On the Spectral Turán Problems for Bipartite Hypergraphs

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Abstract

Given a graph F and a positive integer $r \geqslant 2$, the r-expansion of F, denoted by F^+, is the r-graph obtained from F by enlarging each edge of F with r-2 new vertices disjoint from $V(F)$ such that distinct edges of F are enlarged by distinct vertices. In this paper, we present some sharp bounds on the spectral radius of $K^+_{s,t}$-free linear r-graphs by establishing the connection between the spectral radius and the number of walks in uniform hypergraphs. For $t \geqslant 2$, we show that the spectral radius of a $K^+_{2,t}$-free n-vertex linear r-graph is at most $\frac{\sqrt{t-1}}{r-1}\sqrt{n} + O(1)$, which is close to being asymptotically optimal when $r = 3$. Meanwhile, we prove that the spectral radius of a $K^+_{s,t}$-free n-vertex linear r-graph is $O(n^{1-\frac{1}{s}})$, where $t \geqslant s \geqslant 2$. The exponent of this upper bound is tight for $t > (s-1)!$ and $r = 3$.

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1 Introduction

An r-uniform hypergraph (or r-graph for short) $H = (V(H), E(H))$ consists of a vertex set $V(H)$ and an edge set $E(H)$, where $E(H)$ is a set of r-element subsets of $V(E)$. As usual, a graph is a 2-uniform hypergraph. Throughout this paper, we default to $r \geqslant 2$.

For an r-graph H and a family of r-graphs $\mathcal{F}$, we say H is $\mathcal{F}$-free if H does not contain any member of $\mathcal{F}$ as a subgraph. The Turán number, denoted by $\mathrm{ex}_r(n, \mathcal{F})$, is the maximum possible number of edges of an n-vertex $\mathcal{F}$-free r-graph. Determining the Turán number of r-graphs is one of the central problems in extremal combinatorics. For nonbipartite graphs, the Turán number was asymptotically solved by the celebrated

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Erdős-Stone-Simonovits Theorem [4, 5]. Nevertheless, there is comparatively little understanding of the Turán number of bipartite graphs and hypergraphs. We refer the reader to the surveys [1, 6, 9, 15, 19] for more details.

An r-uniform hypergraph H is linear if $|e_1 \cap e_2| \leqslant 1$ for any $e_1, e_2 \in E(H)$. Similar to the Turán number, for a family of r-graphs $\mathcal{F}$, the linear Turán number $\mathrm{ex}_r^{lin}(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum possible number of edges of an n-vertex $\mathcal{F}$-free linear r-graph. The linear Turán problem is closely related to the famous Brown-Erdős-Sós problem. Let (v, e)-configuration be the collection of 3-graphs with e edges and at most v vertices. Brown-Erdős-Sós [2] conjectured the number of edges of an n-vertex 3-graph without (v, e)-configuration is $o(n^2)$. It was shown by Solymosi [26] that the Brown-Erdős-Sós conjecture is equivalent to proving the linear Turán number of (v, e)-configuration is $o(n^2)$.

In the past twenty years, the linear Turán number of bipartite hypergraphs has attracted considerable attention. Given a graph F, we say that a hypergraph H is an element of Berge-F if there is a bijection $f : E(F) \to E(H)$ such that $e \subset f(e)$ for each $e \in E(F)$. Lazebnik and Verstraëte [17] showed that $ex_3^{lin}(n, \{\text{Berge-}C_3, \text{Berge-}K_{2,2}\}) = \frac{1}{6}n^{3/2} + O(n)$. For $t \geqslant 2$, Timmons [27] showed that $ex_r^{lin}(n, \{\text{Berge-}C_3, \text{Berge-}K_{2,t}\}) \leqslant \frac{\sqrt{t-1}}{r(r-1)}n^{3/2} + O(n)$. Gerbner, Methuku and Vizer [10] improved Timmons' result by showing $ex_r^{lin}(n, \text{Berge-}K_{2,t}) \leqslant \frac{\sqrt{t-1}}{r(r-1)}n^{3/2} + O(n)$. Recently, She, Fan and Kang [25] proved that $ex_r^{lin}(n, \{\text{Berge-}C_3, \text{Berge-}K_{s,t}\}) \leqslant \frac{(t-s-1)^{\frac{1}{s}}}{r(r-1)}n^{2-\frac{1}{s}} + O(n^{2-\frac{1}{s}})$, which is a generalization of Timmons' result. For a graph F and a positive integer $r \geqslant 2$, the r-expansion of F, denoted by F^+, is the r-graph obtained from F by enlarging each edge of F with $r-2$ new vertices disjoint from $V(F)$ such that distinct edges of F are enlarged by distinct vertices. Gao and Chang [11] proved that for all integers $r \geqslant 2$ and $t \geqslant s \geqslant 2$, $ex_r^{lin}(n, K_{s,t}^+) \leqslant \frac{(t-1)^{\frac{1}{s}}}{r(r-1)}n^{2-\frac{1}{s}} + O(n)$.

The purpose of this paper is to consider the spectral analogues of Turán problems. The spectral Turán problem is to maximize the spectral radius of n-vertex r-graphs which do not contain any element of a given hypergraph family $\mathcal{F}$ as a subgraph, where the spectral radius is the maximal absolute value of the eigenvalues of the adjacency tensor of a uniform hypergraph (see the definitions in Section 2). For graphs, the spectral Turán type problem is relatively complete, largely due to a longstanding project of Nikiforov, see [20] for details. Meanwhile, there are several spectral Turán type results in uniform hypergraphs. Keevash, Lenz and Mubayi [16] determined the maximum spectral radius of 3-graphs on n vertices not containing the Fano plane when n is sufficiently large. Hou, Chang and Cooper [14] proved an upper bound for the maximum spectral radius of Berge-C_4-free linear r-graph on n vertices. Gao, Chang and Hou [12] proved that if H is a K_{r+1}^+-free linear r-graph on n vertices, then $\rho(H) \leqslant \frac{n}{r}$, with equality if and only if $r|n$ and H is a transversal design with n vertices and r groups. Subsequently, She, Fan, Kang and Hou [24] generalized Gao-Chang-Hou's result by showing that if H is an F^+-free linear r-graph on n vertices, then $\rho(H) = \frac{1}{r-1}(1 - \frac{1}{\chi(F)-1})n + o(n)$, where F is a color critical graph with chromatic number $\chi(F)$ and $\chi(F) \geqslant r+1 \geqslant 3$. Ni, Liu and Kang [21] determined that the maximum spectral radius of a cancellative hypergraph is achieved by the balanced complete tripartite 3-uniform hypergraph. Recently, She, Fan and Kang [25] proved that for sufficiently large n, the maximum spectral radius of the Berge-$K_{2,t}$-free linear r-graph on n vertices is at most $\frac{\sqrt{t-1}}{r-1}\sqrt{n}+O(n^{\frac{1}{4}})$, where $t \geqslant 2$. Furthermore, they also gave an upper bound for the maximum spectral radius of the Berge-$\{K_{s,t}, C_3\}$-free linear r-graph on n vertices, where $2 \leqslant s \leqslant t$.

In this paper, we consider the maximum spectral radius for $K_{s,t}^+$-free linear r-graphs. Firstly, we give an upper bound for spectral radius of $K_{2,t}^+$-free linear r-graphs, which is close to being asymptotically optimal for $r = 3$.

Theorem 1. *Let H be a $K_{2,t}^+$-free linear r-graph on n vertices, where $t \geqslant 2$. Then*

$$\rho(H) \leqslant \frac{\sqrt{t-1}}{r-1}\sqrt{n} + \frac{2tr^2(3r-4)}{r-1}.$$

Since $K_{2,t}^+ \in$ Berge-$K_{2,t}$, our Theorem 1 implies the Theorem 3.3 of She, Fan and Kang [25], with an improvement on the error term. Moreover, by the result of Cooper and Dutle [3] that the average degree of an r-graph H is less than or equal to $\rho(H)$, we can obtain the following corollary, which recovers a partial result of Gao and Chang in [11].

Corollary 2. *For $t \geqslant 2$,*

$$ex_r^{lin}(n, K_{2,t}^+) \leqslant \frac{\sqrt{t-1}}{r(r-1)}n^{\frac{3}{2}} + \frac{2tr^2(3r-4)}{r(r-1)}n.$$

For $t = 2$, Lazebnik and Verstraëte [17] showed that there is a $K_{2,2}^+$-free linear 3-graph with $(\frac{1}{6}+o(1))n^{\frac{3}{2}}$ edges. Since the average degree of an r-graph is less than or equal to its spectral radius, the upper bound in Theorem 1 is asymptotically optimal for $t = 2$ and $r = 3$.

Corollary 3. *Let H be an n-vertex $K_{2,2}^+$-free linear 3-graph with the maximum spectral radius. Then*

$$\rho(H) = \frac{1}{2}\sqrt{n} + O(1).$$

Gerbner, Methuku and Vizer [10] showed that there exists a $K_{2,t}^+$-free linear 3-graph with $\left(1 - \frac{c}{\sqrt{t-1}}\ln^{\frac{3}{2}}(t-1)\right)\frac{\sqrt{t-1}}{6}n^{\frac{3}{2}}$ edges. Hence, we can obtain the following corollary.

Corollary 4. *Let H be an n-vertex $K_{2,t}^+$-free linear 3-graph with the maximum spectral radius. Then*

$$\rho(H) = (1+o_t(1))\frac{\sqrt{t-1}}{r-1}\sqrt{n}.$$

Furthermore, we give an upper bound for the spectral radius of $K_{s,t}^+$-free linear r-graphs.

Theorem 5. *Let H be a $K_{s,t}^+$-free linear r-graph on n vertices, where $t \geqslant s \geqslant 3$. Then*

$$\rho(H) \leqslant \frac{(s^2r^2t-1)^{\frac{1}{s}}}{r-1}n^{1-\frac{1}{s}} + o(n^{1-\frac{1}{s}}).$$

For $t > (s-1)!$, it was proved by Gao and Chang [11] that there is a $K_{s,t}^{+}$-free linear 3-graph with $\Omega(n^{2-\frac{1}{s}})$ edges. Therefore, the exponent of the upper bound in Theorem 5 is tight for $t > (s-1)!$.

Corollary 6. *For $t > (s-1)!$, let H be an n-vertex $K_{s,t}^{+}$-free linear 3-graph with the maximum spectral radius. Then*

$$\rho(H) = \Theta(n^{1-\frac{1}{s}}).$$

The rest of this paper is organized as follows. In the next section, we introduce the adjacency tensor and spectral radius of uniform hypergraphs. We obtain the connection between spectral radius and the number of walks of the uniform hypergraphs in Section 3. Finally, we prove our main results in Section 4.

2 Eigenvalues of tensors

In this section, we introduce spectral radius of an uniform hypergarph that will be used throughout the paper.

In 2005, Qi [22] and Lim [18] independently introduced the concept of tensor eigenvalues and the spectra of tensors. An order r dimension n real tensor $\mathcal{T} = (\mathcal{T}_{i_1\cdots i_r})$ consists of n^r real entries $\mathcal{T}_{i_1\cdots i_r}$ for all $i_1, i_2, \ldots, i_r \in [n]$, where $[n] = \{1, 2, \ldots, n\}$. Evidently, a vector of dimension n is a tensor of order 1 and matrix is a tensor of order 2. A tensor $\mathcal{T}$ is called symmetric if the value of $\mathcal{T} = (\mathcal{T}_{i_1\cdots i_r})$ is invariant under any permutation of the indices $i_1, i_2, \ldots, i_r$. Given a vector $\mathbf{x} \in \mathbb{C}^n$, we adopt the following notation: $\mathcal{T}\mathbf{x}^r$ is a real number and $\mathcal{T}\mathbf{x}^{r-1}$ is an n-dimensional vector, where $\mathcal{T}\mathbf{x}^r$ and the ith component of $\mathcal{T}\mathbf{x}^{r-1}$ are given by:

$$\mathcal{T}\mathbf{x}^r = \sum_{i_1, i_2, \cdots, i_r \in [n]} \mathcal{T}_{i_1 i_2 \cdots i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

$$(\mathcal{T}\mathbf{x}^{r-1})_i = \sum_{i_2, \cdots, i_r \in [n]} \mathcal{T}_{i i_2 \cdots i_r} x_{i_2} \cdots x_{i_r}.$$

If there exist $\lambda \in \mathbb{C}$ and a nonzero vector $\mathbf{x} \in \mathbb{C}^n$ satisfying

$$\mathcal{T}\mathbf{x}^{r-1} = \lambda \mathbf{x}^{[r-1]},$$

then λ is called an eigenvalue of $\mathcal{T}$ and $\mathbf{x}$ is its corresponding eigenvector, where $\mathbf{x}^{[r-1]} = (x_1^{r-1}, x_2^{r-1}, \ldots, x_n^{r-1})^T \in \mathbb{C}^n \setminus \{0\}$. If $\mathbf{x}$ is a real eigenvector of $\mathcal{T}$, surely the corresponding eigenvalue λ is real. In this case, λ is called an H-eigenvalue and $\mathbf{x}$ is called an H-eigenvector associated with λ. Furthermore, if $\mathbf{x}$ is nonnegative and real, we say λ is an H^+-eigenvalue of $\mathcal{T}$. If $\mathbf{x}$ is positive and real, λ is said to be an H^{++}-eigenvalue of $\mathcal{T}$. Throughout this paper, we will refer H-eigenvalues and H-eigenvectors to eigenvalue and eigenvector, or use both interchangeably. The maximal absolute value of the eigenvalues of $\mathcal{T}$ is called the **spectral radius** of $\mathcal{T}$, denoted by $\rho(\mathcal{T})$.

Let $\mathcal{T} = (\mathcal{T}_{i_1 \dots i_r})$ be a tensor of order r and dimension n. We can associate $\mathcal{T}$ with a directed graph $D(\mathcal{T})$ on vertex set $[n]$ such that (i, j) is an arc of $D(\mathcal{T})$ if and only if there exists a nonzero entry $\mathcal{T}_{ii_2 \dots i_r}$ satisfying $j \in \{i_2, \dots, i_r\}$. Then $\mathcal{T}$ is called weakly irreducible if $D(\mathcal{T})$ is strongly connected; otherwise it is called weakly reducible [7]. A tensor with all nonnegative entries is called a nonnegative tensor.

It is well known that the Perron-Frobenius theorem for nonnegative matrices plays a crucial role in the study of spectral graph theory. As an extension of matrices, the Perron-Frobenius theorem for nonnegative tensors has also been established, see [7, 28].

Theorem 7 (Perron-Frobenius theorem for nonnegative tensors)**.**
(1) (Yang and Yang, 2010 [28]). If $\mathcal{T}$ is a nonnegative tensor of order r and dimension n, then $\rho(\mathcal{T})$ is an H^+-eigenvalue of $\mathcal{T}$.
(2) (Friedland, Gaubert and Han, 2013 [7]). If furthermore $\mathcal{T}$ is weakly irreducible, then $\rho(\mathcal{T})$ is the unique H^{++}-eigenvalue of $\mathcal{T}$, with the unique eigenvector $\mathbf{x} \in \mathbb{R}^n_{++}$, up to a positive scaling coefficient.

In 2012, Cooper and Dutle [3] defined the adjacency tensor $\mathcal{A}(H)$ of an r-graph H. The adjacency tensor $\mathcal{A}(H)$ of H is an order r dimension n symmetric tensor defined by

$$\mathcal{A}_{i_1 \cdots i_r} = \begin{cases} \frac{1}{(r-1)!} & if\ \{i_1, \dots, i_r\} \in E, \\ 0 & otherwise. \end{cases}$$

For an r-graph H, the spectral radius of H is defined as the spectral radius of the adjacency tensor $\mathcal{A}(H)$, denoted by $\rho(H)$. Obviously, the adjacency tensor $\mathcal{A}(H)$ of H is a nonnegative tensor. Its spectral radius $\rho(H)$ is an H^+-eigenvalue of $\mathcal{A}(H)$. Specifically, Friedland et al [7] proved that an uniform hypergraph H is connected if and only if its adjacency tensor $\mathcal{A}(H)$ is weakly irreducible. By above Perron-Frobenius theorem, if H is connected, then the eigenvector corresponding to the spectral radius $\rho(H)$, known as the principal eigenvector, can be chosen to be strictly positive. So this allows us to normalize the principal eigenvector so that the maximum entries of it are 1, and the other entries are in $(0, 1)$.

Let $\mathbf{x}$ be a eigenvector of $\mathcal{A}(\mathcal{H})$ and $U \subset V(H)$. We define $\mathbf{x}^U = \prod_{v_i \in U} x_i$. Clearly, for a vector $\mathbf{x}$, we have

$$\mathcal{A}(\mathcal{H})\mathbf{x}^r = r \sum_{e \in E(H)} \mathbf{x}^e.$$

3 Walks and spectral radius

In this section, we obtain the connection between spectral radius and the number of walks of the uniform hypergraphs.

Let H be an r-graph. For a vertex $v \in V(H)$, the neighborhood of v is defined as $N_H(v) = \{u \in V(H) \backslash \{v\} : \{u, v\} \subset e \in E(H)\}$. The degree of a vertex v, denoted by $d(v)$, is defined as the number of edges containing v. Denote $E_v = \{e \in E(H) : v \in e\}$ for a vertex v of H. A walk of length k, denoted by k-walk, is an alternating sequence of vertices and edges of the form $v_1e_1v_2e_2v_3\cdots v_ke_kv_{k+1}$, where $v_i\neq v_{i+1}$ and $v_iv_{i+1}\subseteq e_i$ for $1\leqslant i\leqslant k$. For a vertex $u\in V(H)$, we use $w_k(u)$ to denote the number of k-walks of H with starting vertex u. For two vertices $u,v\in V(H)$, define $w_k(u,v)$ as the number of k-walks of H starting at u and ending at v. The following two conclusions are obvious.

Proposition 8. *Let H be an r-graph and $v\in V(H)$. Then for $k\geqslant 1$,*

$$w_k(v)=\sum_{u\in V(H)}w_k(u,v)=\sum_{u\in V(H)}w_k(v,u).$$

Proposition 9. *Let H be an r-graph and $u,v\in V(H)$. Then for $k\geqslant 1$,*

$$w_k(u,v)=\sum_{e\in E_u}\sum_{w\in e\setminus\{u\}}w_{k-1}(w,v).$$

The following lemma strengthens the Lemma 2.5 of [12].

Lemma 10. *Let H be an r-graph with n vertices and ρ be the spectral radius of H. Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T$ be a nonnegative eigenvector of H corresponding to ρ. Then for any vertex $u\in V(H)$, we have*

$$\rho^kx_u^{r-1}\leqslant\frac{1}{(r-1)^k}\cdot\sum_{v\in V(H)}w_k(u,v)x_v^{r-1}.$$

Proof. We use the induction on k to prove the lemma. For $k=1$, we have

$$\begin{aligned}\rho x_u^{r-1}&=\sum_{e\in E_u}x^{e\setminus\{u\}}\leqslant\sum_{e\in E_u}\sum_{v\in e\setminus\{u\}}\frac{x_v^{r-1}}{r-1}\\&=\frac{1}{r-1}\cdot\sum_{v\in V(H)}w_1(u,v)x_v^{r-1},\end{aligned}$$

where the first inequality follows from the AM-GM inequality.

Suppose Lemma 10 holds for $k-1$, where $k\geqslant 2$. Then

$$\begin{aligned}\rho^kx_u^{r-1}&=\rho^{k-1}\cdot\rho x_u^{r-1}=\rho^{k-1}\cdot\sum_{e\in E_u}x^{e\setminus\{u\}}\\&\leqslant\rho^{k-1}\cdot\frac{1}{r-1}\sum_{e\in E_u}\sum_{w\in e\setminus\{u\}}x_w^{r-1}=\frac{1}{r-1}\sum_{e\in E_u}\sum_{w\in e\setminus\{u\}}\rho^{k-1}x_w^{r-1}\\&\leqslant\frac{1}{r-1}\sum_{e\in E_u}\sum_{w\in e\setminus\{u\}}\frac{1}{(r-1)^{k-1}}\cdot\sum_{v\in V(H)}w_{k-1}(w,v)x_v^{r-1}\\&=\frac{1}{(r-1)^k}\sum_{e\in E_u}\sum_{w\in e\setminus\{u\}}\sum_{v\in V(H)}w_{k-1}(w,v)x_v^{r-1}\\&=\frac{1}{(r-1)^k}\sum_{v\in V(H)}\sum_{e\in E_u}\sum_{w\in e\setminus\{u\}}w_{k-1}(w,v)x_v^{r-1}\\&=\frac{1}{(r-1)^k}\sum_{v\in V(H)}w_k(u,v)x_v^{r-1},\end{aligned}$$

where the first inequality follows from the AM-GM inequality, the second inequality follows from the induction hypothesis and the last equality follows from Proposition 9. This completes the proof. $\square$

Combining Lemma 10 and Proposition 8, we can obtain the following corollary.

Corollary 11. *Let H be an r-graph with n vertices and ρ be the spectral radius of H. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be a nonnegative eigenvector of H corresponding to ρ. Then*

$$\sum_{u\in V(H)} \rho^k x_u^{r-1} \leqslant \frac{1}{(r-1)^k} \cdot \sum_{u\in V(H)} w_k(u) x_u^{r-1}.$$

Proof. By Lemma 10 and Proposition 8, we have

$$\begin{aligned}
\sum_{u\in V(H)} \rho^k x_u^{r-1} &\leqslant \sum_{u\in V(H)} \frac{1}{(r-1)^k} \cdot \sum_{v\in V(H)} w_k(u,v) x_v^{r-1} \\
&= \frac{1}{(r-1)^k} \cdot \sum_{v\in V(H)} \sum_{u\in V(H)} w_k(u,v) x_v^{r-1} \\
&= \frac{1}{(r-1)^k} \cdot \sum_{v\in V(H)} w_k(v) x_v^{r-1} = \frac{1}{(r-1)^k} \cdot \sum_{u\in V(H)} w_k(u) x_u^{r-1}. \qquad \square
\end{aligned}$$

Corollary 11 implies the connection between spectral radius and the number of walks of a uniform hypergraph. Next we will show that if the number of walks satisfies some certain conditions, then the spectral radius can be bounded by an inequality.

Theorem 12. *Let H be an r-graph with the spectral radius ρ and k be a positive integer. If $\sum_{1\leqslant i\leqslant k} P_i w_i(u) \leqslant P_0$ holds for any vertex $u \in V(H)$, then*

$$\sum_{1\leqslant i\leqslant k} \frac{P_i \rho^i}{(r-1)^{k-i}} \leqslant \frac{P_0}{(r-1)^k},$$

where P_i is a parameter independent of the choice of u for all $0 \leqslant i \leqslant k$.

Proof. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ be a nonnegative eigenvector of H corresponding to the spectral radius ρ. Then

$$\begin{aligned}
\sum_{1\leqslant i\leqslant k} \frac{P_i \rho^i}{(r-1)^{k-i}} \sum_{u\in V(H)} x_u^{r-1} &= \sum_{1\leqslant i\leqslant k} \frac{P_i}{(r-1)^{k-i}} \sum_{u\in V(H)} \rho^i x_u^{r-1} \\
&\leqslant \sum_{1\leqslant i\leqslant k} \frac{P_i}{(r-1)^{k-i}} \cdot \frac{1}{(r-1)^i} \cdot \sum_{u\in V(H)} w_i(u) x_u^{r-1} \\
&= \frac{1}{(r-1)^k} \sum_{u\in V(H)} \sum_{1\leqslant i\leqslant k} P_i w_i(u) x_u^{r-1} \\
&\leqslant \frac{P_0}{(r-1)^k} \sum_{u\in V(H)} x_u^{r-1},
\end{aligned}$$

where the first inequality follows from Corollary 11 and the last equality follows from that the inequality $\sum_{1\leqslant i\leqslant k} P_i w_i(u) \leqslant P_0$ holds for any vertex $u \in V(H)$. Since $\mathbf{x}$ is nonnegative, we have

$$\sum_{1\leqslant i\leqslant k} \frac{P_i\rho^i}{(r-1)^{k-i}} \leqslant \frac{P_0}{(r-1)^k},$$

which completes the proof. □

By taking $k = 2, P_2 = 1, P_1 = -P$ and $P_0 = (r-1)Q$, Theorem 12 recovers the following Lemma, which is proved by She, Fan and Kang[25].

Lemma 13 ([25])**.** *Let H be an r-graph with the spectral radius ρ. If $w_2(u) \leqslant Pw_1(u) + (r-1)Q$ holds for any vertex $u \in V(H)$, then*

$$\rho^2 - \frac{P}{r-1}\rho - \frac{Q}{r-1} \leqslant 0,$$

where P and Q are parameters independent of the choice of u.

Note that for a linear r-graph H and $u \in V(H)$, it is easy to see that $w_1(u) = (r-1)d(u)$ and $w_2(u) = (r-1)\sum_{v\in N_H(u)} d(v)$. Thus, by Lemma 13, we can immediately deduce the following corollary.

Corollary 14 ([25])**.** *Let H be a linear r-graph with the spectral radius ρ. Suppose that $\sum_{v\in N_H(u)} d(v) \leqslant Pd(u) + Q$ holds for any vertex $u \in V(H)$, then*

$$\rho^2 - \frac{P}{r-1}\rho - \frac{Q}{r-1} \leqslant 0,$$

where P and Q are parameters independent of the choice of u.

4 Spectral extremal problems

In this section, we present the proofs of our main results.

Let H be an r-graph. Recall that the neighborhood of a vertex $v \in V(H)$ is defined as $N_H(v) = \{u \in V(H)\backslash\{v\} : \{u, v\} \subset e \in E(H)\}$. For two vertices $\{u, v\} \subset V(H)$, we define $N_H(u, v) = \{w \in V(H) : \{u, v, w\} \subset e \in E(H)\}$. For a set $X \subset V(H)$, let $E_t(X) = \{e : e \in E(H), |e\cap X| = t\}$ and $e_t(X) = |E_t(X)|$. Similarly, let $E_t^v(X) = \{e : e \in E(H), v \in e, |e\cap X| = t\}$ and $e_t^v(X) = |E_t^v(X)|$. For an integer i and a set S, let $\binom{S}{i}$ be the family of all i-subsets of S. In additional, we say $S \subset V(H)$ is an independent set if any pair of vertices of S is not contained in an edge. For a vertex set $S = \{v_1, v_2, \ldots, v_s\} \subset V(H)$, we define $N_1(S) = \cap_{1\leqslant i\leqslant s} N_H(v_i)$ and $N_2(S) = \cup_{1\leqslant i<j\leqslant s} N_H(v_i, v_j)$. In [11], Gao and Chang gave the following Lemma.

Lemma 15 ([11])**.** *Let H be a linear r-graph and $2 \leqslant s \leqslant t$. Let $S \subseteq V(H)$ be a vertex set with s vertices. If $|N_1(S)\backslash N_2(S)| \geqslant s^2r^2t$, then we can find a $K_{s,t}^+ \subset H$.*

4.1 Proof of the Theorem 1

Lemma 16. *Let H be a $K_{2,t}^{+}$-free linear r-graph on n vertices. Then for any $u \in V(H)$,*

$$\sum_{v \in N_H(u)} d(v) \leqslant 2tr^2(3r-4)d(u) + \frac{(t-1)n}{r-1}.$$

Proof. For any $v \in N_H(u)$, it is easy to see that

$$d(v) = e_1^v(N_H(u)) + \sum_{i=2}^{r} e_i^v(N_H(u)). \tag{1}$$

Firstly, for any $v \in N_H(u)$, we will show that

$$\sum_{i=2}^{r} e_i^v(N_H(u)) \leqslant 4tr^2. \tag{2}$$

Suppose for a contradiction that there is a vertex $v \in N_H(u)$ such that $\sum_{i=2}^{r} e_i^v(N_H(u)) \geqslant 4tr^2 + 1$. Since H is linear, it is easy to see that only one hyperedge, say h_0, in $\cup_{i=2}^{r} E_i^v(N_H(u))$ contains u, and let $h_1, h_2, \cdots, h_l$ be the remaining hyperedges, where $l \geqslant 4tr^2$. Clearly, for $1 \leqslant i \leqslant l$, $|h_i \cap (N_H(u) \backslash \{v\})| \geqslant 1$. Let $S = \{u, v\}$. Then, $|N_1(S) \backslash N_2(S)| \geqslant l \geqslant 4tr^2$. Thus, by Lemma 15, we have that H contains a copy of $K_{2,t}^{+}$, a contradiction.

For each vertex $v \in N_H(u)$, let $S_v = \{w \in V(H) \backslash \{v\} : w \in h \in E_1^v(N_H(u))\}$. Clearly, $|S_v| = (r-1)e_1^v(N_H(u))$. By (1) and (2), we have $e_1^v(N_H(u)) = d(v) - \sum_{i=2}^{r} e_i^v(N_H(u)) \geqslant d(v) - 4tr^2$. Hence,

$$|S_v| \geqslant (r-1)d(v) - 4(r-1)tr^2. \tag{3}$$

For $1 \leqslant i \leqslant d(u)$, let $h_i^u = \{u, u_{1,i}, u_{2,i}, \ldots, u_{r-1,i}\}$ be a hyperedge incident to u in H, and define $S_i = \cup_{j=1}^{r-1} S_{u_{j,i}}$. Note that

$$|S_i| = |\cup_{j=1}^{r-1} S_{u_{j,i}}| \geqslant \sum_{j=1}^{r-1} |S_{u_{j,i}}| - \sum_{1 \leqslant p < q \leqslant r-1} |S_{u_{p,i}} \cap S_{u_{q,i}}|. \tag{4}$$

Since H is $K_{2,t}^{+}$-free, by Lemma 15, we can obtain that $|S_{u_{p,i}} \cap S_{u_{q,i}}| \leqslant 4tr^2$. Hence, combining (4), we have

$$|S_i| = |\cup_{j=1}^{r-1} S_{u_{j,i}}| \geqslant \sum_{j=1}^{r-1} |S_{u_{j,i}}| - \binom{r-1}{2} 4tr^2. \tag{5}$$

By (3), we get

$$\begin{aligned} \sum_{v \in N_H(u)} (r-1)d(v) &\leqslant \sum_{v \in N_H(u)} \left(|S_v| + 4t(r-1)r^2\right) \\ &= \sum_{v \in N_H(u)} |S_v| + 4t(r-1)^2 r^2 d(u). \end{aligned} \tag{6}$$

Moreover, by (5), we have

$$\sum_{v\in N_H(u)} |S_v| = \sum_{i=1}^{d(u)}\sum_{j=1}^{r-1} |S_{u_{j,i}}| \leqslant \sum_{i=1}^{d(u)} \left(|S_i| + \binom{r-1}{2} 4tr^2\right). \tag{7}$$

Next, we give an upper bound of $\sum_{i=1}^{d(u)} |S_i|$ to complete the proof.

Claim 17. $\sum_{i=1}^{d(u)} |S_i| \leqslant (t-1)n$.

Proof. It suffices to show that any vertex $v \in V(H)$ belongs to at most $t-1$ of the sets S_i for any $1 \leqslant i \leqslant d(u)$. Suppose for a contradiction that there is a vertex v that is contained in sets $S_{i_1}, S_{i_2}, \ldots, S_{i_t}$ for some distinct $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, d(u)\}$. For notational simplicity, we may assume that $i_1 = 1, i_2 = 2, \ldots,$ and $i_t = t$. This means that there are t hypergraphs $h_1, h_2, \ldots, h_t \in E_1^v(N_H(u))$ containing the pairs $vz_1, vz_2, \ldots, vz_t$, respectively, where $z_j \in h_j^u \backslash \{u\} = \{u_{1,j}, u_{2,j}, \ldots, u_{r-1,j}\}$ for $1 \leqslant j \leqslant t$. It is easy to see that the hyperedges $h_1, h_2, \ldots, h_t$ are distinct and $h_1 \cap h_2 \cap \cdots \cap h_t = \{v\}$. Moreover, for $1 \leqslant i, j \leqslant t$, it is easy to see that $h_i^u \cap h_i = \{z_i\}$ and $h_i^u \cap h_j = \emptyset$. Hence, these $2t$ hyperedges form a $K_{2,t}^+$, a contradiction. $\square$

Using Calim 17 and (7), we have

$$\sum_{v\in N_H(u)} |S_v| \leqslant (t-1)n + \binom{r-1}{2} 4tr^2 d(u). \tag{8}$$

Combining (6) and (8), we get

$$\begin{aligned}\sum_{v\in N_H(u)} (r-1)d(v) &\leqslant (t-1)n + \left(4t(r-1)^2r^2 + \binom{r-1}{2}4tr^2\right) d(u)\\ &= (t-1)n + 2tr^2(r-1)(3r-4)d(u).\end{aligned}$$

Hence,

$$\sum_{v\in N_H(u)} d(v) \leqslant 2tr^2(3r-4)d(u) + \frac{(t-1)n}{r-1},$$

as desired. $\square$

Proof of Threorem 1. By Lemma 16, taking $P = 2tr^2(3r-4)$ and $Q = \frac{(t-1)n}{r-1}$ in Corollary 14, we have

$$\rho^2 - \frac{P}{r-1}\rho - \frac{Q}{r-1} = \rho^2 - \frac{2tr^2(3r-4)}{r-1}\rho - \frac{(t-1)n}{(r-1)^2} \leqslant 0.$$

Hence,

$$
\begin{aligned}
\rho &\leqslant \frac{1}{2}\left(\frac{P}{r-1}+\sqrt{\left(\frac{P}{r-1}\right)^2+4\frac{Q}{r-1}}\right)\\
&\leqslant \frac{1}{2}\left(\frac{P}{r-1}+\sqrt{\left(\frac{P}{r-1}+2\sqrt{\frac{Q}{r-1}}\right)^2}\right)\\
&= \frac{P}{r-1}+\sqrt{\frac{Q}{r-1}}\\
&= \frac{\sqrt{t-1}}{r-1}\sqrt{n}+\frac{2tr^2(3r-4)}{r-1},
\end{aligned}
$$

completing the proof. □

4.2 Proof of the Theorem 5

Lemma 18 ([11]). *Let H be a $K^+_{s,t}$-free linear r-graph with n vertices. For any $v \in V(H)$, the number of edges $h \in E(H)$ with $|h \cap N_H(v)| \geqslant 2$ and $v \notin h$ is $O(d^{2-\frac{1}{s-1}}(v))$.*

Lemma 19 ([8]). *Let $v, k \geqslant 1$ be integers and $c, x_0, x_1, \ldots, x_v$ be reals. If $\sum_{i=1}^{v}\binom{x_i}{k} \leqslant c\binom{x_0}{k}$, then*

$$\sum_{i=1}^{v} x_i \leqslant x_0 c^{\frac{1}{k}} v^{1-\frac{1}{k}} + (k-1)v.$$

As in [13, 23], an r-graph H is called hm-bipartite if its vertex set has a bipartition $V(H) = V_1 \cup V_2$ such that $|e \cap V_1| = 1$ and $|e \cap V_2| = r-1$ for any edge $e \in E(H)$. In the above bipartition, V_1 is called the head part and V_2 is called the mass part. Moreover, if $|V_1| = m$ and $|V_2| = n$, then we say the above H is (m,n)-hm-bipartite. Given a complete bipartite graph $K_{s,t}$, we denote the two parts of its vertex set by S and T, where $|S| = s$ and $|T| = t$. Let $K^+_{s,t}$ be the r-expansion of this $K_{s,t}$. Without any confusion, we define the s-part of $K^+_{s,t}$ as S and the t-part of $K^+_{s,t}$ as T.

Lemma 20. *Let H be an (m,n)-hm-bipartite linear r-graph. If H does not contain a copy of $K^+_{s,t}$ with the s-part contained in the head part of H and the t-part contained in the mass part of H, where $t \geqslant 2$ and $s \geqslant 2$, then*

$$e(H) \leqslant \frac{(s^2r^2t-1)^{\frac{1}{s}}}{r-1} mn^{1-\frac{1}{s}} + \frac{s-1}{r-1}n.$$

Proof. Let H be an (m,n)-hm-bipartite linear r-graph with the head part V_1 and the mass part V_2, where $|V_1| = m$ and $|V_2| = n$. We distinguish the following two cases.

Case 1. $|V_1| \geqslant s$.

Let

$$A = \{(v, \{v_1, v_2, \dots, v_s\}) : v \in V_2, \{v_1, \dots, v_s\} \subset N_H(v) \cap V_1\}.$$

From the definition of hm-bipartite, it is easy to see that V_1 is an independent set, i.e., $|N_2(V_1)| = 0$. Since H is $K_{s,t}^+$-free, by Lemma 15, we have that the number of common neighbors of any s vertices chosen from V_1 is at most s^2r^2t-1. Note that for any $e \in E(H)$, we have $|e \cap V_1| = 1$. Thus, by double counting, we have

$$\sum_{v \in V_2} \binom{d(v)}{s} = |A| = \sum_{U \in \binom{V_1}{s}} |N_1(U)| \leqslant (s^2r^2t - 1)\binom{|V_1|}{s}.$$

By Lemma 19, we can obtain

$$(r-1)e(H) = \sum_{v \in V_2} d(v) \leqslant (s^2r^2t-1)^{\frac{1}{s}}|V_1||V_2|^{1-\frac{1}{s}} + (s-1)|V_2|.$$

Hence,

$$e(H) \leqslant \frac{(s^2r^2t-1)^{\frac{1}{s}}}{r-1} mn^{1-\frac{1}{s}} + \frac{s-1}{r-1}n.$$

Case 2. $0 \leqslant |V_1| \leqslant s-1$.

Since H is an hm-bipartite linear r-graph, for any $v \in V_1$, we have that $d(v) \leqslant \frac{|V_2|}{r-1}$. Hence,

$$e(H) = \sum_{v \in V_1} d(v) \leqslant |V_1|\frac{|V_2|}{r-1} \leqslant \frac{s-1}{r-1}n \leqslant \frac{(s^2r^2t-1)^{\frac{1}{s}}}{r-1} mn^{1-\frac{1}{s}} + \frac{s-1}{r-1}n. \qquad \square$$

Lemma 21. *Let H be a $K_{s,t}^+$-free linear r-graph on n vertices, where $2 \leqslant s \leqslant t$. Then for any $u \in V(H)$,*

$$\sum_{v \in N_H(u)} d(v) \leqslant \left((s^2r^2t-1)^{\frac{1}{s}} n^{1-\frac{1}{s}} + O(n^{1-\frac{1}{s-1}})\right) d(u) + \frac{s-1}{r-1}n.$$

Proof. For any $u \in V(H)$, we have

$$\sum_{v \in N_H(u)} d(v) = \sum_{v \in N_H(u)} e_1^v(N_H(u)) + \sum_{v \in N_H(u)} \sum_{i=2}^{r} e_i^v(N_H(u)). \tag{9}$$

Firstly, we can see that

$$\begin{aligned}
\sum_{v \in N_H(u)} \sum_{i=2}^{r} e_i^v(N_H(u)) &= \sum_{i=2}^{r} i e_i(N_H(u)) \\
&= \sum_{i=2}^{r-2} i e_i(N_H(u)) + r e_r(N_H(u)) \\
&\quad + (r-1)|\{e : e \in E_{r-1}(N_H(u)), u \notin e\}| + (r-1)d(u) \\
&\leqslant r|\{e : |e \cap N_H(u)| \geqslant 2, u \notin e\}| + (r-1)d(u),
\end{aligned}$$

Hence, by Lemma 18,

$$\begin{aligned}\sum_{v\in N_H(u)}\sum_{i=2}^{r} e_i^v(N_H(u)) &\leqslant O(d^{2-\frac{1}{s-1}}(u))+(r-1)d(u)\\ &\leqslant O(n^{1-\frac{1}{s-1}})d(u)+(r-1)d(u)\\ &= O(n^{1-\frac{1}{s-1}})d(u).\end{aligned}\tag{10}$$

Next, we will give the upper bound for $\sum_{v\in N_H(u)} e_1^v(N_H(u))$. Let H_1 be a sub-hypergraph of H such that $V(H_1)=V(H)\backslash\{u\}$ and $E(H_1)=E_1(N_H(u))$. Clearly, H_1 is a hm-bipartite linear r-graph with the head part $V_1=N_H(u)$ and the mass part $V_2=V(H)\backslash(N_H(u)\cup\{u\})$. It is easy to see that $e(H_1)=\sum_{v\in N_H(u)} e_1^v(N_H(u))$, $|V_1|=(r-1)d(u)$ and $|V_2|=n-(r-1)d(u)-1$. Furthermore, H_1 contain no copy of $K_{s,t}^+$ with the s-part contained in V_1 and the t-part contained in V_2. Hence, by Lemma 20, we have

$$\begin{aligned}\sum_{v\in N_H(u)} e_1^v(N_H(u)) = e(H_1) &\leqslant \frac{(s^2r^2t-1)^{\frac{1}{s}}}{r-1}|V_1||V_2|^{1-\frac{1}{s}}+\frac{s-1}{r-1}|V_2|\\ &\leqslant (s^2r^2t-1)^{\frac{1}{s}}n^{1-\frac{1}{s}}d(u)+\frac{s-1}{r-1}n.\end{aligned}\tag{11}$$

Combining (9), (10) and (11), we have

$$\sum_{v\in N_H(u)} d(v)\leqslant \left((s^2r^2t-1)^{\frac{1}{s}}n^{1-\frac{1}{s}}+O(n^{1-\frac{1}{s-1}})\right)d(u)+\frac{s-1}{r-1}n. \qquad \square$$

Proof of Threorem 5. By Lemma 21, taking $P=(s^2r^2t-1)^{\frac{1}{s}}n^{1-\frac{1}{s}}+O(n^{1-\frac{1}{s-1}})$ and $Q=\frac{s-1}{r-1}n$ in Corollary 14, we have

$$\rho^2-\frac{P}{r-1}\rho-\frac{Q}{r-1}=\rho^2-\frac{(s^2r^2t-1)^{\frac{1}{s}}n^{1-\frac{1}{s}}+O(n^{1-\frac{1}{s-1}})}{r-1}\rho-\frac{(s-1)n}{(r-1)^2}\leqslant 0.$$

Hence,

$$\begin{aligned}\rho &\leqslant \frac{1}{2}\left(\frac{P}{r-1}+\sqrt{\left(\frac{P}{r-1}\right)^2+4\frac{Q}{r-1}}\right)\\ &\leqslant \frac{1}{2}\left(\frac{P}{r-1}+\sqrt{\left(\frac{P}{r-1}+2\sqrt{\frac{Q}{r-1}}\right)^2}\right)\\ &= \frac{P}{r-1}+\sqrt{\frac{Q}{r-1}}.\end{aligned}$$

Applying the values of P and Q, we have

$$\begin{aligned}\rho &\leqslant \frac{(s^2r^2t-1)^{\frac{1}{s}}}{r-1}n^{1-\frac{1}{s}} + O(n^{1-\frac{1}{s-1}}) + \frac{\sqrt{s-1}}{r-1}\sqrt{n}\\ &= \frac{(s^2r^2t-1)^{\frac{1}{s}}}{r-1}n^{1-\frac{1}{s}} + o(n^{1-\frac{1}{s}}),\end{aligned}$$

as desired. □

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