Some Orbits of a Two-Vertex Stabilizer in a Grassmann Graph

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Abstract

Let $\mathbb{F}_q$ denote a finite field with q elements. Let n, k denote integers with $n > 2k \geqslant 6$. Let V denote a vector space over $\mathbb{F}_q$ that has dimension n. The vertex set of the Grassmann graph $J_q(n,k)$ consists of the k-dimensional subspaces of V. Two vertices of $J_q(n,k)$ are adjacent whenever their intersection has dimension $k-1$. Let ∂ denote the path-length distance function of $J_q(n,k)$. Pick vertices x, y of $J_q(n,k)$ such that $1 < \partial(x,y) < k$. Let $\mathrm{Stab}(x,y)$ denote the subgroup of $GL(V)$ that stabilizes both x and y. In this paper, we investigate the orbits of $\mathrm{Stab}(x,y)$ acting on the local graph $\Gamma(x)$. We show that there are five orbits. By construction, these five orbits give an equitable partition of $\Gamma(x)$; we find the corresponding structure constants. In order to describe the five orbits more deeply, we bring in a Euclidean representation of $J_q(n,k)$ associated with the second largest eigenvalue of $J_q(n,k)$. By construction, for each orbit its characteristic vector is represented by a vector in the associated Euclidean space. We compute many inner products and linear dependencies involving the five representing vectors.

Mathematics Subject Classifications: 05E30, 05E18

1 Introduction

This paper is about a family of finite undirected graphs known as distance-regular graphs [2, 4, 6, 13]. For any distance-regular graph, there is a construction called a Euclidean representation. In order to motivate our main topic, we now recall this construction. Let Γ denote a distance-regular graph with vertex set X and path-length distance function ∂. According to [12, Definition 6.1], a Euclidean representation of Γ is a nonzero Euclidean space E together with a map $\rho : X \to E$ such that

(i) E is spanned by $\{\rho(x) \mid x \in X\}$;

(ii) for all $x, y \in X$, the inner product $\langle \rho(x), \rho(y) \rangle$ depends only on $\partial(x,y)$;

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(iii) there exists $\vartheta \in \mathbb{R}$ such that for all $x \in X$,

$$\sum_{\substack{z \in X \\ \partial(z,x)=1}} \rho(z) = \vartheta \rho(x).$$

By [12, Section 6], the scalar ϑ is an eigenvalue of Γ. For each eigenvalue θ of Γ, the corresponding eigenspace gives a Euclidean representation of Γ.

In this paper we discuss a particular family of distance-regular graphs called the Grassmann graphs. We briefly recall the definition of a Grassmann graph. Let $\mathbb{F}_q$ denote a finite field with q elements. Fix an integer $n \geqslant 1$. Let V denote a vector space over $\mathbb{F}_q$ that has dimension n. Let the set $P_q(n)$ consist of the subspaces of V. For $0 \leqslant k \leqslant n$ let the set P_k consist of the elements of $P_q(n)$ that have dimension k. For $1 \leqslant k \leqslant n-1$ the vertex set of the Grassmann graph $J_q(n,k)$ is P_k. Two vertices of $J_q(n,k)$ are adjacent whenever their intersection has dimension $k-1$. For more information on the Grassmann graphs, see [5, 8, 9, 10]. For the rest of this section, we assume that Γ is the Grassmann graph $J_q(n,k)$ with $n > 2k \geqslant 6$.

In what follows, we will use the notation

$$[m] = \frac{q^m - 1}{q - 1} \qquad (m \in \mathbb{Z}).$$

By [4, Theorem 9.3.3], the eigenvalues of Γ are:

$$\theta_i = q^{i+1}[k-i][n-k-i] - [i] \qquad (0 \leqslant i \leqslant k).$$

In [12, Section 4], we used $P_q(n)$ to construct a Euclidean representation of Γ associated with θ_1. We now recall this construction. Let E denote a Euclidean space with dimension $[n]-1$ and bilinear form $\langle \; , \; \rangle$. Define a function

$$\begin{aligned} P_1 &\to E \\ s &\mapsto \widehat{s} \end{aligned} \tag{1}$$

that satisfies the following conditions (C1) − (C4):

(C1) $E = \operatorname{Span}\{\widehat{s} \mid s \in P_1\}$;

(C2) for $s \in P_1$, $\|\widehat{s}\|^2 = [n] - 1$;

(C3) for distinct $s, t \in P_1$, $\langle \widehat{s}, \widehat{t} \rangle = -1$;

(C4) $\sum_{s \in P_1} \widehat{s} = 0$.

Next, extend the function (1) to a function

$$\begin{aligned} P_q(n) &\to E \\ u &\mapsto \widehat{u} \end{aligned} \tag{2}$$

such that for all $u \in P_q(n)$,

$$\widehat{u} = \sum_{\substack{s \in P_1 \\ s \subseteq u}} \widehat{s}.$$

By [12, Section 6], the Euclidean space E, together with the restriction of the map (2) to $X = P_k$, gives a Euclidean representation of Γ that is associated with θ_1. By [12, Lemma 4.2], the $GL(V)$-action on $P_q(n)$ induces a $GL(V)$-module structure on E.

We now summarize the results of [12]. For $x \in X$, let $\Gamma(x)$ denote the local graph of x. For the rest of the section, fix $x, y \in X$ such that $1 < \partial(x, y) < k$. Let $\mathrm{Stab}(x, y)$ denote the subgroup of $GL(V)$ that stabilizes both x and y. Let $\mathrm{Fix}(x, y)$ denote the subspace of E consisting of the vectors that are fixed by every element of $\mathrm{Stab}(x, y)$. In [12, Lemma 8.3], we showed that the following vectors form a basis for $\mathrm{Fix}(x, y)$:

$$\widehat{x}, \qquad \widehat{y}, \qquad \widehat{x \cap y}, \qquad \widehat{x + y}. \tag{3}$$

We now describe a second basis for $\mathrm{Fix}(x, y)$. In [12, Definition 9.1], we defined the sets

$$\mathcal{B}_{xy} = \{z \in \Gamma(x) \mid \partial(z, y) = \partial(x, y) + 1\}, \qquad \mathcal{C}_{xy} = \{z \in \Gamma(x) \mid \partial(z, y) = \partial(x, y) - 1\}.$$

By [12, Lemma 9.2, 9.4], the sets $\mathcal{B}_{xy}, \mathcal{C}_{xy}$ are orbits of the $\mathrm{Stab}(x, y)$-action on $\Gamma(x)$. In [12, Definition 9.5], we defined the vectors

$$B_{xy} = \sum_{z \in \mathcal{B}_{xy}} \widehat{z}, \qquad C_{xy} = \sum_{z \in \mathcal{C}_{xy}} \widehat{z}.$$

In [12, Theorem 11.1], we showed that the following vectors form a basis for $\mathrm{Fix}(x, y)$:

$$\widehat{x}, \qquad \widehat{y}, \qquad B_{xy}, \qquad C_{xy}. \tag{4}$$

In [12, Theorem 11.3], we found the transition matrices between the basis (3) and the basis (4). We found the inner products between:

(i) any pair of vectors in the basis (3) [12, Theorem 10.4];

(ii) any pair of vectors in the basis (4) [12, Theorem 10.15];

(iii) any vector in the basis (3) and any vector in the basis (4) [12, Theorem 10.9].

In this paper, we investigate the orbits of $\mathrm{Stab}(x, y)$ acting on $\Gamma(x)$. As we will see, there are five orbits. We already mentioned two of the orbits, namely $\mathcal{B}_{xy}$ and $\mathcal{C}_{xy}$. We now describe the other three orbits.

Define the set

$$\mathcal{A}_{xy} = \{z \in \Gamma(x) \mid \partial(y,z) = \partial(x,y)\}.$$

We partition the set $\mathcal{A}_{xy}$ into the following three sets:

$$\mathcal{A}^+_{xy} = \{z \in \mathcal{A}_{xy} \mid z + x + y \supsetneq x + y,\ z \cap x \cap y = x \cap y\},$$
$$\mathcal{A}^0_{xy} = \{z \in \mathcal{A}_{xy} \mid z + x + y = x + y,\ z \cap x \cap y = x \cap y\},$$
$$\mathcal{A}^-_{xy} = \{z \in \mathcal{A}_{xy} \mid z + x + y = x + y,\ z \cap x \cap y \subsetneq x \cap y\}.$$

We show that the sets

$$\mathcal{A}^+_{xy}, \qquad \mathcal{A}^0_{xy}, \qquad \mathcal{A}^-_{xy}$$

are orbits of the $\mathrm{Stab}(x,y)$-action on $\Gamma(x)$. Hence,

$$\mathcal{B}_{xy}, \qquad \mathcal{C}_{xy}, \qquad \mathcal{A}^+_{xy}, \qquad \mathcal{A}^0_{xy}, \qquad \mathcal{A}^-_{xy} \tag{5}$$

are the five orbits of the $\mathrm{Stab}(x,y)$-action on $\Gamma(x)$. By construction, (5) is a partition of $\Gamma(x)$ that is equitable in the sense of [11, p. 159]. We call this partition the y-partition of $\Gamma(x)$.

Define the vectors

$$A^+_{xy} = \sum_{z \in \mathcal{A}^+_{xy}} \widehat{z}, \qquad A^0_{xy} = \sum_{z \in \mathcal{A}^0_{xy}} \widehat{z}, \qquad A^-_{xy} = \sum_{z \in \mathcal{A}^-_{xy}} \widehat{z}. \tag{6}$$

We show that $A^+_{xy}, A^0_{xy}, A^-_{xy}$ are contained in $\mathrm{Fix}(x,y)$. We write each vector in (6) as a linear combination of the vectors in (3) and also the vectors in (4). We find the inner products between:

(i) any vector in (6) and any vector in the basis (3);

(ii) any vector in (6) and any vector in the basis (4);

(iii) any pair of vectors in (6).

We mentioned that the y-partition of $\Gamma(x)$ is equitable. We compute the corresponding structure constants. In the table below, for each orbit $\mathcal{O}$ in the header column, and each orbit $\mathcal{N}$ in the header row, the $(\mathcal{O},\mathcal{N})$-entry gives the number of vertices in $\mathcal{N}$ that are adjacent to a given vertex in $\mathcal{O}$. Write $i = \partial(x,y)$.

	$\mathcal{B}_{xy}$	$\mathcal{C}_{xy}$	$\mathcal{A}^+_{xy}$	$\mathcal{A}^0_{xy}$	$\mathcal{A}^-_{xy}$
$\mathcal{B}_{xy}$	$q^{i+1}[k-i] + q^{i+1}[n-k-i]-q-1$	0	$q[i]$	0	$q[i]$
$\mathcal{C}_{xy}$	0	$2q[i-1]$	$q^{i+1}[n-k-i]$	$(q-1)\big(2[i]-1\big)$	$q^{i+1}[k-i]$
$\mathcal{A}^+_{xy}$	$q^{i+1}[k-i]$	$[i]$	$q[n-k]-q-1$	$(q-1)[i]$	0
$\mathcal{A}^0_{xy}$	0	$2[i]-1$	$q^{i+1}[n-k-i]$	$(q-1)\big(2[i]-1\big)-1$	$q^{i+1}[k-i]$
$\mathcal{A}^-_{xy}$	$q^{i+1}[n-k-i]$	$[i]$	0	$(q-1)[i]$	$q[k]-q-1$

Let $\mathcal{M}_i$ denote the 5×5 matrix from the table above. We display the eigenvalues for $\mathcal{M}_i$. For each eigenvalue, we give a corresponding row eigenvector and column eigenvector. We show that the eigenvalues of $\mathcal{M}_i$ are the same as the eigenvalues of the local graph $\Gamma(x)$.

This paper is organized as follows. In Sections 2 and 3, we present some preliminaries on the Grassmann graph $J_q(n,k)$ and the projective geometry $P_q(n)$. In Section 4, we represent the elements of $P_q(n)$ as vectors in the Euclidean space E. In Section 5, we present some results about $\mathrm{Stab}(x,y)$ and $\mathrm{Fix}(x,y)$. In Section 6, we find all the orbits of the $\mathrm{Stab}(x,y)$-action on $\Gamma(x)$. In Sections 7 and 8, we define the vectors $A^+_{xy}, A^0_{xy}, A^-_{xy}$ and write these vectors in terms of the basis (3) and the basis (4). We also obtain inner products between the vectors $\widehat{x}, \widehat{y}, B_{xy}, C_{xy} A^+_{xy}, A^0_{xy}, A^-_{xy}$. In Section 9, we use the matrix $\mathcal{M}_i$ to describe the adjacency between the $\mathrm{Stab}(x,y)$-orbits. We find the eigenvalues of $\mathcal{M}_i$ and their corresponding row eigenvectors and column eigenvectors. We also show that the eigenvalues of $\mathcal{M}_i$ are the same as the eigenvalues of $\Gamma(x)$.

2 The Grassmann graph Γ

Let $\Gamma = (X, \mathcal{E})$ denote a finite undirected graph that is connected, without loops or multiple edges, with vertex set X, edge set $\mathcal{E}$, and path-length distance function ∂. Two vertices $x, y \in X$ are said to be adjacent whenever they form an edge. The diameter d of Γ is defined as $d = \max\{\partial(x,y) \mid x, y \in X\}$. For $x \in X$ and an integer $i \geqslant 0$, define the set $\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. The subgraph induced on $\Gamma(x)$ is called the *local graph* of x.

We say that Γ is *regular with valency* κ whenever $\left|\Gamma(x)\right| = \kappa$ for all $x \in X$. We say that Γ is *distance-regular* whenever for all integers h, i, j such that $0 \leqslant h, i, j \leqslant d$ and all $x, y \in X$ such that $\partial(x,y) = h$, the cardinality of the set $\{z \in X \mid \partial(x,z) = i, \partial(y,z) = j\}$ depends only on h, i, j. This cardinality is denoted by $p^h_{i,j}$. For the rest of this section, we assume that Γ is distance-regular with diameter $d \geqslant 3$. Observe that Γ is regular with valency $\kappa = p^0_{1,1}$. Define

$$b_i = p^i_{1,i+1} \quad (0 \leqslant i < d), \qquad a_i = p^i_{1,i} \quad (0 \leqslant i \leqslant d), \qquad c_i = p^i_{1,i-1} \quad (0 < i \leqslant d).$$

Note that $b_0 = \kappa$, $a_0 = 0$, $c_1 = 1$. Also note that

$$b_i + a_i + c_i = \kappa \qquad (0 \leqslant i \leqslant d),$$

where $c_0 = 0$ and $b_d = 0$. We call b_i, a_i, c_i the *intersection numbers of* Γ.

By the *eigenvalues of* Γ we mean the roots of the minimal polynomial of the adjacency matrix. Since Γ is distance-regular, by [4, p. 128], Γ has $d+1$ eigenvalues; we denote these eigenvalues by

$$\theta_0 > \theta_1 > \cdots > \theta_d.$$

By [4, p. 129], $\theta_0 = \kappa$. By the *spectrum of* Γ we mean the set of ordered pairs $\left\{(\theta_i, m_i)\right\}_{i=0}^d$, where $\{\theta_i\}_{i=0}^d$ are the eigenvalues of Γ and m_i the dimension of the θ_i-eigenspace $(0 \leqslant i \leqslant d)$.

This paper is about a class of distance-regular graphs called the Grassmann graphs. These graphs are defined as follows. Let $\mathbb{F} = \mathbb{F}_q$ denote a finite field with q elements, and let n, k denote positive integers such that $n > k$. Let V denote an n-dimensional vector space over $\mathbb{F}$. The Grassmann graph $J_q(n, k)$ has vertex set X consisting of the k-dimensional subspaces of V. Vertices x, y of $J_q(n, k)$ are adjacent whenever $x \cap y$ has dimension $k - 1$.

According to [4, p. 268], the graphs $J_q(n, k)$ and $J_q(n, n-k)$ are isomorphic. Without loss of generality, we may assume $n \geqslant 2k$. Under this assumption, the diameter of $J_q(n, k)$ is equal to k. (See [4, Theorem 9.3.3].) The case $n = 2k$ is somewhat special, so throughout this paper we assume that $n > 2k$. For the rest of this paper, we assume that Γ is the Grassmann graph $J_q(n, k)$ with $k \geqslant 3$.

In what follows, we will use the notation

$$[m] = \frac{q^m - 1}{q - 1} \qquad (m \in \mathbb{Z}).$$

By [4, Theorem 9.3.2], the valency of Γ is

$$\kappa = q[k][n - k].$$

By [4, Theorem 9.3.3], the intersection numbers of Γ are

$$b_i = q^{2i+1}[k - i][n - k - i], \qquad c_i = [i]^2 \qquad (0 \leqslant i \leqslant k). \tag{7}$$

By [4, Theorem 9.3.3], the eigenvalues of Γ are

$$\theta_i = q^{i+1}[k - i][n - k - i] - [i] \qquad (0 \leqslant i \leqslant k). \tag{8}$$

The given ordering of the eigenvalues is known to be Q-polynomial in the sense of [4, Theorem 8.1.1].

3 The projective geometry $P_q(n)$

To study the graph Γ, it is helpful to view its vertex set X as a subset of a certain poset $P_q(n)$, which is defined as follows.

Definition 1. Let the poset $P_q(n)$ consist of the subspaces of V, together with the partial order given by inclusion. This poset $P_q(n)$ is called the *projective geometry*.

For the rest of the paper, we abbreviate $P = P_q(n)$. In this section we present some lemmas about the poset P.

Lemma 2. [1, p. 47] *For $u, v \in P$ we have*

$$\dim u + \dim v = \dim (u \cap v) + \dim (u + v)\,.$$

Lemma 3. [12, Lemma 3.3] *Let $u, v \in P$. Let the subset $\mathcal{R} \subseteq V$ form a basis for $u \cap v$. Extend the basis $\mathcal{R}$ to a basis $\mathcal{R} \cup \mathcal{S}$ for u, and extend the basis $\mathcal{R}$ to a basis $\mathcal{R} \cup \mathcal{T}$ for v. Then $\mathcal{R} \cup \mathcal{S} \cup \mathcal{T}$ forms a basis for the subspace $u + v$.*

For $0 \leqslant \ell \leqslant n$, let the set P_ℓ consist of the ℓ-dimensional subspaces of V. Note that $X = P_k$. Also note that $P_0 = \{0\}$ and $P_n = \{V\}$.

Lemma 4. *For $x, y \in X$ the following (i), (ii) hold:*

(i) [4, p. 269] *the dimension of $x \cap y$ is $k - \partial(x, y)$;*

(ii) [12, Lemma 3.5] *the dimension of $x + y$ is $k + \partial(x, y)$.*

Definition 5. For $u \in P$ define the set

$$\Omega(u) = \{s \in P_1 \mid s \subseteq u\}.$$

Note that $\Omega(V) = P_1$. By [12, Section 3], the following (i)–(ii) hold:

(i) for all $u \in P$,
$$\big|\Omega(u)\big| = [m],$$
where $u \in P_m$;

(ii) $|P_1| = [n]$.

We now comment on the symmetries of P. Recall that the general linear group $GL(V)$ consists of the invertible $\mathbb{F}$-linear maps from V to V. The action of $GL(V)$ on V induces a permutation action of $GL(V)$ on the set P. This permutation action respects the partial order on P. The orbits of the action are P_ℓ for $0 \leqslant \ell \leqslant n$. By [12, Lemma 3.9], the action of $GL(V)$ on X preserves the path-length distance ∂.

4 Representing P using a Euclidean space E

In [12, Section 4] we described how to represent the elements of P as vectors in a Euclidean space. Our goal in this section is to summarize the description. The material in this section will be used to state and prove our main results later in the paper.

There are two stages to representing the elements of P as vectors in a Euclidean space. In the first stage we consider the elements of P_1. Let E denote a Euclidean space with dimension $[n] - 1$ and bilinear form $\langle\ ,\ \rangle$. Recall the notation $\|\nu\|^2 = \langle \nu, \nu \rangle$ for any $\nu \in E$. We define a function

$$\begin{aligned} P_1 &\to E \\ s &\mapsto \widehat{s} \end{aligned} \tag{9}$$

that satisfies the following conditions (C1) $-$ (C4):

(C1) $E = \mathrm{Span}\big\{\widehat{s} \mid s \in P_1\big\}$;

(C2) for $s \in P_1$, $\left\|\widehat{s}\right\|^2 = [n] - 1$;

(C3) for distinct $s, t \in P_1$, $\left\langle \widehat{s}, \widehat{t} \right\rangle = -1$;

(C4) $\displaystyle\sum_{s \in P_1} \widehat{s} = 0$.

Next, we extend the function (9) to a function

$$\begin{aligned} P &\to E \\ u &\mapsto \widehat{u} \end{aligned} \tag{10}$$

such that for all $u \in P$,

$$\widehat{u} = \sum_{s \in \Omega(u)} \widehat{s}.$$

Note that $\widehat{u} = 0$ if $u \in P_0$ or $u \in P_n$.

Next we present a lemma that involves the map (10).

Lemma 6. *The following* (i)–(vi) *hold:*

(i) [12, Lemma 6.2] *for* $u, v \in P$,

$$\left\langle \widehat{u}, \widehat{v} \right\rangle = [n][h] - [i][j],$$

where

$$i = \dim u, \qquad j = \dim v, \qquad h = \dim (u \cap v);$$

(ii) [12, Lemma 6.3] *for* $u \in P$,

$$\left\|\widehat{u}\right\|^2 = q^i [i][n-i],$$

where $i = \dim u$*;*

(iii) [12, Lemma 6.4] *for* $x, y \in X$,

$$\left\langle \widehat{x}, \widehat{y} \right\rangle = [n][k-i] - [k]^2,$$

where $i = \partial(x, y)$*;*

(iv) [12, Lemma 6.5] *for* $x \in X$,

$$\left\|\widehat{x}\right\|^2 = q^k [k][n-k];$$

(v) [12, Lemma 6.6] *for* $x \in X$,

$$\sum_{z \in \Gamma(x)} \widehat{z} = \theta_1 \widehat{x},$$

where θ_1 *is from* (8)*;*

(vi) [12, Lemma 6.7] *the vector space E is spanned by $\{\hat{x} \mid x \in X\}$.*

By [12, Section 6], the Euclidean space E, together with the restriction of the map (10) to X gives a Euclidean representation of Γ in the sense of [12, Definition 6.1]. This representation is associated with the eigenvalue θ_1. By [12, Lemma 4.2], the Euclidean space E becomes a $GL(V)$-module such that for all $u \in P$ and $\sigma \in GL(V)$,

$$\sigma(\hat{u}) = \widehat{\sigma(u)}.$$

By [12, Section 6], the Euclidean space E is irreducible as a $GL(V)$-module.

5 The stabilizer of some elements in X

In this section, we consider some stabilizer subgroups of $GL(V)$. These subgroups are the stabilizer of a vertex in X, and the stabilizer of two distinct vertices in X. We obtain some results about these stabilizers that will be used later in the paper.

For $x \in X$, let $\mathrm{Stab}(x)$ denote the subgroup of $GL(V)$ consisting of the elements that fix x. We call $\mathrm{Stab}(x)$ the *stabilizer of x in $GL(V)$*.

Lemma 7. [12, Lemma 5.1] *For $v, v' \in P$ and $x \in X$, the following are equivalent:*

(i) $\dim v = \dim v'$ *and* $\dim (v \cap x) = \dim(v' \cap x)$;

(ii) *the subspaces v and v' are contained in the same orbit of the* $\mathrm{Stab}(x)$*-action on P.*

Pick distinct $x, y \in X$. Let $\mathrm{Stab}(x, y)$ denote the subgroup of $GL(V)$ consisting of the elements that fix both x and y. We call $\mathrm{Stab}(x, y)$ the *stabilizer of x and y in $GL(V)$*. Let $\mathrm{Fix}(x, y)$ denote the subspace of E consisting of the vectors that are fixed by every element of $\mathrm{Stab}(x, y)$.

Lemma 8. [12, Theorem 8.3] *Pick distinct $x, y \in X$. In the table below, we display vectors that form a basis for* $\mathrm{Fix}(x, y)$.

Case	basis for $\mathrm{Fix}(x, y)$
$1 \leqslant \partial(x, y) < k$	$\hat{x}$, $\hat{y}$, $\widehat{x \cap y}$, $\widehat{x + y}$
$\partial(x, y) = k$	$\hat{x}$, $\hat{y}$, $\widehat{x + y}$

Definition 9. Pick distinct $x, y \in X$. By the *geometric basis for* $\mathrm{Fix}(x, y)$, we mean the basis displayed in Lemma 8.

Note that the case $\partial(x, y) = k$ is special. The case $\partial(x, y) = 1$ is also special; see [12, Definition 9.1, 9.5]. For the rest of the paper, we assume that $1 < \partial(x, y) < k$.

6 The y-partition of $\Gamma(x)$

Pick $x, y \in X$ such that $1 < \partial(x, y) < k$. In this section we describe the orbits of $\operatorname{Stab}(x, y)$ acting on $\Gamma(x)$. We will show that there are five orbits. The partition of $\Gamma(x)$ into these five orbits will be called the y-partition of $\Gamma(x)$.

Definition 10. For $x, y \in X$ such that $1 < \partial(x, y) < k$, define

$$\begin{aligned} \mathcal{B}_{xy} &= \big\{ z \in \Gamma(x) \mid \partial(y, z) = \partial(x, y) + 1 \big\}, \\ \mathcal{C}_{xy} &= \big\{ z \in \Gamma(x) \mid \partial(y, z) = \partial(x, y) - 1 \big\}, \\ \mathcal{A}_{xy} &= \big\{ z \in \Gamma(x) \mid \partial(y, z) = \partial(x, y) \big\}. \end{aligned}$$

Observe that

$$\big|\mathcal{B}_{xy}\big| = b_i, \qquad \big|\mathcal{C}_{xy}\big| = c_i, \qquad \big|\mathcal{A}_{xy}\big| = a_i \qquad \big(i = \partial(x, y)\big).$$

Lemma 11. [12, Lemma 9.2, 9.4] *For $x, y \in X$ such that $1 < \partial(x, y) < k$, the sets $\mathcal{B}_{xy}$ and $\mathcal{C}_{xy}$ are orbits of the* $\operatorname{Stab}(x, y)$*-action on* $\Gamma(x)$.

Definition 12. For $x, y \in X$ such that $1 < \partial(x, y) < k$, define the vectors

$$B_{xy} = \sum_{z \in \mathcal{B}_{xy}} \widehat{z}, \qquad C_{xy} = \sum_{z \in \mathcal{C}_{xy}} \widehat{z}, \qquad A_{xy} = \sum_{z \in \mathcal{A}_{xy}} \widehat{z}.$$

Note that B_{xy}, C_{xy}, A_{xy} are contained in E. We call B_{xy}, C_{xy}, A_{xy} the *characteristic vectors* of $\mathcal{B}_{xy}, \mathcal{C}_{xy}, \mathcal{A}_{xy}$ respectively.

Lemma 13. [12, Theorem 11.1] *For $x, y \in X$ such that $1 < \partial(x, y) < k$, the following vectors form a basis for* $\operatorname{Fix}(x, y)$*:*

$$\widehat{x}, \qquad \widehat{y}, \qquad B_{xy}, \qquad C_{xy}. \tag{11}$$

Definition 14. Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$. By the *combinatorial basis for* $\operatorname{Fix}(x, y)$, we mean the basis formed by the vectors in (11).

Next we focus on the set $\mathcal{A}_{xy}$. This set turns out to be the disjoint union of three orbits of the $\operatorname{Stab}(x, y)$-action on $\Gamma(x)$. Our next general goal is to describe these three orbits.

Definition 15. For $x, y \in X$ such that $1 < \partial(x, y) < k$, define

$$\begin{aligned} \mathcal{A}^{+}_{xy} &= \big\{ z \in \mathcal{A}_{xy} \mid z + x + y \supsetneq x + y, z \cap x \cap y = x \cap y \big\}, \\ \mathcal{A}^{0}_{xy} &= \big\{ z \in \mathcal{A}_{xy} \mid z + x + y = x + y, z \cap x \cap y = x \cap y \big\}, \\ \mathcal{A}^{-}_{xy} &= \big\{ z \in \mathcal{A}_{xy} \mid z + x + y = x + y, z \cap x \cap y \subsetneq x \cap y \big\}. \end{aligned}$$

We are going to show that the three sets in Definition 15 are orbits of $\operatorname{Stab}(x, y)$. First we have a few remarks.

Lemma 16. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the set $\mathcal{A}_{xy}$ is the disjoint union of the sets $\mathcal{A}^+_{xy}, \mathcal{A}^0_{xy}, \mathcal{A}^-_{xy}$.*

Proof. By linear algebra, the set $\big\{z \in \mathcal{A}_{xy} \mid z + x + y \supsetneq x + y,\ z \cap x \cap y \subsetneq x \cap y\big\}$ is empty. The result follows. $\square$

Lemma 17. *For $x, y \in X$ such that $1 < \partial(x,y) < k$,*

$$\big|\mathcal{A}^+_{xy}\big| = q^{i+1}[i][n-k-i], \qquad \big|\mathcal{A}^0_{xy}\big| = (q-1)[i]^2, \qquad \big|\mathcal{A}^-_{xy}\big| = q^{i+1}[i][k-i], \tag{12}$$

where $i = \partial(x,y)$.

Proof. Routine from counting. $\square$

Observe that the values in (12) depend only on $\partial(x,y)$.

Definition 18. We refer to Lemma 17. For $1 < i < k$, define

$$a^+_i = \big|\mathcal{A}^+_{xy}\big|, \qquad a^0_i = \big|\mathcal{A}^0_{xy}\big|, \qquad a^-_i = \big|\mathcal{A}^-_{xy}\big|, \tag{13}$$

where $i = \partial(x,y)$. Note that $a^+_i + a^0_i + a^-_i = a_i$ for $1 < i < k$.

Our next goal is to show that $\mathcal{A}^0_{xy}$ is an orbit of $\mathrm{Stab}(x,y)$.

Lemma 19. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, let $z \in \mathcal{A}^0_{xy}$. Then*

$$x \cap y \subseteq (z + x) \cap y.$$

Moreover,

$$\dim(x \cap y) + 1 = \dim\big((z + x) \cap y\big).$$

Proof. Routine from the definition of $\mathcal{A}^0_{xy}$ and linear algebra. $\square$

Lemma 20. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, let $z \in \mathcal{A}^0_{xy}$. Then there exist vectors*

$$\psi \in (z + x) \cap y, \qquad \eta \in z, \qquad \varrho \in x$$

such that

$$\psi \notin x \cap y, \qquad \eta \notin z \cap x, \qquad \varrho \notin z \cap x,$$
$$\psi = \eta + \varrho.$$

Proof. Pick $\psi \in (z + x) \cap y$ such that $\psi \notin x \cap y$. Note that $\psi \in z + x$. Also note that $\psi \notin x$ and $\psi \notin z$. Hence, ψ is a linear combination of some nonzero vector $\eta \in z$ and some nonzero vector $\varrho \in x$. We assume without loss that $\psi = \eta + \varrho$. Assume that $\eta \in z \cap x$. Then $\psi = \eta + \varrho \in x$, which is a contradiction. Hence, $\eta \notin z \cap x$. Assume that $\varrho \in z \cap x$. Then $\psi = \eta + \varrho \in z$, which is a contradiction. Hence, $\varrho \notin z \cap x$. The result follows. $\square$

Lemma 21. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, let $z \in \mathcal{A}^0_{xy}$. Let the vectors ψ, η, ϱ be from Lemma 20. Then*

$$z + \mathbb{F}\psi = z + x, \qquad z + \mathbb{F}\varrho = z + x, \qquad (z \cap x) + \mathbb{F}\eta = z, \tag{14}$$

$$x + \mathbb{F}\psi = z + x, \qquad x + \mathbb{F}\eta = z + x, \qquad (z \cap x) + \mathbb{F}\varrho = x, \tag{15}$$

$$(x \cap y) + \mathbb{F}\psi = (z + x) \cap y. \tag{16}$$

Moreover, for each equation in (14), (15), (16) *the sum on the left is direct.*

Proof. Immediate from linear algebra. □

Lemma 22. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the set $\mathcal{A}^0_{xy}$ is an orbit of the* $\mathrm{Stab}(x,y)$*-action on* $\Gamma(x)$.

Proof. By Lemma 7, the set $\mathcal{A}^0_{xy}$ is a disjoint union of orbits of $\mathrm{Stab}(x,y)$. We now show that $\mathcal{A}^0_{xy}$ is a single orbit. Let $z, z' \in \mathcal{A}^0_{xy}$. It suffices to show that there exists $\sigma \in \mathrm{Stab}(x,y)$ that sends $z \mapsto z'$. Let the vectors ψ, η, ϱ be from Lemma 20. Let the subset $\mathcal{R} \subseteq V$ form a basis for $x \cap y$. Extend the basis $\mathcal{R}$ for $x \cap y$ to a basis $\mathcal{R} \cup \mathcal{S}$ for $z \cap x$. By the third equation in (14), $\mathcal{R} \cup \mathcal{S} \cup \{\eta\}$ forms a basis for z. By the third equation in (15), $\mathcal{R} \cup \mathcal{S} \cup \{\varrho\}$ forms a basis for x. By (16), $\mathcal{R} \cup \{\psi\}$ forms a basis for $(z+x) \cap y$. By the first equation in (15), $\mathcal{R} \cup \mathcal{S} \cup \{\psi, \varrho\}$ forms a basis for $z + x$. Extend the basis $\mathcal{R} \cup \{\psi\}$ for $(z+x) \cap y$ to a basis $\mathcal{R} \cup \mathcal{Q} \cup \{\psi\}$ for y. By Lemma 3, $\mathcal{R} \cup \mathcal{S} \cup \mathcal{Q} \cup \{\psi, \varrho\}$ forms a basis for $x + y$. Extend the basis $\mathcal{R} \cup \mathcal{S} \cup \mathcal{Q} \cup \{\psi, \varrho\}$ for $x + y$ to a basis $\mathcal{R} \cup \mathcal{S} \cup \mathcal{Q} \cup \mathcal{W} \cup \{\psi, \varrho\}$ for V.

Recall the element $z' \in \mathcal{A}^0_{xy}$. Consider the corresponding vectors ψ', η', ϱ' from Lemma 20. Extend the basis $\mathcal{R}$ for $x \cap y$ to a basis $\mathcal{R} \cup \mathcal{S}'$ for $z' \cap x$. By the third equation in (14), $\mathcal{R} \cup \mathcal{S}' \cup \{\eta'\}$ forms a basis for z'. By the third equation in (15), $\mathcal{R} \cup \mathcal{S}' \cup \{\varrho'\}$ forms a basis for x. By (16), $\mathcal{R} \cup \{\psi'\}$ forms a basis for $(z' + x) \cap y$. By the first equation in (15), $\mathcal{R} \cup \mathcal{S}' \cup \{\psi', \varrho'\}$ forms a basis for $z' + x$. Extend the basis $\mathcal{R} \cup \{\psi'\}$ for $(z' + x) \cap y$ to a basis $\mathcal{R} \cup \mathcal{Q}' \cup \{\psi'\}$ for y. By Lemma 3, $\mathcal{R} \cup \mathcal{S}' \cup \mathcal{Q}' \cup \{\psi', \varrho'\}$ forms a basis for $x + y$. Extend the basis $\mathcal{R} \cup \mathcal{S}' \cup \mathcal{Q}' \cup \{\psi', \varrho'\}$ for $x + y$ to a basis $\mathcal{R} \cup \mathcal{S}' \cup \mathcal{Q}' \cup \mathcal{W}' \cup \{\psi', \varrho'\}$ for V.

By linear algebra, there exists $\sigma \in GL(V)$ that sends $\mathcal{S} \mapsto \mathcal{S}'$, $\mathcal{Q} \mapsto \mathcal{Q}'$, $\mathcal{W} \mapsto \mathcal{W}'$, $\psi \mapsto \psi'$, $\varrho \mapsto \varrho'$ and acts as the identity on $\mathcal{R}$. By construction, σ is contained in $\mathrm{Stab}(x,y)$ and sends $z \mapsto z'$. The result follows. □

Lemma 23. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the sets $\mathcal{A}^+_{xy}, \mathcal{A}^-_{xy}$ are orbits of the* $\mathrm{Stab}(x,y)$*-action on* $\Gamma(x)$.

Proof. Similar to Lemma 22. □

Theorem 24. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, the following sets are orbits of the* $\mathrm{Stab}(x,y)$*-action on* $\Gamma(x)$:

$$\mathcal{B}_{xy}, \qquad \mathcal{C}_{xy}, \qquad \mathcal{A}^+_{xy}, \qquad \mathcal{A}^0_{xy}, \qquad \mathcal{A}^-_{xy}. \tag{17}$$

Furthermore, these orbits form a partition of $\Gamma(x)$.

Proof. For the first assertion, combine Lemmas 11, 22, 23. The second assertion is immediate from Lemma 16 and the fact that the disjoint union of $\mathcal{B}_{xy}, \mathcal{C}_{xy}, \mathcal{A}_{xy}$ is equal to $\Gamma(x)$. □

Definition 25. Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$, and consider the partition of $\Gamma(x)$ given in (17). By construction, this partition is equitable in the sense of [11, p. 159]. We call this partition the *y-partition of* $\Gamma(x)$.

7 The vectors $A^+_{xy}, A^0_{xy}, A^-_{xy}$

Pick $x, y \in X$ such that $1 < \partial(x, y) < k$. Recall the sets $\mathcal{A}^+_{xy}, \mathcal{A}^0_{xy}, \mathcal{A}^-_{xy}$ from Definition 15. In this section we use these sets to define some vectors $A^+_{xy}, A^0_{xy}, A^-_{xy}$ in the Euclidean space E. We show that $A^+_{xy}, A^0_{xy}, A^-_{xy}$ are contained in $\mathrm{Fix}(x, y)$. We write $A^+_{xy}, A^0_{xy}, A^-_{xy}$ in terms of the geometric basis for $\mathrm{Fix}(x, y)$ and also the combinatorial basis for $\mathrm{Fix}(x, y)$.

Definition 26. For $x, y \in X$ such that $1 < \partial(x, y) < k$, define the vectors

$$A^+_{xy} = \sum_{z \in \mathcal{A}^+_{xy}} \widehat{z}, \qquad A^0_{xy} = \sum_{z \in \mathcal{A}^0_{xy}} \widehat{z}, \qquad A^-_{xy} = \sum_{z \in \mathcal{A}^-_{xy}} \widehat{z}. \tag{18}$$

Note that $A^+_{xy}, A^0_{xy}, A^-_{xy}$ are contained in E. We call $A^+_{xy}, A^0_{xy}, A^-_{xy}$ the *characteristic vectors* of $\mathcal{A}^+_{xy}, \mathcal{A}^0_{xy}, \mathcal{A}^-_{xy}$ respectively. By Lemma 16, $A_{xy} = A^+_{xy} + A^0_{xy} + A^-_{xy}$.

Lemma 27. *For $x, y \in X$ such that $1 < \partial(x, y) < k$, the vectors $A^+_{xy}, A^0_{xy}, A^-_{xy}$ are contained in* $\mathrm{Fix}(x, y)$.

Proof. Pick $\sigma \in \mathrm{Stab}(x, y)$. Since $\mathcal{A}^+_{xy}, \mathcal{A}^0_{xy}, \mathcal{A}^-_{xy}$ are orbits of the $\mathrm{Stab}(x, y)$-action on $\Gamma(x)$, the map σ fixes $\mathcal{A}^+_{xy}, \mathcal{A}^0_{xy}, \mathcal{A}^-_{xy}$. The result follows. □

Our next goal is to write $A^+_{xy}, A^0_{xy}, A^-_{xy}$ in terms of the geometric basis for $\mathrm{Fix}(x, y)$. To do this, we recall the inner products that involve the vectors in the geometric basis for $\mathrm{Fix}(x, y)$.

Lemma 28. [12, Theorem 10.4] *Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u, v)-entry of the table gives the inner product $\langle u, v\rangle$. Write $i = \partial(x, y)$.*

$\langle\ ,\ \rangle$	$\widehat{x}$	$\widehat{y}$	$\widehat{x \cap y}$	$\widehat{x + y}$
$\widehat{x}$	$q^k[k][n-k]$	$[n][k-i]-[k]^2$	$q^k[k-i][n-k]$	$q^{k+i}[k][n-k-i]$
$\widehat{y}$	$[n][k-i]-[k]^2$	$q^k[k][n-k]$	$q^k[k-i][n-k]$	$q^{k+i}[k][n-k-i]$
$\widehat{x \cap y}$	$q^k[k-i][n-k]$	$q^k[k-i][n-k]$	$q^{k-i}[k-i][n-k+i]$	$q^{k+i}[k-i][n-k-i]$
$\widehat{x + y}$	$q^{k+i}[k][n-k-i]$	$q^{k+i}[k][n-k-i]$	$q^{k+i}[k-i][n-k-i]$	$q^{k+i}[k+i][n-k-i]$

For $1 < i < k$ let M_i denote the matrix of inner products in Lemma 28.

Lemma 29. [12, Lemma 10.10] *For $1 < i < k$ the inverse of the matrix M_i is given by*

$$M_i^{-1} = \frac{1}{q^{k-i}(q-1)[i]^2[n]} \begin{pmatrix} q^i & 1 & -q^i & -1 \\ 1 & q^i & -q^i & -1 \\ -q^i & -q^i & \frac{q^i[k]-[i]}{[k-i]} & 1 \\ -1 & -1 & 1 & \frac{q^i[n-k]-[i]}{q^{2i}[n-k-i]} \end{pmatrix}.$$

Next we find inner products that involve the vectors $A^+_{xy}, A^0_{xy}, A^-_{xy}$.

Lemma 30. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\left\langle A^+_{xy}, \widehat{x} \right\rangle = q^{i+1}[i][n-k-i]\Big([n][k-1]-[k]^2\Big), \tag{19}$$

$$\left\langle A^0_{xy}, \widehat{x} \right\rangle = (q-1)[i]^2\Big([n][k-1]-[k]^2\Big), \tag{20}$$

$$\left\langle A^-_{xy}, \widehat{x} \right\rangle = q^{i+1}[i][k-i]\Big([n][k-1]-[k]^2\Big), \tag{21}$$

where $i = \partial(x,y)$.

Proof. We first prove (19). Using the first equation in (18), we obtain

$$\left\langle A^+_{xy}, \widehat{x} \right\rangle = \sum_{z \in \mathcal{A}^+_{xy}} \langle \widehat{z}, \widehat{x} \rangle. \tag{22}$$

Pick $z \in \mathcal{A}^+_{xy}$. By the definition of $\mathcal{A}^+_{xy}$ and Lemma 6(iii),

$$\langle \widehat{z}, \widehat{x} \rangle = [n][k-1]-[k]^2. \tag{23}$$

By the above comments,

$$\left\langle A^+_{xy}, \widehat{x} \right\rangle = \left|\mathcal{A}^+_{xy}\right|\Big([n][k-1]-[k]^2\Big). \tag{24}$$

In (24), we evaluate $\left|\mathcal{A}^+_{xy}\right|$ using (12); this yields (19).

We have now verified (19). Equations (20) and (21) are obtained in a similar fashion. □

Lemma 31. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\left\langle A^+_{xy}, \widehat{y} \right\rangle = q^{i+1}[i][n-k-i]\Big([n][k-i]-[k]^2\Big), \tag{25}$$

$$\left\langle A^0_{xy}, \widehat{y} \right\rangle = (q-1)[i]^2\Big([n][k-i]-[k]^2\Big), \tag{26}$$

$$\left\langle A^-_{xy}, \widehat{y} \right\rangle = q^{i+1}[i][k-i]\Big([n][k-i]-[k]^2\Big), \tag{27}$$

where $i = \partial(x,y)$.

Proof. We first prove (25). Using the first equation in (18), we obtain

$$\left\langle A^+_{xy}, \widehat{y} \right\rangle = \sum_{z \in \mathcal{A}^+_{xy}} \langle \widehat{z}, \widehat{y} \rangle. \tag{28}$$

Pick $z \in \mathcal{A}^+_{xy}$. By the definition of $\mathcal{A}^+_{xy}$ and Lemma 6(iii),

$$\langle \widehat{z}, \widehat{y} \rangle = [n][k-i] - [k]^2. \tag{29}$$

By the above comments,

$$\left\langle A^+_{xy}, \widehat{y} \right\rangle = \left|\mathcal{A}^+_{xy}\right| \Big([n][k-i] - [k]^2 \Big). \tag{30}$$

In (30), we evaluate $\left|\mathcal{A}^+_{xy}\right|$ using (12); this yields (25).

We have now verified (25). Equations (26) and (27) are obtained in a similar fashion. □

Lemma 32. *For $x, y \in X$ such that $1 < \partial(x, y) < k$, we have*

$$\left\langle A^+_{xy}, \widehat{x \cap y} \right\rangle = q^{k+i+1}[i][n-k-i][k-i][n-k], \tag{31}$$

$$\left\langle A^0_{xy}, \widehat{x \cap y} \right\rangle = q^k(q-1)[i]^2[k-i][n-k], \tag{32}$$

$$\left\langle A^-_{xy}, \widehat{x \cap y} \right\rangle = q^{i+1}[i][k-i]\Big([n][k-i-1] - [k-i][k] \Big), \tag{33}$$

where $i = \partial(x, y)$.

Proof. We first prove (31). Using the first equation in (18), we obtain

$$\left\langle A^+_{xy}, \widehat{x \cap y} \right\rangle = \sum_{z \in \mathcal{A}^+_{xy}} \left\langle \widehat{z}, \widehat{x \cap y} \right\rangle. \tag{34}$$

Pick $z \in \mathcal{A}^+_{xy}$. By the definition of $\mathcal{A}^+_{xy}$ and Lemma 6(i),

$$\left\langle \widehat{z}, \widehat{x \cap y} \right\rangle = [n][k-i] - [k][k-i] = q^k[k-i][n-k]. \tag{35}$$

By the above comments,

$$\left\langle A^+_{xy}, \widehat{x \cap y} \right\rangle = \left|\mathcal{A}^+_{xy}\right| q^k[k-i][n-k]. \tag{36}$$

In (36), we evaluate $\left|\mathcal{A}^+_{xy}\right|$ using (12); this yields (31).

We have now verified (31). Equation (32) is obtained in a similar fashion.

Next we prove (33). Using the last equation in (18), we obtain

$$\left\langle A^-_{xy}, \widehat{x \cap y} \right\rangle = \sum_{z \in \mathcal{A}^-_{xy}} \left\langle \widehat{z}, \widehat{x \cap y} \right\rangle. \tag{37}$$

Pick $z \in \mathcal{A}_{xy}^-$. By the definition of $\mathcal{A}_{xy}^-$ and Lemma 2,

$$\dim (z \cap x \cap y) = k - i - 1. \tag{38}$$

By (38) and Lemma 6(i),

$$\left\langle \widehat{z}, \widehat{x \cap y} \right\rangle = [n][k-i-1] - [k-i][k]. \tag{39}$$

By the above comments,

$$\left\langle A_{xy}^-, \widehat{x \cap y} \right\rangle = \left|\mathcal{A}_{xy}^-\right| \Big([n][k-i-1] - [k-i][k] \Big). \tag{40}$$

In (40), we evaluate $\left|\mathcal{A}_{xy}^-\right|$ using (12); this yields (33). $\square$

Lemma 33. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\left\langle A_{xy}^+, \widehat{x+y} \right\rangle = q^{i+1}[i][n-k-i]\Big([n][k-1] - [k][k+i] \Big), \tag{41}$$

$$\left\langle A_{xy}^0, \widehat{x+y} \right\rangle = q^{k+i}(q-1)[i]^2[k][n-k-i], \tag{42}$$

$$\left\langle A_{xy}^-, \widehat{x+y} \right\rangle = q^{k+2i+1}[i][k-i][k][n-k-i], \tag{43}$$

where $i = \partial(x,y)$.

Proof. We first prove (41). Using the first equation in (18), we obtain

$$\left\langle A_{xy}^+, \widehat{x+y} \right\rangle = \sum_{z \in \mathcal{A}_{xy}^+} \left\langle \widehat{z}, \widehat{x+y} \right\rangle. \tag{44}$$

Pick $z \in \mathcal{A}_{xy}^+$. By the definition of $\mathcal{A}_{xy}^+$ and Lemma 2,

$$\dim \big(z \cap (x+y)\big) = k - 1. \tag{45}$$

By (45) and Lemma 6(i),

$$\left\langle \widehat{z}, \widehat{x+y} \right\rangle = [n][k-1] - [k][k+i]. \tag{46}$$

By the above comments,

$$\left\langle A_{xy}^+, \widehat{x+y} \right\rangle = \left|\mathcal{A}_{xy}^+\right| \Big([n][k-1] - [k][k+i] \Big). \tag{47}$$

In (47), we evaluate $\left|\mathcal{A}_{xy}^+\right|$ using (12); this yields (41).

Next we prove (42). Using the second equation in (18), we obtain

$$\left\langle A_{xy}^0, \widehat{x+y} \right\rangle = \sum_{z \in \mathcal{A}_{xy}^0} \left\langle \widehat{z}, \widehat{x+y} \right\rangle. \tag{48}$$

Pick $z \in \mathcal{A}^0_{xy}$. By the definition of $\mathcal{A}^0_{xy}$,

$$z \cap (x+y) = z. \tag{49}$$

By (49) and Lemma 6(i),

$$\left\langle \widehat{z}, \widehat{x \cap y} \right\rangle = [n][k] - [k][k+i] = q^{k+i}[k][n-k-i]. \tag{50}$$

By the above comments,

$$\left\langle A^0_{xy}, \widehat{x+y} \right\rangle = \left| \mathcal{A}^0_{xy} \right| q^{k+i}[k][n-k-i]. \tag{51}$$

In (51), we evaluate $\left|\mathcal{A}^0_{xy}\right|$ using (12); this yields (42).

We have now verified (42). Equation (43) is obtained in a similar fashion. $\square$

Theorem 34. *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u,v)-entry of the table gives the inner product $\langle u, v\rangle$. Write $i = \partial(x,y)$.*

$\langle\ ,\ \rangle$	A^+_{xy}	A^0_{xy}	A^-_{xy}
$\widehat{x}$	$q^{i+1}[i][n-k-i]\big([n][k-1]-[k]^2\big)$	$(q-1)[i]^2\big([n][k-1]-[k]^2\big)$	$q^{i+1}[i][k-i]\big([n][k-1]-[k]^2\big)$
$\widehat{y}$	$q^{i+1}[i][n-k-i]\big([n][k-i]-[k]^2\big)$	$(q-1)[i]^2\big([n][k-i]-[k]^2\big)$	$q^{i+1}[i][k-i]\big([n][k-i]-[k]^2\big)$
$\widehat{x \cap y}$	$q^{k+i+1}[i][n-k-i][k-i][n-k]$	$q^k(q-1)[i]^2[k-i][n-k]$	$q^{i+1}[i][k-i]([n][k-i-1]-[k-i][k])$
$\widehat{x+y}$	$q^{i+1}[i][n-k-i]([n][k-1]-[k][k+i])$	$q^{k+i}(q-1)[i]^2[k][n-k-i]$	$q^{k+2i+1}[i][k-i][k][n-k-i]$

Proof. Combine Lemmas 30–33. $\square$

In the next result, we write $A^+_{xy}, A^0_{xy}, A^-_{xy}$ in terms of the geometric basis for $\mathrm{Fix}(x,y)$.

Theorem 35. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$A^+_{xy} = q^{i+1}[n-k-i][i-1]\widehat{x} + q^{2i}[n-k-i]\widehat{x \cap y} - [i]\widehat{x+y}, \tag{52}$$

$$A^0_{xy} = \big(q^i[i-1]-[i]\big)\widehat{x} - q^{i-1}\widehat{y} + q^{2i-1}\widehat{x \cap y} + q^{i-1}\widehat{x+y}, \tag{53}$$

$$A^-_{xy} = q^{i+1}[k-i][i-1]\widehat{x} - q^i[i]\widehat{x \cap y} + q^i[k-i]\widehat{x+y}, \tag{54}$$

where $i = \partial(x,y)$.

Proof. Write

$$A^+_{xy} = \alpha\widehat{x} + \beta\widehat{y} + \gamma\widehat{x\cap y} + \delta\widehat{x+y}, \tag{55}$$

$$A^0_{xy} = \alpha'\widehat{x} + \beta'\widehat{y} + \gamma'\widehat{x\cap y} + \delta'\widehat{x+y}, \tag{56}$$

$$A^-_{xy} = \alpha''\widehat{x} + \beta''\widehat{y} + \gamma''\widehat{x\cap y} + \delta''\widehat{x+y}, \tag{57}$$

for $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta', \alpha'', \beta'', \gamma'', \delta'' \in \mathbb{R}$. Let N_i denote the matrix of inner products from Theorem 34. In each of (55), (56), (57) we take the inner product of either side with each of $\widehat{x}, \widehat{y}, \widehat{x\cap y}$, $\widehat{x+y}$ to obtain

$$M_i \begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \\ \delta & \delta' & \delta'' \end{pmatrix} = N_i.$$

The matrix M_i is invertible by Lemma 29, so

$$\begin{pmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \\ \delta & \delta' & \delta'' \end{pmatrix} = M_i^{-1} N_i.$$

Using Lemma 29 and matrix multiplication we obtain

$$M_i^{-1} N_i = \begin{pmatrix} q^{i+1}[n-k-i][i-1] & q^i[i-1]-[i] & q^{i+1}[k-i][i-1] \\ 0 & -q^{i-1} & 0 \\ q^{2i}[n-k-i] & q^{2i-1} & -q^i[i] \\ -[i] & q^{i-1} & q^i[k-i] \end{pmatrix}.$$

The result follows. □

Fix $x, y \in X$ such that $1 < \partial(x,y) < k$. Our next goal is to write $A^+_{xy}, A^0_{xy}, A^-_{xy}$ in terms of the combinatorial basis for $\mathrm{Fix}(x,y)$. To do this, we write $\widehat{x\cap y}, \widehat{x+y}$ in terms of the combinatorial basis for $\mathrm{Fix}(x,y)$.

Lemma 36. [12, Theorem 11.4] *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\begin{aligned}
\widehat{x\cap y} = {} & \frac{[k-i][n-k-1]}{q^{k-1}[n-2k]}\widehat{x} + \frac{[k-i]}{q^{k-i+1}[i-1][n-2k]}\widehat{y} \\
& - \frac{1}{q^{k+i}[n-2k]}B_{xy} - \frac{[k-i]}{q^k[i-1][n-2k]}C_{xy}, \\
\widehat{x+y} = {} & -\frac{[k-1][n-k-i]}{q^{k-i-1}[n-2k]}\widehat{x} - \frac{[n-k-i]}{q^{k-2i+1}[i-1][n-2k]}\widehat{y} \\
& + \frac{1}{q^k[n-2k]}B_{xy} + \frac{[n-k-i]}{q^{k-i}[i-1][n-2k]}C_{xy},
\end{aligned}$$

where $i = \partial(x,y)$.

Theorem 37. *For $x, y \in X$ such that $1 < \partial(x,y) < k$, we have*

$$\begin{aligned} A^+_{xy} = \frac{[k-1][n-k-i][n-k]}{q^{k-i-1}[n-2k]}\widehat{x} + \frac{[k][n-k-i]}{q^{k-2i+1}[i-1][n-2k]}\widehat{y} \\ - \frac{[n-k]}{q^k[n-2k]}B_{xy} - \frac{[k][n-k-i]}{q^{k-i}[i-1][n-2k]}C_{xy}, \end{aligned} \tag{58}$$

$$A^0_{xy} = -[i]\widehat{x} - \frac{q^{i-1}[i]}{[i-1]}\widehat{y} + \frac{q^{i-1}}{[i-1]}C_{xy}, \tag{59}$$

$$\begin{aligned} A^-_{xy} = -\frac{[k-i][k][n-k-1]}{q^{k-i-1}[n-2k]}\widehat{x} - \frac{[k-i][n-k]}{q^{k-2i+1}[i-1][n-2k]}\widehat{y} \\ + \frac{[k]}{q^k[n-2k]}B_{xy} + \frac{[k-i][n-k]}{q^{k-i}[i-1][n-2k]}C_{xy}, \end{aligned} \tag{60}$$

where $i = \partial(x,y)$.

Proof. We first prove (58). In the equation (52), eliminate $\widehat{x \cap y}$ and $\widehat{x+y}$ using Lemma 36 and simplify the result.

We have now verified (58). Equations (59) and (60) are obtained in a similar fashion. □

8 Some inner products involving the y-partition of $\Gamma(x)$

Pick $x, y \in X$ such that $1 < \partial(x,y) < k$. In this section we calculate the inner products between the vectors $\widehat{x}, \widehat{y}, B_{xy}, C_{xy}, A^+_{xy}, A^0_{xy}, A^-_{xy}$. We begin by recalling some inner products from [12].

Lemma 38. [12, Theorem 10.9] *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u,v)-entry of the table gives the inner product $\langle u, v\rangle$. Write $i = \partial(x,y)$.*

$\langle\ ,\ \rangle$	$\widehat{x}$	$\widehat{y}$	$\widehat{x \cap y}$	$\widehat{x+y}$
$\widehat{x}$	$q^k[k][n-k]$	$[n][k-i]-[k]^2$	$q^k[k-i][n-k]$	$q^{k+i}[k][n-k-i]$
$\widehat{y}$	$[n][k-i]-[k]^2$	$q^k[k][n-k]$	$q^k[k-i][n-k]$	$q^{k+i}[k][n-k-i]$
B_{xy}	$q^{2i+1}[k-i][n-k-i]\cdot \big([n][k-1]-[k]^2\big)$	$q^{2i+1}[k-i][n-k-i]\cdot \big([n][k-i-1]-[k]^2\big)$	$q^{2i+1}[k-i][n-k-i]\cdot ([n][k-i-1]-[k-i][k])$	$q^{2i+1}[k-i][n-k-i]\cdot ([n][k-1]-[k][k+i])$
C_{xy}	$[i]^2\big([n][k-1]-[k]^2\big)$	$[i]^2\big([n][k-i+1]-[k]^2\big)$	$q^k[i]^2[k-i][n-k]$	$q^{k+i}[i]^2[k][n-k-i]$

Lemma 39. [12, Theorem 10.15] *Let $x, y \in X$ satisfy $1 < \partial(x,y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u,v)-entry of the table gives the inner product $\langle u, v\rangle$. Write $i = \partial(x,y)$.*

$\langle\ ,\ \rangle$	$\widehat{x}$	$\widehat{y}$	B_{xy}	C_{xy}
$\widehat{x}$	$q^k[k][n-k]$	$[n][k-i]-[k]^2$	$q^{2i+1}[k-i][n-k-i]\cdot \big([n][k-1]-[k]^2\big)$	$[i]^2\big([n][k-1]-[k]^2\big)$
$\widehat{y}$	$[n][k-i]-[k]^2$	$q^k[k][n-k]$	$q^{2i+1}[k-i][n-k-i]\cdot \big([n][k-i-1]-[k]^2\big)$	$[i]^2\big([n][k-i+1]-[k]^2\big)$
B_{xy}	$q^{2i+1}[k-i][n-k-i]\cdot \big([n][k-1]-[k]^2\big)$	$q^{2i+1}[k-i][n-k-i]\cdot \big([n][k-i-1]-[k]^2\big)$	$q^{4i+2}[k-i][n-k-i]\cdot \Big(q^{k-i-2}[n]([k-i]+[n-k-i])+ [k-i][n-k-i]\big([n][k-2]-[k]^2\big)\Big)$	$q^{2i+1}[k-i][n-k-i]\cdot [i]^2\big([n][k-2]-[k]^2\big)$
C_{xy}	$[i]^2\big([n][k-1]-[k]^2\big)$	$[i]^2\big([n][k-i+1]-[k]^2\big)$	$q^{2i+1}[k-i][n-k-i]\cdot [i]^2\big([n][k-2]-[k]^2\big)$	$[i]^2\Big(q^{k-2}[n](2q[i-1]+q+1)+ [i]^2\big([n][k-2]-[k]^2\big)\Big)$

Theorem 40. *Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u, v)-entry of the table gives the inner product $\langle u, v\rangle$. Write $i = \partial(x, y)$.*

$\langle\ ,\ \rangle$	A^+_{xy}	A^0_{xy}	A^-_{xy}
$\widehat{x}$	$q^{i+1}[i][n-k-i]\big([n][k-1]-[k]^2\big)$	$(q-1)[i]^2\big([n][k-1]-[k]^2\big)$	$q^{i+1}[i][k-i]\big([n][k-1]-[k]^2\big)$
$\widehat{y}$	$q^{i+1}[i][n-k-i]\big([n][k-i]-[k]^2\big)$	$(q-1)[i]^2\big([n][k-i]-[k]^2\big)$	$q^{i+1}[i][k-i]\big([n][k-i]-[k]^2\big)$
B_{xy}	$q^{2i+2}[i][k-i][n-k-i] \Big(\big(q^i[n-k-i]-1\big)\big([n][k-2]-[k]^2\big) +\big([n][k-1]-[k]^2\big)\Big)$	$q^{2i+1}(q-1)[k-i][n-k-i] [i]^2\big([n][k-2]-[k]^2\big)$	$q^{2i+2}[i][k-i][n-k-i] \Big(\big(q^i[k-i]-1\big)\big([n][k-2]-[k]^2\big) +\big([n][k-1]-[k]^2\big)\Big)$
C_{xy}	$q^{i+1}[n-k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$	$(q-1)[i]^2\Big(q^{k-2}[n](2[i]-1) +[i]^2\big([n][k-2]-[k]^2\big)\Big)$	$q^{i+1}[k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$

Proof. The entries in the first two rows are immediate from Theorem 34. Next we calculate the inner product $\left\langle B_{xy}, A^+_{xy}\right\rangle$. Using (52),

$$\left\langle B_{xy}, A^+_{xy}\right\rangle = q^{i+1}[n-k-i][i-1]\big\langle B_{xy}, \widehat{x}\big\rangle + q^{2i}[n-k-i]\left\langle B_{xy}, \widehat{x\cap y}\right\rangle - [i]\left\langle B_{xy}, \widehat{x+y}\right\rangle.$$

In the above equation, evaluate the right-hand side using Lemma 38.

We have now calculated the inner product $\left\langle B_{xy}, A^+_{xy}\right\rangle$. For the other inner products the calculations are similar, and omitted. □

Theorem 41. *Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$. In the following table, for each vector u in the header column, and each vector v in the header row, the (u, v)-entry of the table gives the inner product $\langle u, v\rangle$. Write $i = \partial(x, y)$.*

$\langle\ ,\ \rangle$	A^+_{xy}	A^0_{xy}	A^-_{xy}
A^+_{xy}	$q^{2i+2}[i][n-k-i]\Big(q^{k-i-2}[n][n-k] +[i][n-k-i]\big([n][k-2]-[k]^2\big)\Big)$	$q^{i+1}(q-1)[n-k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$	$q^{2i+2}[k-i][n-k-i] [i]^2\big([n][k-2]-[k]^2\big)$
A^0_{xy}	$q^{i+1}(q-1)[n-k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$	$(q-1)[i]^2\Bigg(q^{k-2}[n]\Big(2(q-1)[i]+1\Big) +(q-1)[i]^2\big([n][k-2]-[k]^2\big)\Bigg)$	$q^{i+1}(q-1)[k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$
A^-_{xy}	$q^{2i+2}[k-i][n-k-i] [i]^2\big([n][k-2]-[k]^2\big)$	$q^{i+1}(q-1)[k-i][i]^2 \Big(q^{k-2}[n]+[i]\big([n][k-2]-[k]^2\big)\Big)$	$q^{2i+2}[i][k-i]\Big(q^{k-i-2}[n][k] +[i][k-i]\big([n][k-2]-[k]^2\big)\Big)$

Proof. We will calculate the inner product $\left\langle A^+_{xy}, A^+_{xy}\right\rangle$. Using (52),

$$\left\langle A^+_{xy}, A^+_{xy}\right\rangle = q^{i+1}[n-k-i][i-1]\left\langle A^+_{xy}, \widehat{x}\right\rangle + q^{2i}[n-k-i]\left\langle A^+_{xy}, \widehat{x\cap y}\right\rangle - [i]\left\langle A^+_{xy}, \widehat{x+y}\right\rangle.$$

In the above equation, evaluate the right-hand side using Theorem 34.

We have now calculated the inner product $\left\langle A^+_{xy}, A^+_{xy}\right\rangle$. For the other inner products the calculations are similar, and omitted. □

9 Some combinatorics and algebra involving the y-partition of $\Gamma(x)$

Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$. In (17) we partitioned $\Gamma(x)$ into five orbits for $\mathrm{Stab}(x, y)$. In this section, we describe the edges between pairs of orbits in this partition. In this description we use a 5 by 5 matrix. We find the eigenvalues of this matrix. For each eigenvalue we display a row eigenvector and column eigenvector.

Theorem 42. *Let $x, y \in X$ satisfy $1 < \partial(x, y) < k$, and consider the orbits of* $\mathrm{Stab}(x, y)$ *on $\Gamma(x)$. Referring to the table below, for each orbit $\mathcal{O}$ in the header column, and each orbit $\mathcal{N}$ in the header row, the $(\mathcal{O}, \mathcal{N})$-entry gives the number of vertices in $\mathcal{N}$ that are adjacent to a given vertex in $\mathcal{O}$. Write $i = \partial(x, y)$.*

	$\mathcal{B}_{xy}$	$\mathcal{C}_{xy}$	$\mathcal{A}^+_{xy}$	$\mathcal{A}^0_{xy}$	$\mathcal{A}^-_{xy}$
$\mathcal{B}_{xy}$	$q^{i+1}[k-i] +q^{i+1}[n-k-i]-q-1$	0	$q[i]$	0	$q[i]$
$\mathcal{C}_{xy}$	0	$2q[i-1]$	$q^{i+1}[n-k-i]$	$(q-1)\big(2[i]-1\big)$	$q^{i+1}[k-i]$
$\mathcal{A}^+_{xy}$	$q^{i+1}[k-i]$	$[i]$	$q[n-k]-q-1$	$(q-1)[i]$	0
$\mathcal{A}^0_{xy}$	0	$2[i]-1$	$q^{i+1}[n-k-i]$	$(q-1)\big(2[i]-1\big)-1$	$q^{i+1}[k-i]$
$\mathcal{A}^-_{xy}$	$q^{i+1}[n-k-i]$	$[i]$	0	$(q-1)[i]$	$q[k]-q-1$

Proof. We will verify the $\big(\mathcal{B}_{xy}, \mathcal{B}_{xy}\big)$-entry of the table. Pick a vertex $w \in \mathcal{B}_{xy}$. Let $\#$ denote the number of vertices in $\mathcal{B}_{xy}$ that are adjacent to w. Note that $\#$ is independent of the choice of w, because the partition (17) is equitable.

We now compute $\#$. By construction, each vertex in $\mathcal{B}_{xy}$ is at distance at most 2 from w. Using Lemma 6(iii), (iv) we obtain

$$\begin{aligned} \big\langle \widehat{w}, B_{xy} \big\rangle = \sum_{z \in \mathcal{B}_{xy}} \big\langle \widehat{w}, \widehat{z} \big\rangle = q^k[k][n-k] + \#\Big([n][k-1] - [k]^2\Big) \\ + \Big(\big|\mathcal{B}_{xy}\big| - \# - 1\Big)\Big([n][k-2] - [k]^2\Big). \end{aligned} \tag{61}$$

By construction,

$$\big\langle B_{xy}, B_{xy} \big\rangle = \big|\mathcal{B}_{xy}\big| \big\langle \widehat{w}, B_{xy} \big\rangle. \tag{62}$$

We now evaluate (62). The left-hand side is evaluated using the $\big(B_{xy}, B_{xy}\big)$-entry in the table of Lemma 39. The right-hand side is evaluated using (61) and $b_i = \big|\mathcal{B}_{xy}\big|$; the value of b_i is given in (7). After evaluating (62), we solve the resulting equation for $\#$; this yields the $\big(\mathcal{B}_{xy}, \mathcal{B}_{xy}\big)$-entry of the table. The other entries are obtained in a similar fashion. $\square$

Definition 43. For $1 < i < k$ let $\mathcal{M}_i$ denote the 5×5 matrix in Theorem 42.

Note that $\mathcal{M}_i$ is not symmetric. We now give the transpose $\mathcal{M}_i^t$.

Lemma 44. *For $1 < i < k$, we have $\mathcal{M}_i^t = D\mathcal{M}_i D^{-1}$, where $D = \operatorname{diag}\big(b_i, c_i, a_i^+, a_i^0, a_i^-\big)$. Recall b_i, c_i from (7) and a_i^+, a_i^0, a_i^- from (13).*

Proof. Immediate. $\square$

Our next goal is to find the eigenvalues of $\mathcal{M}_i$. For each eigenvalue we display a row eigenvector and a column eigenvector.

Lemma 45. *For $1 < i < k$, the eigenvalues of the matrix $\mathcal{M}_i$ are*

$$a_1, \qquad q[n-k] - q - 1, \qquad q[k] - q - 1, \qquad -1, \qquad -q-1,$$

where $a_1 = q[k] + q[n-k] - q - 1$.

Proof. Routine. $\square$

Lemma 46. *For $1 < i < k$ we consider the matrix $\mathcal{M}_i$. In the table below, for each eigenvalue of $\mathcal{M}_i$, we display a corresponding row eigenvector and column eigenvector. Recall b_i, c_i from (7) and a_i^+, a_i^0, a_i^- from (13).*

Eigenvalue of $\mathcal{M}_i$	*corresponding row eigenvector*	*corresponding column eigenvector*
a_1	$\left(b_i, c_i, a_i^+, a_i^0, a_i^-\right)$	$(1,1,1,1,1)^t$
$q[n-k]-q-1$	$\left(a_i^+, -c_i, -a_i^+, -a_i^0, qc_i\right)$	$\left(qc_i, -a_i^-, -a_i^-, -a_i^-, qc_i\right)^t$
$q[k]-q-1$	$\left(a_i^-, -c_i, qc_i, -a_i^0, -a_i^-\right)$	$\left(qc_i, -a_i^+, qc_i, -a_i^+, -a_i^+\right)^t$
-1	$(0,1,0,-1,0)$	$(0, q-1, 0, -1, 0)^t$
$-q-1$	$(q, 1, -q, q-1, -q)$	$\left(qc_i, b_i, -a_i^-, b_i, -a_i^+\right)^t$

Proof. Routine. □

Remark 47. [3, 5, 7] For $x \in X$ the spectrum of the local graph $\Gamma(x)$ is given in the table below. Recall $a_1 = q[k] + q[n-k] - q - 1$.

Eigenvalue	Multiplicity
a_1	1
$q[n-k]-q-1$	$[k]-1$
$q[k]-q-1$	$[n-k]-1$
-1	$(q-1)[k][n-k]$
$-q-1$	$q^2[k-1][n-k-1]$

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