Turán number for bushes

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Abstract

Let $a, b \in \mathbf{Z}^+$, $r = a+b$, and let T be a tree with color classes $U = \{u_1, u_2, \ldots, u_s\}$ and $V = \{v_1, v_2, \ldots, v_t\}$. Let $A_1, \ldots, A_s$ and $B_1, \ldots, B_t$ be disjoint sets, such that $|A_i| = a$ and $|B_j| = b$ for all i, j. The (a, b)-*blowup* of T is the r-uniform hypergraph with edge set $\{A_i \cup B_j : u_i v_j \in E(T)\}$.

We use the Δ-systems method to prove the following Turán-type result. Suppose $a, b, t \in \mathbf{Z}^+$, $r = a + b \geqslant 3$, $a \geqslant 2$, and T is a fixed tree of diameter 4 in which the degree of the center vertex is t. Then there exists a $C = C(r, t, T) > 0$ such that $|E(\mathcal{H})| \leqslant (t-1)\binom{n}{r-1} + Cn^{r-2}$ for every n-vertex r-uniform hypergraph $\mathcal{H}$ not containing an (a, b)-blowup of T. This is asymptotically exact when $t \leqslant |V(T)|/2$. A stability result is also presented.

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1 Introduction

1.1 Basic definitions and notation

An r-uniform hypergraph (an r-*graph*, for short) is a family of r-element subsets of a finite set. We associate an r-graph $\mathcal{H}$ with its edge set and call its vertex set $V(\mathcal{H})$. Often we take $V(\mathcal{H}) = [n]$, where $[n] := \{1, 2, 3, \ldots, n\}$. Given an r-graph $\mathcal{F}$, let *the Turán number* of $\mathcal{F}$, $\mathrm{ex}_r(n, \mathcal{F})$, denote the maximum number of edges in an r-graph on n vertices that does not contain a copy of $\mathcal{F}$.

Since a (graph) tree is connected and bipartite, it uniquely defines the parts in its bipartition. So, we say a tree T is an (s, t)-*tree* if one part of $V(T)$ has s vertices and the other has t vertices.

Let $s, t, a, b > 0$ be integers, $r = a + b$, and let $T = T(U, V)$ be an (s, t)-tree with parts $U = \{u_1, u_2, \ldots, u_s\}$ and $V = \{v_1, v_2, \ldots, v_t\}$. Let $A_1, \ldots, A_s$ and $B_1, \ldots, B_t$ be pairwise disjoint sets, such that $|A_i| = a$ and $|B_j| = b$ for all i, j. So $|\bigcup A_i \cup B_j| = as + bt$. The (a, b)-*blowup* of T, denoted by $\mathcal{T}(T, a, b)$, is the r-uniform hypergraph with edge set $\mathcal{T}(T, a, b) := \{A_i \cup B_j : u_i v_j \in E(T)\}$, see Fig. 1.

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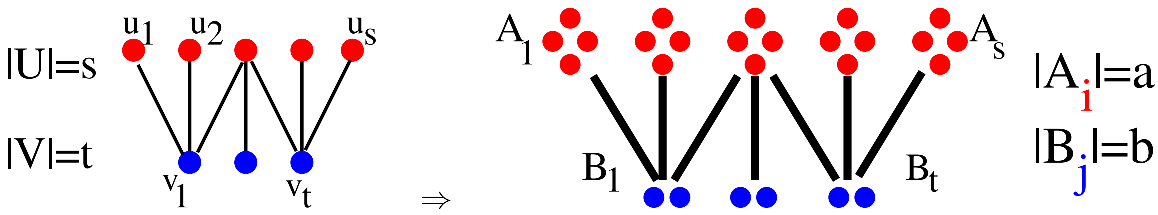

Figure 1: An example of a $(4,2)$-blowup.

The goal of this paper is to find the asymptotics of the Turán number for (a,b)-blowups of many trees of radius 2 using the Δ-systems method. Earlier, (a,b)-blowups of different classes of trees and different pairs (a,b) were considered in [7]. The main result in [7] is the following.

Theorem 1 ([7])**.** *Suppose $r \geqslant 3$, $s,t \geqslant 2$, $a+b=r$, $b<a<r$. Let T be an (s,t)-tree and let $\mathcal{T}=\mathcal{T}(T,a,b)$ be its (a,b)-blowup. Then (as $n\to\infty$) any $\mathcal{T}$-free n-vertex r-graph $\mathcal{H}$ satisfies*

$$|\mathcal{H}| \leqslant (t-1)\binom{n}{r-1} + o(n^{r-1}).$$

This is asymptotically sharp whenever $t \leqslant s$.

This theorem asymptotically settles about a half of possible cases, but when $t>s$ it is expected that the asymptotic is different. More is known on (a,b)-blowups of paths.

Let P_ℓ denote the (graph) path with ℓ edges. The first edge of the path corresponds to $A_1\cup B_1$, the second edge to $B_1\cup A_2$, etc. The case of P_2 was resolved asymptotically by Frankl [3] (for $b=1$) and by Frankl and Füredi [4] (for all $1\leqslant a\leqslant r-2$ and $b=r-a$):

$$\mathrm{ex}_r(n,\mathcal{T}(P_2,a,b)) = \Theta\left(n^{\max\{a-1,b\}}\right).$$

The case of P_3 was fully solved for large n by Füredi and Özkahya [8]. They showed that for fixed $1\leqslant a,b<r$ with $r=a+b\geqslant 3$ and for $n>n_0(r)$,

$$\mathrm{ex}_r(n,\mathcal{T}(P_3,a,b)) = \binom{n-1}{r-1}.$$

For longer paths, the following was proved in [7].

Theorem 2 (Theorem 1 in [7])**.** *Let $a+b=r$, $a,b\geqslant 1$ and $\ell\geqslant 3$. Suppose further that (i) ℓ is odd, or (ii) ℓ is even and $a>b$, or (iii) $(\ell,a,b)=(4,1,2)$. Then*

$$\mathrm{ex}_r(n,\mathcal{T}(P_\ell,a,b)) = \left\lfloor \frac{\ell-1}{2} \right\rfloor \binom{n}{r-1} + o(n^{r-1}).$$

So, the situation with blowups of P_ℓ is not resolved for the case when $\ell\geqslant 4$ is even and $a\leqslant b$ apart from the case $(\ell,a,b)=(4,1,2)$.

In this paper, we consider (a, b)-blowups of trees of radius 2.

A *graph bush* $B_{t,h}$ is the radius 2 tree obtained from the star $K_{1,t}$ by joining each vertex in the t-part of $K_{1,t}$ to h new vertices. So $B_{t,h}$ has $1 + t + th$ vertices. Let $s = 1 + th$. Then $B_{t,h}$ is an (s, t)-tree with $s > t$.

Suppose that a, b, t, h are positive integers, $a + b = r$ and $t \geqslant 2$. We will call the (a, b)-blowup of $B_{t,h}$ an (a, b, t, h)-*bush* and denote it by $\mathcal{B}_{t,h}(a, b)$. This means the center vertex of $B_{t,h}$ is replaced by an a-set A, its neighbors by the b-sets $B_1, \dots, B_t$ and its second neighbors by a-sets $A_{i,j}$, $i \in [t]$, $j \in [h]$. In particular, the (a, b)-blowup of the path P_4 is the $(a, b, 2, 1)$-bush $\mathcal{B}_{2,1}(a, b)$.

1.2 New results, bushes and shadows

Since $\mathcal{B}_{t,h}(a, b)$ has t disjoint edges $B_i \cup A_{i,1}$ for $i = 1, \dots, t$, the example of the r-uniform hypergraph with vertex set $[n]$ in which every edge intersects the set $[t - 1]$ shows that

$$\mathrm{ex}_r(n, \mathcal{B}_{t,h}(a, b)) \geqslant \binom{n}{r} - \binom{n-t+1}{r} \sim (t-1)\binom{n}{r-1}. \tag{1}$$

We will use the Δ-systems approach to show that this is asymptotically correct in many cases. For $a > b \geqslant 2$ the asymptotic equality follows from Theorem 1. In this paper we deal with *all* cases and also present a somewhat refined result by considering shadows of hypergraphs.

For an r-graph $\mathcal{H}$ the *shadow*, $\partial\mathcal{H}$, is the collection of $(r - 1)$-sets that lie in some edge of $\mathcal{H}$. The *codegree*, $\deg_{\mathcal{H}}(Y)$, is the number of hyperedges of $\mathcal{H}$ containing the set Y. (In case of $|Y| = 1$, we use the word *degree*). With this terminology $\partial\mathcal{H} := \{Y : |Y| = r - 1, \deg(Y) > 0\}$. Our first main result is the following.

Theorem 3. *Suppose that a, b, t, h are positive integers, $r = a + b \geqslant 3$. Also suppose that in case of $(a, b) = (1, r - 1)$ we have $h = 1$. Then there exists a $C = C(r, t, h) > 0$ such that every n-vertex r-uniform family $\mathcal{H}$ satisfying*

$$|\mathcal{H}| > (t-1)|\partial\mathcal{H}| + Cn^{r-2} \tag{2}$$

contains the bush $\mathcal{B}_{t,h}(a, b)$.

This implies that (1) is asymptotically exact in these cases as r, t, h are fixed and $n \to \infty$. The research on the relations between the shadow and the cardinality of an $\mathcal{F}$-free hypergraph $\mathcal{H}$ was started by Katona [10] who proved $|\mathcal{H}| \leqslant |\partial\mathcal{H}|$ for any r-graph containing no two disjoint hyperedges. In our case, an extra additive term in (2) cannot be removed. Indeed, if $\mathcal{H}$ is the complete r-graph on rt vertices then it is (a, b, t, h)-bush-free for all a, b, t, h such that $a+b = r$ and $t, h \geqslant 1$, but $|\mathcal{H}|-(t-1)|\partial\mathcal{H}| = \binom{rt}{r}-(t-1)\binom{rt}{r-1} > 0$.

For $(a, b) = (1, r - 1)$ our proof works only for $h = 1$. In fact, in this case the example below shows that the theorem does not hold for $h \geqslant 2$. Recall the definition of the block design $S_\lambda(v, k, \ell)$. It is a k-uniform hypergraph $\mathcal{S}$ on v elements such that $\deg_{\mathcal{S}}(Y) = \lambda$ for each ℓ-subset $Y \subset [v]$. Such designs exist for all sufficiently large v ($v > v_0(r, \lambda)$) when some simple divisibility conditions hold, see Keevash [11], and also Glock, Kühn, Lo and Osthus [9].

Construction 4. Define a $\mathcal{B}_{t,2}(1, r-1)$-free hypergraph $\mathcal{H}$ as follows. Let $V(\mathcal{H}) = [n]$ and $\mathcal{H} = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1$ is the set of r-subsets of $[n]$ meeting the set $[t-1]$ and $\mathcal{E}_2$ is a block design $S_{h-1}(n-t+1, r, r-1)$ on $[n] \setminus [t-1]$.

This $\mathcal{H}$ has asymptotically $(t-1+\frac{h-1}{r})\binom{n}{r-1}$ edges. It does not contain $\mathcal{B}_{t,h}(1, r-1)$ because the sets $B_i \cup_{1 \leqslant j \leqslant h} A_{i,j}$ are pairwise disjoint (here $1 \leqslant i \leqslant t$) so at least one of them avoids $[t-1]$.

McLennan [15] proved that in the graph case ($a = b = 1$), $\mathrm{ex}_r(n, B_{t,h}) = \frac{1}{2}(t + th - 1)n + O(1)$. Vertex-disjoint unions of complete graphs K_{t+th} are extremal. For the case $t = 1$, restriction (2) is too strong, $\mathrm{ex}_r(n, \mathcal{B}_{1,h}(a,b)) = O(n^{r-2})$ is known for $a \geqslant 2$. Even better bounds were proved in [5]. So we suppose that $t \geqslant 2$, $r \geqslant 3$.

Since each tree of diameter 4 with the degree of the center equal to t is a subgraph of a graph bush $B_{t,h}$ for some h, Theorem 3 yields the following more general result.

Corollary 5. *Suppose $a, b, t \in \mathbf{Z}^+$, $r = a + b \geqslant 3$ and $a \geqslant 2$. Let T be a fixed tree of diameter 4 in which the degree of the center vertex is t. Then there exists a $C = C(r, t, T) > 0$ such that every n-vertex r-graph $\mathcal{H}$ satisfying*

$$|\mathcal{H}| > (t-1)\binom{n}{r-1} + Cn^{r-2}$$

contains an (a,b)-blowup of T. The coefficient $(t-1)$ is best possible (as $n \to \infty$).

We also use the Δ-systems approach to show the following theorem.

Theorem 6. *Suppose that a, b, t, h are positive integers, $a, b \geqslant 2$ and $r = a + b \geqslant 5$. Then for any $C_0 > 0$ there exist $n_0 > 0$ and $C_1 > 0$ such that the following holds. If $n > n_0$, $\mathcal{H}$ is an n-vertex r-uniform family not containing a bush $\mathcal{B}_{t,h}(a,b)$ and $|\mathcal{H}| > (t-1)\binom{n}{r-1} - C_0 n^{r-2}$, then there are $t-1$ vertices in $[n]$ each of which is contained in at least $\binom{n}{r-1} - C_1 n^{r-2}$ edges of $\mathcal{H}$.*

This is a stability-type result describing an approximate structure of hypergraphs "close" to extremal. It shows that for $a, b \geqslant 2$ and $a + b = r$, such an r-uniform hypergraph without $\mathcal{B}_{t,h}(a,b)$ contains vertices of "large" degrees. Note that stability theorems often are more useful than the extremal result themselves (cf., Erdős-Simonovits Stability Theorem vs. Turán Theorem).

The structure of this paper is as follows. In the next section, we discuss the Δ-system method and present a lemma from [6] that will be our main tool. In Section 3 we describe properties of so called intersection structures. It allows us to prove the main case of Theorem 3 (the case $a \geqslant 2$) in Section 4 and the case of $a = 1$ and $h = 1$ in Section 5. In Section 6 we prove Theorem 6.

2 Definitions for the Δ-system method and a lemma

The idea of the Δ-system method is that any "dense" r-graph contains a quite structured subgraph that still has reasonably many edges. The existence of large delta systems in a hypergraph $\mathcal{F}$ allows us to embed tree-like structures into it, see Lemma 7 below. The uniformity of the intersection structure of the hyperedges made the Δ-system method one of the basic tools to tackle Turán-type hypergraph problems (see the recent survey by A. Kupavskii [12]), especially in the Erdős-Ko-Rado range.

A family of sets $\{F_1, \dots, F_q\}$ is a *q-star* or a *Δ-system* or a *q-sunflower with kernel A*, if $F_i \cap F_j = A$ for all $1 \leqslant i < j \leqslant q$. The sets $F_i \setminus A$ are called *petals*.

For a member F of a family $\mathcal{F}$, let the *intersection structure of F relative to $\mathcal{F}$* be

$$\mathcal{I}(F, \mathcal{F}) = \{F \cap F' : F' \in \mathcal{F} \setminus \{F\}\}.$$

An r-uniform family $\mathcal{F} \subseteq \binom{[n]}{r}$ is *r-partite* if there exists a partition $(X_1, \dots, X_r)$ of the vertex set $[n]$ such that $|F \cap X_i| = 1$ for each $F \in \mathcal{F}$ and each $i \in [r]$. For a partition $(X_1, \dots, X_r)$ of $[n]$ and a set $S \subseteq [n]$, the *pattern* $\Pi(S)$ is the set $\{i \in [r] : S \cap X_i \neq \emptyset\}$. Naturally, for a family $\mathcal{L}$ of subsets of $[n]$,

$$\Pi(\mathcal{L}) = \{\Pi(S) : S \in \mathcal{L}\} \subseteq 2^{[r]}.$$

Lemma 7 (The intersection semilattice lemma (Füredi [6]))**.** *For any positive integers q and r, there exists a positive constant $c(r, q)$ such that every family $\mathcal{F} \subseteq \binom{[n]}{r}$ contains a subfamily $\mathcal{F}^* \subseteq \mathcal{F}$ satisfying*

1. $|\mathcal{F}^*| \geqslant c(r, q)|\mathcal{F}|$.

2. $\mathcal{F}^*$ is r-partite, together with an r-partition $(X_1, \dots, X_r)$.

3. There exists a family $\mathcal{J}$ of proper subsets of $[r]$ such that $\Pi(\mathcal{I}(F, \mathcal{F}^*)) = \mathcal{J}$ holds for all $F \in \mathcal{F}^*$.

4. $\mathcal{J}$ is closed under intersection, i.e., for all $A, B \in \mathcal{J}$ we have $A \cap B \in \mathcal{J}$, as well.

5. For any $F \in \mathcal{F}^*$ and each $A \in \mathcal{I}(F, \mathcal{F}^*)$, there is a q-star in $\mathcal{F}^*$ containing F with kernel A.

Remark 8. The proof of Lemma 7 in [6] yields that if $\mathcal{F}$ itself is r-partite with an r-partition $(X_1, \dots, X_r)$, then the r-partition in the statement can be taken the same.

Remark 9. By definition, if for some $M \subset [r]$ none of the members of the family $\mathcal{J}$ of proper subsets of $[r]$ in Lemma 7 contains M, then for any two sets $F_1, F_2 \in \mathcal{F}^*$, their intersections with $\bigcup_{j \in M} X_j$ are distinct. It follows that if $|M| = m$, then $|\mathcal{F}^*| \leqslant \prod_{j \in M} |X_j| \leqslant \left(\frac{n-(r-m)}{m}\right)^m$. Thus, if $|\mathcal{F}^*| > \left(\frac{n-r+m}{m}\right)^m$, then every m-element subset of $[r]$ is contained in some $B \in \mathcal{J}$.

Call a family $\mathcal{J}$ of proper subsets of $[r]$ *m-covering* if every m-element subset of $[r]$ is contained in some $B \in \mathcal{J}$. In these terms, Remark 9 says that

$$\text{if } |\mathcal{F}^*| > \left(\tfrac{n-r+m}{m}\right)^m, \text{ then the corresponding } \mathcal{J} \text{ is } m\text{-covering.} \tag{3}$$

For $k = 0, 1, \dots, r$, define the family $\mathcal{J}^{(k)}$ of proper subsets of $[r]$ as follows. It contains
(a) the sets $[r] \setminus \{i\}$ for $1 \leqslant i \leqslant k$,
(b) all $(r-2)$-element subsets of $[r]$ containing $\{1, 2, \dots, k\}$, and
(c) all the intersections of these subsets.

By definition, each $\mathcal{J}^{(k)}$ is $(r-2)$-covering. Moreover,

$$\text{each } (r-2)\text{-covering family of proper subsets of } [r] \text{ closed under intersections contains a subfamily isomorphic to some } \mathcal{J}^{(k)}. \tag{4}$$

Indeed, let $\mathcal{J}$ be any $(r-2)$-covering family of proper subsets of $[r]$ and let k be the number of sets of size $r-1$ in $\mathcal{J}$. Since $\mathcal{J}$ is $(r-2)$-covering, every $(r-2)$-element subset of $[r]$ not contained in these k sets must be in $\mathcal{J}$ by itself. So properties (a) and (b) of the definition hold. Part (c) follows since $\mathcal{J}$ is closed under intersections.

3 General claims on intersection structures.

Call a set B a (b,q)-*kernel* in a set system $\mathcal{F}$ if B is the kernel of size b in a sunflower with q petals formed by members of $\mathcal{F}$.

Lemma 10. *If $a+b=r$ and an r-uniform family $\mathcal{H}$ does not contain $\mathcal{B}_{t,h}(a,b)$, then there do not exist disjoint sets $A_0, B_1, B_2, \dots, B_t$ with $|A_0|=a$, $|B_1|=\ldots=|B_t|=b$ such that all $B_1,\dots,B_t$ are (b,thr)-kernels in $\mathcal{H}$ and the sets $A_0\cup B_1,\dots,A_0\cup B_t$ are edges of $\mathcal{H}$.*

Proof. Suppose, there are such disjoint sets $A_0, B_1, B_2, \dots, B_t$. Let $D_0 = A_0 \cup \bigcup_{j=1}^t B_j$. For $i=1,\dots,t$, at Step i we shall define an auxiliary set $D_i \supset D_{i-1}$ with $|D_i| = a+tb+iha$ as follows. Since B_i is a (b,thr)-kernel and $|D_{i-1}\setminus B_i| = a+(t-1)b+(i-1)ha \leqslant thr-h$, there exist h petals $A_{i,j}$ $(1\leqslant j\leqslant h)$ of a thr-sunflower with kernel B_i that are disjoint from D_{i-1}. Let $D_i = D_{i-1}\cup\bigcup_{j=1}^h A_{i,j}$. After t steps, we find a $\mathcal{B}_{t,h}(a,b)$ whose edges are $A_0\cup B_i$ and $B_i\cup A_{i,j}$ for $i=1,\dots,t$, $j=1,2,\dots,h$. □

Suppose $a+b=r$ and $\mathcal{G}\subset\binom{[n]}{r}$ with $|\mathcal{G}| > \frac{1}{c(r,thr)}n^{r-2}$ does not contain $\mathcal{B}_{t,h}(a,b)$. By Lemma 7 and (3), there is $\mathcal{G}^*\subseteq\mathcal{G}$ satisfying the lemma such that the corresponding family $\mathcal{J}$ of proper subsets of $[r]$ is $(r-2)$-covering. Let $(X_1,\dots,X_r)$ be the corresponding partition.

Lemma 11. *If $\mathcal{G},\mathcal{G}^*$ and $\mathcal{J}$ are as in the paragraph above, then $\mathcal{J}$ does not contain disjoint members A and B such that $|A|=a$ and $|B|=b$.*

Proof. Suppose, it does. By renaming the elements of $\mathcal{J}$, we may assume that $A = \{1,\dots,a\}$ and $B=\{a+1,\dots,r\}$. Let $X=\{x_1,\dots,x_r\}\in\mathcal{G}^*$, where $x_i\in X_i$ for all i. Since $A\in\mathcal{J}$, $\{x_1,\dots,x_a\}$ is an (a,thr)-kernel in $\mathcal{G}^*$. Let $B_1,\dots,B_t$ be some t petals in the sunflower with kernel $\{x_1,\dots,x_a\}$. As $[r]\setminus[a]=B\in\mathcal{J}$, each of $B_1,\dots,B_t$ is a (b,thr)-kernel in $\mathcal{G}^*$, contradicting Lemma 10. □

Lemma 12. *If $a+b=r$ and $2\leqslant a,b\leqslant r-2$, then for each $0\leqslant k\leqslant r-2$ and for $k=r$ the family $\mathcal{J}^{(k)}$ defined at the end of Section 2 has disjoint members A and B such that $|A|=a$ and $|B|=b$, unless $(r,a,b,k)=(4,2,2,1)$.*

If $(a,b)=(1,r-1)$ or $(a,b)=(r-1,1)$ and $r\geqslant 3$, then for each $1\leqslant k\leqslant r-2$ and for $k=r$ the family $\mathcal{J}^{(k)}$ has disjoint members A and B such that $|A|=a$ and $|B|=b$.

Proof. The case $k = r$ is obvious, since $\mathcal{J}^{(r)} = 2^{[r]} \setminus \{[r]\}$. From now on, we may suppose that $k \leqslant r-2$. If $k \geqslant a$, then we let $A = [a]$, $B = [r] \setminus [a]$, and represent them as follows:

$$A = \bigcap_{k+1 \leqslant i < i' \leqslant r} ([r] \setminus \{i, i'\}) \cap \bigcap_{a+1 \leqslant i \leqslant k} ([r] \setminus \{i\}), \quad B = \bigcap_{1 \leqslant i \leqslant a} ([r] \setminus \{i\}).$$

If $k \geqslant b$, then we can switch the definitions of A and B. In particular, this proves the claim for $(a, b) = (1, r-1)$ and $(a, b) = (r-1, 1)$.

If $k \leqslant a-2$, then we again let $A = [a]$, $B = [r] \setminus [a]$, but represent them as follows (using that $a \leqslant r-2$ and $k \leqslant a-2$):

$$A = \bigcap_{a+1 \leqslant i < i' \leqslant r} ([r] \setminus \{i, i'\}), \quad B = \bigcap_{1 \leqslant i \leqslant k} ([r] \setminus \{i\}) \cap \bigcap_{k+1 \leqslant i < i' \leqslant a} ([r] \setminus \{i, i'\}).$$

By symmetry, the only remaining case is that $k = a-1 = b-1$. So r is even, and $a = b = r/2 = k+1$. In this case, if $r > 4$, then we let $A = [k-1] \cup \{k+1, k+2\}$, $B = ([r] \setminus [k+2]) \cup \{k\}$, and represent them as follows:

$$A = \bigcap_{k+3 \leqslant i < i' \leqslant r} ([r] \setminus \{i, i'\}) \cap ([r] \setminus \{k\}), \quad B = \bigcap_{1 \leqslant i \leqslant k-1} ([r] \setminus \{i\}) \cap ([r] \setminus \{k+1, k+2\}).$$

□

4 Proof of the main Theorem for $a > 1$

In this section we prove the main part of Theorem 3: the case of $2 \leqslant a \leqslant r-1$. The case of $(a, h) = (1, 1)$ will be considered in Section 5. In Subsection 4.1 we describe a procedure of partitioning $\mathcal{H}$ into structured subfamilies, and in the next three subsections we use this partition to find the required bush or to get a contradiction to (2). We distinguish three cases: (i) $a, b \geqslant 2$ and $r \geqslant 5$ (discussed in Subsection 4.2), (ii) $(a, b) = (2, 2)$ and $r = 4$ (Subsection 4.3), and (iii) $(a, b) = (r-1, 1)$ and $r \geqslant 3$ (Subsection 4.4).

4.1 Basic procedure

Let $2 \leqslant a \leqslant r-1$. Define $C = C(r, t, h) := 1/c(r, thr)$, where c is from Lemma 7. Assume that an n-vertex r-uniform family $\mathcal{H}$ satisfies (2) but does not contain $\mathcal{B}_{t,h}(a, b)$.

For any r-uniform family $\mathcal{G}$, let $\mathcal{G}^*$ denote a family satisfying Lemma 7 and $\mathcal{J}(\mathcal{G}^*) \subset 2^{[r]}$ denote the corresponding intersection structure.

Do the following procedure. Let $\mathcal{H}_1 = \mathcal{H}^*$ and $\mathcal{J}_1 = \mathcal{J}(\mathcal{H}^*)$. For $i = 1, 2, \dots,$ if $|\mathcal{H} \setminus \bigcup_{j=1}^{i} \mathcal{H}_j| \leqslant Cn^{r-2}$, then stop and let $m := i$ and $\mathcal{H}_0 = \mathcal{H} \setminus \bigcup_{j=1}^{i} \mathcal{H}_j$; otherwise, let $\mathcal{H}_{i+1} := (\mathcal{H} \setminus \bigcup_{j=1}^{i} \mathcal{H}_j)^*$ and $\mathcal{J}_{i+1} = \mathcal{J}((\mathcal{H} \setminus \bigcup_{j=1}^{i} \mathcal{H}_j)^*)$.

This procedure provides a partition of $\mathcal{H}$, $\mathcal{H} = \bigcup_{i=0}^{m} \mathcal{H}_i$. Let $\widehat{\mathcal{H}}$ denote $\bigcup_{i=1}^{m} \mathcal{H}_i$. By definition, $|\mathcal{H}_0| \leqslant Cn^{r-2}$, so we get

$$|\widehat{\mathcal{H}}| + Cn^{r-2} \geqslant |\mathcal{H}|. \tag{5}$$

4.2 Case of $a, b \geqslant 2$ and $r \geqslant 5$

Here $2 \leqslant a, b \leqslant r-2$ and $(a, b) \neq (2, 2)$. By Lemma 11 and Lemma 12, for each $1 \leqslant i \leqslant m$, $\mathcal{J}(\mathcal{H}_i)$ does not contain isomorphic copies of $\mathcal{J}^{(k)}$ for any $k \in \{0, 1, \ldots, r-2, r\}$. Therefore, by (4) $\mathcal{J}(\mathcal{H}_i)$ has exactly $r-1$ $(r-1)$-subsets and contains an isomorphic copy of $\mathcal{J}^{(r-1)}$. Recall that $\mathcal{H}_i$ is r-partite and by Part 5 of Lemma 7 for each hyperedge $E \in \mathcal{H}_i \subset \widehat{\mathcal{H}}$ $(1 \leqslant i \leqslant m)$ there exists an element $c(E) \in E$ (from the same part of the r-partition) such that each proper subset of E containing $c(E)$ is a kernel of a thr-star in $\mathcal{H}_i$. Beware of the fact that although each $\mathcal{H}_i$ is r-partite, the partitions might differ for different values of i. This does not cause any problem in our argument, we only need the existence of the element $c(E)$.

Define the function α on $\binom{[n]}{r-1}$ as follows: For each $(r-1)$-set $Y \subset [n]$, let $\alpha(Y)$ be the number of edges $E \in \widehat{\mathcal{H}}$ with $Y = E \setminus \{c(E)\}$.

Claim 13. *$\alpha(Y) \leqslant t-1$, for each $(r-1)$-subset Y of $[n]$.*

This claim implies

$$(t-1)|\partial\widehat{\mathcal{H}}| \geqslant \sum\nolimits_{Y \in \binom{[n]}{r-1}} \alpha(Y) = |\widehat{\mathcal{H}}|.$$

This, together with (5) contradicts (2) and thus completes the proof of Theorem 3 in this case.

We prove Claim 13 in two steps in a stronger form which will be useful in Section 6 to handle the stability of the extremal systems (i.e., Theorem 6). For every $Y \subset [n]$, let $\mathcal{U}(Y)$ be the set of vertices $v \in [n] \setminus Y$ such that there is an edge $E \in \widehat{\mathcal{H}}$ containing Y with $c(E) = v$.

Claim 14. *Suppose $Y \subset [n]$, $a \leqslant |Y| \leqslant r-1$, $v, v' \in \mathcal{U}(Y)$, $v \neq v'$ and the edges $E, E' \in \widehat{\mathcal{H}}$ are such that $v = c(E)$, $v' = c(E')$ and $Y \subseteq E \cap E'$. If $E \in \mathcal{H}_i$ and $E' \in \mathcal{H}_{i'}$, then $i \neq i'$.*

Proof. Suppose $v \neq v'$, but $i = i'$. We may assume that the partition of $[n]$ corresponding to $\mathcal{H}_i$ is $(X_1, \ldots, X_r)$ and $v, v' \in X_r$. Let $Z := \{j \in [r] : X_j \cap E \cap E' \neq \emptyset\}$. By symmetry, we may also assume that $E \cap E' \subset X_1 \cup \ldots \cup X_{|Z|}$. So $Z \in \mathcal{J}_i$ and we know that $a \leqslant |Z| \leqslant r-1$. The family $\mathcal{J}_i$ contains $\mathcal{J}^{(r-1)}$, namely $[r] \setminus \{j\} \in \mathcal{J}_i$ for each $1 \leqslant j \leqslant r-1$. Since $\mathcal{J}_i$ is intersection closed it must contain every subset of Z, e.g., $[a] \in \mathcal{J}_i$, and it also contains every subset containg the element r, e.g., $[r] \setminus [a] \in \mathcal{J}_i$. This contradicts Lemma 11. □

Claim 15. *Suppose $Y \subset [n]$, $a \leqslant |Y| \leqslant r-1$. Then $|\mathcal{U}(Y)| \leqslant t-1$.*

Proof. Suppose to the contrary that there are t distinct $v_1, \ldots, v_t \in [n]$ and distinct $E_1, \ldots, E_t \in \widehat{\mathcal{H}}$ such that $Y \subseteq E_1 \cap \ldots \cap E_t$ and $v_i = c(E_i)$ for $i = 1, \ldots, t$. Let $E_1 \in \mathcal{H}_{i_1}, \ldots, E_t \in \mathcal{H}_{i_t}$. By Claim 14, $i_1, \ldots, i_t$ are all distinct. By relabelling we may suppose that $E_i \in \mathcal{H}_i$.

We will find $t+1$ disjoint sets $A_0, B_1, B_2, \ldots, B_t$ contradicting Lemma 10 using induction as follows. Fix a subset A_0 of Y with $|A_0| = a$ and let $D_0 := A_0 \cup \{c(E_1), \ldots, c(E_t)\}$. We have $|D_0| = a + t$. We define the sets E_i', D_i, B_i step by step as follows. We will have $D_i := D_0 \cup \bigcup_{j \leqslant i} E_j'$ and $|D_i| = a + t + i(r - 1 - a)$. For $i = 1, 2, \ldots, t$ consider the family $\mathcal{H}_i$ and its member E_i in it. By the intersection structure of $\mathcal{H}_i$, the set $A_0 \cup \{c(E_i)\}$ is an $(a+1, thr)$-kernel in $\mathcal{H}_i$. One of the thr petals of the sunflower in $\mathcal{H}_i$ with kernel $A_0 \cup \{c(E_i)\}$ should be disjoint from D_{i-1}; let E_i' be the corresponding set in $\mathcal{H}_i$. Since $c(E_i) \in E_i'$, and Claim 14 gives $c(E_i') = c(E_i)$, the set $B_i := E_i' \setminus A_0$ is a (b, thr)-kernel. $\square$

4.3 The case $(a, b) = (2, 2)$

Lemmas 11 and 12 imply that for each $1 \leqslant i \leqslant m$, either
— $\mathcal{J}^{(3)}$ is contained in $\mathcal{J}(\mathcal{H}_i)$, it has exactly three 3-subsets, so $[4] \setminus \{1\}$, $[4] \setminus \{2\}$, and $[4] \setminus \{3\}$ are in $\mathcal{J}$ and $\mathcal{J}$ also contains all subsets containing the element 4 but it does not contain $\{1, 2, 3\}$, or
— $\mathcal{J}(\mathcal{H}_i)$ is of $\mathcal{J}^{(1)}$ type, it has a unique 3-subset, $\{2, 3, 4\}$, and $\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\} \subset \mathcal{J}$.

Note that in both cases $\mathcal{J}(\mathcal{H}_i)$ does not contain other triples or pairs.

Call $\mathcal{H}_i$ (and its edges) *type* α if $\mathcal{J}(\mathcal{H}_i)$ has three 3-subsets. Each of these edges $E \in \mathcal{H}_i$ has an element $c(E) \in E$ such that each proper subset of E containing $c(E)$ is a kernel of a $4 \cdot t \cdot h$-star in $\mathcal{H}_i$. The union of the families $\mathcal{H}_i$ of type α is $\widehat{\mathcal{H}}_\alpha$. Call $\mathcal{H}_i$ (and its edges) *type* β if $\mathcal{J}(\mathcal{H}_i)$ has a unique 3-subset. Each of these edges $E \in \mathcal{H}_i$ has an element $b(E) \in E$ such that each set $K \subset E$ of the form $\{b(E), x\}$ $(x \in E \setminus \{b(E)\})$ and the set $E \setminus \{b(E)\}$ is a kernel of a $4 \cdot t \cdot h$-star in $\mathcal{H}_i$. The union of the families $\mathcal{H}_i$ of type β is $\widehat{\mathcal{H}}_\beta$.

Define the function α on $\binom{[n]}{3}$ as follows: Given a 3-set Y, let $\alpha(Y)$ be the number of edges $E \in \widehat{\mathcal{H}}_\alpha$ with $Y = E \setminus \{c(E)\}$. Define the function β on $\binom{[n]}{3}$ as follows: Given a 3-set Y, let $\beta(Y)$ be the number of edges $E \in \widehat{\mathcal{H}}_\beta$ with $b(E) \in Y \subset E$.

Claim 16. $\alpha(Y) + \frac{1}{3}\beta(Y) \leqslant t - 1$, *for all* 3*-subsets* Y *of* $[n]$.

Proof. For brevity, let $\alpha = \alpha(Y)$ and $\beta = \beta(Y)$. Let $E_1 \in \mathcal{H}_{i_1}, \ldots, E_\alpha \in \mathcal{H}_{i_\alpha}$ be the α distinct edges E with $E \in \widehat{\mathcal{H}}_\alpha$ and $Y = E \setminus \{c(E)\}$ and let $E_{\alpha+1} \in \mathcal{H}_{i_{\alpha+1}}, \ldots, E_{\alpha+\beta} \in \mathcal{H}_{i_{\alpha+\beta}}$ be the β distinct edges E with $E \in \widehat{\mathcal{H}}_\beta$ and $b(E) \in Y \subset E$. If $i_j = i_{j'}$ for some $j \neq j'$, then the intersection structure of $\mathcal{H}_{i_j}$ would contain the 3-set Y, a contradiction. Thus $i_1, i_2, \ldots$ are all distinct. By relabelling we may suppose that $E_i \in \mathcal{H}_i$.

Suppose that $\alpha + \frac{1}{3}\beta > t - 1$, so $\alpha + \lceil \beta/3 \rceil \geqslant t$. Since $|Y| = 3$, one can find an element $y_0 \in Y$ such that $b(E_j) = y_0$ at least $\lceil \beta/3 \rceil$ times. So we may suppose that there are t distinct $E_i \in \mathcal{H}_i$ such that the elements $c(E_1), \ldots, c(E_\alpha)$ and $d(E_j) := E_j \setminus Y$ for $\alpha < j \leqslant t$ are all distinct and $E_i = Y \cup \{c(E_i)\}$, $b(E_j) = y_0$.

Let $A_0 := Y \setminus \{y_0\}$. Then $|A_0| = 2$ and we can find $t+1$ disjoint sets $A_0, B_1, B_2, \ldots, B_t$ contradicting Lemma 10 using induction in the same way as we did in the proof of Claim 15. $\square$

Claim 16 implies

$$(t-1)|\partial\widehat{\mathcal{H}}| \geqslant \sum\nolimits_{Y\in\binom{[n]}{3}}\left(\alpha(Y)+\frac{1}{3}\beta(Y)\right) = |\widehat{\mathcal{H}}_\alpha| + |\widehat{\mathcal{H}}_\beta|.$$

This, together with (5) contradicts (2) and thus completes the proof of Theorem 3 in this case.

4.4 The case $(a, b) = (r-1, 1)$

Call $\mathcal{H}_i$ (as above) of *type* α if $\mathcal{J}(\mathcal{H}_i)$ has $r-1$ $(r-1)$-subsets. Each of these edges $E \in \mathcal{H}_i$ has an element $c(E) \in E$ such that each proper subset of E containing $c(E)$ is a kernel of a thr-star in $\mathcal{H}_i$. The union of these families $\mathcal{H}_i$ is $\widehat{\mathcal{H}}_\alpha$. Call $\mathcal{H}_j$ (and its edges) *type* β if $\mathcal{J}(\mathcal{H}_j)$ has no $(r-1)$-subset. Note that each element y of an edge $E \in \mathcal{H}_j$ is a kernel of a thr-star in $\mathcal{H}_j$. The union of these $\mathcal{H}_j$ families is $\widehat{\mathcal{H}}_\beta$.

As in the previous subsections, for each $Y \in \binom{[n]}{r-1}$ let $\alpha(Y)$ be the number of edges $E \in \widehat{\mathcal{H}}_\alpha$ with $Y = E \setminus \{c(E)\}$. The definition of β on $\binom{[n]}{r-1}$ is even simpler: $\beta(Y)$ is the number of edges $E \in \widehat{\mathcal{H}}_\beta$ with $Y \subset E$. If $\alpha(Y)+\beta(Y) > t-1$, then taking $A_0 := Y$ and $B_i := E_i \setminus Y$, each B_i is a kernel of a thr-star, contradicting Lemma 10. Therefore, $\alpha(Y)+\beta(Y) \leqslant t-1$ for each $Y \in \binom{[n]}{r-1}$, and

$$(t-1)|\partial\widehat{\mathcal{H}}| \geqslant \sum_{Y\in\binom{[n]}{r-1}} (\alpha(Y)+\beta(Y)) = |\widehat{\mathcal{H}}_\alpha| + r|\widehat{\mathcal{H}}_\beta| \geqslant |\widehat{\mathcal{H}}|,$$

contradicting (2).

5 Hypergraphs without a bush $\mathcal{B}_{t,1}(1, r-1)$

In this section we consider r-graphs not containing $\mathcal{B}_{t,1}(1, r-1)$. We will prove this case for $C = C(r,t) := 1/c(r,q)$, where $q := 8rt^2$ and c is from Lemma 7. Suppose $\mathcal{H}$ is a counter-example with the fewest edges. So $\mathcal{H} \subset \binom{[n]}{r}$, it is bush-free and $|\mathcal{H}| - |\partial\mathcal{H}|$ satisfies the lower bound (2). In particular, $|\mathcal{H}| > Cn^{r-2}$.

Call an r-graph $\mathcal{G}$ *t-normal* if it has no $(r-1)$-tuples of vertices whose codegree is positive but less than t. If $\mathcal{H}$ is not t-normal, then choose an $(r-1)$-tuple Y of vertices whose codegree is positive but less than t and let $\mathcal{H}'$ be obtained from $\mathcal{H}$ by deleting the edges containing Y. Then $|\mathcal{H}'| - (t-1)|\partial\mathcal{H}'| \geqslant |\mathcal{H}| - (t-1)|\partial\mathcal{H}| > Cn^{r-2}$, so $\mathcal{H}'$ satisfies (2) and is $\mathcal{B}_{t,1}(1, r-1)$-free. This contradicts the minimality of $|\mathcal{H}|$. From now on, we suppose that $\mathcal{H}$ is t-normal.

For every edge $Y \in \mathcal{H}$ and any $u \in Y$, let $Q(Y,u) = \{z \in V(\mathcal{H})\setminus Y : Y\setminus\{u\}\cup\{z\} \in \mathcal{H}\}$. Since $\mathcal{H}$ is t-normal, $|Q(Y,u)| \geqslant t-1$ for every edge $Y \in \mathcal{H}$ and every $u \in Y$. For each $u \in Y \in \mathcal{H}$, fix a subset $Q'(Y,u)$ of $Q(Y,u)$ with $|Q'(Y,u)| = \min\{t, |Q(Y,u)|\}$. Call a subfamily $\mathcal{P}' \subset \mathcal{H}$ with $u \in \bigcap \mathcal{P}'$ *separable* if $(\bigcup_{P\in\mathcal{P}'} Q'(P,u)) \cap (\bigcup \mathcal{P}') = \emptyset$.

Claim 17. *Suppose that u is a kernel of a star $\mathcal{P}$ in $\mathcal{H}$, i.e., $\mathcal{P} \subset \mathcal{H}$ such that $P_1 \cap P_2 = u$ for all $P_1, P_2 \in \mathcal{P}$ whenever $P_1 \neq P_2$. Then there exists a separable $\mathcal{P}' \subset \mathcal{P}$ with $|\mathcal{P}'| \geqslant |\mathcal{P}|/(2t+1)$.*

Proof. Let G be the auxiliary directed graph with vertex set $\mathcal{P}$ where the pair $\{P_1, P_2\} \subset \mathcal{P}$ is an arc if $P_2 \cap Q'(P_1, u) \neq \emptyset$. Let G' be the underlying undirected graph of G.

Since $Q'(P, u)$ can meet at most t members of $\mathcal{P}$, the outdegree of each vertex in G is at most t. So G' is $2t$-degenerate and hence $(2t+1)$-colorable. In particular, G' has an independent set of size at least $|\mathcal{P}|/(2t+1)$. An independent set in G' corresponds to a separable subfamily of $\mathcal{P}$. □

Claim 18. *Suppose that u is a center of a separable star $\mathcal{P}'$ in $\mathcal{H}$. If $|\mathcal{P}'| \geqslant r+2t-2$ then there exists a unique $(t-1)$-element set $T(u)$ such that $Q(P, u) = T(u)$ for all $P \in \mathcal{P}'$. Moreover, $Q(Y, u) = T(u)$ for each $u \in Y \in \mathcal{H}$ with $Y \cap T(u) = \emptyset$. I.e.,*

$$Y \setminus \{u\} \cup \{z\} \in \mathcal{H} \text{ for all } z \in T(u) \cup \{u\}. \tag{6}$$

Let $T^+(u) := T(u) \cup \{u\}$. Then $T^+(z)$ is defined for each $z \in T^+(u)$ and coincides with $T^+(u)$.

Proof. If there are t members of $\mathcal{P}'$, say $P_1, \ldots, P_t$ such that $|\bigcup_{1 \leqslant i \leqslant t} Q'(P_i, u)| \geqslant t$, then the family $\{Q'(P_1, u), \ldots, Q'(P_t, u)\}$ satisfies Hall's condition. So, there exists a set of distinct representatives $\{z_1, \ldots, z_t\}$ such that $z_i \in Q'(P_i, u)$ for $1 \leqslant i \leqslant t$. Then the sets $P_1, \ldots, P_t$ together with the hyperedges $P_1 \setminus \{u\} \cup \{z_1\}, \ldots, P_t \setminus \{u\} \cup \{z_t\}$ form a bush $\mathcal{B}_{t,1}(1, r-1)$ with central element u, a contradiction.

Hence $|\bigcup_{1 \leqslant i \leqslant t} Q'(P_i, u)| \leqslant t-1$ for any t distinct $P_1, \ldots, P_t \in \mathcal{P}'$. This implies $|Q'(P_i, u)| = t-1$ for all i, so $|Q(P_i, u)| = t-1$. It also follows that $Q(P_i, u) = Q(P_j, u)$ for $1 \leqslant i \leqslant j \leqslant t$. This holds for any pair from $\mathcal{P}'$, so we get $Q(P, u) = Q(P_1, u)$ for all $P \in \mathcal{P}'$. Define $T(u) := Q(P_1, u)$.

Consider $Y \in \mathcal{H}$ with $u \in Y$ and $Y \cap T(u) = \emptyset$. The set $(Y \setminus \{u\}) \cup Q'(Y, u)$ can meet at most $(r-1+t)$ members of $\mathcal{P}'$. Since $|\mathcal{P}'| \geqslant (r-1) + (2t-1)$ we can still find $P_1, \ldots, P_{t-1} \in \mathcal{P}'$ such that $P_1, \ldots, P_{t-1}$ and Y form a separable star. This yields $Q(Y, u) = T(u)$ and we are done.

The uniqueness of $T(u)$ follows from the fact that having a $T_2(u)$ with similar properties it can meet at most $t-1$ members of $\mathcal{P}'$. So one can find a $P \in \mathcal{P}'$ avoiding it. Hence $T_2(u) = Q(P, u) = T(u)$.

To prove the last statement, choose $z \in T(u)$. Equation (6) implies that the family $\{P \setminus \{u\} \cup \{z\} : P \in \mathcal{P}'\}$ is a separable star (we have $T^+(u) \setminus \{z\} \subset Q(P \setminus \{u\} \cup \{z\}, z)$). So the first part of Claim 18 implies that $Q(P \setminus \{u\} \cup \{z\}, z) = T^+(u) \setminus \{z\} = T(z)$. □

Do the following procedure. Apply Lemma 7 for $\mathcal{H}$ to get $\mathcal{H}_1 = \mathcal{H}^*$ with the corresponding intersection structure $\mathcal{J}_1 \subset 2^{[r]}$. For $i = 1, 2, \ldots$, if $|\mathcal{H} \setminus \bigcup_{j=1}^{i} \mathcal{H}_j| \leqslant C \cdot n^{r-2}$, then stop, let $m := i$ and $\widehat{\mathcal{H}} := \bigcup_{j=1}^{i} \mathcal{H}_j$ and $\mathcal{H}_0 = \mathcal{H} \setminus \widehat{\mathcal{H}}$; otherwise, let $\mathcal{H}_{i+1} := (\mathcal{H} \setminus \bigcup_{j=1}^{i} \mathcal{H}_j)^*$. We have $|\mathcal{H}_1| > n^{r-2}$ because $|\mathcal{H}| > Cn^{r-2}$ and by the choice of C. Similarly, $|\mathcal{H}_i| > n^{r-2}$ for each $1 \leqslant i \leqslant m$. Recall that in our case $a = 1$. By Lemmas 11 and 12, $\mathcal{J}_i$ contains a family isomorphic to $\mathcal{J}^{(r-1)}$, or to $\mathcal{J}^{(0)}$. In both cases, $\mathcal{J}_i$ contains a singleton, so $\mathcal{H}_i$ contains q-stars with singleton kernels.

Case 1. There exists a $\mathcal{J}_i$ containing a family isomorphic to $\mathcal{J}^{(r-1)}$.
We may assume that $\mathcal{J}_i$ contains all proper subsets of $[r]$ containing 1 and $X_1, \ldots, X_r$ are the parts of $\mathcal{H}_i$. There is an element $y_1 \in X_1$ such that $\{y_1\}$ is a kernel of a q-star in $\mathcal{H}_i$. Claims 17 and 18 imply the existence of $T(y_1)$ since $\frac{q}{2t+1} \geqslant r + 2t - 2$. We can choose a $Y_1 = \{y_1, \ldots, y_r\} \in \mathcal{H}_i$ where $y_j \in X_j$ for $j \in [r]$ with $Y \cap T(y_1) = \emptyset$.

Since $\mathcal{J}_i$ contains all proper subsets of $[r]$ containing 1, $\{1, 2\} \in \mathcal{J}_i$. So, $\{y_1, y_2\}$ is the kernel of a $2t$-star $Y_1, \ldots, Y_{2t} \in \mathcal{H}_i$ (i.e, the sets $Y_j \setminus \{y_1, y_2\}$ are pairwise disjoint for $j \in [2t]$). At most $t - 1$ of them intersect $T(y_1)$. So we may assume, e.g., $Y_1, \ldots, Y_t$ are disjoint from $T(y_1)$. Since $Y_j \setminus \{y_2\}$ is a kernel of a q-star for each $j \in [t]$, one can find distinct elements $x_1, \ldots, x_t$ from X_2 such that none of them lies in $T(y_1)$ and $Y_j \setminus \{y_2\} \cup \{x_j\} \in \mathcal{H}_i$. Let $T^+(y_1) := T(y_1) \cup \{y_1\} = \{z_1, \ldots, z_t\}$ and apply (6) for $Y_j \setminus \{y_2\} \cup \{x_j\}$. We obtain that $Y_j \setminus \{y_1, y_2\} \cup \{z_j, x_j\} \in \mathcal{H}$.

Apply (6) for Y_j (again with $T^+(y_1)$). We get that $Y_j \setminus \{y_1\} \cup \{z_j\} \in \mathcal{H}$. These edges, together with the edges $Y_j \setminus \{y_1, y_2\} \cup \{z_j, x_j\}$ form a bush $\mathcal{B}_{t,1}(1, r-1)$ with the central vertex y_2. This contradiction leads to the last case.

Case 2. $\mathcal{J}_i$ contains a family isomorphic to $\mathcal{J}^{(0)}$ for each $1 \leqslant i \leqslant m$. Since $\mathcal{J}^{(0)}$ contains all subsets of $[r]$ of size at most $r - 2$, by Part 5 of Lemma 7, each $v \in V(\mathcal{H}_i)$ is a kernel of a q-star in $\mathcal{H}_i$. So by Claims 17 and 18 for each $u \in \bigcup \widehat{\mathcal{H}}$, $T^+(u)$ is well defined and $|T^+(u)| = t$.

Since $\mathcal{J}^{(0)}$ does not contain sets of size $r - 1$, each $(r-1)$-element set $Y \in \partial\mathcal{H}_i$ is only in one set in $\mathcal{H}_i$, thus $|\partial\mathcal{H}_i| = r|\mathcal{H}_i|$. As $|\mathcal{H}_0| \leqslant Cn^{r-2}$, equation (2) implies $|\widehat{\mathcal{H}}| > (t-1)|\partial\mathcal{H}| \geqslant (t-1)|\partial\widehat{\mathcal{H}}|$. We obtain $\sum_i |\partial\mathcal{H}_i| = r\sum_i |\mathcal{H}_i| = r|\widehat{\mathcal{H}}| > r(t-1)|\partial\widehat{\mathcal{H}}|$. Hence some $(r-1)$-tuple S belongs to at least $r(t-1)+1$ shadow families $\partial\mathcal{H}_i$. Say, $S \cup \{z_i\} \in \mathcal{H}_i$ for $i \in [r(t-1)+1]$. Let $Z = \{z_1, \ldots, z_{r(t-1)+1}\}$.

For every $y \in S$, $|T(y)| = t - 1$; thus there is $z_j \in Z \setminus \bigcup_{y \in S} T^+(y)$, say z_1. Let $Y_1 := S \cup \{z_1\}$. We got that $T(z_1)$ is disjoint from Y_1 while $z_1 \in Y_1 \in \mathcal{H}_1$. So Claim 18 implies that $Q(Y_1, z_1) = T(z_1)$. However $Q(Y_1, z_1)$ contains $Z \setminus \{z_1\}$ whose size is $(r-1)t$, and not $t - 1$.

This final contradiction implies that the minimal counterexample $\mathcal{H}$ does not exist, completing the proof of Theorem 3.

6 Stability: Proof of Theorem 6

6.1 Inequalities

In this subsection we recall some useful inequalities. For $n \geqslant r^2$ and $r \geqslant 5$ we have

$$\binom{n}{r-1} = \frac{n^{r-1}}{(r-1)!} \times \prod_{0 \leqslant i \leqslant r-2} \left(1 - \frac{i}{n}\right) > \frac{n^{r-1}}{2(r-1)!}. \tag{7}$$

Similarly we have

$$\binom{n}{r-2} > \frac{n^{r-2}}{2(r-2)!}. \tag{8}$$

For integers $n \geqslant b \geqslant 0$, $n \geqslant r^2$, $r \geqslant 5$ we also have

$$b\frac{n^{r-2}}{(r-2)!} \geqslant \binom{n}{r-1} - \binom{n-b}{r-1} \geqslant b\frac{n^{r-2}}{2(r-1)!}. \tag{9}$$

Indeed, the middle part in (9) is $\sum_{n-1\geqslant m\geqslant n-b}\binom{m}{r-2}$, these terms are monotone decreasing, so this sum is at most $b\binom{n-1}{r-2}$. On the other hand, these are the largest b terms in the sum $\binom{n}{r-1} = \sum_{n-1\geqslant m\geqslant 0}\binom{m}{r-2}$, so it is at least $(b/n)\times\binom{n}{r-1}$. Then (7) completes the lower bound.

6.2 Start of proof of Theorem 6

Let $C = C(r,t,h) := 1/c(r,thr)$, where c is from Lemma 7. In our proof of Theorem 6 we may suppose that $C_0 \geqslant 1$. Let $b = \lceil 3(r-1)!(C+C_0)\rceil$ and define $n_0 := \max\{r^2, b\}$.

Suppose that $n > n_0$ and let $\mathcal{H}$ be an n-vertex r-uniform family not containing a bush $\mathcal{B}_{t,h}(a,b)$ with

$$|\mathcal{H}| > (t-1)\binom{n}{r-1} - C_0 n^{r-2}. \tag{10}$$

Define m, $\mathcal{H}_0,\dots,\mathcal{H}_m$ and $\widehat{\mathcal{H}}$ as in Subsection 4.1. By (10) and the definition of $\mathcal{H}_0$,

$$|\widehat{\mathcal{H}}| > (t-1)\binom{n}{r-1} - (C+C_0)n^{r-2}. \tag{11}$$

As in Subsection 4.2, for each $1 \leqslant i \leqslant m$, the intersection structure $\mathcal{J}(\mathcal{H}_i)$ contains $\mathcal{J}^{(r-1)}$. So, again for each hyperedge $E \in \mathcal{H}_i \subset \widehat{\mathcal{H}}$ $(1 \leqslant i \leqslant m)$ there is an element $c(E) \in E$ such that each proper subset of E containing $c(E)$ is a kernel of an thr-star in $\mathcal{H}_i$.

For $v \in [n]$, let $\widehat{\mathcal{H}}(v) = \{E \in \widehat{\mathcal{H}} : v = c(E)\}$ and $\mathcal{G}(v) = \{E \setminus \{v\} : E \in \widehat{\mathcal{H}}(v)\}$. Let $\mathcal{G} = \bigcup_{v\in[n]}\mathcal{G}(v)$. Since all edges in $\widehat{\mathcal{H}}(v)$ contain v, $|\mathcal{G}(v)| = |\widehat{\mathcal{H}}(v)|$ for each $v \in [n]$; in particular, $\sum_v |\mathcal{G}(v)| = |\widehat{\mathcal{H}}|$. Furthermore, Claim 15 implies that

each $(r-2)$-subset Y of $[n]$ is in the shadow of at most $t-1$ families $\mathcal{G}(v)$. (12)

Recall the Lovász's form [14] of the Kruskal-Katona Theorem:

Theorem 19 ([14])**.** *If $1 \leqslant k < n$, $\mathcal{F} \subseteq \binom{[n]}{k}$ and $x \geqslant k$ is a positive real such that $|\mathcal{F}| = \binom{x}{k}$, then $|\partial\mathcal{F}| \geqslant \binom{x}{k-1}$.*

Here $\binom{x}{k}$ is a non-negative real convex function defined as a degree k polynomial $x(x-1)\dots(x-k+1)/k!$ for $x \geqslant k$.

For every $j \in [n]$ with a nonempty $\mathcal{G}(j)$, there is a real $x_j \geqslant r-1$ such that $|\mathcal{G}(j)| = \binom{x_j}{r-1}$. Inequality (11) gives

$$\sum_{i: x_i \geqslant r-1} \binom{x_i}{r-1} > (t-1)\binom{n}{r-1} - (C+C_0)n^{r-2}, \tag{13}$$

so (12) and Lovász Theorem give

$$(t-1)\binom{n}{r-2} \geqslant \sum_{i: x_i \geqslant r-1} |\partial \mathcal{G}(j)| \geqslant \sum_i \binom{x_i}{r-2}. \tag{14}$$

Lemma 20. *Suppose that $n > n_0$ and $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n$. Then inequalities* (13) *and* (14) *for $n, r, x_1, \ldots, x_n$ imply $x_1, \ldots, x_{t-1} > n - b$.*

Proof. For brevity let $y := x_t$. If $y < r-1$, then the left hand side of (13) has at most $t-1$ nonzero terms. Hence $\binom{x_{t-1}}{r-1} > \binom{n}{r-1} - (C+C_0)n^{r-2}$. So, if $x_{t-1} \leqslant n-b$, then the lower bound in (9) gives

$$(C+C_0)n^{r-2} \geqslant \binom{n}{r-1} - \binom{n-b}{r-1} \geqslant b\frac{n^{r-2}}{2(r-1)!}.$$

This yields $2(r-1)!(C+C_0) \geqslant b$, contradicting the definition of b. From now on, we suppose that $y \geqslant r-1$.

Multiply (14) by $(y-r+2)$ and add it to (13) multiplied by $r-1$. We get

$$\begin{aligned}(y-r+2)(t-1)\binom{n}{r-2} &+ (r-1)\sum_{i: x_i \geqslant r-1}\binom{x_i}{r-1} \\ &> (t-1)(r-1)\binom{n}{r-1} - (r-1)(C+C_0)n^{r-2} + (y-r+2)\sum_{i: x_i \geqslant r-1}\binom{x_i}{r-2}.\end{aligned}$$

Using $(r-1) \times \binom{x}{r-1} = (x-r+2) \times \binom{x}{r-2}$ (for all reals $x \geqslant r-1 \geqslant 1$) after rearrangements we get

$$\begin{aligned}(r-1)(C+C_0)n^{r-2} \;&>\; (t-1)(n-y)\binom{n}{r-2} + \sum_{i: x_i \geqslant r-1}(y-x_i)\binom{x_i}{r-2} \\ &= \sum_{i \geqslant t:\ x_i \geqslant r-1}(y-x_i)\binom{x_i}{r-2} \\ &\quad + \sum_{1 \leqslant i \leqslant t-1}(n-x_i)\binom{n}{r-2} \\ &\quad + \sum_{1 \leqslant i \leqslant t-1}(x_i-y)\left(\binom{n}{r-2} - \binom{x_i}{r-2}\right).\end{aligned}$$

After the equation sign the first and the third rows are non-negative. So, neglecting them and switching the sides of the inequality, we obtain

$$\binom{n}{r-2} \sum_{1 \leqslant i \leqslant t-1} (n - x_i) \leqslant (r-1)(C + C_0)n^{r-2}.$$

Now (8) yields $\sum_{1 \leqslant i \leqslant t-1}(n - x_i) \leqslant 2(r-1)!(C + C_0)$. This is less than b, the lemma holds. ☐

Now we are ready to finish the proof of the theorem. By Lemma 20, for every $1 \leqslant i \leqslant t-1$,

$$|\widehat{\mathcal{H}}(i)| = \binom{x_i}{r-1} \geqslant \binom{n-b}{r-1}.$$

The left inequality in (9) gives $\binom{n-b}{r-1} \geqslant \binom{n}{r-1} - b\frac{n^{r-2}}{(r-2)!}$. This completes the proof of Theorem 6 for any $C_1 \geqslant \frac{b}{(r-2)!}$, we can take $C_1 := 4(r-1)(C + C_0)$.

7 Concluding remarks

This manuscript takes a small step toward a general theory of Turán type hypergraph problems in the most promising case that we call Erdős-Ko-Rado range (i.e., $\mathrm{ex}(n, \mathcal{F}) = \Theta(n^{r-1})$, and $\mathcal{F}$ is "tree-like").

Let $\mathcal{S}_b^r(h)$ denote an r-uniform star with h petals and kernel of size b. Theorem 6 could be the first step in the proof of our next conjecture.

Conjecture 21. Given a, b, h, t positive integers, $a + b = r > 2$ and $n > n_0(r, h, t)$

$$\mathrm{ex}(n, \mathcal{B}_{t,h}(a, b)) = \binom{n}{r} - \binom{n-t+1}{r} + \mathrm{ex}(n-t+1, \mathcal{S}_b^r(h)).$$

The lower bound is a generalization of Construction 4 ($\mathcal{E}_2$ can be any $\mathcal{S}_b^r(h)$-free r-graph on $[n] \setminus [t-1]$. The proof of this is the same as for Construction 4).

Note that since Duke and Erdős [2] proposed the problem of determining $\mathrm{ex}(n, \mathcal{S}_b^r(h))$ in 1977, there were many remarkable results. E.g., in [5] an asymptotic bound was proved when b, r, h are fixed, $r \geqslant 2b + 3$, and $n \to \infty$. In particular, it is $\Theta(n^{\max\{b, r-b-1\}})$ for all r and b. This was recently extended by Bradač, Bucić, and Sudakov [1] who showed $\mathrm{ex}(n, \mathcal{S}_b^r(h)) = \Theta(n^{r-b-1}h^{b+1})$ when h is arbitrary and $n > n_0(r)$ (and $r \geqslant 2b + 1$). Note that these bounds were proved although we do not know the Erdős-Rado function $\phi(h, r)$, the size of the largest r-family that does not contain an h-star (with arbitrary core size). For newest developments, see Kupavskii and Noskov [13].

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