The subsequence α is in $\mathfrak{A}_{p-1,j,i}$ for some $j<m$ and $i<f$ (for the same reason as in the proof of Theorem 61). The subsequence γ is a $\{010,110\}$-avoiding word of length $n-p$ over the alphabet $\Sigma=[0,m]\backslash\mathbf{Forb}(\alpha)$ of size $m+1-i$. Additionally, if a letter m appears in γ, then all letters to its right are also m; otherwise σ would contain the pattern 110. We distinguish two cases for γ.

- If γ does not contain the letter m, then γ is a $\{010,110\}$-avoiding word of length $n-p$ over an alphabet of size $m-i$. Let q be the largest repeated letter in γ, or $q=0$ if there is no such letter. From Remark 63, we observe that the forbidden values of σ are $\mathbf{Forb}(\sigma)=\mathbf{Forb}(\alpha)\sqcup\{\ell\in\Sigma\ :\ \ell<q\}\sqcup\{\ell\in\mathbf{Vals}(\gamma)\ :\ \ell\geqslant q\}\sqcup\{m\}$. In particular, $|\{\ell\in\Sigma\ :\ \ell<q\}\sqcup\{\ell\in\mathbf{Vals}(\gamma)\ :\ \ell\geqslant q\}|=f-i-1$. Replacing the letters of Σ by $[0,m-i]$ (while preserving their order) yields a bijection between the words γ and the words of $\mathfrak{B}_{n-p,m-i,f-i-1}$.
- If γ contains the letter m, then γ can be written as $\gamma'\cdot(m)^k$, where γ' does not contain the letter m, and $k>0$. The number of such words γ' is $\mathfrak{c}_{n-p-1,m-i}$. In this case, m is the largest repeated letter in σ as well as the maximum of σ. This occurs if and only if $f=m+1$ (i.e. all values $[0,m]$ are forbidden by σ). ☐

By definition,

$$\mathfrak{C}_{n,k}=\coprod_{\ell=0}^{n}\coprod_{f=0}^{k}\mathfrak{B}_{\ell,k,f},\qquad\text{hence}\qquad\mathfrak{c}_{n,k}=\sum_{\ell=0}^{n}\sum_{f=0}^{k}\mathfrak{b}_{\ell,k,f}.$$

We count the words of $\mathfrak{B}_{n,k,f}$ in Lemma 66. In order to do so, we first construct the words of $\overline{\mathcal{W}}_{n,k}(010, 110)$ by inserting their letters in increasing order from 0 to $k-1$ (all such letters necessarily appearing), and inserting repeats of a letter from left to right. For all $\omega \in \overline{\mathcal{W}}_{n,k}(010, 110)$, let $\mathbf{Sites}(\omega) = \{i \in [1, n+1] \ : \ \mathbf{Vals}((\omega_j)_{j<i}) \cap \mathbf{Vals}((\omega_j)_{j\geqslant i}) = \emptyset\}$ be the set of positions (called *active sites*) where the letter k may be inserted in ω without creating an occurrence of the pattern 010. In particular, 1 and $n+1$ are always active sites of ω. Note also that inserting the letter k in ω cannot create an occurrence of the pattern 110: the letter k cannot take the role of the letter 0 in the pattern 110 since it is greater than all letters of ω, and it cannot take the role of the letter 1 since it only appears once.

Let $\mathfrak{D}_{n,k,s} = \{\omega \in \overline{\mathcal{W}}_{n,k}(010, 110) \ : \ s = |\mathbf{Sites}(\omega)|\}$ be the set of $\{010, 110\}$-avoiding words of length n containing all letters of the alphabet $[0, k-1]$ and having s active sites. Let $\mathfrak{D}^{(1)}_{n,k,s}$ be the subset of $\mathfrak{D}_{n,k,s}$ of words containing exactly one letter $k-1$, and $\mathfrak{D}^{(2)}_{n,k,s}$ be the subset of remaining words (i.e. words containing at least two letters $k-1$ if $k \geqslant 1$, or the empty word if $(n, k, s) = (0, 0, 1)$), so that $\mathfrak{D}_{n,k,s} = \mathfrak{D}^{(1)}_{n,k,s} \sqcup \mathfrak{D}^{(2)}_{n,k,s}$. Let $\mathfrak{d}_{n,k,s} = |\mathfrak{D}_{n,k,s}|$, $\mathfrak{d}^{(1)}_{n,k,s} = |\mathfrak{D}^{(1)}_{n,k,s}|$, and $\mathfrak{d}^{(2)}_{n,k,s} = |\mathfrak{D}^{(2)}_{n,k,s}|$.

Lemma 65. *For all $n, k, s \geqslant 1$,*

$$\mathfrak{d}^{(1)}_{n,k,s} = (s-1)\mathfrak{d}_{n-1,k-1,s-1}.$$

For all $n, s \geqslant 2, k \geqslant 1$,

$$\mathfrak{d}^{(2)}_{n,k,s} = \mathfrak{d}^{(2)}_{n,k,s+1} + \mathfrak{d}^{(2)}_{n-1,k,s} - \mathfrak{d}^{(2)}_{n-1,k,s+1} + \mathfrak{d}_{n-2,k-1,s-1}.$$

Proof. Let $n, k, s \geqslant 0$, $\omega \in \mathfrak{D}_{n,k,s}$, and let $(p_1, \ldots, p_s)$ be the values of $\mathbf{Sites}(\omega)$ in increasing order (in particular, $p_1 = 1$ and $p_s = n+1$). We consider three different ways in which the word ω can "grow".

1. Inserting a letter k. Let $i \in [1, s]$, and let ω' be the word obtained by inserting the letter k at position p_i in ω. Then $\mathbf{Sites}(\omega') = \{p_j \ : \ j \in [1, i]\} \sqcup \{p_j + 1 \ : \ j \in [i, s]\}$, so $\omega' \in \mathfrak{D}^{(1)}_{n+1,k+1,s+1}$.

2. Inserting two occurrences of the letter k. This is similar to the previous case, since the rightmost letter k can only be inserted at the end of the word in order to avoid the pattern 110. Let $i \in [1, s]$, and let ω' be the word obtained by inserting one letter k at position p_i in ω, and one letter k at the end of the resulting word (at position $n+2$). Then $\mathbf{Sites}(\omega') = \{p_j \ : \ j \in [1, i]\} \sqcup \{n+3\}$, so $\omega' \in \mathfrak{D}^{(2)}_{n+2,k+1,i+1}$.

3. Inserting a repeat of letter $k-1$, to the right of the rightmost letter $k-1$, and only if ω already contains at least two occurrences of the letter $k-1$ (i.e. $\omega \in \mathfrak{D}^{(2)}_{n,k,s}$ and $k > 0$). In that case, every letter $k-1$ in ω except (possibly) the leftmost $k-1$ must be in a single factor at the end of ω, in order to avoid 110. Let ω' be the word obtained by inserting the letter $k-1$ at position $n+1$ in ω. Then $\mathbf{Sites}(\omega') = \big(\mathbf{Sites}(\omega) \backslash \{n+1\}\big) \sqcup \{n+2\}$, so $\omega' \in \mathfrak{D}^{(2)}_{n+1,k,s}$.

It can be seen that any word in a set $\mathfrak{D}_{n,k,s}$ for some $n,k,s \geqslant 0$ can be obtained in exactly one way from the construction described above, starting from the empty word ε. In other words, we described a combinatorial generating tree for the class $\coprod_{n,k\geqslant 0} \overline{\mathcal{W}}_{n,k}(010,110)$.

From item 1, we have for all $n,k,s \geqslant 1$,

$$\mathfrak{d}^{(1)}_{n,k,s} = (s-1)\mathfrak{d}_{n-1,k-1,s-1}.$$

From items 2 and 3, we have for all $n,s \geqslant 2, k \geqslant 1$,

$$\mathfrak{d}^{(2)}_{n,k,s} = \mathfrak{d}^{(2)}_{n-1,k,s} + \sum_{j=s-1}^{n-1} \mathfrak{d}_{n-2,k-1,j},$$

which may be rewritten

$$\mathfrak{d}^{(2)}_{n,k,s} - \mathfrak{d}^{(2)}_{n,k,s+1} = \mathfrak{d}^{(2)}_{n-1,k,s} - \mathfrak{d}^{(2)}_{n-1,k,s+1} + \mathfrak{d}_{n-2,k-1,s-1}. \qquad \square$$

For any integer sequence σ, let $\mathbf{rep}(\sigma) = \mathbf{max}(v \in \mathbf{Vals}(\sigma) \ : \ \exists i \neq j,\ \sigma_i = \sigma_j = v)$ be the largest repeated value in σ, with the convention $\mathbf{rep}(\sigma) = -1$ if σ does not contain any repeated value. Let $\mathbf{top}(\sigma) = |\{i \in \mathbf{Vals}(\sigma) \ : \ i > \mathbf{rep}(\sigma)\}|$ be the number of values of σ which are greater than its largest repeated value (if no value is repeated, then $\mathbf{top}(\sigma)$ is the number of distinct values of σ, or equivalently the size of σ).

We call *unused letters* of a word $\omega \in \mathcal{W}_{n,k}$ the letters in the set $[0,k-1]\backslash\mathbf{Vals}(\omega)$. Note this definition relies not only on ω, but also on the alphabet considered (e.g. each word of $\mathcal{W}_{n,k}$ is also in $\mathcal{W}_{n,k+1}$, but has different unused letters).

Lemma 66. *For all $n,k,f \geqslant 0$,*

$$\mathfrak{b}_{n,k,f} = \sum_{t=0}^{f} \binom{t+k-f}{k-f} \sum_{b=0}^{f-t} \binom{f-t-1}{b} \sum_{s=1}^{n-t+1} \frac{(s+t-1)!}{(s-1)!} \mathfrak{d}^{(2)}_{n-t,f-b-t,s}.$$

Proof. Let $\omega \in \mathfrak{B}_{n,k,f}$. In particular, by Remark 63, $f = \mathbf{rep}(\omega) + \mathbf{top}(\omega) + 1$, and $k-f$ is the number of unused letters of ω greater than $\mathbf{rep}(\omega)$. Let $t = \mathbf{top}(\omega)$, $d = \mathbf{dist}(\omega)$, and $b = f-d$ be the number of unused letters in ω less than $\mathbf{rep}(\omega)$. To summarize, over the alphabet $[0,k-1]$, ω has:

- t letters greater than $\mathbf{rep}(\omega)$,
- $k-f$ unused letters greater than $\mathbf{rep}(\omega)$,
- b unused letters less than $\mathbf{rep}(\omega)$,
- $\mathbf{rep}(\omega) - b = f-t-1-b$ letters less than $\mathbf{rep}(\omega)$, if $\omega \neq \varepsilon$.

Replacing the d letters of $\mathbf{Vals}(\omega)$ by $[0,d-1]$ while preserving their order (by shifting the values so that there are no more unused letters) yields a word $\omega' \in \overline{\mathcal{W}}_{n,d}$. Further removing all letters greater than $\mathbf{rep}(\omega')$ from ω' yields a word $\omega'' \in \mathfrak{D}^{(2)}_{n-t,f-b-t,s}$ for some $s \in [1, n-t+1]$.

For any n,k,f,t,b,s, and $\omega'' \in \mathfrak{D}^{(2)}_{n-t,f-b-t,s}$, there are $\binom{t+k-f}{k-f}\binom{f-t-1}{b}\frac{(s+t-1)!}{(s-1)!}$ words $\omega \in \mathfrak{B}_{n,k,f}$ whose image under the above construction is ω''. Indeed, there are

- $\binom{t+k-f}{k-f}$ possible sets of unused letters greater than $\mathbf{rep}(\omega)$,
- $\binom{f-t-1}{b}$ possible sets of unused letters less than $\mathbf{rep}(\omega)$ (this still holds if $\omega = \varepsilon$),
- $\frac{(s+t-1)!}{(s-1)!}$ possible placements for the t letters greater than $\mathbf{rep}(\omega)$. This can be seen by inserting each of those t letters in increasing order: inserting a letter greater than the maximum in a word corresponds to item 1 in the proof of Lemma 65, so there are s possible positions for the smallest inserted letter, $s+1$ for the next letter, and so on. □

5 Shifted inversion sequences

5.1 Method

For all $n, s \in \mathbb{N}$, let $\mathcal{I}_n^s = \{\sigma \in \mathbb{N}^n \ : \ \sigma_i < i + s \quad \forall i \in [1, n]\}$ be the set of s-*shifted* inversion sequences of size n. In particular, $\mathcal{I}_n^s \subseteq \mathcal{I}_n^{s+1}$, and $\mathcal{I}_n^0 = \mathcal{I}_n$. An s-shifted inversion sequence of size n can be seen as an inversion sequence of size $n+s$ whose first s entries were removed. More precisely, for all $n, s \geqslant 0$, $\mathcal{I}_{n+s} = \{\sigma \cdot \tau \ : \ (\sigma, \tau) \in \mathcal{I}_s \times \mathcal{I}_n^s\}$. For any set of patterns P, we denote by $\mathcal{I}_n^s(P)$ the set of P-avoiding s-shifted inversion sequences of size n.

In this section, we study some patterns for which it is easier to split sequences around their minimum. We use a decomposition of sequences around their first minimum, similar to that of Section 4, although a shifted inversion sequence now naturally appears on the right side when this decomposition is applied to an inversion sequence.

In Section 4, the right side of the decomposition was a word, since the value of the maximum was fixed. A similar event occurs in the upcoming cases, when decomposing sequences around their first minimum: the sequences of Sections 5.2 and 5.3 avoid the pattern 102, so when the left side of the decomposition is nonempty (i.e. the first value of the sequence is not the minimum), all values of the right side must be less than or equal to all values of the left side, hence the right side is a word on a fixed alphabet, once again. It follows that shifted inversion sequences only appear on the right side of this decomposition when the left side is empty. Naturally, the left side of the decomposition around the first minimum of an inversion sequence is always empty, since the leftmost value of a nonempty inversion sequence is always 0. It follows that words only appear when this decomposition is recursively applied to a shifted inversion sequence.

For instance, applying this decomposition to $(0, 0, 0, 0, 4, 4, 5, 0, 3, 4, 2, 3, 0) \in \mathcal{I}_{13}(102)$ will first remove each leading zero, yielding the sequence $(4, 4, 5, 0, 3, 4, 2, 3, 0) \in \mathcal{I}_9^4(102)$, then split it into a shifted inversion sequence $(4, 4, 5)$ and a word $(3, 4, 2, 3, 0)$.

5.2 The pair {010, 102}

For the generating function of $\mathcal{I}(010, 102)$, see [30].

Let $\mathfrak{A}_{n,s} = \mathcal{I}_n^s(010, 102)$ be the set of $\{010, 102\}$-avoiding s-shifted inversion sequences of size n. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(010, 102) \ : \ \mathbf{max}(\omega) = k - 1\}$ be the set of $\{010, 102\}$-

avoiding words of length n over the alphabet $[0, k-1]$ which contain the letter $k-1$. Let $\mathfrak{a}_{n,s} = |\mathfrak{A}_{n,s}|$ and $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$. In particular, $|\mathcal{I}_n(010, 102)| = \mathfrak{a}_{n,0}$.

Theorem 67. *For all $n \geqslant 1, s \geqslant 0$,*

$$\mathfrak{a}_{n,s} = (\sum_{z=0}^{n} \mathfrak{a}_{n-z,s+z-1}) + (\sum_{z=1}^{n-1} \mathfrak{a}_{n-z,s-1}) + \sum_{r=1}^{n-2} \sum_{m=1}^{s} \mathfrak{b}_{r,m} \sum_{\ell=0}^{n-r-1} (n - r - \ell - \delta_{\ell,0}) \mathfrak{a}_{\ell,s-m-1}.$$

Proof. Let $n \geqslant 1, s \geqslant 0$, and $\sigma \in \mathfrak{A}_{n,s}$. Since σ avoids the pattern 010, all occurrences of the value 0 in σ must be consecutive. Let $z = |\{i \in [1, n] \ : \ \sigma_i = 0\}|$ be the number of occurrences of the value 0 in σ. Let σ' be the sequence obtained by removing every occurrence of the value 0 from σ, and subtracting 1 from all remaining values.

- If $z = 0$ or $\sigma_1 = 0$, then $\sigma' \in \mathfrak{A}_{n-z,s+z-1}$. This is a bijection since $\sigma = (0)^z \cdot (\sigma' + 1)$.
- If $\sigma_1 \neq 0$ and $\sigma_n = 0$, then $\sigma' \in \mathfrak{A}_{n-z,s-1}$. This is a bijection since $\sigma = (\sigma' + 1) \cdot (0)^z$.
- If $z > 0$, $\sigma_0 \neq 0$, and $\sigma_n \neq 0$, let α and β be the (uniquely defined) integer sequences such that $\sigma = \alpha \cdot (0)^z \cdot \beta$. In particular, α and β are both nonempty, and their values are positive. Let r be the size of β, m be the maximum of β, and $\beta' \in \mathfrak{B}_{r,m}$ be the word obtained by subtracting 1 from each letter of β.

  Since σ avoids 102, every value of α is greater than or equal to m (in particular, $m \leqslant \alpha_1 \leqslant s$). All occurrences of the value m in α must be in a single factor at the end of α; otherwise, a value greater than m would appear to the right of a value m in α, creating an occurrence of the pattern 010 in σ (since β also contains m). Let q be the number of occurrences of the value m in α, and let $\ell = |\alpha| - q$ be the number of entries of α of value greater than m. Let $\alpha' \in \mathfrak{A}_{\ell,s-m-1}$ be the sequence obtained by removing every occurrence of the value m from α and subtracting $m+1$ from the remaining values.

  Note that any choice of q and z such that $q + z = n - \ell - r$ does not affect α' and β'. We have $q \in [0, n - r - \ell - 1]$ if α' is nonempty, and $q \in [1, n - r - \ell - 1]$ if α' is empty (since α is not empty), so there are $n - r - \ell - \delta_{\ell,0}$ possible values for q. For each possible value of q, there is one value of z such that $q + z = n - \ell - r$.

  This decomposition is bijective, since for any choice of

  - $m \in [1, s]$,
  - $\ell, q, z, r \in \mathbb{N}$ such that $\ell + q, z, r \geqslant 1$ and $\ell + q + z + r = n$,
  - $\alpha' \in \mathfrak{A}_{\ell,s-m-1}$,
  - $\beta' \in \mathfrak{B}_{r,m}$,

  we have $(\alpha' + m + 1) \cdot (m)^q \cdot (0)^z \cdot (\beta' + 1) \in \mathfrak{A}_{n,s}$. Specifically, this construction cannot create any occurrence of a pattern 010 or 102 since all values of $\alpha' + m + 1$ are greater than all values of $\beta' + 1$. □

Let $F : (x,y) \mapsto \sum_{\ell,k\geqslant 0} |\mathcal{W}_{\ell,k}(010,102)|x^\ell y^k$ be the ordinary generating function of $\{010,102\}$-avoiding words. The enumeration of $\{010,102\}$-avoiding words was solved in [18], with the following expression:

$$F(x,y) = \frac{(1-x)^2 - (1-2x)y - \sqrt{(1-x)^4 - 2(1-x)^2 y + (1-4x^2+4x^3)y^2}}{2xy(1-y)}.$$

We easily observe that $\mathfrak{B}_{n,k} = \mathcal{W}_{n,k}(010,102) \backslash \mathcal{W}_{n,k-1}(010,102)$, so the generating function of $\coprod_{n,k\in\mathbb{N}} \mathfrak{B}_{n,k}$ is $(x,y) \mapsto (1-y)F(x,y)$. We can then extract the numbers $\mathfrak{b}_{n,k}$ from this generating function, as in Section 4.8. The function F is actually the same as in Section 4.8, since $\{010,102\}$-avoiding words and $\{101,120\}$-avoiding words are in bijection via the complement map $\mathcal{W}_{n,k}(010,102) \to \mathcal{W}_{n,k}(101,120)$, $\omega \mapsto (k-1-\omega_i)_{i\in[1,n]}$.

5.3 The pair {100, 102}

Let $\mathfrak{A}_{n,s} = \mathcal{I}_n^s(100,102)$ be the set of $\{100,102\}$-avoiding s-shifted inversion sequences of size n. Let $\mathfrak{A}'_{n,s} = \{\sigma \in \mathfrak{A}_{n,s} \ : \ 0 \in \mathbf{Vals}(\sigma) \text{ and } \sigma_1 \neq 0\}$ be the subset of sequences which contain a 0 but do not begin by 0. Let $\mathfrak{B}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(102) \ : \ \omega_i = \omega_j \implies i = j\}$ be the set of 102-avoiding words of length n over the alphabet $[0,k-1]$ which do not contain any repeated letter. Let $\mathfrak{C}_{n,k} = \{\omega \in \mathcal{W}_{n,k}(102) \ : \ \omega_i = \omega_j \neq k-1 \implies i = j\}$ be the set of 102-avoiding words of length n over the alphabet $[0,k-1]$ which do not contain any non-maximal repeated letter. Let $\mathfrak{a}_{n,s} = |\mathfrak{A}_{n,s}|$, $\mathfrak{a}'_{n,s} = |\mathfrak{A}'_{n,s}|$, $\mathfrak{b}_{n,k} = |\mathfrak{B}_{n,k}|$, and $\mathfrak{c}_{n,k} = |\mathfrak{C}_{n,k}|$. In particular, $\mathcal{I}_n(100,102) = \mathfrak{a}_{n,0}$.

Theorem 68. *For all $n \geqslant 1, s \geqslant 0$,*

$$\mathfrak{a}_{n,s} = \mathfrak{a}_{n,s-1} + \mathfrak{a}_{n-1,s+1} + \mathfrak{a}'_{n,s},$$

$$\mathfrak{a}'_{n,s} = \sum_{p=2}^{n} \sum_{m=1}^{s} \left(\mathfrak{c}_{n-p,m} + \sum_{r=1}^{p-2} \mathfrak{a}_{p-1-r,s+r-m-1} \cdot \mathfrak{b}_{n-p,m} + \sum_{r=0}^{p-2} \mathfrak{a}'_{p-1-r,s+r-m} \cdot \mathfrak{b}_{n-p,m-1} \right).$$

Proof. Let $n \geqslant 1, s \geqslant 0$, and $\sigma \in \mathfrak{A}_{n,s}$.

- If σ does not contain the value 0, then subtracting 1 from every value of σ yields a sequence in $\mathfrak{A}_{n,s-1}$, and this is a bijection.
- If $\sigma_1 = 0$, then removing the first term from σ yields a sequence in $\mathfrak{A}_{n-1,s+1}$, and this is a bijection.
- Otherwise, $\sigma \in \mathfrak{A}'_{n,s}$.

This proves the first equality. Let us now turn to the second one.

Let $n \geqslant 1, s \geqslant 0$, and $\sigma \in \mathfrak{A}'_{n,s}$. Since σ avoids the pattern 100 and $\sigma_1 > 0$, there is exactly one term of value 0 in σ. Let p be the position of the only value 0 in σ. Let $\alpha = (\sigma_i)_{i\in[1,p-1]}$ and $\beta = (\sigma_i)_{i\in[p+1,n]}$ be the two subsequences such that $\sigma = \alpha \cdot 0 \cdot \beta$. In particular, all values of α and β are positive. By construction, α is nonempty, but β may be empty. Let m be the minimum of α. Since σ avoids the pattern 102, all values of β must be less than or equal to m, so β is a word of length $n-p$ over the alphabet $[1, m]$. Since σ avoids the pattern 100, each value in $[1, m-1]$ appears at most once in β. Let β' be the word obtained by subtracting 1 from every letter of β.

- If $\alpha = (m)^{p-1}$ is constant, then β' can be any word of $\mathfrak{C}_{n-p,m}$.
- Otherwise, α contains at least one value greater than m. Since σ avoids the pattern 100, the letter m can appear at most once in β. In other words, all letters of β must be distinct, so $\beta' \in \mathfrak{B}_{n-p,m}$. Let $r \in [0, p-2]$ be the number of consecutive occurrences of the value m at the beginning of α.
  - If all values m in α appear in the factor $(m)^r$ at the beginning of α, let $\alpha' \in \mathfrak{A}_{p-1-r,s+r-m-1}$ be the sequence obtained by removing this factor from α, and subtracting $m+1$ from all remaining values. Recall α must contain the value m at least once, so $r > 0$. This defines a bijection, since for any $\alpha' \in \mathfrak{A}_{p-1-r,s+r-m-1}$ and $\beta' \in \mathfrak{B}_{n-p,m}$, the sequence $(m)^r \cdot (\alpha'+m+1) \cdot 0 \cdot (\beta'+1)$ is in $\mathfrak{A}'_{n,s}$.
  - Otherwise, there is exactly one value m in α outside of the factor $(m)^r$ at the beginning of α, since having a second value m outside of this factor would imply that the subsequence (α_{r+1}, m, m) is an occurrence of the pattern 100. The same reasoning shows that β cannot contain the letter m, so $\beta' \in \mathfrak{B}_{n-p,m-1}$. Let $\alpha' \in \mathfrak{A}'_{p-1-r,s+r-m}$ be the sequence obtained by removing the factor $(m)^r$ at the beginning of α, and subtracting m from all remaining values. This defines a bijection, since for any $\alpha' \in \mathfrak{A}'_{p-1-r,s+r-m}$ and $\beta' \in \mathfrak{B}_{n-p,m-1}$, the sequence $(m)^r \cdot (\alpha' + m) \cdot 0 \cdot (\beta' + 1)$ is in $\mathfrak{A}'_{n,s}$. □

Words avoiding the pairs of patterns $\{100, 102\}$ or $\{011, 120\}$ are trivially in bijection via the complement map $\mathcal{W}_{n,k}(100, 102) \to \mathcal{W}_{n,k}(011, 120)$, $\omega \mapsto (k-1-\omega_i)_{i \in [1,n]}$. We studied the pair of patterns $\{011, 120\}$ in Section 4.3, and it can easily be seen that the complement map defines bijections between the sets $\mathfrak{B}_{n,k}$, resp. $\mathfrak{C}_{n,k}$, and their counterparts of Section 4.3. Therefore, the following identities are deduced from Proposition 38 and Lemma 39:

- for all $n, k \geqslant 0$,
$$\mathfrak{b}_{n,k} = \frac{\binom{k}{n}\binom{2n}{n}}{n+1},$$
- for all $n \geqslant 1, k \geqslant 2$,
$$\mathfrak{c}_{n,k} = 2\mathfrak{c}_{n,k-1} + \mathfrak{c}_{n-1,k} - \mathfrak{c}_{n-1,k-1} - \frac{\binom{k-2}{n}\binom{2n}{n}}{n+1}.$$

6 Generating functions

In the literature, a succession rule describing a generating tree is often used to calculate the generating function of the associated combinatorial class. Indeed, a succession rule can be turned into a system of functional equations satisfied by the generating function. These equations often involve *catalytic* variables, which correspond to some statistics of the combinatorial objects (the same statistics that label the generating tree). A common tool for solving these equations is the *kernel method* [6].

The constructions of pattern-avoiding inversion sequences we presented allow us to compute many terms of their enumeration sequences (usually a few hundred, see Table 2 for more precise data). From those initial terms, Pantone has conjectured algebraic expressions for three of their generating functions, using his software [26].

Conjecture 69. The ordinary generating function F of $\mathcal{I}(000, 102)$ is algebraic with minimal polynomial

$$x^4F(x)^4 - 2x^3(x-1)F(x)^3 + x(x^3 - 2x^2 + 4x - 1)F(x)^2 - (2x^2 - 2x + 1)F(x) + 1.$$

Conjecture 70. The ordinary generating function F of $\mathcal{I}(102, 201)$ is algebraic with minimal polynomial

$$\begin{aligned}&x(x-1)^2(x-2)^2(2x-1)^2F(x)^2 + (x-1)(2x-1)(4x^4 - 9x^3 + 5x^2 + 4x - 2)F(x)\\&- x^5 + 9x^4 - 22x^3 + 25x^2 - 12x + 2.\end{aligned}$$

Conjecture 71. The ordinary generating function F of $\mathcal{I}(102, 210)$ is algebraic with minimal polynomial

$$\begin{aligned}&(4x-1)(x-1)^4x^3F(x)^2 - (4x-1)(4x^4 - 22x^3 + 25x^2 - 9x + 1)(x-1)^2F(x)\\&+ 4x^7 - 44x^6 + 165x^5 - 254x^4 + 194x^3 - 75x^2 + 14x - 1.\end{aligned}$$

We prove Conjectures 70 and 71 by calculating the generating functions of $\mathcal{I}(102, 201)$ and $\mathcal{I}(102, 210)$, in Theorems 74 and 76 below. In a personal communication, Pantone proves Conjecture 69 by calculating the generating function of $\mathcal{I}(000, 102)$, using a system of equations we derived from the succession rule $\Omega_{\{000,102\}}$ of Theorem 17. We present this proof in the appendix.

Pantone also has a conjecture in [27] about inversion sequences avoiding the patterns 010 and 102.

Conjecture 72. The ordinary generating function F of $\mathcal{I}(010, 102)$ is algebraic with minimal polynomial

$$\begin{aligned}&x(x^2 - x + 1)(x-1)^2F(x)^3 + 2x(x-1)(2x^2 - 2x + 1)F(x)^2\\&- (x^4 - 8x^3 + 11x^2 - 6x + 1)F(x) - (2x-1)(x-1)^2.\end{aligned}$$

Conjecture 72 is now proved in [30], using yet another generating tree construction of inversion sequences.

The software [26] could not guess an algebraic, D-finite, or D-algebraic generating function for any other classes of pattern-avoiding inversion sequences we studied in this article.

We now calculate the generating functions of $\mathcal{I}(102, 201)$ and $\mathcal{I}(102, 210)$ from our succession rules, using the kernel method.

6.1 The generating function of $\mathcal{I}(102,\ 201)$

We define generating functions corresponding to the numbers $\mathfrak{a}, \mathfrak{b}^{(i)}, \mathfrak{c}$ of Section 2.3. Let $A(x, y) = \sum_{n,m\geqslant 0} \mathfrak{a}_{n,m} x^n y^m$, let $B^{(i)}(x, y) = \sum_{n,s\geqslant 0} \mathfrak{b}^{(i)}_{n,s} x^n y^s$ for $i \in \{1, 2, 3\}$, and let $C(x, y) = \sum_{n,\ell\geqslant 0} \mathfrak{c}_{n,\ell} x^n y^\ell$. Let also $F(x) = \sum_{n\geqslant 0} |\mathcal{I}_n(102, 201)|x^n$ be the ordinary generating function of $\mathcal{I}(102, 201)$.

Recall the succession rule $\Omega_{\{102,201\}}$ from Remark 12:

$$\Omega_{\{102,201\}} = \begin{cases} \quad (a,0,0) & & & \\ (a,n,m) & \rightsquigarrow & (a,n+1,i) & \text{for} \quad i \in [m,n] \\ & & (b^{(1)},i)^{n-i} & \text{for} \quad i \in [m,n-1] \\ & & (c,i) & \text{for} \quad i \in [0,m-1] \\ (b^{(1)},s) & \rightsquigarrow & (b^{(1)},s)\,(b^{(2)},s) & \\ (b^{(2)},s) & \rightsquigarrow & (b^{(2)},s)\,(b^{(3)},s) & \\ (b^{(3)},s) & \rightsquigarrow & (b^{(3)},s)^2 & \\ & & (c,i) & \text{for} \quad i \in [0,s-1] \\ (c,\ell) & \rightsquigarrow & (c,i) & \text{for} \quad i \in [0,\ell]. \end{cases}$$

Proposition 73. *The functions $A, B^{(1)}, B^{(2)}, B^{(3)}$, C, and F satisfy the following equations.*

$$A(x,y) = 1 + \frac{x}{1-y}\big(A(x,y) - yA(xy,1)\big)$$
$$B^{(1)}(x,y) = xB^{(1)}(x,y) + \frac{x}{(1-y)}\left(x\frac{\partial A}{\partial x}(x,y) - y\frac{\partial A}{\partial y}(x,y) + \frac{y}{1-y}\big(A(xy,1) - A(x,y)\big)\right)$$
$$B^{(2)}(x,y) = x\big(B^{(1)}(x,y) + B^{(2)}(x,y)\big)$$
$$B^{(3)}(x,y) = x\big(B^{(2)}(x,y) + 2B^{(3)}(x,y)\big)$$
$$C(x,y) = \frac{x}{1-y}\big(C(x,1) - yC(x,y) + A(x,1) - A(x,y) + B^{(3)}(x,1) - B^{(3)}(x,y)\big)$$
$$F(x) = A(x,1) + B^{(3)}(x,1) + C(x,1)$$

Proof. We turn the succession rule $\Omega_{\{102,201\}}$ into equations relating the generating functions.

$$\begin{aligned}
A(x,y) &= 1 + \sum_{n,m\geqslant 0} \mathfrak{a}_{n,m} x^{n+1} \sum_{i=m}^{n} y^i \\
&= 1 + \sum_{n,m\geqslant 0} \mathfrak{a}_{n,m} x^{n+1} \frac{y^m - y^{n+1}}{1-y} \\
&= 1 + \frac{x}{1-y}\big(A(x,y) - yA(xy,1)\big)
\end{aligned}$$

$$\begin{aligned}
B^{(1)}(x,y) &= \left(\sum_{n,s\geqslant 0} \mathfrak{b}^{(1)}_{n,s} x^{n+1} y^s\right) + \left(\sum_{n,m\geqslant 0} \mathfrak{a}_{n,m} x^{n+1} \sum_{i=m}^{n-1} (n-i) y^i\right) \\
&= xB^{(1)}(x,y) + \sum_{n,m\geqslant 0} \mathfrak{a}_{n,m} \frac{x^{n+1}}{1-y}\left(ny^m - my^m + \frac{y^{n+1} - y^{m+1}}{1-y}\right) \\
&= xB^{(1)}(x,y) + \frac{x}{(1-y)}\left(x\frac{\partial A}{\partial x}(x,y) - y\frac{\partial A}{\partial y}(x,y) + \frac{y}{1-y}\big(A(xy,1) - A(x,y)\big)\right)
\end{aligned}$$

$$\begin{aligned}
B^{(2)}(x,y) &= \sum_{n,s\geqslant 0} (\mathfrak{b}^{(1)}_{n,s} + \mathfrak{b}^{(2)}_{n,s}) x^{n+1} y^s \\
&= x\big(B^{(1)}(x,y) + B^{(2)}(x,y)\big)
\end{aligned}$$

$$\begin{aligned}
B^{(3)}(x,y) &= \sum_{n,s\geqslant 0} (\mathfrak{b}^{(2)}_{n,s} + 2\mathfrak{b}^{(3)}_{n,s}) x^{n+1} y^s \\
&= x\big(B^{(2)}(x,y) + 2B^{(3)}(x,y)\big)
\end{aligned}$$

$$\begin{aligned}
C(x,y) &= \left(\sum_{n,\ell\geqslant 0} \mathfrak{c}_{n,\ell} x^{n+1} \sum_{i=0}^{\ell} y^i\right) + \left(\sum_{n,k\geqslant 0} (\mathfrak{a}_{n,k} + \mathfrak{b}^{(3)}_{n,k}) x^{n+1} \sum_{i=0}^{k-1} y^i\right) \\
&= \left(\sum_{n,\ell\geqslant 0} \mathfrak{c}_{n,\ell} x^{n+1} \frac{1-y^{\ell+1}}{1-y}\right) + \left(\sum_{n,k\geqslant 0} (\mathfrak{a}_{n,k} + \mathfrak{b}^{(3)}_{n,k}) x^{n+1} \frac{1-y^k}{1-y}\right) \\
&= \frac{x}{1-y}\big(C(x,1) - yC(x,y) + A(x,1) - A(x,y) + B^{(3)}(x,1) - B^{(3)}(x,y)\big)
\end{aligned}$$

The equation $F(x) = A(x,1) + B^{(3)}(x,1) + C(x,1)$ is an immediate consequence of Theorem 10. $\square$

Theorem 74. *The generating function of* $\mathcal{I}(102, 201)$ *is*

$$F(x) = \frac{-8x^4 + 18x^3 - 10x^2 - 8x + 4 + 2(2x-1)(x^2-2x+2)\sqrt{(5x-1)(x-1)}}{4x(2x-1)(x-1)(x-2)^2}.$$

Proof. From Proposition 73, we have

$$(1 - y - x)A(x, y) = 1 - y - xyA(xy, 1). \tag{1}$$

We cancel the kernel $(1 - y - x)$ by defining $Y(x) = 1 - x$ and replacing y by $Y(x)$:

$$0 = 1 - (1 - x) - x(1 - x)A(x(1 - x), 1),$$

and obtain

$$A(x - x^2, 1) = \frac{1}{1 - x}.$$

This determines a unique formal power series $A(x, 1)$:

$$A(x, 1) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Replacing $A(x, 1)$ by this expression in (1) yields

$$A(x, y) = \frac{1 - 2y + \sqrt{1 - 4xy}}{2 - 2x - 2y}.$$

From Proposition 73, we have

$$B^{(1)}(x, y) = \frac{x}{(1-x)(1-y)} \left(x\frac{\partial A}{\partial x}(x, y) - y\frac{\partial A}{\partial y}(x, y) + \frac{y}{1-y}\big(A(xy, 1) - A(x, y)\big) \right),$$

$$B^{(2)}(x, y) = \frac{x}{1-x} B^{(1)}(x, y),$$

$$B^{(3)}(x, y) = \frac{x}{1-2x} B^{(2)}(x, y),$$

$$(1 - y + xy)C(x, y) = x\big(C(x, 1) + A(x, 1) - A(x, y) + B^{(3)}(x, 1) - B^{(3)}(x, y)\big).$$

We easily obtain expressions for $B^{(1)}(x, y)$, $B^{(2)}(x, y)$, and $B^{(3)}(x, y)$ from that of $A(x, y)$. As for C, we apply the kernel method again. The kernel is $1 - y + xy$, we cancel it by setting $y = \frac{1}{1-x}$. We obtain an expression for $C(x, 1)$:

$$C(x, 1) = A\left(x, \frac{1}{1-x}\right) - A(x, 1) + B^{(3)}\left(x, \frac{1}{1-x}\right) - B^{(3)}(x, 1).$$

Finally, we can write an expression for $F(x)$:

$$\begin{aligned} F(x) &= A(x, 1) + B^{(3)}(x, 1) + C(x, 1) \\ &= \frac{-8x^4 + 18x^3 - 10x^2 - 8x + 4 + 2(2x-1)(x^2-2x+2)\sqrt{(5x-1)(x-1)}}{4x(2x-1)(x-1)(x-2)^2}. \end{aligned}$$ □

6.2 The generating function of $\mathcal{I}(102,\ 210)$

We define generating functions corresponding to the numbers $\mathfrak{a}, \mathfrak{b}^{(i)}, \mathfrak{c}^{(i)}$ of Section 3.3. Let $A(x,y) = \sum_{n,k\geqslant 0} \mathfrak{a}_{n,k} x^n y^k$, $B^{(i)}(x,y) = \sum_{n,k\geqslant 0} \mathfrak{b}^{(i)}_{n,k} x^n y^k$, $C^{(i)}(x,y) = \sum_{n,k\geqslant 0} \mathfrak{c}^{(i)}_{n,k} x^n y^k$ for $i \in \{1,2\}$. Let $F(x) = \sum_{n\geqslant 0} |\mathcal{I}_n(102,210)| x^n$ be the ordinary generating function of $\mathcal{I}(102,210)$.

Recall the succession rule $\Omega_{\{102,210\}}$ from Theorem 20:

$$\Omega_{\{102,210\}} = \begin{cases} (a,0) & & & \\ (a,k) & \rightsquigarrow & (a,i) & \text{for} \quad i \in [1,k+1] \\ & & (b^{(1)},i)^{k-i} & \text{for} \quad i \in [1,k-1] \\ (b^{(1)},k) & \rightsquigarrow & (b^{(1)},k) & \\ & & (b^{(2)},i) & \text{for} \quad i \in [1,k+1] \\ & & (c^{(1)},i) & \text{for} \quad i \in [1,k-1] \\ (b^{(2)},k) & \rightsquigarrow & (b^{(2)},i) & \text{for} \quad i \in [1,k+1] \\ (c^{(1)},k) & \rightsquigarrow & (c^{(1)},k) & \\ & & (c^{(2)},i) & \text{for} \quad i \in [1,k] \\ (c^{(2)},k) & \rightsquigarrow & (c^{(2)},i) & \text{for} \quad i \in [1,k]. \end{cases}$$

Proposition 75. *The functions $A, B^{(1)}, B^{(2)}, C^{(1)}, C^{(2)}$, and F satisfy the following equations.*

$$\begin{aligned}
&A(x,y) = 1 + \frac{xy}{1-y}\big(A(x,1) - yA(x,y)\big) \\
&B^{(1)}(x,y) = xB^{(1)}(x,y) + \frac{xy}{1-y}\left(\frac{\partial A}{\partial y}(x,1) + \frac{1}{1-y}\big(A(x,y) - A(x,1)\big)\right) \\
&B^{(2)}(x,y) = \frac{xy}{1-y}\left(B^{(1)}(x,1) + B^{(2)}(x,1) - y\big(B^{(1)}(x,y) + B^{(2)}(x,y)\big)\right) \\
&C^{(1)}(x,y) = xC^{(1)}(x,y) + \frac{x}{1-y}\left(yB^{(1)}(x,1) - B^{(1)}(x,y)\right) \\
&C^{(2)}(x,y) = \frac{xy}{1-y}\left(C^{(1)}(x,1) + C^{(2)}(x,1) - C^{(1)}(x,y) - C^{(2)}(x,y)\right). \\
&F(x) = A(x,1) + B^{(1)}(x,1) + B^{(2)}(x,1) + C^{(1)}(x,1) + C^{(2)}(x,1)
\end{aligned}$$

Proof. We turn the succession rule $\Omega_{\{102,210\}}$ into equations relating the generating functions.

$$\begin{aligned}
A(x,y) &= 1 + \sum_{n,k\geqslant 0} \mathfrak{a}_{n,k} x^{n+1} \sum_{i=1}^{k+1} y^i \\
&= 1 + \sum_{n,k\geqslant 0} \mathfrak{a}_{n,k} x^{n+1} \frac{y - y^{k+2}}{1-y} \\
&= 1 + \frac{xy}{1-y}\big(A(x,1) - yA(x,y)\big)
\end{aligned}$$

$$\begin{aligned}
B^{(1)}(x,y) &= \left(\sum_{n,k\geqslant 0} \mathfrak{b}^{(1)}_{n,k} x^{n+1} y^k\right) + \left(\sum_{n,k\geqslant 0} \mathfrak{a}_{n,k} x^{n+1} \sum_{i=1}^{k-1} (k-i) y^i\right) \\
&= xB^{(1)}(x,y) + \sum_{n,k\geqslant 0} \mathfrak{a}_{n,k} \frac{x^{n+1} y}{1-y} \left(k + \frac{y^k - 1}{1-y}\right) \\
&= xB^{(1)}(x,y) + \frac{xy}{1-y} \left(\frac{\partial A}{\partial y}(x,1) + \frac{1}{1-y}\big(A(x,y) - A(x,1)\big)\right)
\end{aligned}$$

$$\begin{aligned}
B^{(2)}(x,y) &= \sum_{n,k\geqslant 0} \left(\mathfrak{b}^{(1)}_{n,k} + \mathfrak{b}^{(2)}_{n,k}\right) x^{n+1} \sum_{i=1}^{k+1} y^i \\
&= \sum_{n,k\geqslant 0} \left(\mathfrak{b}^{(1)}_{n,k} + \mathfrak{b}^{(2)}_{n,k}\right) x^{n+1} \frac{y - y^{k+2}}{1-y} \\
&= \frac{xy}{1-y} \left(B^{(1)}(x,1) + B^{(2)}(x,1) - y\big(B^{(1)}(x,y) + B^{(2)}(x,y)\big)\right)
\end{aligned}$$

$$\begin{aligned}
C^{(1)}(x,y) &= \left(\sum_{n,k\geqslant 0} \mathfrak{c}^{(1)}_{n,k} x^{n+1} y^k\right) + \left(\sum_{n,k\geqslant 0} \mathfrak{b}^{(1)}_{n,k} x^{n+1} \sum_{i=1}^{k-1} y^i\right) \\
&= xC^{(1)}(x,y) + \sum_{n,k\geqslant 0} \mathfrak{b}^{(1)}_{n,k} x^{n+1} \frac{y - y^k}{1-y} \\
&= xC^{(1)}(x,y) + \frac{x}{1-y} \left(yB^{(1)}(x,1) - B^{(1)}(x,y)\right)
\end{aligned}$$

$$\begin{aligned}
C^{(2)}(x,y) &= \sum_{n,k\geqslant 0} \left(\mathfrak{c}^{(1)}_{n,k} + \mathfrak{c}^{(2)}_{n,k}\right) x^{n+1} \sum_{i=1}^{k} y^i \\
&= \sum_{n,k\geqslant 0} \left(\mathfrak{c}^{(1)}_{n,k} + \mathfrak{c}^{(2)}_{n,k}\right) x^{n+1} \frac{y - y^{k+1}}{1-y} \\
&= \frac{xy}{1-y} \left(C^{(1)}(x,1) + C^{(2)}(x,1) - C^{(1)}(x,y) - C^{(2)}(x,y)\right).
\end{aligned}$$

By definition, $F(x) = A(x,1) + B^{(1)}(x,1) + B^{(2)}(x,1) + C^{(1)}(x,1) + C^{(2)}(x,1)$. □

Theorem 76. *The generating function $F(x)$ of $\mathcal{I}(102, 210)$ is*

$$\frac{(4x-1)(4x^4-22x^3+25x^2-9x+1)-(2x-1)(x^2-5x+1)(2x^2-4x+1)\sqrt{1-4x}}{2x^3(4x-1)(x-1)^2}.$$

Proof. From Proposition 75, we have

$$(1-y+xy^2)A(x,y) = 1-y+xyA(x,1) \tag{2}$$

$$B^{(1)}(x,y) = \frac{xy}{(1-x)(1-y)}\left(\frac{\partial A}{\partial y}(x,1)+\frac{1}{1-y}\big(A(x,y)-A(x,1)\big)\right) \tag{3}$$

$$(1-y+xy^2)B^{(2)}(x,y) = xy\left(B^{(1)}(x,1)+B^{(2)}(x,1)-yB^{(1)}(x,y)\right) \tag{4}$$

$$C^{(1)}(x,y) = \frac{x}{(1-x)(1-y)}\left(yB^{(1)}(x,1)-B^{(1)}(x,y)\right) \tag{5}$$

$$(1-y+xy)C^{(2)}(x,y) = xy\left(C^{(1)}(x,1)+C^{(2)}(x,1)-C^{(1)}(x,y)\right). \tag{6}$$

First, we derive an expression of $A(x,y)$ from (2). To cancel the kernel $(1-y+xy^2)$, there are two solution in y:

$$Y_1(x) = \frac{1-\sqrt{1-4x}}{2x}, \qquad Y_2(x) = \frac{1+\sqrt{1-4x}}{2x}.$$

Only Y_1 defines a formal power series. We replace y by $Y_1(x)$ in (2), and obtain

$$A(x,1) = \frac{1-\sqrt{1-4x}}{2x}.$$

Replacing $A(x,1)$ by this expression in (2) yields

$$A(x,y) = \frac{2-y-y\sqrt{1-4x}}{2(1-y+xy^2)}.$$

We can obtain expressions for $B^{(1)}(x,y)$ and $C^1(x,y)$ by substituting our expression of $A(x,y)$ in (3) and (5).

Next, we look for an expression of $B^{(2)}(x,1)$ from (4). The kernel is the same as for A. Replacing y by $\frac{1-\sqrt{1-4x}}{2x}$ in (4) yields

$$B^{(2)}(x,1) = \frac{1-\sqrt{1-4x}}{2x}B^{(1)}\left(x, \frac{1-\sqrt{1-4x}}{2x}\right) - B^{(1)}(x,1).$$

If we tried to directly evaluate $B^{(1)}\left(x, \frac{1-\sqrt{1-4x}}{2x}\right)$ from our expressions of $B^{(1)}(x,y)$ and $A(x,y)$, we would obtain the fraction $\frac{0}{0}$. Instead, we can write down an expression for $A\left(x, \frac{1-\sqrt{1-4z}}{2z}\right)$, then consider its limit as z tends to x, to show that

$$A\left(x, \frac{1-\sqrt{1-4x}}{2x}\right) = \frac{1}{2}\left(1+\frac{1}{\sqrt{1-4x}}\right).$$

We can then obtain an expression of $B^{(1)}\left(x, \frac{1-\sqrt{1-4x}}{2x}\right)$, and therefore of $B^{(2)}(x,1)$.

Finally, we look for an expression of $C^{(2)}(x,1)$ from (6). The kernel is $(1-y+xy)$. We cancel it by setting $y = \frac{1}{1-x}$, and obtain

$$C^{(2)}(x,1) = C^{(1)}\left(x, \frac{1}{1-x}\right) - C^{(1)}(x,1).$$

The generating function of $\mathcal{I}(102, 210)$ is

$$\begin{aligned}&A(x,1) + B^{(1)}(x,1) + B^{(2)}(x,1) + C^{(1)}(x,1) + C^{(2)}(x,1)\\ =&\frac{(4x-1)(4x^4-22x^3+25x^2-9x+1)-(2x-1)(x^2-5x+1)(2x^2-4x+1)\sqrt{1-4x}}{2x^3(4x-1)(x-1)^2}. \quad \square\end{aligned}$$

7 Asymptotics

Looking back at the first few terms of the enumeration sequences in Table 2, it could appear that among the pairs of patterns we have studied, the one avoided by the most inversion sequences is {101, 210}, and the one avoided by the fewest inversion sequences is {000, 010}. Interestingly, it can be shown that the enumeration sequence of $\mathcal{I}(101, 210)$ is bounded above by an exponential function, and that of $\mathcal{I}(000, 010)$ is super-exponential (see Propositions 77 and 78 below). In fact, computation indicates there are more inversion sequences of size n avoiding {000, 010} than {101, 210} starting at $n = 41$.

Marcus and Tardos [21, Theorem 9] prove that the number of $n \times n$ 0-1 matrices avoiding a permutation matrix P is bounded above by c_P^n for some constant c_P. The subset of $n \times n$ 0-1 matrices which contain exactly one 1-entry in each column is clearly in bijection with the set $\mathcal{W}_{n,n}$ of words of length n over the alphabet $[0, n-1]$. For any permutation π, this restricts to a bijection between matrices which avoid the permutation matrix of π and words avoiding the pattern π. Observing that $\mathcal{W}_{n,n}$ includes the set $\mathcal{I}_n$ of inversion sequences of size n, we obtain the following proposition.

Proposition 77. *For any permutation π, for all $n \geqslant 0$, we have $|\mathcal{I}_n(\pi)| \leqslant c_\pi^n$ for some constant c_π.*

In particular, the growth of $\mathcal{I}(101, 210)$ is at most exponential since 210 is a permutation pattern (i.e. a pattern without any repeated values).

Proposition 78 below implies that every class of inversion sequences avoiding only patterns with repeated values studied in this article have a super-exponential growth: $\mathcal{I}(010)$, $\mathcal{I}(000, 010)$, $\mathcal{I}(000, 100)$, $\mathcal{I}(100, 110)$, $\mathcal{I}(100, 101)$, and $\mathcal{I}(010, 110)$.

Proposition 78. *The enumeration sequence of $\mathcal{I}(000, 010, 100, 101, 110)$ grows super-exponentially.*

Proof. Let $\mathfrak{A}_{n,k} = \{\sigma \in \mathcal{I}_n : \sigma = \alpha\cdot(\beta+k),\ \alpha = (0,0,1,1,\ldots,k-1,k-1),\ \beta \in \mathcal{I}^k_{n-2k}(00)\}$ be the set of inversion sequences of length n which begin by $(0,0,1,1,\ldots,k-1,k-1)$, followed by $n-2k$ distinct values greater than or equal to k. Let $\mathfrak{A}_n = \coprod_{k\geqslant 0} \mathfrak{A}_{n,k}$.

For all $k \geqslant 0$ and $n \geqslant 2k$, we have $|\mathfrak{A}_{n,k}| = (k+1)^{n-2k}$, hence for any $k \geqslant 0$, $|\mathfrak{A}_n|$ is asymptotically larger than k^n. Since this holds for an arbitrarily large value of k, $|\mathfrak{A}_n|$ has a super-exponential asymptotic behavior. By construction, every sequence in $\mathfrak{A}_n$ avoids the patterns 000, 010, 100, 101, and 110. □

Constructions similar to the above proof can be used to show that many classes of inversion sequences avoiding patterns with repeated letters have a super-exponential growth. However this is not always the case: for instance, [12, Theorem 10] proves that $|\mathcal{I}_n(001)| = 2^{n-1}$ for all $n \geqslant 1$. Remarkably, 001 is the only pattern ρ of size 3 with repeated values such that the growth of $\mathcal{I}(\rho)$ is at most exponential.

Using his software [25], Pantone provided us with precise conjectures about the asymptotic behavior of some of the enumeration sequences, based on the initial terms we computed. We present these conjectures in Table 3. In particular, it seems that the classes $\mathcal{I}(010, 201)$ and $\mathcal{I}(101, 210)$ have the same growth rate, and the classes $\mathcal{I}(100, 120)$ and $\mathcal{I}(110, 201)$ as well.

Class	μ	α	β
$\mathcal{I}(000, 201)$	~ 10.9282032	~ -7.3906	?
$\mathcal{I}(010, 201)$	~ 10.1566572	~ -7.6168360	?
$\mathcal{I}(100, 102)$	$\sim 5.066130494716195596699600$	$-3/2$	0
$\mathcal{I}(100, 120)$	~ 7.72334814688	$-3/2$	0
$\mathcal{I}(101, 210)$	~ 10.156657	~ -6.273831	0
$\mathcal{I}(110, 201)$	$\sim 7.7233481468847308370$	$-3/2$	0
$\mathcal{I}(120, 201)$	$\sim 7.4563913226671221339$	$-3/2$	0

Table 3: Conjectured asymptotic behavior of the form $C\mu^n n^\alpha \log(n)^\beta$ for the enumeration sequences of some classes of pattern-avoiding inversion sequences.

Appendix

In this appendix, we present a proof of Conjecture 69 provided by Jay Pantone in a personal communication.

Recall the succession rule $\Omega_{\{000,102\}}$ from Section 3.2:

$$\Omega_{\{000,102\}} = \begin{cases} (1) & \\ (s) \overset{1}{\rightsquigarrow} (j)^{s+1-j} & \text{for} \quad j \in [1, s] \\ (s) \overset{2}{\rightsquigarrow} (j+1)^{s+1-j}(j)^{\binom{s+1-j}{2}} & \text{for} \quad j \in [1, s]. \end{cases}$$

Let $\mathfrak{a}_{n,s} = |\{\sigma \in \mathcal{I}_n(000, 102) \ : \ \mathbf{sites}(\sigma) = s\}|$ where **sites** is defined in Section 3.2 and corresponds to the parameter of the rule $\Omega_{\{000,102\}}$. Let $A(x, y) = \sum_{n,s \geqslant 0} \mathfrak{a}_{n,s} x^n y^s$.

From the succession rule $\Omega_{\{000,102\}}$, it can be seen that the generating function of the children of a sequence of $\mathcal{I}(000,102)$ having size n and s active sites is

$$\begin{aligned}
&(x^{n+1}+x^{n+2}y)\left(\sum_{j=1}^{s}(s+1-j)y^j\right)+x^{n+2}\left(\sum_{j=1}^{s}\binom{s+1-j}{2}y^j\right)\\
&=(x^{n+1}y+x^{n+2}y^2)\frac{y^{s+1}-y+s(1-y)}{(1-y)^2}\\
&+x^{n+2}y\frac{2y-2y^{s+1}-2s(1-y)+s(s+1)(1-y)^2}{2(1-y)^3}.
\end{aligned}$$

Seeing that

$$\frac{\partial A}{\partial y}(x,1)=\sum_{n,s\geqslant 0} s\,\mathfrak{a}_{n,s}x^n,$$

$$\frac{\partial^2(y\cdot A)}{\partial y^2}(x,1)=\sum_{n,s\geqslant 0} s(s+1)\mathfrak{a}_{n,s}x^n,$$

we can apply this transformation to each monomial $\mathfrak{a}_{n,s}x^ny^s$, and obtain an equation which characterizes $A(x,y)$.

$$\begin{aligned}
A(x,y)&=y+\frac{xy+x^2y^2}{(1-y)^2}\left(y(A(x,y)-A(x,1))+(1-y)\frac{\partial A}{\partial y}(x,1)\right)\\
&+\frac{x^2y}{2(1-y)^3}\left(2y(A(x,1)-A(x,y))-2(1-y)\frac{\partial A}{\partial y}(x,1)+(1-y)^2\frac{\partial^2(y\cdot A)}{\partial y^2}(x,1)\right)\\
&=y-xy^2\frac{(1+xy)(1-y)-x}{(1-y)^3}(A(x,1)-A(x,y))\\
&+xy\frac{(1+xy)(1-y)-x}{(1-y)^2}\frac{\partial A}{\partial y}(x,1)+\frac{x^2y}{2(1-y)}\frac{\partial^2(y\cdot A)}{\partial y^2}(x,1)
\end{aligned}$$

To simplify this expression, we introduce two series $B(x,y)$ and $C(x,y)$.

$$B(x,y)=y\frac{A(x,1)-A(x,y)}{1-y}=\sum_{n,s\geqslant 0}\mathfrak{a}_{n,s}x^n\sum_{i=1}^{s}y^i$$

$$C(x,y)=\frac{B(x,1)-B(x,y)}{1-y}=\sum_{n,s\geqslant 0}\mathfrak{a}_{n,s}x^n\sum_{i=1}^{s}\sum_{j=0}^{i-1}y^j$$

From the definitions of $B(x,y)$ and $C(x,y)$, it can be seen that

$$\begin{aligned}
&B(x,1)=\frac{\partial A}{\partial y}(x,1),\\
&C(x,1)=\sum_{n,s\geqslant 0}\mathfrak{a}_{n,s}x^n\sum_{i=1}^{s}\sum_{j=0}^{i-1}1=\sum_{n,s\geqslant 0}\frac{s(s+1)}{2}\mathfrak{a}_{n,s}x^n=\frac{1}{2}\frac{\partial^2(y\cdot A)}{\partial y^2}(x,1).
\end{aligned}$$

We can now rewrite our equation for $A(x,y)$ in terms of B and C, without partial derivatives.

$$\begin{aligned}
A(x,y) &= y + xy\frac{(1+xy)(1-y)-x}{(1-y)^2}(B(x,1)-B(x,y)) + \frac{x^2y}{1-y}C(x,1)\\
&= y + xy\frac{(1+xy)(1-y)-x}{1-y}C(x,y) + \frac{x^2y}{1-y}C(x,1)
\end{aligned}$$

In summary, $A(x,y)$ is defined by the following system of equations involving x, y, the three series $A(x,y)$, $B(x,y)$, $C(x,y)$, and their evaluations for $y=1$.

$$\begin{cases}
0 = (y-1)A(x,y) + (1-y)y + xy((1+xy)(1-y)-x)C(x,y) + x^2yC(x,1)\\
0 = (y-1)B(x,y) + y(A(x,1)-A(x,y))\\
0 = (y-1)C(x,y) + (B(x,1)-B(x,y))
\end{cases}$$

This can be seen as a system of three polynomial equations in eight variables (x, y, $A(x,1)$, $B(x,1)$, $C(x,1)$, $A(x,y)$, $B(x,y)$, and $C(x,y)$). Using Gröbner basis computations[8], we can eliminate $B(x,y)$ and $C(x,y)$ from this system, and obtain an equation involving only six variables:

$$\begin{aligned}
&(-x^2y^4 + x^2y^3 - x^2y^2 - xy^3 + xy^2 + y^3 - 3y^2 + 3y - 1)A(x,y)\\
&+ xy^2(xy^2 - xy + x + y - 1)A(x,1) + xy(y-1)(xy^2 - xy + x + y - 1)B(x,1)\\
&+ x^2y(y-1)^2C(x,1) - y(y-1)^3 = 0.
\end{aligned} \tag{7}$$

This single equation uniquely defines the series $A(x,y)$, $A(x,1)$, $B(x,1)$, and $C(x,1)$, under the assumption that $A(x,y)$ is a formal power series in x whose coefficients are polynomials in y. Next, we use the approach from [9] generalizing the kernel method to solve this equation. The kernel of (7) is

$$-x^2y^4 + x^2y^3 - x^2y^2 - xy^3 + xy^2 + y^3 - 3y^2 + 3y - 1.$$

The coefficient of x^0 is a polynomial of degree 3 in y, therefore the kernel has three roots $Y_1(x)$, $Y_2(x)$, $Y_3(x)$ that are fractional power series in x, by [9, Theorem 2]. Now, for $i \in \{1,2,3\}$, we have

$$-x^2Y_i(x)^4 + x^2Y_i(x)^3 - x^2Y_i(x)^2 - xY_i(x)^3 + xY_i(x)^2 + Y_i(x)^3 - 3Y_i(x)^2 + 3Y_i(x) - 1 = 0$$

and

$$\begin{aligned}
&xY_i(x)^2(xY_i(x)^2 - xY_i(x) + x + Y_i(x) - 1)A(x,1)\\
&+ xY_i(x)(Y_i(x)-1)(xY_i(x)^2 - xY_i(x) + x + Y_i(x) - 1)B(x,1)\\
&+ x^2Y_i(x)(Y_i(x)-1)^2C(x,1) - Y_i(x)(Y_i(x)-1)^3 = 0.
\end{aligned}$$

[8] We use the Maple package PolynomialIdeals.

This forms a system of 6 equations. We add another indeterminate Z and a seventh equation

$$Z(Y_1(x) - Y_2(x))(Y_1(x) - Y_3(x))(Y_2(x) - Y_3(x)) - 1 = 0,$$

to ensure that the system has a solution only if the series $Y_1(x)$, $Y_2(x)$, and $Y_3(x)$ are distinct. Fortunately they are distinct, and using Gröbner basis computations again, we can find an equation involving only x and $A(x, 1)$.

$$x^4A(x,1)^4 - 2x^3(x-1)A(x,1)^3 + x(x^3-2x^2+4x-1)A(x,1)^2 + (-2x^2+2x-1)A(x,1) + 1 = 0$$

The left-hand side is the minimal polynomial of $A(x, 1)$, and this proves Conjecture 69. We can compute the minimal polynomials of $B(x, 1)$ and $C(x, 1)$ in the same way.

$$x^6B(x,1)^4 - 3x^4B(x,1)^3 + x^2(x^2 - x + 3)B(x,1)^2 + (-x^2 + x - 1)B(x,1) + 1 = 0$$

$$x^8C(x,1)^4 + x^5(3x+1)C(x,1)^3 + 2x^3(2x+1)C(x,1)^2 + (3x^2 + x - 1)C(x,1) + 1 = 0$$

Going back to equation (7), we can now see that $A(x, y)$ is the sum of three algebraic functions of (x, y) (of degree 4 each), so $A(x, y)$ is itself algebraic. Its minimal polynomial is difficult to obtain, since asking a computer algebra program to directly eliminate $A(x, 1)$, $B(x, 1)$ and $C(x, 1)$ from a system of four equations consisting in (7) and the minimal polynomials of $A(x, 1)$, $B(x, 1)$, and $C(x, 1)$ would yield a polynomial that is too big to compute (the minimal polynomial of $A(x, y)$ is a very small factor of this huge polynomial). Pantone still managed to eliminate the variables $A(x, 1)$, $B(x, 1)$, and $C(x, 1)$ "by hand" one at a time, using resultants. He found that the function $A(x, y)$ is also algebraic of degree 4, and its minimal polynomial is

$$\begin{aligned}
&x^2\big((y-1)(x^2y^3 + xy^2 + 3y) + x^2y^2 - y^3 + 1\big)A(x,y)^4\\
&- xy\big((y-1)(2x^3y^2 + 4x^2y - xy^2 + 2x + 2y - 2) - 2x^2y^3 + 2x^3y\big)A(x,y)^3\\
&+ y^2\big((y-1)(x^4y - 2x^3y + x^3 + 2x^2 - 1)\\
&+ (x-y)(-2x^2 - 2xy + 3x) + x^4 + x^2y^2\big)A(x,y)^2\\
&- y^3\big(x^2y + x^2 - 2x - y + 2\big)A(x,y) + y^4.
\end{aligned}$$

The statistic **sites** associated with the catalytic variable y is essentially the same as the statistic **rank** studied in [14] (to be precise, $\textbf{sites}(\sigma) = \textbf{rank}(\sigma) + 1$ for any $\sigma \in \mathcal{I}(000, 102)$), so this answers the question of enumerating $\{000, 102\}$-avoiding inversion sequences according to their rank, that was left open in [14].

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