

On Vertex-Girth-Regular Graphs: (Non-)Existence, Bounds and Enumeration

Robert Jajcay^a Jorik Jooken^b István Porupsánszki^c

Submitted: Aug 26, 2024; Accepted: Oct 6, 2025; Published: Nov 14, 2025

© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

A vertex-girth-regular $\text{vgr}(n, k, g, \lambda)$ -graph is a k -regular graph of girth g and order n in which every vertex belongs to exactly λ cycles of length g . While all vertex-transitive graphs are necessarily vertex-girth-regular, the majority of vertex-girth-regular graphs are not vertex-transitive. Similarly, while many of the smallest k -regular graphs of girth g , the so-called (k, g) -cages, are vertex-girth-regular, infinitely many vertex-girth-regular graphs of degree k and girth g exist for many pairs k, g . Due to these connections, the study of vertex-girth-regular graphs promises insights into the relations between the classes of extremal, highly symmetric, and locally regular graphs of given degree and girth. This paper lays the foundation to such study by investigating the fundamental properties of $\text{vgr}(n, k, g, \lambda)$ -graphs, specifically the relations necessarily satisfied by the parameters n, k, g and λ to admit the existence of a corresponding vertex-girth-regular graph, by presenting constructions of infinite families of $\text{vgr}(n, k, g, \lambda)$ -graphs, and by establishing lower bounds on the number n of vertices in a $\text{vgr}(n, k, g, \lambda)$ -graph. It also includes computational results determining the orders of smallest cubic and quartic graphs of small girths.

Mathematics Subject Classifications: 05C07, 05C35, 05C38

1 Introduction

The motivation for considering vertex-girth-regular graphs comes from two seemingly disconnected areas.

The first is the Cage Problem; a part of Extremal Graph Theory where one searches for k -regular graphs of girth g , (k, g) -graphs, of the smallest possible order; called (k, g) -cages [7]. Even though this problem has a large number of practical applications (e.g.,

^aDepartment of Algebra and Geometry, Faculty of Mathematics, Physics and Informatics, Mlynská dolina, 842 48 Bratislava, Slovakia (robert.jajcay@fmph.uniba.sk).

^bDepartment of Computer Science, KU Leuven Kulak, 8500 Kortrijk, Belgium (jorik.jooken@kuleuven.be).

^cDepartment of Geometry and MTA-ELTE Geometric and Algebraic Combinatorics Research Group, Eötvös Loránd University, 1117 Budapest, Pázmány s. 1/c, Hungary (rupsansz@gmail.com).

in Network Design) and has been studied since the 1940's, very few orders of (k, g) -cages are known. This is due, among other reasons, to our lack of understanding of the structure of cages as well as the considerable size of the search spaces associated with searching for smallest graphs of given parameters k, g . To fill the existing void, it is often useful to rely on heuristics – observations based on the structure of known cages or smallest known (k, g) -graphs. One such insight is that small (k, g) -graphs often exhibit a high level of regularity and tend to look similar with respect to each of their vertices. Transforming this observation into a structural description suggests for example that each vertex of a small (k, g) -graph is contained in a similar number of *girth-cycles*; cycles of length g . This ultimately leads to the study of (k, g) *vertex-girth-regular graphs* which are k -regular graphs of girth g in which every vertex is contained in the same number of girth-cycles; which we shall usually denote by λ . Since our partial aim is to shed light on structural properties of small (k, g) -graphs, we also address the question of smallest orders of vertex-girth-regular graphs for a given triple k, g and λ . Similar questions concerning girth-regularity of regular graphs have been studied in a series of papers focusing on the orders of smallest k -regular graphs of girth g in which every *edge* is contained in the same number of girth-cycles, called *edge-girth-regular graphs* $\text{egr}(n, k, g, \lambda)$ [5, 10], and in the paper [16] where the authors consider *girth-regular graphs* $\text{gr}(n, k, g, \mathbf{a})$: k -regular graphs of girth g having the property that the *signature* of every vertex is the same, where the signature $\mathbf{a} = \{a_1, a_2, \dots, a_k\}$ of a vertex u represents the number of times the k edges adjacent to u are contained in girth-cycles. Clearly, both of the above classes of graphs are also vertex-girth-regular. However, in our definition, we do not make any assumptions about the distribution of girth-cycles among the edges adjacent to a vertex. It is also worth noting that the number of girth-cycles through any vertex in an edge-girth-regular graph is necessarily a multiple of half of the degree, $\frac{k}{2}$.

The second source of inspiration for our study of vertex-girth-regular graphs is the class of vertex-transitive graphs which are necessarily vertex-girth-regular (and much more; since they have the property that the number of cycles of any specific length through each vertex is the same). Thus, in some sense, the study of vertex-girth-regular graphs is the study of the connection between vertex-transitivity and girth-regularity much the same way as the study of edge-girth-regular graphs is connected to edge-transitivity. It is important to note that neither edge- nor vertex-girth-regularity imply edge- or vertex-transitivity; as amply exhibited by our example of an edge-girth-regular tetravalent graph on 20 vertices which has a trivial automorphism group (see Fig. 1).

2 Preliminaries and notation

We first recall the Moore bound $M(k, g)$, which provides a lower bound for the order of a (k, g) -graph:

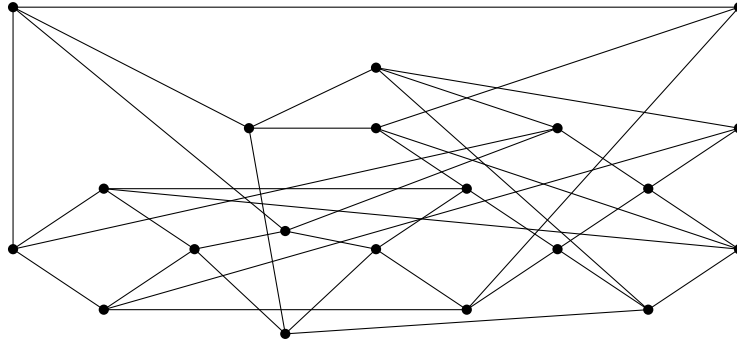


Figure 1: An $\text{egr}(20, 4, 4, 1)$ -graph which is asymmetric (it only has the trivial automorphism).

Observation 1 (Folklore). *Let G be a (k, g) -graph of order n . Then, we have:*

$$n \geq M(k, g) = \begin{cases} 1 + \sum_{i=0}^{(g-3)/2} k(k-1)^i & \text{if } g \text{ is odd,} \\ 2 \sum_{i=0}^{(g-2)/2} (k-1)^i & \text{if } g \text{ is even.} \end{cases}$$

If a (k, g) -graph attains this bound, we call it a *Moore graph*. This lower bound holds because every (k, g) -graph contains a (k, g) -Moore tree as a subgraph, which we will now define. Alongside the definition of a (k, g) -Moore tree, we will also define some other notation that we will later refer to.

Definition 2 ((k, g) -Moore tree). Let G be a (k, g) -graph. We now consider two cases: (i) g is odd and (ii) g is even.

(i): Consider a vertex $u_1 \in V(G)$ and let $\mathcal{T}_{k, \frac{g-1}{2}}^{u_1}$ be a subgraph of G such that $V(\mathcal{T}_{k, \frac{g-1}{2}}^{u_1})$ consists of all vertices which are at distance at most $\frac{g-1}{2}$ from u_1 and $E(\mathcal{T}_{k, \frac{g-1}{2}}^{u_1})$ contains all edges of $E(G)$ between vertices of $V(\mathcal{T}_{k, \frac{g-1}{2}}^{u_1})$ except for edges between vertices at distance $\frac{g-1}{2}$ from u_1 . Since G has girth g , $\mathcal{T}_{k, \frac{g-1}{2}}^{u_1}$ is a tree (called a (k, g) -Moore tree) and $|V(\mathcal{T}_{k, \frac{g-1}{2}}^{u_1})| = M(k, g)$.

(ii): Consider an edge $u_1 u_2 \in E(G)$ and let $\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2}$ be the subtree of G consisting of the edge $u_1 u_2$ and two disjoint trees rooted at u_1 and u_2 , respectively, such that the leaves of the two trees are at distance $\frac{g}{2} - 1$ from u_1 and u_2 , respectively. We call $V(\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2})$ a (k, g) -Moore tree (note that $|V(\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2})| = M(k, g)$). Let D_{u_1} and D_{u_2} be the sets of leaves of $\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2}$ at distance $\frac{g}{2} - 1$ from u_1 and u_2 , respectively. Let $v_1, v_2, \dots, v_{k-1}$ be the $k-1$ neighbors of u_1 distinct from u_2 and let $D_{u_1,1}, D_{u_1,2}, \dots, D_{u_1,k-1}$ be the sets of leaves at distance $\frac{g}{2} - 2$ from $v_1, v_2, \dots, v_{k-1}$, respectively. The set of girth-cycles containing u_1 can be partitioned in three sets $\mathcal{A}_{u_1}, \mathcal{B}_{u_1}$ and $\mathcal{C}_{u_1}$, where $\mathcal{A}_{u_1}$ is the set of girth-cycles that contain exactly one edge with an endpoint in D_{u_1} and one endpoint in D_{u_2} , $\mathcal{B}_{u_1}$ is the set of girth-cycles that contain exactly two edges with one of their endpoints in D_{u_1} and a

shared endpoint in D_{u_2} , and $\mathcal{C}_{u_1}$ is the set of girth-cycles that contain exactly two edges with one endpoint in D_{u_1} and one endpoint in $V(G) \setminus V(\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2})$.

As we already stated in the Introduction, for integers n , k , g and λ , a vertex-girth-regular $\text{vgr}(n, k, g, \lambda)$ -graph is a k -regular graph with girth g on n vertices such that every vertex is contained in exactly λ cycles of length g .

It is important to note that not every triple (k, g, λ) admits the existence of a $\text{vgr}(n, k, g, \lambda)$ -graph. For example, we can derive the following obvious upper bounds on λ . Several of the proofs in this paper will follow the same setup as the proof of the following proposition.

Proposition 3. *Let G be a $\text{vgr}(n, k, g, \lambda)$ -graph. Then $\lambda \leq \frac{k(k-1)^{\lfloor \frac{g}{2} \rfloor}}{2}$ with equality if and only if G is a Moore graph.*

Proof. We first prove the result for (i) g odd and then for (ii) g even.

(i): Let u_1 and $\mathcal{T}_{k, \frac{g-1}{2}}^{u_1}$ be chosen as in Definition 2. Now there is a one-to-one correspondence between girth-cycles of G containing u_1 and edges whose endpoints are both leaves of $\mathcal{T}_{k, \frac{g-1}{2}}^{u_1}$. Since each vertex of G has degree k and $\mathcal{T}_{k, \frac{g-1}{2}}^{u_1}$ has $k(k-1)^{\frac{g-3}{2}}$ leaves, we obtain

that the number of edges between two leaves of $\mathcal{T}_{k, \frac{g-1}{2}}^{u_1}$ is bounded from above by $\frac{k(k-1)^{\frac{g-1}{2}}}{2}$ and this bound is attained if and only if G is a Moore graph.

(ii): The even girth case uses a similar setup as the odd girth case, but requires different arguments.

Let $u_1 u_2$, $\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2}$, $\mathcal{A}_{u_1}$, $\mathcal{B}_{u_1}$, $\mathcal{C}_{u_1}$, D_{u_1} , D_{u_2} , $D_{u_1,1}$, $D_{u_1,2}, \dots, D_{u_1, k-1}$ be chosen as in Definition 2. Note that each girth-cycle in $\mathcal{A}_{u_1}$ contains the edge $u_1 u_2$, whereas none of the girth-cycles in $\mathcal{B}_{u_1}$ and $\mathcal{C}_{u_1}$ contain $u_1 u_2$. Since every vertex in G has degree k and $|D_{u_1}| = (k-1)^{\frac{g-2}{2}}$, we obtain $|\mathcal{A}_{u_1}| \leq (k-1)^{\frac{g}{2}}$. Since G has girth g , each vertex $v \in (V(G) \setminus V(\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2})) \cup D_{u_2}$ is adjacent to at most one vertex in $D_{u_1, i}$, for each integer $1 \leq i \leq k-1$. Therefore, every vertex $v \in (V(G) \setminus V(\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2})) \cup D_{u_2}$ is adjacent to at most $k-1$ vertices in D_{u_1} . Since each vertex in G has degree k , we obtain $|\mathcal{B}_{u_1}| + |\mathcal{C}_{u_1}| \leq (k-1)^{\frac{g-2}{2}} \frac{(k-1)(k-2)}{2}$. This yields $\lambda = |\mathcal{A}_{u_1}| + |\mathcal{B}_{u_1}| + |\mathcal{C}_{u_1}| \leq (k-1)^{\frac{g}{2}} + (k-1)^{\frac{g-2}{2}} \frac{(k-1)(k-2)}{2} = \frac{k(k-1)^{\frac{g}{2}}}{2}$ and if equality occurs, $|\mathcal{A}_{u_1}| = (k-1)^{\frac{g}{2}}$ and therefore $|\mathcal{B}_{u_1}| = (k-1)^{\frac{g-2}{2}} \frac{(k-1)(k-2)}{2}$ and $|\mathcal{C}_{u_1}| = 0$, which implies that G is a Moore graph. Conversely, if G is a Moore graph, we have $|\mathcal{A}_{u_1}| = (k-1)^{\frac{g}{2}}$, $|\mathcal{B}_{u_1}| = (k-1)^{\frac{g-2}{2}} \frac{(k-1)(k-2)}{2}$, $|\mathcal{C}_{u_1}| = 0$ and $\lambda = \frac{k(k-1)^{\frac{g}{2}}}{2}$. $\square$

Beside the upper bounds on λ derived from the properties of cages stated in Proposition 3, vertex-girth-regular graphs also do not exist in cases when λ is close to the upper bounds stated in there.

For example, when considering cubic vertex-girth-regular graphs of girth 3, it is easy to see that K_4 is a $\text{vgr}(4, 3, 3, 3)$ -graph. Moreover, it is the unique 3-regular graph of girth 3 and $\lambda = 3$, which is the maximal λ in any $\text{vgr}(n, 3, 3, \lambda)$ -graph. Also, it is not very hard to construct a 3-regular graph of girth 3 and $\lambda = 1$, as shown in Fig. 2.

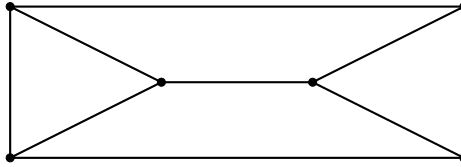


Figure 2: A $\text{vgr}(6, 3, 3, 1)$ -graph.

However, there is no cubic vertex-girth-regular graph of girth 3 in which every vertex belongs to 2 girth-cycles. This is a consequence of the following lemma.

Lemma 4. *Let $k \geq 3$. There is no integer n such that a $\text{vgr}(n, k, 3, \binom{k}{2} - 1)$ -graph exists.*

Proof. It is easy to see that the complete graph K_{k+1} is a $\text{vgr}(k+1, k, 3, \binom{k}{2})$ -graph, and that $\lambda \leq \binom{k}{2}$ in a $\text{vgr}(n, k, 3, \lambda)$ -graph.

Consider a potential $\text{vgr}(n, k, 3, \binom{k}{2} - 1)$ -graph G . Let u be a vertex in G , and let $v_1, v_2, \dots, v_k$ be the neighbors of u . Similarly as in K_{k+1} , each pair of distinct neighbors of u , except for one, must be adjacent (recall that u is assumed to be contained in $\binom{k}{2} - 1$ triangles). Without loss of generality, we may assume that v_1 and v_2 are the neighbors of u that are non-adjacent. As v_1 is assumed to be of degree k , it must have a neighbor w distinct from the vertices $u, v_1, v_2, \dots, v_k$. Furthermore, note that each triangle containing v_1 must contain two adjacent neighbors of v_1 , while w is not adjacent to any of the vertices $u, v_3, \dots, v_k$ as they are already assumed to be of degree k . Note that w may be adjacent to v_2 , but this does not yield a triangle containing v_1 . This means that v_1 is contained in at most $\binom{k-1}{2}$ triangles. Since $k \geq 3$, $\binom{k-1}{2} < \binom{k}{2} - 1$. Therefore G is not a $\text{vgr}(n, k, 3, \binom{k}{2} - 1)$ -graph; which completes the argument. $\square$

Thus, unlike the case of (k, g) -graphs which exist for every pair of parameters $k, g \geq 3$ [17], the question of the existence of at least one $\text{vgr}(n, k, g, \lambda)$ -graph for a given triple (k, g, λ) necessarily precedes the question of the order of a smallest such graph. That is why we begin our paper with Section 3 in which we show the existence of $\text{vgr}(n, k, g, \lambda)$ -graphs for large classes of triples (k, g, λ) . This is followed by Section 4 where we derive several natural lower bounds on the orders of $\text{vgr}(n, k, g, \lambda)$ -graphs. In analogy to the Cage Problem, we define $n(k, g, \lambda)$ to be the smallest integer n such that a $\text{vgr}(n, k, g, \lambda)$ -graph exists (or ∞ otherwise), and similarly, $n_2(k, g, \lambda)$ to be the smallest integer n such that a bipartite $\text{vgr}(n, k, g, \lambda)$ exists (and ∞ otherwise). We then present further non-existence results in Section 5, and conclude the paper with a number of computational results in which we determine the orders of smallest vertex-girth-regular graphs for various sets of small parameter triples (k, g, λ) .

Before exiting this section, let us revisit Lemma 4. As shown by the existence of a $\text{vgr}(6, 3, 3, 1)$ -graph, at least in case of $k = 3$, the lemma cannot be strengthened. Even though we were unable to find a general proof for the claim that no $\text{vgr}(n, k, 3, \binom{k}{2} - \epsilon)$ -graphs exist for $0 < \epsilon < \frac{k-1}{2}$, the result appears feasible (especially in view of the analogous Theorem 17). However, as shown by the existence of a $\text{vgr}(6, 4, 3, 4)$ -graph, the upper

bound $\epsilon < \frac{k-1}{2}$ is sharp and cannot be replaced by $\epsilon \leq \frac{k-1}{2}$ appearing in Theorem 17. Finally note that $\text{vgr}(2k, k, 3, \binom{k}{2} - (k-1))$ -graphs exist for all $k \geq 3$. They can be constructed in the same way as the $\text{vgr}(6, 3, 3, 1)$ -graph depicted above by joining two copies of the complete graph K_k via a perfect matching connecting each vertex in one of the copies to exactly one vertex in the other.

3 Existence results

In contrast to Lemma 4, we will show in this section that $\text{vgr}(n, k, g, \lambda)$ -graphs exist in many cases. We start by showing, based on the idea of generalized truncation, that one can construct vertex-girth-regular graphs of increasing degree starting from an arbitrary $\text{vgr}(n, k, g, \lambda)$ -graph (based on a similar idea as the one discussed in [17]).

Proposition 5. *If a $\text{vgr}(n, k, g, \lambda)$ -graph exists, then there exist infinitely many integers n' such that $\text{vgr}(n', k+1, g, \lambda)$ -graphs also exist.*

Proof. Let G be a $\text{vgr}(n, k, g, \lambda)$ -graph. Consider an n -regular graph G' of girth strictly larger than $g/2$ (such a graph exists for any choice of n and g as shown in [17]). Construct the graph H by generalized truncation: H is obtained by replacing every vertex in G' by a copy of the graph G (if u is a vertex of G' with neighbors $w_1, w_2, \dots, w_n$ and $x_1, x_2, \dots, x_n$ are the vertices of G in an arbitrary order, then the vertex u is replaced by the graph G and the edges $w_1x_1, w_2x_2, \dots, w_nx_n$ are added). An example with $G = C_4$ and $G' = K_5$ is given in Fig. 3.

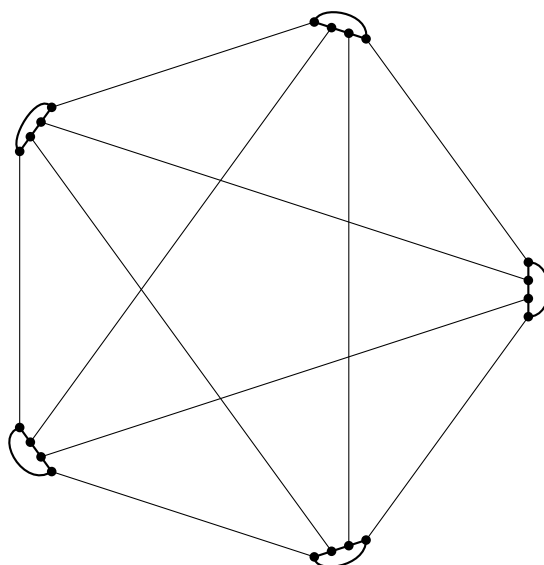


Figure 3: A graph H obtained by generalized truncation by taking $G = C_4$ and $G' = K_5$.

Now each cycle in H corresponds either to a cycle in G or to a cycle in G' in which each vertex is replaced by a path consisting of at least two vertices. Therefore H is a $(k+1)$ -regular graph with girth g such that each vertex is contained in λ girth-cycles. $\square$

Since every girth-cycle of a graph containing a vertex u contains exactly two edges incident with u , every edge-girth-regular graph is also vertex-girth-regular.

Observation 6. *If G is an $\text{egr}(n, k, g, \lambda)$ -graph, then G is a $\text{vgr}(n, k, g, \frac{k\lambda}{2})$ -graph.*

We now recall three theorems from [10] concerning existence of edge-girth-regular graphs that we will use later:

Theorem 7 (Th. 3.4 in [10]). *For every $k \geq 3$ and every $g \geq 6$, there exist infinitely many $\text{egr}(n, k, g, 2)$ -graphs.*

Theorem 8 (Th. 4.1 in [10]). *The Cartesian product of an $\text{egr}(n_1, k_1, 3, \lambda)$ -graph and an $\text{egr}(n_2, k_2, 3, \lambda)$ -graph is an $\text{egr}(n_1 n_2, k_1 + k_2, 3, \lambda)$ -graph.*

Theorem 9 (Th. 4.3 in [10]). *For every $r \geq 2$ and $g \geq 3$, there exist infinitely many $\text{egr}(n, 2^r, g, 1)$ -graphs.*

We are now ready to prove the main existence theorem of this section which shows that vertex-girth-regular graphs exist for many k, g and λ :

Theorem 10. *There are infinitely many integers n such that a $\text{vgr}(n, k, g, \lambda)$ -graph exists:*

- (i) *for $\lambda = 1$ and all integers $k, g \geq 3$;*
- (ii) *for $\lambda = 2$ and all integers $k \geq 4, g \geq 3$;*
- (iii) *for all integers $\lambda \geq 3, k \geq \lambda, g \geq 3, g \notin \{4, 5\}$.*

Proof. (i) By repeatedly applying Proposition 5 to cycles (i.e., $\text{vgr}(g, 2, g, 1)$ -graphs), we obtain the existence of infinitely many $\text{vgr}(n, k, g, 1)$ -graphs for all integers $k, g \geq 3$.

(ii) By applying Theorem 9 for $r = 2$, we obtain the existence of infinitely many $\text{egr}(n, 4, g, 1)$ -graphs for all $g \geq 3$ and thus infinitely many $\text{vgr}(n, 4, g, 2)$ -graphs by applying Observation 6. Finally, repeatedly applying Proposition 5 to these graphs proves (ii).

(iii) We first deal with the case $g = 3$. As discussed in [8], there exist an $\text{egr}(4, 3, 3, 2)$ -graph G_1 , an $\text{egr}(6, 4, 3, 2)$ -graph G_2 and an $\text{egr}(12, 5, 3, 2)$ -graph G_3 (these are respectively the 1-skeleta of the tetrahedron, the octahedron and the icosahedron). For all integers $k \geq 6$, one can construct the graph G_{k-2} by taking the Cartesian product of G_1 and G_{k-5} . Due to Theorem 8, the graphs G_{k-2} ($k \geq 3$) are $\text{egr}(n, k, 3, 2)$ -graphs for some n , and by Observation 6, they are also $\text{vgr}(n, k, 3, k)$ -graphs. By repeatedly applying Proposition 5 to these graphs, we obtain infinitely many $\text{vgr}(n, k, 3, \lambda)$ -graphs and we are done with the girth 3 case.

We can combine Theorem 7 and Observation 6 to obtain the existence of a $\text{vgr}(n, k, g, k)$ -graph for each $g \geq 6$ and $k \geq 3$. Finally, by repeatedly applying Proposition 5 to these graphs, we obtain the existence of infinitely many $\text{vgr}(n, k, g, \lambda)$ -graphs for each choice of (k, g, λ) with $k \geq \lambda \geq 3$ and $g \geq 6$. $\square$

4 Lower bounds on $n(k, g, \lambda)$

In this section, we present lower bounds on the order of extremal vertex-girth-regular graphs (i.e., vertex-girth-regular graphs of the smallest order). We get the lower bounds by generalizing already existing bounds for edge-girth-regular and girth-regular graphs [1, 5, 13]. We will use the fact that every vertex of a $\text{vgr}(n, k, g, \lambda)$ -graph has an edge that is contained in at least/at most $\frac{2\lambda}{k}$ distinct girth-cycles. In order to prove this fact, we choose an arbitrary vertex u and denote its signature by $\mathbf{a} = \{a_1, \dots, a_k\}$, where $a_1 \geq a_2 \geq \dots \geq a_k$. The sum of these numbers is two times the number of girth-cycles through u , which is equal to λ by definition. Hence, the average of the signature is $\frac{2\lambda}{k}$. Therefore

$$a_k \leq \left\lfloor \frac{2\lambda}{k} \right\rfloor \leq \left\lceil \frac{2\lambda}{k} \right\rceil \leq a_1. \quad (1)$$

First, we present a generalization of the combinatorial lower bounds.

Theorem 11. *Let G be a $\text{vgr}(n, k, g, \lambda)$ -graph, where $g = 2h$ is an even number and $k \geq 3$. Then*

$$n \geq 2 \frac{(k-1)^h - 1}{k-2} + \left\lceil \frac{2(k-1)^h - 2 \left\lfloor \frac{2\lambda}{k} \right\rfloor}{k} \right\rceil.$$

Moreover, if G is bipartite then

$$n \geq 2 \frac{(k-1)^h - 1}{k-2} + 2 \left\lceil \frac{(k-1)^h - \left\lfloor \frac{2\lambda}{k} \right\rfloor}{k} \right\rceil.$$

Proof. There is an edge that is contained in $\Lambda \leq \left\lfloor \frac{2\lambda}{k} \right\rfloor$ distinct girth-cycles. For that particular edge, we follow the proof of Theorem 2.3 and Theorem 5.1 in [5], and we immediately get the lower bound: we consider the set of vertices that are at distance $h-1$ from the chosen edge. The number of edges with both endpoints in the set of these vertices equals the number of girth-cycles through the chosen edge, which is precisely Λ . Hence, the number of edges that leave the Moore tree is $2(k-1)^h - 2\Lambda$. The graph is k -regular, so we can give a lower bound on the number of vertices outside the Moore tree:

$$\begin{aligned} n &\geq 2 \frac{(k-1)^h - 1}{k-2} + \left\lceil \frac{2(k-1)^h - 2\Lambda}{k} \right\rceil \\ &\geq 2 \frac{(k-1)^h - 1}{k-2} + \left\lceil \frac{2(k-1)^h - 2 \left\lfloor \frac{2\lambda}{k} \right\rfloor}{k} \right\rceil. \end{aligned}$$

In the bipartite case, there is a slight improvement on the lower bound. There are $2(k-1)^h - 2\Lambda$ edges that leave the Moore tree. Half of them have an endpoint in one part of the graph, and the rest of them have an endpoint in the other part. Hence

$$\begin{aligned} n &\geq 2 \frac{(k-1)^h - 1}{k-2} + 2 \left\lceil \frac{(k-1)^h - \Lambda}{k} \right\rceil \\ &\geq 2 \frac{(k-1)^h - 1}{k-2} + 2 \left\lceil \frac{(k-1)^h - \left\lfloor \frac{2\lambda}{k} \right\rfloor}{k} \right\rceil. \end{aligned}$$

□

Theorem 12. Let G be a $\text{vgr}(n, k, g, \lambda)$ -graph, where $g = 2h + 1$ is an odd number and $k \geq 3$. Then

$$n \geq n(k, g, \lambda) \geq \frac{k(k-1)^h - 2}{k-2} + \left\lceil \frac{k(k-1)^h - 2\lambda}{k} \right\rceil.$$

Proof. We can apply the same argument as in Theorem 2.3 in [5]: we choose an arbitrary vertex v and consider the set of vertices that are at distance h from v . Then, the number of girth-cycles through v equals the number of edges with both endpoints in this set, which is exactly λ . Hence, we can give a lower bound on the number of vertices outside the Moore tree. Thus, the order of the graph is at least

$$n \geq n(k, g, \lambda) \geq \frac{k(k-1)^h - 2}{k-2} + \left\lceil \frac{k(k-1)^h - 2\lambda}{k} \right\rceil. \quad \square$$

The next lower bound for even girth is a combinatorial one that gives a lower bound on the number of vertices outside the Moore tree. It is a straightforward generalization of Theorem 4.4 in [1]. The proof is based on counting the number of girth-cycles through an arbitrary vertex avoiding one of its edges.

Theorem 13. Let G be a $\text{vgr}(n, k, g, \lambda)$ -graph, where $g = 2h$ is an even number, $k \geq 3$ and G is not a Moore graph. Suppose that there exists an edge that is contained in Λ distinct girth-cycles. Then

$$n \geq 2 \frac{(k-1)^h - 1}{k-2} + \left\lceil \frac{((k-1)^h - \Lambda)^2}{2\lambda - 3\Lambda + (k-1)^h - 2 \max\left(0, \left\lceil \frac{\Lambda^2}{2(k-1)^{(h-1)}} - \frac{\Lambda}{2} \right\rceil\right)} \right\rceil.$$

Moreover, we have

$$\Lambda \leq \max\left((k-1)^{h-1}, (k-1)^{h-1} \left(\sqrt{\frac{1}{4} + \frac{2\lambda}{(k-1)^{h-1}}} - \frac{1}{2}\right)\right).$$

Proof. We follow the same argument as in the proof of Theorem 4.4 in [1]. Choose an arbitrary edge uv that is contained in exactly Λ distinct g -cycles. We define the set D_u of vertices as follows: $w \in D_u$ if and only if the length of the shortest uw -path is $h-1$, and the length of the shortest vw -path is h (i.e., the distance of w from v is h and at least one of the shortest paths from w to v contains the edge uv). Similarly, $w \in D_v$ if and only if the length of the shortest vw -path is $h-1$, and the length of the shortest uw -path is h . The number of edges between D_u and D_v is exactly Λ . Hence, the number of girth-cycles through u that do not contain the edge uv is $\lambda - \Lambda$. There are two types of such cycles: the ones that contain a vertex in D_v and the ones that contain a vertex at a distance h from u and outside of D_v . Denote the latter set of such vertices with M (see Fig. 4), and note that the vertices in M do not belong to the Moore tree containing uv , and therefore

$$n \geq M(k, g) + |M|, \quad (2)$$

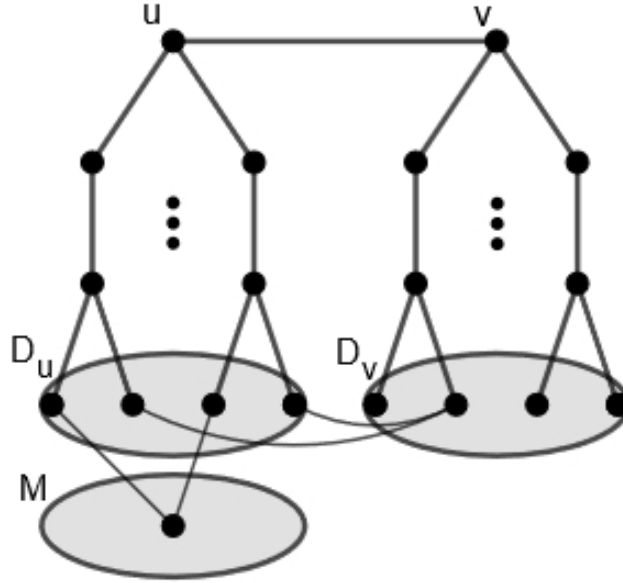


Figure 4: The sets D_u , D_v and M .

where $M(k, g) = 2^{\frac{(k-1)^h-1}{k-2}}$ is the Moore bound. In what follows, we let m denote the cardinality $|M|$.

We derive a lower bound for the number of the first type of the above discussed cycles to give an upper bound for the number of the second type of such cycles. There are $(k-1)^{h-1}$ vertices in D_v . For each vertex $w \in D_v$, let y_w be the number of neighbors that w has in D_u . Clearly, $\sum_{w \in D_v} y_w = \Lambda$. For each vertex $w \in D_v$, we have $\binom{y_w}{2}$ possible choices to form a girth-cycle containing both u and w , but not containing the edge uv . Hence, the number of girth-cycles of the first type is

$$\sum_{w \in D_v} \binom{y_w}{2} = \frac{1}{2} \sum_{w \in D_v} y_w^2 - \frac{1}{2} \sum_{w \in D_v} y_w = \frac{1}{2} \sum_{w \in D_v} y_w^2 - \frac{\Lambda}{2}.$$

The inequality between the arithmetic and quadratic means for the multiset $\{y_w \mid w \in D_v\}$ gives a lower bound on the number of such cycles:

$$\frac{1}{2} \sum_{w \in D_v} y_w^2 - \frac{\Lambda}{2} \geq \frac{\Lambda^2}{2(k-1)^{h-1}} - \frac{\Lambda}{2}.$$

For small values of Λ , this lower bound is negative. Hence, we have the following lower bound:

$$\sum_{w \in D_v} \binom{y_w}{2} \geq \max \left(0, \left\lceil \frac{\Lambda^2}{2(k-1)^{h-1}} - \frac{\Lambda}{2} \right\rceil \right).$$

For each vertex $w \in M$, let x_w be the number of neighbors of w in the set D_u . We obtain a girth-cycle if we choose such a vertex w , two of its neighbors, w_1 and w_2 in D_u and their unique $(h-1)$ -paths to the vertex u . Therefore the number of girth-cycles of the second type is exactly $\sum_{w \in M} \binom{x_w}{2}$. Now, we have the following upper bound on the number of such cycles:

$$\sum_{w \in M} \binom{x_w}{2} = \lambda - \Lambda - \sum_{w \in D_v} \binom{y_w}{2} \leq \lambda - \Lambda - \max \left(0, \left\lceil \frac{\Lambda^2}{2(k-1)^{(h-1)}} - \frac{\Lambda}{2} \right\rceil \right).$$

We use this inequality to give a lower bound for m , but first, we need to rearrange the terms. We also use the fact that $\sum_{w \in M} x_w = (k-1)^h - \Lambda$. Now, we have that

$$\begin{aligned} \sum_{w \in M} x_w^2 &= 2 \sum_{w \in M} \binom{x_w}{2} + \sum_{w \in M} x_w \\ &\leq (k-1)^h + 2\lambda - 3\Lambda - 2 \max \left(0, \left\lceil \frac{\Lambda^2}{2(k-1)^{(h-1)}} - \frac{\Lambda}{2} \right\rceil \right). \end{aligned}$$

By using the inequality between the arithmetic and quadratic mean again, we get a lower bound for m :

$$m \geq \frac{(\sum_{w \in M} x_w)^2}{\sum_{w \in M} x_w^2} \geq \frac{((k-1)^h - \Lambda)^2}{2\lambda - 3\Lambda + (k-1)^h - 2 \max \left(0, \left\lceil \frac{\Lambda^2}{2(k-1)^{(h-1)}} - \frac{\Lambda}{2} \right\rceil \right)}.$$

Note that the denominator is strictly positive, because G is not a Moore graph and therefore M is not empty. Combining this lower bound with inequality (2) yields the desired lower bound stated in the first part of our theorem.

For the second part of the theorem, recall that:

$$0 \leq \sum_{w \in M} \binom{x_w}{2} = \lambda - \Lambda - \sum_{w \in D_v} \binom{y_w}{2} \leq \lambda - \Lambda - \max \left(0, \left\lceil \frac{\Lambda^2}{2(k-1)^{(h-1)}} - \frac{\Lambda}{2} \right\rceil \right).$$

If $\Lambda \leq (k-1)^{h-1}$, then clearly the upper bound stated in the theorem holds. Otherwise we have

$$0 \leq \lambda - \Lambda - \left\lceil \frac{\Lambda^2}{2(k-1)^{(h-1)}} - \frac{\Lambda}{2} \right\rceil \leq \lambda - \Lambda - \frac{\Lambda^2}{2(k-1)^{(h-1)}} + \frac{\Lambda}{2}.$$

Since the quadratic function in Λ appearing on the right side of the above inequality is concave, any Λ for which its value is non-negative must lie between the two roots of the corresponding quadratic equation, and must therefore be smaller than the larger of the two roots. Hence,

$$\Lambda \leq (k-1)^{h-1} \left(\sqrt{\frac{1}{4} + \frac{2\lambda}{(k-1)^{h-1}}} - \frac{1}{2} \right). \quad \square$$

By definition, we do not have any assumption on the signature of a vertex-girth-regular graph. We only know the average of the signature that we can use to obtain a general lower bound on the order of vertex-girth-regular graphs.

Theorem 14. *Let G be a $\text{vgr}(n, k, g, \lambda)$ -graph, where $g = 2h$ is an even number, $k \geq 3$ and G is not a Moore graph. Then*

$$n \geq 2 \frac{(k-1)^h - 1}{k-2} + \max_{\Lambda \in \{\lfloor \frac{2\lambda}{k} \rfloor, \lceil \frac{2\lambda}{k} \rceil\}} \left\lceil \frac{((k-1)^h - \Lambda)^2}{2\lambda - 3\Lambda + (k-1)^h - 2 \max\left(0, \frac{\Lambda^2}{2(k-1)^{h-1}} - \frac{\Lambda}{2}\right)} \right\rceil.$$

Proof. First of all, recall that G must contain an edge which is contained in at most $\lfloor \frac{2\lambda}{k} \rfloor$ girth cycles and an edge which is contained in at least $\lceil \frac{2\lambda}{k} \rceil$ girth cycles. Consider the second term of the lower bound in Theorem 13 as a function of Λ :

$$g(x) := \left\lceil \frac{((k-1)^h - x)^2}{2\lambda - 3x + (k-1)^h - 2 \max\left(0, \left\lceil \frac{x^2}{2(k-1)^{h-1}} - \frac{x}{2} \right\rceil\right)} \right\rceil.$$

Now also consider

$$f(x) := \frac{((k-1)^h - x)^2}{2\lambda - 3x + (k-1)^h - 2 \max\left(0, \frac{x^2}{2(k-1)^{h-1}} - \frac{x}{2}\right)}.$$

Our proof is based on a careful investigation of the properties of $f(x)$ and $g(x)$ on the intervals containing admissible values of Λ . We clearly have $f(x) \leq g(x)$ for all real x for which both denominators are strictly positive. Let $b = (k-1)^{h-1} \left(\sqrt{\frac{1}{4} + \frac{2\lambda}{(k-1)^{h-1}}} - \frac{1}{2} \right)$. Since Theorem 13 asserts that every edge is contained in at most $\max((k-1)^{h-1}, b)$ girth cycles and every edge is clearly also contained in at most λ girth cycles, we are interested in the behaviour of $f(x)$ on the interval $[0, \min(\lambda, \max((k-1)^{h-1}, b))]$. For $x \in [0, (k-1)^{h-1}]$, we have:

$$f(x) = f_1(x) := \frac{((k-1)^h - x)^2}{2\lambda - 3x + (k-1)^h}.$$

In this case, the second derivative is given by:

$$f_1''(x) = \frac{8((k-1)^h - \lambda)^2}{((k-1)^h + 2\lambda - 3x)^3}.$$

Since $f_1''(x) \geq 0$ for all $x \in [0, \min(\lambda, (k-1)^{h-1})]$, $f(x)$ is convex in this interval. If $\lambda \leq (k-1)^{h-1}$ or $b \leq (k-1)^{h-1}$, we can conclude that f is convex on the whole interval $[0, \min(\lambda, \max((k-1)^{h-1}, b))]$.

Otherwise (if $(k-1)^{h-1} < \min(\lambda, b)$), we also consider the case when $x \in [(k-1)^{h-1}, \min(\lambda, b)]$, in which case we have:

$$f(x) = f_2(x) := \frac{((k-1)^h - x)^2}{2\lambda - 2x + (k-1)^h - \frac{x^2}{(k-1)^{h-1}}}$$

The second derivative $f_2''(x)$ is given by a very tedious expression (which can be explicitly computed using Wolfram Alpha for example) such that $f_2''(x) \geq 0$ for all $(k-1)^{h-1} \leq x \leq \min(\lambda, b)$. Hence, $f(x)$ is convex on the interval $[(k-1)^{h-1}, \min(\lambda, b)]$. Now the first derivative of $f_1(x)$ is given by:

$$f_1'(x) = \frac{((k-1)^h - x)((k-1)^h + 3x - 4\lambda)}{((k-1)^h - 3x + 2\lambda)^2}$$

and the first derivative of $f_2(x)$ is given by:

$$f_2'(x) = \frac{2(k-1)^{2h}((k-1)^h - x)(kx - 2\lambda)}{(2(k-1)^hx + (k-1)x^2 - (k-1)^h((k-1)^h + 2\lambda))^2}$$

Since $f_1'((k-1)^{h-1}) \leq f_2'((k-1)^{h-1})$, we can again conclude that $f(x)$ is convex on the whole interval $[0, \min(\lambda, \max((k-1)^{h-1}, b))]$.

In both cases, $f(x)$ is convex on the whole interval and therefore

$$n \geq 2 \frac{(k-1)^h - 1}{k-2} + \max_{\Lambda \in \{\lfloor \frac{2\lambda}{k} \rfloor, \lceil \frac{2\lambda}{k} \rceil\}} f(\Lambda)$$

Finally, we conclude by rounding up $f(x)$, since n is always an integer. $\square$

Next, using spectral graph theory, we generalize a lower bound for the order of edge-girth-regular graphs that appeared in [13]. Consider the adjacency matrix of a $\text{vgr}(n, k, g, \lambda)$ -graph and denote its eigenvalues by $\lambda_1 \geq \dots \geq \lambda_n$. Since the graph is k -regular, the largest eigenvalue is k . If the graph is bipartite, then the smallest eigenvalue is $-k$. Moreover, the sum of the ℓ -th powers of the eigenvalues is the sum of the numbers of closed walks of length ℓ rooted at the vertices of the graph (summed through all vertices of the graph). We note that in a k -regular graph of girth g , the number of *cycle-free* closed walks of length $\ell \leq g$ rooted at any vertex is independent of the choice of the vertex. We denote the number of cycle-free closed walks of length $\ell \leq g$ rooted at a(ny) vertex by $c(\ell, k)$, and observe that $c(\ell, k) = 0$ for odd $\ell \leq g$. For even ℓ , it is an $\frac{\ell}{2}$ -th degree polynomial of k . For example, the first four $c(\ell, k)$ polynomials (for even lengths ℓ) can be easily shown to be equal to the following:

$$\begin{aligned} c(2, k) &= k, \\ c(4, k) &= 2k^2 - k, \\ c(6, k) &= 5k^3 - 6k^2 + 2k, \\ c(8, k) &= 14k^4 - 28k^3 + 20k^2 - 5k. \end{aligned}$$

In particular, using the polynomials $c(\ell, k)$, if g is even, the number of closed walks of length $g/2$ in a (k, g) -graph is

$$\sum_{i=1}^n \lambda_i^{\frac{g}{2}} = n \cdot c\left(\frac{g}{2}, k\right),$$

and the number of closed walks of length g in a $\text{vgr}(n, k, g, \lambda)$ -graph is equal to

$$\sum_{i=1}^n \lambda_i^g = n \cdot (c(g, k) + 2\lambda).$$

Next, we apply the inequality between the quadratic and arithmetic means to the set $\{\lambda_2^{g/2}, \dots, \lambda_n^{g/2}\}$ to obtain a lower bound on the order n of a $\text{vgr}(n, k, g, \lambda)$ -graph of even girth. For the bipartite case, we repeat the process for the set $\{\lambda_2^{g/2}, \dots, \lambda_{n-1}^{g/2}\}$ because $\lambda_n = -k$. We obtain the following theorem, which is a direct generalization of Theorem 3.14 in [13].

Theorem 15. *Let G be a $\text{vgr}(n, k, g, \lambda)$ -graph, where g is even and $k \geq 3$. If $g \equiv 0 \pmod{4}$, then*

$$n(k, g, \lambda) \geq \frac{c(g, k) + 2\lambda + k^g - 2c(\frac{g}{2}, k)k^{\frac{g}{2}}}{c(g, k) - c^2(\frac{g}{2}, k) + 2\lambda},$$

$$n_2(k, g, \lambda) \geq 2 \frac{c(g, k) + 2\lambda + k^g - 2c(\frac{g}{2}, k)k^{\frac{g}{2}}}{c(g, k) - c^2(\frac{g}{2}, k) + 2\lambda}.$$

If $g \equiv 2 \pmod{4}$, then

$$n(k, g, \lambda) \geq \frac{c(g, k) + 2\lambda + k^g}{c(g, k) + 2\lambda},$$

$$n_2(k, g, \lambda) \geq \frac{2k^g}{c(g, k) + 2\lambda}.$$

Finally, we remark that these lower bounds can often be improved by noticing that for a $\text{vgr}(n, k, g, \lambda)$ -graph, nk is even because of the handshaking lemma and $n\lambda$ must be a multiple of g since each vertex is contained in precisely λ girth-cycles and each girth-cycle contains precisely g vertices. This leads to the following observation.

Observation 16. *There are $\frac{n\lambda}{g}$ cycles of length g in a $\text{vgr}(n, k, g, \lambda)$ -graph.*

In other words, if $n(k, g, \lambda) \geq n$, but n does not satisfy these divisibility conditions, then $n(k, g, \lambda) \geq n + 1$ (and we can repeatedly apply this argument).

5 Non-existence results

Our first result in this section is a generalization of Lemma 4 and is reminiscent of a similar result proven in [11] for k -regular graphs in which all vertices have the same signature; and hence also for edge-girth-regular graphs. Since vertex-girth-regular graphs are not necessarily signature-regular, our result requires an independent proof which is based on ideas different from those in [11]. Moreover, we prove both the odd and the even girth cases, whereas [11] only covers the even girth case.

For $k \geq 3$ and $g = 2s + 1 \geq 3$, let $\text{nc}(k, g) = \frac{k(k-1)^s}{2}$. Then, the number of girth-cycles through a vertex of a (k, g) -graph is at most $\text{nc}(k, g)$, and this number is equal to $\text{nc}(k, g)$ if and only if the graph is a (k, g) -Moore graph (see Proposition 3).

Theorem 17. *Let $k \geq 3$, $g = 2s + 1 \geq 7$, and $0 < \epsilon \leq \frac{k-1}{2}$ be integers. Then there is no $\text{vgr}(n, k, g, \text{nc}(k, g) - \epsilon)$ -graph.*

Proof. Suppose k, g and ϵ satisfy the above requirements, and, by means of contradiction, let us assume that G is a $\text{vgr}(n, k, g, \text{nc}(k, g) - \epsilon)$ -graph. Let u be an arbitrary vertex of G , and let G_u^s be the subgraph of G induced by the union of the sets $N_G(u, i) = \{v \in V(G) \mid d_G(u, v) = i\}$, $0 \leq i \leq s$. Let us call the edges of G_u^s connecting two vertices at distance s from u *horizontal with respect to u* and note that the number of g -cycles through u is equal to the number of such horizontal edges with respect to u . Since the number of g -cycles through u is assumed to be equal to $\text{nc}(k, g) - \epsilon$, the number of horizontal edges with respect to u is also $\text{nc}(k, g) - \epsilon$. Since G is assumed to be k -regular, there must exist 2ϵ edges connecting vertices at distance s from u to vertices at distance $s + 1$, i.e., edges between $N_G(u, s)$ and $N_G(u, s + 1)$. To simplify our arguments, let us call the vertices of G_u^s black, and the rest of the vertices of G white. Thus G contains exactly 2ϵ black-white edges (i.e., having different colored endpoints) and all the other edges are either black (both ends are black) or white (both ends are white). Let v be a black vertex incident to a black-white edge, and consider the induced subgraph G_v^s induced by the union of the sets $N_G(v, i)$, $0 \leq i \leq s$. Once again, the number of g -cycles containing v is equal to the number of horizontal edges with respect to v (i.e., edges connecting two vertices from $N_G(v, s)$). Our aim is to show that the number of horizontal edges with respect to v in G_v^s is smaller than the required $\text{nc}(k, g) - \epsilon$.

Let $w_1, w_2, \dots, w_k$ be the neighbors of v . Let G_{v, w_i}^s , $1 \leq i \leq k$, denote the induced subgraphs of G_v^s ‘rooted’ at the vertices w_i . These are vertex disjoint subgraphs induced by the unions of vertices in $N_G(v, j) \cap N_G(w_i, j - 1)$, $1 \leq j \leq s$, or, in other words, subgraphs each consisting of a root w_i and vertices at distance at least 2 and at most s from v whose shortest path toward v contains w_i . We may assume without loss of generality that the vertices $w_1, w_2, \dots, w_\ell$, where $1 \leq \ell$, are white (since v was chosen to be incident with at least one black-white edge), and that the vertices $w_{\ell+1}, w_{\ell+2}, \dots, w_k$ are black, where $\ell < k$ (since v is at distance s from u , and therefore adjacent to at least one black vertex).

To complete our argument, let us consider the neighbors $w_{1,1}, w_{1,2}, \dots, w_{1,k-1}$ of the white vertex w_1 distinct from v . Each of the vertices $w_{1,1}, w_{1,2}, \dots, w_{1,k-1}$ determines an

induced subgraph $G_{v,w_1,w_{1,i}}^s$ of G_{v,w_1}^s consisting of w_1 , the edge $w_1w_{1,i}$, and the subgraph of G_{v,w_1}^s induced by the subset of vertices of G_{v,w_1}^s comprised of vertices at distance at least 2 from v whose shortest path to v contains $w_{1,i}$. Each two of these subgraphs share exactly one vertex, namely the vertex w_1 .

There are the total of $2\epsilon \leq k-1$ black-white edges contained in G . Let r denote the number of subgraphs $G_{v,w_1,w_{1,i}}^s$, $1 \leq i \leq k-1$, that contain no black-white edges. Note that $r \geq 1$ as otherwise the subgraphs $G_{v,w_1,w_{1,i}}^s$, $1 \leq i \leq k-1$, would contain at least $k-1$ black-white edges (recall that these graphs only share one vertex) which together with the black-white vw_1 would make for at least k black-white edges. Since w_1 is white and it is contained in each of these subgraphs, the $r \geq 1$ subgraphs $G_{v,w_1,w_{1,i}}^s$, $1 \leq i \leq k-1$, that do not contain a black-white edge must consist entirely of white vertices. Each of them contains $(k-1)^{s-2}$ vertices at distance s from v , and hence there are at least $r(k-1)^{s-2}$ white vertices at distance s from v whose shortest path to v contains w_1 .

Next, let us apply this same kind of argument to the $k-1$ subgraphs G_{v,w_i}^s , $2 \leq i \leq k$. Since v is black and vw_1 is black-white, at least one of these subgraphs must not contain a black-white edge, and must consist entirely of black vertices (if at least one vertex in G_{v,w_i}^s is white, either vw_i is black-white or an edge in the subgraph G_{v,w_i}^s is black-white). Let us denote the number of subgraphs G_{v,w_i}^s , $2 \leq i \leq k$, that contain exclusively black vertices by t . We have argued that $1 \leq t$.

Let us inspect the number of horizontal edges with respect to v contained in G_v^s . Since the girth of G is assumed to be equal to $2s+1 \geq 7$, for each vertex x of the at least $r(k-1)^{s-2}$ white vertices from $N_G(v, s) \cap N_G(w_1, s-1)$, there can be at most one edge between x and the subgraph G_{v,w_i}^s (for each $2 \leq i \leq k$, with $t \geq 1$ of them consisting entirely of black vertices). The number of horizontal edges with respect to v in G_v^s is bounded from above by $\text{nc}(k, g) = \frac{k(k-1)^s}{2}$, which is precisely half the number of pairs (x, j) such that $x \in N_G(v, s) \cap N_G(w_i, s-1)$ and $1 \leq i, j \leq k$, $i \neq j$ (i.e., it counts the number of potential arcs with an endpoint $x \in V(G_{v,w_i}^s)$ and another endpoint in $V(G_{v,w_j}^s)$). Let H be the set of such pairs (x, j) where x is white and G_{v,w_j}^s consists entirely of black vertices, and let H' be the set of pairs $(x, j) \in H$ such that there is an edge between x and G_{v,w_j}^s . Since there are at least $r(k-1)^{s-2}$ white vertices in $N_G(v, s) \cap N_G(w_1, s-1)$ and there are t subgraphs G_{v,w_i}^s ($2 \leq i \leq k$) consisting entirely of black vertices, we have $|H| \geq tr(k-1)^{s-2}$. Furthermore, since there are at least $(k-1-r)$ black-white edges in G_{v,w_1}^s , at least $(k-1-t)$ black-white edges among vw_i and the edges of G_{v,w_i}^s ($2 \leq i \leq k$) and the edge vw_1 is black-white, it follows that $|H'| \leq 2\epsilon - (k-1-r) - (k-1-t) - 1$. Therefore, the number of horizontal edges with respect to v in G_v^s is at most

$$\text{nc}(k, g) - |H \setminus H'| \leq \frac{k(k-1)^s}{2} - tr(k-1)^{s-2} + 2\epsilon - (k-1-r) - (k-1-t) - 1.$$

This means that the total number of horizontal edges with respect to v contained in G_v^s is smaller than $\frac{k(k-1)^s}{2}$ by at least $tr(k-1)^{s-2} - 2\epsilon + (k-1-r) + (k-1-t) + 1$.

Recall that we assume that $2s+1 \geq 7$, and hence $s \geq 3$ and $s-2 \geq 1$. Since $s, k \geq 3$, $r, t \geq 1$, and $2\epsilon \leq k-1$, the expression

$$tr(k-1)^{s-2} - 2\epsilon + (k-1-r) + (k-1-t) + 1$$

is minimized when $s = 3$, $2\epsilon = k - 1$ and $r, t = 1$. Therefore, we obtain

$$\begin{aligned} |H \setminus H'| &\geq tr(k-1)^{s-2} - 2\epsilon + (k-1-r) + (k-1-t) + 1 \\ &\geq (k-1) - (k-1) + (k-2) + (k-2) + 1 = 2k-3 \geq k > \epsilon. \end{aligned}$$

The final contradiction now follows from the fact that the number of girth-cycles through v is the number of horizontal edges with respect to v , which is smaller than $\text{nc}(k, g) - \epsilon$. This contradicts the assumption that G is a $\text{vgr}(n, k, g, \text{nc}(k, g) - \epsilon)$ -graph. $\square$

Next, consider $k \geq 3$ and $g = 2s \geq 4$. Once again, the number of girth-cycles through a vertex of a (k, g) -graph is at most $\text{nc}(k, g) = \frac{k(k-1)^s}{2}$, and it is equal to $\text{nc}(k, g)$ if and only if the graph is a (k, g) -Moore graph.

Theorem 18. *Let $k \geq 3$, $g = 2s \geq 4$, and $0 < \epsilon < k - 1$ be integers. Then there is no $\text{vgr}(n, k, g, \text{nc}(k, g) - \epsilon)$ -graph.*

Proof. In similarity to the proof of Proposition 3, the argument for the even girth case is quite different from the argument used in the proof of Theorem 17. For the sake of obtaining a contradiction, let G be a $\text{vgr}(n, k, g, \text{nc}(k, g) - \epsilon)$ -graph and $0 < \epsilon < k - 1$. Let $u_1 u_2$, $\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2}$, D_{u_1} , D_{u_2} , $D_{u_1, 1}, D_{u_1, 2}, \dots, D_{u_1, k-1}$, $\mathcal{A}_{u_1}$, $\mathcal{B}_{u_1}$ and $\mathcal{C}_{u_1}$ be chosen as in Definition 2. For each vertex $w \in D_{u_2}$, let y_w be the number of neighbors that w has in D_{u_1} . For each vertex $w \in V(G) \setminus V(\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2})$, let x_w be the number of neighbors of w in the set D_{u_1} . Let H be the set of pairs (x, y) such that $x \in D_{u_2}$, $y \in \{1, \dots, k-1\}$ and x has no neighbor in $D_{u_1, y}$.

Since $\epsilon > 0$, H is non-empty and therefore $|\mathcal{A}_{u_1}| < (k-1)^s$ and $|\mathcal{B}_{u_1}| \leq ((k-1)^{s-1} - 1) \binom{k-1}{2} + \binom{k-2}{2}$. We now claim that the maximum of the sum $|\mathcal{A}_{u_1}| + |\mathcal{B}_{u_1}| + |\mathcal{C}_{u_1}|$ is attained when $|H| = 1$, in which case, $|\mathcal{A}_{u_1}| = (k-1)^s - 1$, $|\mathcal{B}_{u_1}| = ((k-1)^{s-1} - 1) \binom{k-1}{2} + \binom{k-2}{2}$, and $|\mathcal{C}_{u_1}| = 0$.

This can be seen as follows: the number of girth cycles containing u_1 is given by $|\mathcal{A}_{u_1}| + |\mathcal{B}_{u_1}| + |\mathcal{C}_{u_1}| = \sum_{w \in D_{u_2}} y_w + \sum_{w \in D_{u_2}} \binom{y_w}{2} + \sum_{w \in V(G) \setminus V(\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2})} \binom{x_w}{2}$. Note that $x_w = k$ for some $w \in V(G) \setminus V(\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2})$ is impossible, because of the girth constraint. Since every vertex in G has degree k , we have $x_w \leq k-1$ (for $w \in V(G) \setminus V(\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2})$) and $y_w \leq k-1$ (for $w \in D_{u_2}$). Since $|H| \geq 1$, we have $\sum_{w \in D_{u_2}} y_w \leq (k-1)^s - 1$. Finally, since all $(k-1)^{s-1}$ vertices in D_{u_1} also have degree k , we have $\sum_{w \in D_{u_2}} y_w + \sum_{w \in V(G) \setminus V(\mathcal{T}_{k, \frac{g}{2}-1}^{u_1, u_2})} x_w = (k-1)^s$. Now indeed we see that $|\mathcal{A}_{u_1}| + |\mathcal{B}_{u_1}| + |\mathcal{C}_{u_1}|$ subject to the previous inequalities is maximized when $y_w = k-1$ for all except for one $w \in D_{u_2}$ (and in this exceptional case we need $y_w = k-2$).

Thus, the number of girth-cycles containing u_1 is bounded from above by $(k-1)^s - 1 + ((k-1)^{s-1} - 1) \binom{k-1}{2} + \binom{k-2}{2} = \text{nc}(k, g) - (k-1)$. This leads to a contradiction with the assumption that $\epsilon < k-1$, and therefore no such G can exist. $\square$

Interestingly, we could not prove Theorem 17 for the case $s = 2$. Consulting the tables at the end of our article suggests that no $\text{vgr}(n, 3, 5, \lambda)$ -graphs exist for $\lambda = 4$ or $\lambda = 5$,

i.e., for $\epsilon = 1$ or $\epsilon = 2$ (where $1 \leq \frac{3-\epsilon}{2}$). This suggests that Theorem 17 might hold for $s = 2$ (i.e., $g = 5$) as well. In addition, Observation 6 suggests that the theorem may even hold for $s = 1$ (i.e., $g = 3$). Maybe even more intriguingly, inspecting our computational results summarized at the end of this article opens space for further non-existence results for cases when λ is not particularly close to $\text{nc}(k, g)$. Similar results exist in case of edge-girth-regular graphs, where, for example, it has been shown in [6] that no $(3, 7, 6)$ and no $(3, 8, 14)$ edge-girth-regular graphs exist. This raises the possibility of finding further arithmetic conditions that would yield the non-existence of $\text{vgr}(n, k, g, \lambda)$ -graphs. Having brought up the connections between edge-girth-regular, girth-regular, and vertex-girth-regular graphs, we also wish to point out that non-existence results concerning vertex-girth-regular graphs yield the non-existence of corresponding edge-girth-regular and girth-regular graphs as well.

6 Exhaustive generation algorithm

Goedgebeur and the second author [8] described an algorithm to exhaustively generate all $\text{egr}(n, k, g, \lambda)$ -graphs for given integers n, k, g and λ . In the current paper, we adapt this algorithm to generate all $\text{vgr}(n, k, g, \lambda)$ -graphs and add a different heuristic and pruning rule that speed up the algorithm without affecting the exhaustiveness guarantee.

The algorithm (pseudo code shown in Algorithm 1 and 2) expects as input four integers n, k, g and λ and works as follows: the algorithm starts from a (k, g) -Moore tree and adds isolated vertices until there are n vertices in total (note that this graph occurs as a subgraph of every $\text{vgr}(n, k, g, \lambda)$ -graph). The algorithm then recursively adds edges to this graph in all possible ways such that no $\text{vgr}(n, k, g, \lambda)$ -graphs are excluded from the search space.

In order to obtain an efficient algorithm, a heuristic is used for the order in which the edges are added as well as several pruning rules that allow the algorithm to backtrack as soon as a graph is encountered for which the algorithm can decide that it cannot occur as a subgraph of any $\text{vgr}(n, k, g, \lambda)$ -graph. For each edge that the algorithm considers, it will branch into two possibilities (adding the edge in the first branch and not adding the edge in the second branch) and the algorithm keeps track of which edges can potentially be added to the graph. The new heuristic that the algorithm employs is to choose the next edge to consider as an edge which is incident to the vertex u for which the difference between the number of potential valid edges that could be added incident with u and k minus the degree of u is minimized. For example, if $k = 5$ and u has degree 3 and there are only 2 possibilities for edges that could potentially be added incident with u , then the difference will be equal to 0 and the algorithm will add these two missing edges as soon as possible. This heuristic works well, because these two edges must be added eventually and adding them sooner further constrains the search space. After each iteration, the algorithm will mark potential edges to add as invalid if their addition would result in either:

- a graph with girth smaller than g ;

Algorithm 1 recursivelyAddEdges(Integer n , Integer k , Integer g , Integer λ , Graph $G = (V, E)$, Set validEdgesToBeAdded)

```

1: // Each call adds one edge to  $G$ 
2: if One of the pruning rules can be applied then
3:   return
4: end if
5: //  $G$  has the right number of edges
6: if  $|E| = \frac{nk}{2}$  then
7:   if  $G$  is a  $\text{vgr}(n, k, g, \lambda)$ -graph then
8:     Output  $G$ 
9:   end if
10:  return
11: end if
12: // Apply heuristic for choosing the next edge  $e$  to consider
13:  $u \leftarrow \arg \min_{w \in V(G), \deg_G(w) < k} (|\{e \in \text{validEdgesToBeAdded} \text{ and } e \text{ incident with } w\}| - (k - \deg_G(w)))$ 
14:  $e \leftarrow$  arbitrary edge from validEdgesToBeAdded incident with  $u$ 
15: // Option 1: add this edge to  $G$ 
16:  $G' \leftarrow (V, E \cup \{e\})$ 
17: newValidEdgesToBeAdded  $\leftarrow$  update(validEdgesToBeAdded,  $G'$ )
18: recursivelyAddEdges( $n, k, g, \lambda, G'$ , newValidEdgesToBeAdded)
19: // Option 2: do not add this edge to  $G$ 
20: recursivelyAddEdges( $n, k, g, \lambda, G$ , validEdgesToBeAdded  $\setminus \{e\}$ )

```

Algorithm 2 generateAllVertexGirthRegularGraphs(Integer n , Integer k , Integer g , Integer λ)

```

1: // This function generates all  $\text{vgr}(n, k, g, \lambda)$ -graphs
2:  $T \leftarrow (k, g)$ -Moore tree
3: // There are no  $\text{vgr}(n, k, g, \lambda)$ -graphs if  $n$  is too small
4: if  $n < |V(T)|$  then
5:   return
6: end if
7: // Add  $n - |V(T)|$  isolated vertices
8:  $G \leftarrow \text{addIsolatedVertices}(T, n - |V(T)|)$ 
9: validEdgesToBeAdded  $\leftarrow$  calculateValidEdgesToBeAdded( $G$ )
10: recursivelyAddEdges( $n, k, g, \lambda, G$ , validEdgesToBeAdded)

```

- a graph in which some vertex has degree larger than k ;
- a graph in which some vertex is contained in more than λ girth-cycles.

In each of these cases, the resulting graph can be excluded from the search space, because adding more edges will never lead to a $\text{vgr}(n, k, g, \lambda)$ -graph.

Additionally, the algorithm prunes the current graph if it is isomorphic with a graph that was encountered previously during the search, because this cannot result in any new $\text{vgr}(n, k, g, \lambda)$ -graphs which were not previously generated. Moreover, the algorithm prunes the current graph if it has a vertex u of degree less than k for which the number of potential edges that can be added incident with u is strictly smaller than k minus the degree of u , because this means that u cannot obtain degree k anymore. Finally, we also employ a new pruning rule based on the following proposition.

Proposition 19. *Let G be a graph and let $\text{cyc}(G, g, u)$ denote the number of cycles in G of length g containing the vertex u . If, for given integers g, k and λ , there is a vertex $u \in V(G)$ with degree k such that*

$$(k - 2)\lambda + 2\text{cyc}(G, g, u) - \sum_{u' \in N_G(u)} \text{cyc}(G, g, u') < 0,$$

or

$$(k - 2)\lambda + \text{cyc}(G, g, u) - \sum_{u' \in N_G(u)} \text{cyc}(G, g, u') + \min_{u' \in N_G(u)} (\text{cyc}(G, g, u')) < 0,$$

then G does not occur as a subgraph of any $\text{vgr}(n, k, g, \lambda)$ -graph.

Proof. Suppose for the sake of obtaining a contradiction that $u \in V(G)$ is vertex with degree k satisfying the first or the second condition from the proposition and G occurs as a subgraph of a $\text{vgr}(n, k, g, \lambda)$ -graph G' . There are precisely $\lambda - \text{cyc}(G, g, u)$ cycles of length g in G' which contain u and contain some edge from $E(G') \setminus E(G)$ (i.e., they are not cycles of G). Clearly, each of these cycles contains precisely two neighbors of u . Since each vertex in G' is contained in precisely λ girth-cycles, this implies that $2(\lambda - \text{cyc}(G, g, u)) \leq \sum_{u' \in N_G(u)} (\lambda - \text{cyc}(G, g, u'))$ and thus $(k - 2)\lambda + 2\text{cyc}(G, g, u) - \sum_{u' \in N_G(u)} \text{cyc}(G, g, u') \geq 0$. Hence, if the first condition of the lemma is satisfied, we immediately obtain a contradiction. Thus let us assume that the second condition holds. Each girth-cycle containing u in G' contains precisely two neighbors of u . There are at most $\sum_{u' \in N_G(u)} (\lambda - \text{cyc}(G, g, u')) - \max_{u' \in N_G(u)} (\lambda - \text{cyc}(G, g, u'))$ girth-cycles in G' containing u which are not cycles of G . Hence, we have $\lambda - \text{cyc}(G, g, u) \leq \sum_{u' \in N_G(u)} (\lambda - \text{cyc}(G, g, u')) - \max_{u' \in N_G(u)} (\lambda - \text{cyc}(G, g, u'))$ and so

$$(k - 2)\lambda + \text{cyc}(G, g, u) - \sum_{u' \in N_G(u)} \text{cyc}(G, g, u') + \min_{u' \in N_G(u)} (\text{cyc}(G, g, u')) \geq 0.$$

We again obtain a contradiction by assuming that the second condition holds. □

7 Computational lower and upper bounds for $n(k, g, \lambda)$

We implemented the algorithm from Section 6 to exhaustively generate vertex-girth-regular graphs and used it to obtain lower and upper bounds for $n(k, g, \lambda)$. More specifically, if the algorithm does not generate any $\text{vgr}(n', k, g, \lambda)$ -graphs for all $n' < n$, then

$n(k, g, \lambda) \geq n$ since the algorithm is exhaustive. On the other hand, if the algorithm generates at least one $\text{vgr}(n, k, g, \lambda)$ -graph, then clearly $n(k, g, \lambda) \leq n$. We ran the algorithm for $k = 3$, $3 \leq g \leq 8$ and $k = 4$, $3 \leq g \leq 6$. The bounds that we obtained from the computations described in this section are summarized in Tables 1 (cubic case) and 2 (quartic case). Bold values indicate cases where the lower bound is equal to the upper bound.

Apart from running the algorithm from Section 6, we also ran the algorithm GENREG [12] for generating all connected k -regular graphs of girth g on n vertices and filtered out those graphs which are vertex-girth-regular. The lower bounds that we obtained in this way could sometimes also be further improved by applying Observation 16 (i.e., the order n must be such that $n\lambda$ is a multiple of the girth g). Moreover, we also applied the lower bounds from Section 4 and filled the table with best lower bounds among the previously discussed methods.

For improving the upper bounds in cases where our algorithm and GENREG were unable to find any graphs, we also filtered out vertex-girth-regular graphs from known lists of regular graphs (with additional symmetry properties). More specifically, we consulted the list of all vertex-transitive graphs until order 47 [9], cubic vertex-transitive graphs until order 1280 [14], cubic arc-transitive graphs until order 2048 [2] and quartic arc-transitive graphs until order 640 [14, 15]. Additionally, we also applied the construction from Proposition 5 to obtain upper bounds. The total CPU-time for all computations in this paper amounts to approximately 2 CPU-years (the computations were executed on the hardware of the Flemish Supercomputer Center). We make all code and data related to this paper publicly available at <https://github.com/JorikJooker/vertexGirthRegularGraphs>. The graphs that we found can also be downloaded from the House of Graphs [4] by searching for the term “vertex-girth-regular”.

We now briefly discuss some observations and remarkable graphs that we found based on these computations. From Tables 1 and 2 it is clear that often $n(k, g, \lambda) \geq n(k, g, \lambda + 1)$, as one could intuitively expect (e.g. Moore graphs occur for the maximal value of λ). However, this inequality does not always hold. For example, we showed that $n(3, 8, 8) = 42 < n(3, 8, 9) = 48$. The corresponding graphs achieving these bounds are shown in Fig. 5. These two graphs are also among the largest graphs for which we were able to prove that they are extremal.

We also remark that several famous graphs appear as (extremal) vertex-girth-regular graphs. Among others we mention all Moore graphs and several cages [7], the Platonic solids, several incidence graphs, the Pappus graph (a $\text{vgr}(18, 3, 6, 6)$ -graph), the Coxeter graph (a $\text{vgr}(28, 3, 7, 6)$ -graph), the burnt pancake graph $\text{BP}(3)$ (a $\text{vgr}(48, 3, 8, 6)$ -graph) and the generalized Petersen graph $G(13, 5)$, which is a $\text{vgr}(26, 3, 7, 7)$ -graph (see Fig. 6) and appears for example in [3] as the cubic graph of girth 7 on 26 vertices with the most connected induced subgraphs among all such graphs.

For the cage problem, all known cages of even girth are bipartite and an important open question asks whether this is always the case [7]. We remark that there are several non-bipartite extremal vertex-girth-regular graphs of even girth (e.g. the $\text{vgr}(42, 3, 8, 8)$ -graph shown in Fig. 5 is such an example). Hence, the analogous question for vertex-

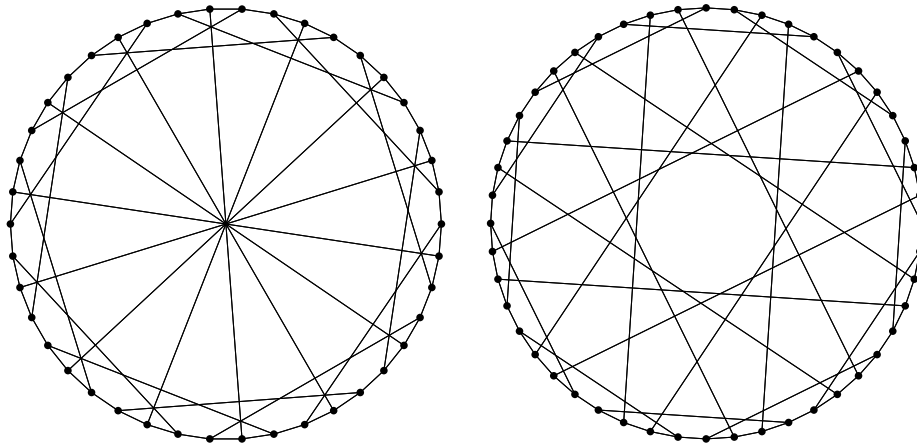


Figure 5: An extremal $\text{vgr}(42, 3, 8, 8)$ -graph (left) and an extremal $\text{vgr}(48, 3, 8, 9)$ -graph (right).

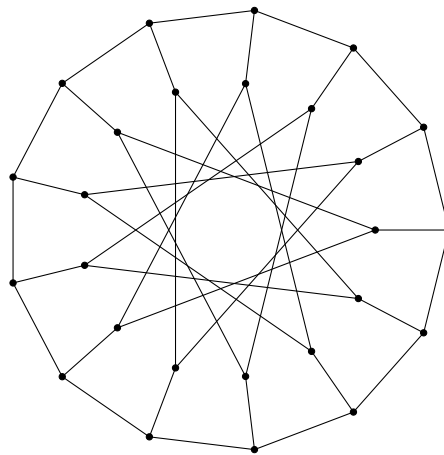


Figure 6: The generalized Petersen graph $G(13, 5)$ is a $\text{vgr}(26, 3, 7, 7)$ -graph.

girth-regular graphs has a negative answer. In fact, there are plenty of non-bipartite extremal vertex-girth-regular graphs of even girth, for example 25 pairwise non-isomorphic $\text{vgr}(20, 4, 4, 1)$ -graphs.

In Section 1 we mentioned that several subclasses of vertex-girth-regular graphs have received attention before in the literature (e.g. vertex-transitive graphs [9], edge-girth-regular graphs [5, 10] and graphs in which each vertex has the same signature [16]). We remark that many of the extremal vertex-girth-regular graphs that we found do not belong to any of these subclasses. For example, the graph shown in Fig. 7 is an extremal vertex-girth-regular graph in which 20 vertices have signature $\{5, 4, 4, 3\}$ and 5 vertices have signature $\{4, 4, 4, 4\}$. In total, we found 98 extremal vertex-girth-regular graphs which were not vertex-transitive (91 of these were not edge-girth-regular and 47 of these contained vertices with different signatures). Another interesting example to mention is again related to the cage problem. The smallest known cubic graph of girth 13 has 272

vertices [7]. This graph is not edge-girth-regular, but it is vertex-girth-regular (and in fact even vertex-transitive; every vertex has signature $\{18, 18, 16\}$).

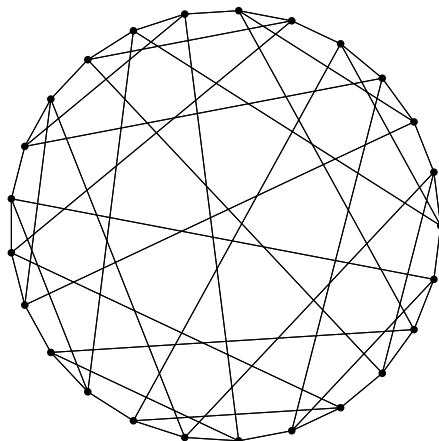


Figure 7: An extremal $\text{vgr}(25, 4, 5, 8)$ -graph for which 20 vertices have signature $\{5, 4, 4, 3\}$ and 5 vertices have signature $\{4, 4, 4, 4\}$.

7.1 Sanity checks

The bounds obtained in this section rely on the outcome of the authors' implementation of the algorithm from Section 6. Therefore, it is very important to take extra measures to ensure the correctness of our implementation, since an incorrect implementation would invalidate the obtained bounds. We first remark that, as expected, the bounds obtained by all different methods described in the previous paragraphs are in complete agreement with each other. Moreover, we also compared the outcome of our algorithm with the outcome of filtering the vertex-girth-regular graphs from all graphs produced by GENREG for orders that are larger than $n(k, g, \lambda)$ (where often there are many $\text{vgr}(n, k, g, \lambda)$ -graphs for a fixed n) and obtained exactly the same graphs in each case. We were also able to independently find many graphs that belong to subclasses of vertex-girth-regular that received attention in the literature before [8, 16].

8 Conclusion

Throughout our paper, we have repeatedly referred to the tables below. They can be simultaneously viewed as a list of best results obtained in the area so far, but also as a source of ideas and inspiration.

For example, consulting the information concerning $n(3, 7, 3)$, we see a gap between the lower bound of 42 on the order of any $\text{vgr}(n, 3, 7, 3)$ -graph and the order of a smallest $\text{vgr}(n, 3, 7, 3)$ -graph found to day; equal to 56. While we do not know whether a smaller $\text{vgr}(n, 3, 7, 3)$ -graph exists, we note that the $\text{vgr}(56, 3, 7, 3)$ -graph is a celebrated graph in topological graph theory. It is the underlying graph of the *Klein map*, a regular

k	g	λ	$n(k, g, \lambda) \geq$	$n(k, g, \lambda) \leq$
3	3	1	6	6
3	3	2	∞	∞
3	3	3	4	4
3	4	1	12	12
3	4	2	8	8
3	4	3	8	8
3	4	4	30	∞
3	4	5	∞	∞
3	4	6	6	6
3	5	1	20	20
3	5	2	20	20
3	5	3	20	20
3	5	4	40	∞
3	5	5	32	∞
3	5	6	10	10
3	6	1	24	24
3	6	2	24	24
3	6	3	24	24
3	6	4	24	24
3	6	5	36	∞
3	6	6	18	18
3	6	7	18	18
3	6	8	36	∞
3	6	9	16	16
3	6	10	36	∞
3	6	11	∞	∞
3	6	12	14	14

(a) $3 \leq g \leq 6$

k	g	λ	$n(k, g, \lambda) \geq$	$n(k, g, \lambda) \leq$
3	7	1	42	56
3	7	2	42	∞
3	7	3	42	56
3	7	4	42	∞
3	7	5	42	∞
3	7	6	28	28
3	7	7	26	26
3	7	8	42	∞
3	7	9	42	∞
3	7	10	42	∞
3	7	11	∞	∞
3	7	12	∞	∞
3	8	1	56	64
3	8	2	52	64
3	8	3	48	64
3	8	4	48	50
3	8	5	48	64
3	8	6	48	48
3	8	7	48	∞
3	8	8	42	42
3	8	9	48	48
3	8	10	44	44
3	8	11	40	40
3	8	12	40	40
3	8	13	48	∞
3	8	14	48	∞
3	8	15	48	∞
3	8	16	48	∞
3	8	17	48	∞
3	8	18	48	∞
3	8	19	48	∞
3	8	20	48	∞
3	8	21	48	∞
3	8	22	48	∞
3	8	23	∞	∞
3	8	24	30	30

(b) $7 \leq g \leq 8$

Table 1: An overview of the best lower and upper bounds for $n(3, g, \lambda)$.

k	g	λ	$n(k, g, \lambda) \geq$	$n(k, g, \lambda) \leq$
4	3	1	9	9
4	3	2	9	9
4	3	3	7	7
4	3	4	6	6
4	3	5	∞	∞
4	3	6	5	5
4	4	1	20	20
4	4	2	18	18
4	4	3	16	16
4	4	4	13	13
4	4	5	12	12
4	4	6	14	14
4	4	7	24	24
4	4	8	11	11
4	4	9	12	12
4	4	10	10	10
4	4	11	24	∞
4	4	12	10	10
4	4	13	24	∞
4	4	14	22	∞
4	4	15	24	∞
4	4	16	∞	∞
4	4	17	∞	∞
4	4	18	8	8
4	5	1	35	420
4	5	2	30	30
4	5	3	30	40
4	5	4	30	30
4	5	5	28	30
4	5	6	30	55
4	5	7	30	∞
4	5	8	25	25
4	5	9	25	25
4	5	10	24	24
4	5	11	30	30
4	5	12	20	20
4	5	13	30	∞
4	5	14	30	∞
4	5	15	28	∞
4	5	16	30	∞
4	5	17	30	∞
4	5	18	∞	∞

(a) $3 \leq g \leq 5$

k	g	λ	$n(k, g, \lambda) \geq$	$n(k, g, \lambda) \leq$
4	6	1	54	1152
4	6	2	51	84
4	6	3	50	1152
4	6	4	48	96
4	6	5	48	∞
4	6	6	46	60
4	6	7	48	648
4	6	8	45	60
4	6	9	44	512
4	6	10	42	81
4	6	11	42	∞
4	6	12	40	64
4	6	13	42	∞
4	6	14	39	60
4	6	15	38	∞
4	6	16	39	48
4	6	17	42	∞
4	6	18	35	35
4	6	19	36	∞
4	6	20	36	48
4	6	21	36	44
4	6	22	36	42
4	6	23	36	42
4	6	24	36	40
4	6	25	36	42
4	6	26	36	42
4	6	27	36	38
4	6	28	36	42
4	6	29	36	∞
4	6	30	35	35
4	6	31	36	36
4	6	32	36	36
4	6	33	36	36
4	6	34	36	36
4	6	35	36	∞
4	6	36	32	32
4	6	37	36	∞
4	6	38	36	∞
4	6	39	32	32
4	6	40	36	∞
4	6	41	36	∞
4	6	42	30	30
4	6	43	30	30
4	6	44	36	∞
4	6	45	36	∞
4	6	46	36	∞
4	6	47	36	∞
4	6	48	28	28
4	6	49	36	∞
4	6	50	36	∞
4	6	51	36	∞
4	6	52	∞	∞
4	6	53	∞	∞
4	6	54	26	26

(b) $g = 6$

Table 2: An overview of the best lower and upper bounds for $n(4, g, \lambda)$.

polyhedron of type $(7, 3)$ and genus 3, whose polyhedral representation was described by Schulte and Wills in 1985 [18]. The relevance of the polyhedral representation lies in the fact that locally each vertex v of the map is adjacent to three faces of the polyhedron, all of which are of length 7 and their borders constitute the only 7-cycles passing through v . Since 7 is also the girth of the underlying graph of the Klein map, the graph represents a specific example of a connection between $\text{vgr}(n, k, g, k)$ -graphs and maps of type (g, k) . This connection certainly deserves further investigation.

Acknowledgements

The authors would like to thank the anonymous reviewers for valuable comments and suggestions. The authors are also grateful to Jan Goedgebeur for helping to make the graphs publicly available on the House of Graphs. The computational resources and services used in this work were provided by the VSC (Flemish Supercomputer Centre), funded by the Research Foundation Flanders (FWO) and the Flemish Government - Department EWI. Robert Jajcay is partially supported by VEGA 1/0437/23, APVV-23-0076, and SK-AT-23-0019. Jorik Jooken is supported by a Postdoctoral Fellowship of the Research Foundation Flanders (FWO) with grant number 1222524N. The research of István Porupsánszki was supported in part by the Hungarian National Research, Development and Innovation Office OTKA grant no. SNN 132625.

References

- [1] G. Araujo-Pardo, Gy. Kiss, I. Porupsánszki, On extremal (almost) edge-girth-regular graphs. *Art Discrete Appl. Math.* 8, No. 3, Paper No. P3.05, 21 p., 2025.
- [2] M. Conder and P. Dobcsányi. Trivalent symmetric graphs on up to 768 vertices. *J. Comb. Math. & Comb. Comp.*, 40:41–63, 2002.
- [3] S. Cambie, J. Goedgebeur, and J. Jooken. The maximum number of connected sets in regular graphs. *Electron. J. Combin.*, 32(1):#P1.20, 2025.
- [4] K. Coolsaet, S. D’hondt, and J. Goedgebeur. House of Graphs 2.0: A database of interesting graphs and more. *Discrete Appl. Math.*, 325:97–107, 2023. Available at <https://houseofgraphs.org/>.
- [5] A. Z. Drglin, S. Filipovski, R. Jajcay and T. Raiman. Extremal edge-girth-regular graphs. *Graphs and Combin.*, 37:2139–2154, 2021.
- [6] L. Droogendijk. Nonexistence of certain edge-girth-regular graphs. arXiv:2403.20049 [math.CO], 2024.
- [7] G. Exoo and R. Jajcay. Dynamic cage survey. *Electron. J. Combin.*, #DS16, 2008.
- [8] J. Goedgebeur and J. Jooken. Exhaustive generation of edge-girth-regular graphs. *Experim. Math.*, 1–13, 2025.

- [9] D. Holt and G. Royle. A census of small transitive groups and vertex-transitive graphs. *J. Symbolic Comput.*, 101:51–60, 2020.
- [10] R. Jajcay, Gy. Kiss, and Š. Miklavič. Edge-girth-regular graphs. *European J. Combin.*, 72:70–82, 2018.
- [11] Gy. Kiss, Š. Miklavič, and T. Szőnyi. A stability result for girth-regular graphs with even girth. *J. Graph Theory*, 100 (1): 163–181, 2022.
- [12] M. Meringer. Fast generation of regular graphs and construction of cages. *J. Graph Theory*, 30(2):137–146, 1999.
- [13] I. Porupsánszki, *On edge-girth-regular graphs: lower bounds and new families.* [arXiv:2305.17014](https://arxiv.org/abs/2305.17014), 2023.
- [14] P. Potočník, P. Spiga and G. Verret. Cubic vertex-transitive graphs on up to 1280 vertices. *J. Symbolic Comput.*, 50:465–477, 2013.
- [15] P. Potočník, P. Spiga and G. Verret. Bounding the order of the vertex-stabiliser in 3-valent vertex transitive and 4-valent arc-transitive graphs. *J. Combin. Theory Ser. B*, 111:148–180, 2015.
- [16] P. Potočník and J. Vidali. Girth-regular graphs. *Ars Math. Contemp.*, 17 (2): 349–368, 2019.
- [17] H. Sachs. Regular graphs with given girth and restricted circuits. *J. London Math. Soc.*, 38:423–429, 1963.
- [18] E. Schulte and J. M. Wills. A polyhedral realization of Felix Klein’s map $\{3, 7\}_8$ on a Riemann surface of genus 3. *J. Lond. Math. Soc.*, II. 32:539–547, 1985.