How Many Random Edges Make an Almost-Dirac Graph Hamiltonian?

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Abstract

We study Hamiltonicity in the union of an n-vertex graph H with high minimum degree and a binomial random graph on the same vertex set. In particular, we consider the case when H has minimum degree close to $n/2$. We determine the perturbed threshold for Hamiltonicity in this setting.

To be precise, let $\eta \coloneqq n/2 - \delta(H)$. For $\eta = \omega(1)$, we show that it suffices to add $\Theta(\eta)$ random edges to H to a.a.s. obtain a Hamiltonian graph; for $\eta = \Theta(1)$, we show that $\omega(1)$ edges suffice. In fact, when $\eta = o(n)$ and $\eta = \omega(1)$, we show that $(8 + o(1))\eta$ random edges suffice, which is best possible up to the error term. This determines the sharp perturbed threshold for Hamiltonicity in this range of degrees.

We also obtain analogous results for perfect matchings, showing that, in this range of degrees, the sharp perturbed thresholds for Hamiltonicity and for perfect matchings differ by a factor of 2.

Mathematics Subject Classifications: 05C35, 05C80.

1 Introduction

The study of random graphs focuses on understanding the (likely) properties of the "average" graph (on a given probability space). In a similar way, the study of *randomly perturbed graphs* can be seen as the study of the properties of the "average" *supergraph* of a given graph H. Since the seminal work of Bohman, Frieze and Martin [12] on Hamiltonicity of randomly perturbed graphs, these have received much attention, especially during the last decade. Most of this research has considered supergraphs of graphs with some minimum degree condition.

More precisely, the framework we consider is the following. Given an (integer) function $d = d(n)$, we take some sequence of n-vertex graphs $\{H_n\}_{n\in\mathbb{N}}$ with minimum degree

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$\delta(H_n) \geqslant d$. We also consider the *binomial random graph* $G(n,p)$ (which is an n-vertex graph sampled by adding each of the $\binom{n}{2}$ possible edges independently with probability p) on the same vertex set as H_n. We are interested in understanding the likely properties of $H_n \cup G(n,p)$ (we only consider *simple* graphs). To simplify statements, we say that an event occurs *asymptotically almost surely* (a.a.s.) if the probability that it does tends to 1 as n tends to infinity.

Much of the research into random graphs has focused on understanding the thresholds for different properties. Roughly speaking, a *threshold* refers to a range of values for the parameter p where the random graph suddenly transitions from a.a.s. not satisfying a property to a.a.s. satisfying it. This notion extends to the study of randomly perturbed graphs in a natural way. Formally, in this paper we consider the following definition.

Definition 1. Given a nontrivial monotone graph property $\mathcal{P}$ and an integer-valued function $d = d(n)$, a function $p^* = p^*(n)$ is a *d-threshold* for $\mathcal{P}$ if the following two statements hold:

(0) There exists a graph sequence $\{H_n\}_{n\in\mathbb{N}}$ with $\delta(H_n) \geqslant d$ such that, if $p = o(p^*)$, then a.a.s. $H_n \cup G(n,p) \notin \mathcal{P}$.

(1) For every graph sequence $\{H_n\}_{n\in\mathbb{N}}$ with $\delta(H_n) \geqslant d$, if $p = \omega(p^*)$, then a.a.s. $H_n \cup G(n,p) \in \mathcal{P}$.

Moreover, we say that $p^* = p^*(n)$ is a *sharp d-threshold* for $\mathcal{P}$ if, for every fixed $\varepsilon > 0$, the following two statements hold:

(0) There exists a graph sequence $\{H_n\}_{n\in\mathbb{N}}$ with $\delta(H_n) \geqslant d$ such that, if $p \leqslant (1-\varepsilon)p^*$, then a.a.s. $H_n \cup G(n,p) \notin \mathcal{P}$.

(1) For every graph sequence $\{H_n\}_{n\in\mathbb{N}}$ with $\delta(H_n) \geqslant d$, if $p \geqslant (1+\varepsilon)p^*$, then a.a.s. $H_n \cup G(n,p) \in \mathcal{P}$.

While the (sharp) d-thresholds for different properties are not unique, we will follow the custom of referring to one such d-threshold (usually, the one with the simplest expression) as *the* (sharp) d-threshold for $\mathcal{P}$. Moreover, when every sequence of graphs $\{H_n\}_{n\in\mathbb{N}}$ with $\delta(H_n) \geqslant d$ satisfies that $H_n \in \mathcal{P}$ (for all sufficiently large n), we abuse the definition and say that the (sharp) d-threshold for $\mathcal{P}$ is 0. Naturally, the results about thresholds in binomial random graphs correspond to 0-thresholds. While these definitions are built on graph sequences, in practice we will omit the sequences from our statements, and they will be implicit in the use of n-vertex graphs H_n.

1.1 Hamiltonicity

In this paper, we focus mainly on Hamiltonicity (that is, the property of containing a spanning cycle). A classical theorem of Dirac [22] ensures that, if $d \geqslant n/2$, then the (sharp) d-threshold for Hamiltonicity is 0. On the opposite extreme, Pósa [47] and Koršunov [38] independently showed that the 0-threshold for Hamiltonicity is $\log n/n$

(in fact, Koršunov [38] showed that $\log n/n$ is the *sharp* 0-threshold for Hamiltonicity). For $d = \alpha n$ with $\alpha \in (0, 1/2)$ fixed, Bohman, Frieze and Martin [12] showed that the d-threshold for Hamiltonicity is $1/n$. Together, these results give a complete "macroscopic" picture of the thresholds for Hamiltonicity: for every $\alpha \in [0, 1)$, if we set $d = \alpha n$, the d-threshold for Hamiltonicity is known. However, in this macroscopic picture we observe two jumps in the behaviour of the threshold as a function of α. The first such jump occurs at $\alpha = 0$, and the second, at $\alpha = 1/2$. For any property $\mathcal{P}$ and any $\alpha \in [0, 1]$ where such a jump occurs, we shall refer to the range $d = (1 \pm o(1))\alpha n$ as a *critical window*. (This should not be confused with the probabilistic critical windows, which refer to the range around the threshold for some property.) One should expect that, when considering these critical windows more carefully, the d-threshold will interpolate between the two thresholds of the macroscopic picture, to some extent.

The behaviour of Hamiltonicity around the critical window when $\alpha = 0$ (which corresponds to random perturbation of sparse graphs) is fairly well understood: indeed, already Bohman, Frieze and Martin [12] showed that, for $1 \leqslant d = o(n)$, the function $\log(n/d)/n$ is a d-threshold for Hamiltonicity, and Hahn-Klimroth, Maesaka, Mogge, Mohr and Parczyk [30] improved the implicit constant to show that $p = (6 + o(1)) \log(n/d)/n$ suffices for $H_n \cup G(n, p)$ to a.a.s. contain a Hamilton cycle. However, the critical window around $\alpha = 1/2$ has not been studied at all. The main goal of this note is to provide the d-threshold for Hamiltonicity in this critical window, thus completing the picture of the d-thresholds for Hamiltonicity. Throughout the paper, whenever we write $n/2 - \eta$, it is assumed that this is an integer.

Theorem 2. *Let $d = n/2 - \eta$, where $1/2 \leqslant \eta \leqslant n/64$. The d-threshold for Hamiltonicity is η/n^2.*

Our proof of Theorem 2 avoids the use of the rotation-extension technique. In the range where η is linear, this leads to a new proof of the main result of Bohman, Frieze and Martin [12], in addition to the subsequent proofs of Krivelevich, Kwan and Sudakov [40] and Hahn-Klimroth, Maesaka, Mogge, Mohr and Parczyk [30]. Our proof does not extend to all the dense range of degrees (the upper bound on η in the statement is not best possible but, since we cannot hope to get close to $n/2$ with our approach, we have made no effort to optimise it). However, for the range that we consider, our proof yields a better upper bound on the sharp d-threshold for Hamiltonicity than that in [30], and thus improves on all previously known bounds (see Theorem 7 (b)). In fact, for a smaller range of the function η (but covering essentially all the critical window), our technique provides us with the *sharp* d-threshold for Hamiltonicity.

Theorem 3. *Let $d = n/2 - \eta$, where $\eta = \omega(1)$ and $\eta = o(n)$. The sharp d-threshold for Hamiltonicity is $16\eta/n^2$.*

This is the first (non-trivial) result about sharp d-thresholds with $d = \omega(1)$ in the literature. We remark that, when $\eta = \Theta(1)$, there is no sharp d-threshold for Hamiltonicity, so the lower bound on η in Theorem 3 is necessary. The extremal graph witnessing the

(0)-statement is the standard one for the Hamiltonicity problem: a complete bipartite graph H_n with parts of size d and $n-d$, respectively; see Section 2 for more details.

As mentioned above, our proof avoids the use of the rotation-extension technique pioneered by Pósa [47], and is in fact elementary. However, it uses some ideas which are reminiscent of this technique. In particular, in non-Hamiltonian graphs, we consider the existence of large sets of non-edges which, if added to the graph, lead to containing a longer cycle; see Lemma 9. Such non-edges are usually called *boosters* in the literature, and have been central to many proofs about Hamilton cycles. A standard tool to use in this setup would be a lemma that ensures that expander graphs have many boosters (see, e.g., the survey of Krivelevich [39, Corollary 2.8]). This could be used to shorten our proof of Theorem 2; however, a more precise counting of such boosters is needed in order to derive the sharp d-threshold in Theorem 3.

As a corollary of Theorem 3, we also obtain the sharp d-threshold for pancyclicity (that is, the property of containing a cycle of every length between 3 and n) for a slightly more restricted range of d. This property has been considered in randomly perturbed graphs by Krivelevich, Kwan and Sudakov [40] and Aigner-Horev, Hefetz and Krivelevich [2] as well as Allin and Espuny Díaz [3]. In particular, the following simple result of Allin and Espuny Díaz [3] can be used in our setting.

Theorem 4 ([3, Theorem 4])**.** *Let H_n be a graph on n vertices containing a Hamilton cycle. Then a.a.s. $H_n \cup G(n, 4\log n/e(H_n))$ is pancyclic.*

The following is an immediate consequence of Theorems 3 and 4.

Corollary 5. *Let $d = n/2 - \eta$, where $\eta = \omega(\log n)$ and $\eta = o(n)$. The sharp d-threshold for pancyclicity is $16\eta/n^2$.*

1.2 Perfect matchings

Let us now assume that n is an even integer (and so all the sequences of graphs are restricted to even values of n). The property of containing a perfect matching (that is, a set of $n/2$ pairwise disjoint edges) is one of the most studied in graph theory. Since a Hamilton cycle contains a perfect matching, Dirac's theorem [22] ensures that, for $d \geqslant n/2$, the (sharp) d-threshold for containing a perfect matching is 0. A classical result of Erdős and Rényi [25] shows that the sharp 0-threshold for perfect matchings is $\log n/n$. Note that these coincide with the respective sharp d-thresholds for Hamiltonicity (in the case of the 0-threshold, a difference can be seen by analysing smaller order terms). For randomly perturbed graphs with $1 \leqslant d < n/2$, the results of Bohman, Frieze and Martin [12], together with Theorem 2 and the standard extremal graph, immediately imply that the d-threshold for perfect matchings also coincides with the d-threshold for Hamiltonicity. However, it turns out that the *sharp* d-threshold for perfect matchings does *not* coincide with that for Hamiltonicity. Our techniques allow us to determine this sharp threshold too, in the same range of d as for Hamiltonicity.

Theorem 6. *Let $d = n/2 - \eta$, where $\eta = \omega(1)$ and $\eta = o(n)$. The sharp d-threshold for containing a perfect matching is $8\eta/n^2$.*

The extremal graph witnessing the (0)-statement for this sharp threshold is the same as for Hamiltonicity. Our proof of Theorem 6 also works when n is odd to show that the sharp d-threshold for containing a matching of size $\lfloor n/2 \rfloor$ is $8\eta/n^2$.

1.3 Related work

Hamiltonicity is likely the property that has been studied the most in randomly perturbed graphs. In addition to the papers we already mentioned (for graphs), it has also been studied in directed graphs [8, 12, 40], hypergraphs [32, 40, 43] and subgraphs of the hypercube [21]. In graphs, Hamiltonicity has also been considered when the perturbation comes from a random regular graph [23, 27, 34] or a random geometric graph [26, 28]. A variant of the problem considers random colourings of randomly perturbed graphs and aims to find rainbow Hamilton cycles [1, 4, 37]. Some recent results include the colour-bias of Hamilton cycles [18] or the Hamilton cycle space [33]. The Hamiltonicity Maker-Breaker [19] and Waiter-Client [20] games played on randomly perturbed graphs have been considered as well. It would be interesting to understand these problems at the appropriate critical windows for Hamiltonicity.

In addition to Hamiltonicity, multiple other spanning properties have been studied in the context of randomly perturbed graphs. For instance, the full "macroscopic" behaviour of the d-thresholds is known for connectivity [11], spanning bounded degree trees [13, 41] and (almost) unbounded degree trees [35] (for which, analogously to Hamiltonicity, there are two critical windows); cycle-factors [10, 15, 16, 31] and 2-universality [17, 46] (for which there are three critical windows), and squares of Hamilton cycles [17, 24] (for which there is an infinite number of critical windows). For K_r-factors with $r \geqslant 4$, the "macroscopic" d-threshold is known for all but finitely many points [7, 10, 31], particularly showing that there are r critical windows. We believe that studying the d-thresholds for these properties in the respective critical windows would be a very interesting problem. Additionally, for some properties, the d-thresholds are known for some range of the values of d. For example, there are results about different F-factors [10] and higher powers of Hamilton cycles [5, 6, 14, 24, 44, 45], as well as some results for general bounded degree spanning graphs [14].

2 Proofs

We begin this section by proving the (0)-statement for our different results. As already mentioned in the introduction, these follow from a construction that is standard in the area; we include the details for the benefit of the unfamiliar reader. Let $d = n/2 - \eta$, where $1/2 \leqslant \eta = \eta(n) \leqslant n/64$. Let H_n be a complete bipartite graph with parts A and B of size d and $n - d$, respectively.

Suppose first that we wish to obtain a graph which contains a matching of size $\lfloor n/2 \rfloor$ on the vertex set $A \cup B$. As $|A| < |B|$, any such matching must have at least $\lfloor n/2 \rfloor - |A| \geqslant \eta - 1/2$ edges contained in B. The expected number of edges in B in the random graph $G(n,p)$ is $\mathbb{E}[|E(G(n,p)[B])|] = \binom{n/2+\eta}{2}p = \Theta(n^2 p)$. Thus, by Markov's inequality, if $p = o(\eta/n^2)$, then a.a.s. $G(n,p)[B]$ contains $o(\eta)$ edges. Since any Hamilton cycle contains a matching of size $\lfloor n/2 \rfloor$, this completes the proof of the (0)-statement for Theorem 2. (There is in fact one missing case, when n is odd and $\eta = 1/2$; in this case, clearly any Hamilton cycle must contain at least one edge in B, and this will not occur if $p = o(n^{-2})$, again by Markov's inequality.)

Suppose now that $\eta = \omega(1)$ and $\eta = o(n)$. Then $\mathbb{E}[|E(G(n,p)[B])|] = \binom{n/2+\eta}{2}p = (1+o(1))n^2p/8$. Since $|E(G(n,p)[B])|$ is a binomial random variable, for any fixed $\varepsilon > 0$, if $p \leqslant (1-\varepsilon)8\eta/n^2$, it follows by Chernoff's inequality that a.a.s. $|E(G(n,p)[B])| \leqslant (1-\varepsilon/2)\eta$, and so $H_n \cup G(n,p)$ does not contain a matching of size $\lfloor n/2 \rfloor$. This completes the (0)-statement for Theorem 6.

Lastly, consider any cycle on vertex set $A \cup B$. Each such cycle can be mapped to a (cyclic) word consisting of n symbols, each being an A or a B depending on which set each vertex of the cycle belongs to. Due to the difference in sizes of the sets, each such word must have at least $|B| - |A| = 2\eta$ pairs of consecutive B's. In other words, in order for $H_n \cup G(n,p)$ to contain a Hamilton cycle, a necessary condition is that $G(n,p)[B]$ must contain at least 2η edges. Arguing like above with Chernoff's inequality, we conclude that a.a.s. this does not hold if $p \leqslant (1-\varepsilon)16\eta/n^2$, thus completing the (0)-statement for Theorem 3.

From now on, we focus on the proofs of the (1)-statements. For technical reasons, it will be convenient to step away from the binomial random graph model and instead consider uniform random graphs. Given an integer $m \in [\binom{n}{2}]$, the random graph $G_{n,m}$ is an n-vertex graph with exactly m edges chosen uniformly at random among all such graphs. A well-known coupling argument allows us to work with this model and transfer the results we obtain to the binomial random graph model. Indeed, assuming $m = \omega(1)$, there exist $p = (1+o(1))m/\binom{n}{2}$ and a coupling (G_1, G_2) such that $G_1 \sim G_{n,m}$, $G_2 \sim G(n,p)$ and a.a.s. $G_1 \subseteq G_2$. As such, in order to conclude our proofs, it suffices to study $H_n \cup G_{n,m}$.

2.1 Hamiltonicity

Given that the (0)-statements have already been proved, Theorems 2 and 3 are an immediate consequence of the following theorem.

Theorem 7. *Let H_n be a graph on n vertices with $\delta(H_n) \geqslant n/2 - \eta$, where $1/2 \leqslant \eta \leqslant n/64$.*

(*a*) *If $m = \omega(\eta)$, then a.a.s. $H_n \cup G_{n,m}$ is Hamiltonian.*

(*b*) *If $\eta = \omega(1)$, $\lambda = \omega(\eta^{1/2})$ and $m \geqslant \frac{2\eta}{1/4 - 8\eta/n} + \lambda$, then a.a.s. $H_n \cup G_{n,m}$ is Hamiltomian.*

One of the main tools for our proof of Theorem 7 is the following classical result of Dirac, which ensures that, if H_n is 2-connected, then it contains an almost spanning cycle.

Lemma 8 (Dirac [22, Theorem 4])**.** *Any 2-connected graph G on n vertices with $\delta(G) = d$, where $1 < d \leqslant n/2$, contains a cycle of length at least $2d$.*

The next lemma ensures that, for any 2-connected non-Hamiltonian graph H_n with high minimum degree, there is a large number of non-edges (essentially as many as in an unbalanced complete bipartite graph) which, if added to H_n, would result in a graph containing longer cycles than H_n.

Lemma 9. *Let H_n be an n-vertex 2-connected graph with $\delta(H_n) \geqslant n/2 - \eta > 1$. Suppose H_n is not Hamiltonian and let $\mathcal{C}$ be a longest cycle in H_n. Then, there exists a set $E \subseteq E(K_n)$ of size $|E| \geqslant n^2/8 - 4n\eta$ such that, for any $e \in E$, $H_n \cup \{e\}$ contains a cycle longer than $\mathcal{C}$.*

Proof. For each $v \in V(H_n)$, let $N_{\mathcal{C}}(v) := \{w \in V(\mathcal{C}) \mid vw \in E(H_n)\}$. By Lemma 8, the cycle $\mathcal{C}$ has length at least $n - 2\eta$. From the definition of $N_{\mathcal{C}}(v)$ and the minimum degree condition, for any $v \in V(H_n)$, we have that

$$|N_{\mathcal{C}}(v)| \geqslant n/2 - 3\eta. \tag{1}$$

Let us now fix an orientation of the cycle $\mathcal{C}$ and, for each vertex $w \in V(\mathcal{C})$, denote by w^- and w^+ the predecessor and the successor of w in the orientation of $\mathcal{C}$, respectively. Then, fix an arbitrary vertex $v \in V(H_n) \setminus V(\mathcal{C})$. Note that, if $w \in N_{\mathcal{C}}(v)$, then $w^- \notin N_{\mathcal{C}}(v)$, as otherwise $\mathcal{C} \cup \{w^-v, vw\} \setminus \{w^-w\}$ would form a cycle longer than $\mathcal{C}$ (see Figure 1 for reference). Analogously, $w^+ \notin N_{\mathcal{C}}(v)$. From this fact, one can easily see that $N_{\mathcal{C}}(v)$ decomposes $\mathcal{C}$ into paths of length at least 2 (where the *length* of a path is its number of edges). For each $i \geqslant 2$, denote by X_i the number of paths of length i in $\mathcal{C}$ which result from this decomposition of $\mathcal{C}$, so we have that

$$\sum_{i \geqslant 2} X_i = |N_{\mathcal{C}}(v)| \geqslant n/2 - 3\eta.$$

It follows that

$$|V(\mathcal{C})| = \sum_{i \geqslant 2} iX_i \geqslant 2X_2 + 3\left(\sum_{i \geqslant 2} X_i - X_2\right) \geqslant 2X_2 + 3\left(\frac{n}{2} - 3\eta - X_2\right),$$

which implies that

$$X_2 \geqslant \frac{n}{2} - 9\eta. \tag{2}$$

Let $W_v := \{w \in V(\mathcal{C}) \setminus N_{\mathcal{C}}(v) \mid w^-, w^+ \in N_{\mathcal{C}}(v)\}$ and note that $|W_v| = X_2$. Observe that, if $w \in W_v$ and $u \in N_{\mathcal{C}}(w) \setminus \{w^+\}$, then $u^- \notin N_{\mathcal{C}}(w)$, as otherwise

$$\mathcal{C} \setminus \{w^-w, ww^+, u^-u\} \cup \{w^-v, vw^+\} \cup \{u^-w, wu\}$$

would form a cycle longer than $\mathcal{C}$ (see Figure 1). In other words, for every $w \in W_v$ and $u \in N_{\mathcal{C}}(w) \setminus \{w^+\}$, we have that $wu^- \notin E(H_n)$ and $H_n \cup \{wu^-\}$ would have a cycle longer than $\mathcal{C}$. Thus, using (1) and (2), there exist at least $(n/2-9\eta)(n/2-3\eta-1)/2 \geqslant n^2/8-4n\eta$ edges $e \in E(K_n) \setminus E(H_n)$ such that $H_n \cup \{e\}$ contains a longer cycle than $\mathcal{C}$ (where we divide by 2 to avoid double counting edges). $\square$

Our last tool is a particular case of a result of Bohman, Frieze, Krivelevich and Martin [11] which ensures that, if H_n has high minimum degree, then a.a.s. we need to add very few random edges for it to become 2-connected, so that Lemmas 8 and 9 apply.

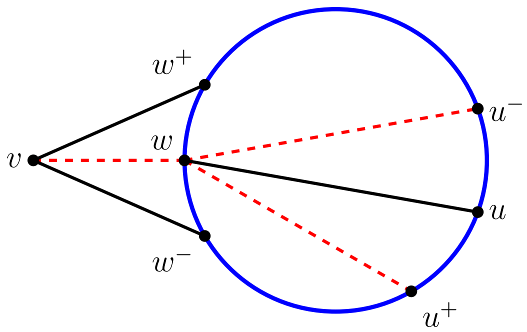

Figure 1: A depiction for the proof of Lemma 9. The circle represents a longest cycle $\mathcal{C}$ in H_n, and $v \in V(H_n) \setminus V(\mathcal{C})$. The neighbours of v in $\mathcal{C}$ split it into paths, most of which have length 2 and are of the form w^-ww^+, where $w \notin N_{\mathcal{C}}(v)$. For each neighbour u of one such vertex w in $\mathcal{C}$, both its predecessor and successor along $\mathcal{C}$ cannot be neighbours of w, and thus can be elements of E.

Lemma 10 (Bohman, Frieze, Krivelevich and Martin [11, Theorem 6])**.** *Let H_n be an n-vertex graph with $\delta(H_n) \geqslant \alpha n$, where $\alpha \in (0,1)$ is fixed. If $m = \omega(1)$, then a.a.s. $H_n \cup G_{n,m}$ is 2-connected.*

With these results in hand, we can already prove Theorem 7. The main idea is to add random edges in rounds to the graph, each round ending when a longer cycle can be found. This is iterated until a Hamilton cycle appears; this process is analogous to the one used by Bohman, Frieze and Martin [12]. Since the starting graph is almost Hamiltonian, the number of rounds that is needed is small. Moreover, since at every step there is a large set of "good" non-edges, each round is likely to finish rather quickly. This leads to the improved bounds.

Proof of Theorem 7. First, let $m_0 = \omega(1)$ grow arbitrarily slowly. Let $G_0 \coloneqq H_n \cup G_{n,m_0}$. By Lemma 10, a.a.s. G_0 is 2-connected. Condition on this event. Let $\mathcal{C}$ be a longest cycle in G_0 and $k \coloneqq |V(H_n) \setminus V(\mathcal{C})|$. Note that $k \leqslant 2\eta$ by Lemma 8. Let E_0 be the set of all edges $e \in E(K_n) \setminus E(G_0)$ such that $G_0 \cup \{e\}$ has a longer cycle than G_0.

Now, consider a sequence of random graphs $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_k$ defined as follows. For each $i \in [k]$, if $G_{i-1}^0 \coloneqq G_{i-1}$ is not Hamiltonian, we start sampling edges $e_i^{(1)}, e_i^{(2)}, \ldots \in E(K_n)$ uniformly at random with replacement and, for each $j \in \mathbb{N}$, define $G_{i-1}^j \coloneqq G_{i-1}^{j-1} \cup \{e_i^{(j)}\}$. In this case, we let Y_i be the minimum $j \in \mathbb{N}$ such that $e_i^{(j)} \in E_{i-1}$ and, then, set $G_i \coloneqq G_{i-1}^{Y_i}$. Moreover, we define E_i to be the set of all edges $e \in E(K_n) \setminus E(G_i)$ such that $G_i \cup \{e\}$ has a longer cycle than G_i. If instead G_{i-1} is Hamiltonian, we let $G_i = G_{i+1} = \cdots = G_k \coloneqq G_{i-1}$ and $Y_i = Y_{i+1} = \cdots = Y_k \coloneqq 0$.

Let h be the minimum $i \in [k]$ for which G_i is Hamiltonian. Note that this is well defined by the definition of k, since each subsequent G_i either is Hamiltonian or contains a longer cycle than G_{i-1}. The random variables $Y_1, Y_2, \ldots, Y_h$ are independent geometric random variables with parameter $p_i = |E_{i-1}|/\binom{n}{2} \geqslant 1/4 - 8\eta/n$, where the inequality holds by Lemma 9. We set $Y \coloneqq \sum_{i=1}^k Y_i = \sum_{i=1}^h Y_i$ to be the total number of random edges added until G_k is Hamiltonian, and let G^Y denote the n-vertex graph consisting of all these random edges. Note that there is a trivial coupling between G_{n,m_0}, G^Y and G_{n,m_0+Y} such that $G_{n,m_0} \cup G^Y \subseteq G_{n,m_0+Y}$, so it suffices to bound Y to reach the desired conclusion.

In order to prove (a), since $Y_1, Y_2, \dots, Y_h$ are geometric random variables and $k \leqslant 2\eta$, note that

$$\mathbb{E}[Y] = \sum_{i=1}^{k} \mathbb{E}[Y_i] = \sum_{i=1}^{h} \frac{1}{p_i} \leqslant \frac{2\eta}{1/4 - 8\eta/n} \leqslant 16\eta,$$

where the second inequality holds by the upper bound on η. Thus, by Markov's inequality and since m_0 is chosen to grow arbitrarily slowly, if $m = \omega(\eta)$, then a.a.s. $Y \leqslant m$, that is, a.a.s. adding $m_0 + m = \omega(\eta)$ random edges to H_n suffices to obtain a Hamiltonian graph.

In order to prove (b), assume that $\eta = \omega(1)$. Recall that $\mathbb{E}[Y] \leqslant \frac{2\eta}{1/4-8\eta/n}$. Moreover, since the variables $Y_1, \dots, Y_k$ are independent, again using the upper bound on η we have that

$$\operatorname{Var}(Y) = \sum_{i=1}^{k} \operatorname{Var}(Y_i) = \sum_{i=1}^{h} \frac{1-p_i}{p_i^2} \leqslant 2\eta \frac{3/4 + 8\eta/n}{(1/4 - 8\eta/n)^2} \leqslant 2\eta \frac{7/8}{(1/8)^2} = 112\eta.$$

By Chebyshev's inequality, for any $\lambda = \omega(\eta^{1/2})$, we have that $\mathbb{P}[|Y - \mathbb{E}[Y]| \geqslant \lambda/2] = o(1)$, so a.a.s. $Y \leqslant \mathbb{E}[Y] + \lambda/2 \leqslant \frac{2\eta}{1/4-8\eta/n} + \lambda/2$. Since m_0 is chosen to grow arbitrarily slowly, we may take $m_0 \leqslant \lambda/2$, and it follows that a.a.s. adding $m_0 + Y \leqslant \frac{2\eta}{1/4-8\eta/n} + \lambda$ random edges to H_n suffices to obtain a Hamiltonian graph. $\square$

2.2 Perfect matchings

For simplicity, throughout this section we assume that n is even. Our goal is to complete the proof of Theorem 6. Since we already showed its (0)-statement, it suffices to prove the following result. (For the sake of simplifying the calculations, we state it only for $\eta = o(n)$, which suffices for Theorem 6; for larger values of η, a statement similar to Theorem 7 (b) holds as well.)

Theorem 11. *Let H_n be a graph on n vertices with minimum degree at least $n/2 - \eta$. If $\eta = o(n)$ with $\eta = \omega(1)$, $\lambda = \omega(\eta^{1/2})$ and $m \geqslant 4\eta + \lambda$, then a.a.s. $H_n \cup G_{n,m}$ contains a perfect matching.*

The proof of Theorem 11 follows along similar lines as that of Theorem 7. The main part is to show that, if H_n is 2-connected, then there are *many* non-edges that, if added to H_n, would result in a graph with a larger matching than H_n.

Lemma 12. *Let H_n be an n-vertex 2-connected graph with $\delta(H_n) \geqslant n/2 - \eta > 1$. Assume that H_n does not contain a perfect matching and let $\mathcal{M}$ be a largest matching in H_n. Then, there exists a set $M \subseteq E(K_n)$ of size at least $n^2/8 - 4\eta n$ such that, for any $e \in M$, $H_n \cup \{e\}$ contains a matching which is larger than $\mathcal{M}$.*

Proof. By Lemma 8, H_n contains a cycle of length at least $n - 2\eta$, so H_n contains a matching of size at least $n/2 - \eta$. Let $\mathcal{M}$ be a largest matching in H_n and, for each $w \in V(\mathcal{M})$, let $w^{\mathcal{M}}$ denote the vertex $w' \in V(\mathcal{M})$ such that $ww' \in \mathcal{M}$. Moreover, for each $u \in V(H_n) \setminus V(\mathcal{M})$, let $N_{\mathcal{M}}(u) \coloneqq \{w \in V(\mathcal{M}) \mid w^{\mathcal{M}} \in N(u)\}$, where $N(u)$ represents the set of all neighbours of u in H_n. An $\mathcal{M}$-*augmenting path* P is a path whose endpoints are not in $\mathcal{M}$ and whose edges alternate between $E(H_n) \setminus \mathcal{M}$ and $\mathcal{M}$; in particular, the edge set $\mathcal{M}_P \coloneqq (\mathcal{M} \setminus E(P)) \cup (E(P) \setminus \mathcal{M})$ is a larger matching than $\mathcal{M}$.

Note that, for any pair of distinct vertices $u, v \in V(H_n) \setminus V(\mathcal{M})$, H_n cannot contain a (u, v)-path of length 5 using two edges of $\mathcal{M}$, as otherwise this would be an $\mathcal{M}$-augmenting path. It follows that, for any pair of distinct $u, v \in V(H_n) \setminus V(\mathcal{M})$ and all $x \in N_{\mathcal{M}}(u)$ and $y \in N_{\mathcal{M}}(v)$ with $x \neq y$, we must have that $e = xy \notin E(H_n)$, since otherwise we would have one such path of length 5. In particular, each such edge is a potential edge that, if added to H_n, creates a graph with a larger matching than $\mathcal{M}$. From the minimum degree condition and the size of $\mathcal{M}$, we conclude that $|N_{\mathcal{M}}(u)| \geqslant n/2-3\eta$ and $|N_{\mathcal{M}}(v)| \geqslant n/2-3\eta$. Hence, there exist at least $\binom{n/2-3\eta}{2} \geqslant n^2/8 - 4\eta n$ such potential edges. $\square$

The proof of Theorem 11 is now essentially the same as the proof of Theorem 7 (b), using Lemma 12 instead of Lemma 9. The main difference is that, through the sequence of random graphs, each subsequent largest matching contains at least two more vertices than the previous, and so one may take $k \coloneqq |V(H_n) \setminus V(\mathcal{M})|/2 \leqslant \eta$ (where $\mathcal{M}$ is a largest matching in G_0). This smaller number of random graphs in the sequence leads to the improved bound on m. We omit the details of the proof.

3 Open problems

With Theorems 3 and 6, we have showed that, if $d = n/2 - o(n)$, the sharp d-thresholds for Hamiltonicity and perfect matchings coincide with what is needed for a randomly perturbed unbalanced complete bipartite graph to contain a Hamilton cycle or perfect matching. It seems plausible that unbalanced complete bipartite graphs should also witness the sharp d-threshold for perfect matchings for smaller values of d. Let $d = \alpha n$ for some fixed constant $\alpha \in (0, 1/2)$, and let H_n be a complete bipartite graph with parts A and B of size d and $n - d$, respectively. Note that H_n contains a matching of size d, where, moreover, its d vertices in B can be chosen arbitrarily. We know, from the work of Bohman, Frieze and Martin [12] and the extremal example of the complete bipartite graph, that the sharp d-threshold for the containment of a perfect matching (if it exists) must be of the form C/n, where $C = C(\alpha)$ is a constant. We conjecture that this threshold should coincide with the threshold for $G(n, p)[B] \sim G(n - d, p)$ to contain a matching of size $n/2 - d$ (which can then be completed using edges of H_n). The (likely) size of a largest matching in sparse random graphs was determined by Karp and Sipser [36] (see also [9, Theorem 4] for a concrete expression). Using their work, we propose the following conjecture.

Conjecture 13. Let $\alpha \in (0, 1/2)$ be fixed, and let $d = \alpha n$. The sharp d-threshold for containing a perfect matching is C/n, where $C = C(\alpha)$ is the solution to the equation

$$1 - \frac{\gamma_* + \gamma^* + \gamma_*\gamma^*}{(2-2\alpha)C} = \frac{1-2\alpha}{2-2\alpha}, \tag{3}$$

where γ_* is the smallest root of the equation $x = (1-\alpha)C\exp(-(1-\alpha)C\mathrm{e}^{-x})$ and $\gamma^* = (1-\alpha)C\mathrm{e}^{-\gamma_*}$.

(To see where the constant in (3) comes from, note from [9, Theorem 4] that, for $p = C/n$, a.a.s. the largest matching of $G(n,p)$ has size

$$(1 \pm o(1))\left(1 - \frac{\theta_* + \theta^* + \theta_*\theta^*}{2C}\right)n,$$

where θ_* is the smallest root of the equation $x = C\exp(-C\mathrm{e}^{-x})$ and $\theta^* = C\mathrm{e}^{-\theta_*}$. Naturally, we are interested in the maximum matching in $G((1-\alpha)n, p) \sim G(n', (1-\alpha)C/n')$ as n' tends to infinity. This yields the expression on the left in (3). The expression on the right comes from wanting this matching to have size at least $n/2 - d = (1-2\alpha)n'/(2-2\alpha)$.)

It also seems plausible that complete bipartite graphs should be the extremal example for Hamiltonicity for the entire range of d. Clearly, the complete bipartite graph H_n defined above contains a cycle of length $2d$, and no longer cycles. If an edge with both endpoints in B is added to H_n, this can be used to construct a longer cycle. In this case, however, it is not only isolated edges that are useful for constructing longer cycles: indeed, any path (of length at most $n-2d$) contained in B can be incorporated into a cycle. Thus, a linear forest (that is, a collection of vertex-disjoint paths) containing $n-2d$ edges within B can be used to construct a Hamilton cycle.

Conjecture 14. Let $\alpha \in (0, 1/2)$ be fixed, and let $d = \alpha n$. The sharp d-threshold for Hamiltonicity coincides with the sharp threshold for $G(n-d, p)$ to contain a linear forest of size $n - 2d$.

In this case, we do not propose an explicit expression for the threshold since the size of the largest linear forest in a sparse random graph has not been considered in the literature. Studying this problem may be of independent interest. We expect the sharp d-threshold for Hamiltonicity to differ from that for perfect matchings by a constant factor, with this constant factor depending on α and tending to 1 as α tends to 0.

We also believe Conjecture 14 should hold for pancyclicity. Moreover, we have no reason to believe that the lower bound on d in Corollary 5 is necessary, and think that a statement analogous to Theorem 3 should hold for pancyclicity as well. Naturally, we also believe that determining the sharp d-threshold for perfect matchings, Hamiltonicity and pancyclicity in the critical window $d = o(n)$ is a problem of interest.

Lastly, we want to consider the extension of our results to randomly perturbed *directed* graphs (or *digraphs* for short), where we allow up to two edges between each pair of vertices, one in each direction. We define thresholds analogously as above, where now the binomial random digraph $D(n,p)$ is obtained by adding each of the possible $n(n-1)$ edges independently at random with probability p, and instead of the minimum degree of a graph we consider the minimum *semidegree* of the digraph, which is the minimum, over all vertices, of the minimum between the number of edges leaving and the number of edges arriving at each vertex. A classical result of Ghouila-Houri [29] shows that, if $\delta^0(D) = d \geqslant n/2$, then the (sharp) d-threshold for Hamiltonicity is 0. The sharp 0-threshold is again $\log n/n$ (as follows by a general coupling argument of McDiarmid [42]). The Hamiltonicity of randomly perturbed digraphs when $d = \alpha n$ with $\alpha \in (0, 1/2)$ fixed was studied by Bohman, Frieze and Martin [12], who showed that the d-threshold in this case is also $1/n$ (later, Krivelevich, Kwan and Sudakov [40] provided a new proof of this fact). Just like in graphs, the thresholds present two critical windows around $\alpha = 0$ and $\alpha = 1/2$, and it is thus natural to consider the d-thresholds in these regimes. The extension of Theorem 2 to digraphs remains open.

Conjecture 15. Let $d = n/2 - \eta$, where $1/2 \leqslant \eta = \eta(n) = o(n)$. The d-threshold for Hamiltonicity in randomly perturbed directed graphs is η/n^2.

Very recently, Araujo, Balogh, Krueger, Piga and Treglown [8] considered different orientations of Hamilton cycles in randomly perturbed digraphs. It would also be interesting to extend their work to the corresponding critical windows.

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