Paths with two blocks in oriented graphs of large minimum semi-degree

Bin Chen[a]   Xinmin Hou[b,c]   Xinyu Zhou[b]

Abstract

Stein (2020) conjectured that for any positive integer k, every oriented graph of minimum semi-degree greater than $k/2$ contains every oriented path of length k. This conjecture is true for directed paths by a result from Jackson (JGT, 1981). In this paper, we establish the validity of Stein's conjecture specifically for any oriented path with two blocks, where, a block of an oriented path P refers to a maximal directed subpath within P.

Mathematics Subject Classifications: 05C20; 05C38

1 Introduction

Throughout this paper, all digraphs considered have no loops or parallel arcs. Let $D = (V, A)$ be a digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. For an arc $(u, v) \in A(D)$, we call u the *tail* and v the *head* of the arc (u, v), and we also say u *dominates* v or v is *dominated* by u. Let $\delta^+(D)$ and $\delta^-(D)$ be the *minimum out-degree* and *minimum in-degree* of D, respectively. By $\delta^0(D)$, we denote the *minimum semi-degree* of D, which is the smallest value between $\delta^+(D)$ and $\delta^-(D)$. Two different vertices are *adjacent* if there is an arc connecting them, and two different arcs are adjacent if they share a common vertex.

An *orientation* of a graph G is an assignment of exactly one direction to every edge of G. A *directed path* (resp., *anti-directed path*) is an oriented path such that any two adjacent arcs have the same orientation (resp., opposite orientations). A *directed cycle* (resp., *anti-directed cycle*) is an oriented cycle such that every pair of two adjacent arcs have the same orientation (resp., opposite orientations). A *Hamiltonian oriented path* (resp.,

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[a]School of Mathematics and Statistics, Fuzhou University, Fujian, China.

[b]School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, Anhui, China.

[c]CAS Key Laboratory of Wu Wen-Tsun Mathematics, University of Science and Technology of China, Hefei 230026, Anhui, China.

Hamiltonian oriented cycle) is an oriented path (resp., oriented cycle) that passes through each vertex of a given digraph.

A digraph is *traceable* (resp., *anti-traceable*) if there is a Hamiltonian directed path (resp., Hamiltonian anti-directed path). A digraph is *Hamiltonian* if it contains a Hamiltonian directed cycle. A digraph D is *strongly connected*(*strong* for short) if and only if for every pair of distinct vertices $u, v \in V(D)$, there is a directed path from u to v and a directed path from v to u.

Paths and cycles are one of the most fundamental objects in graph theory. Given a directed graph, a natural question is to find sufficient conditions for the existence of oriented paths or cycles of specific types and lengths within this digraph.

Rédei [13] proved that any tournament is traceable and contains an odd number of Hamiltonian directed paths. In [1], Camion showed that every strong tournament is Hamiltonian. Extending Camion's result, Moon [12] further proved that every strong tournament is vertex pancyclic. Let RT_n be a regular tournament on n vertices and PT_n be a Paley tournament on n vertices (A Paley tournament is constructed from the members of a suitable finite field by connecting pairs of elements that differ by a quadratic residue). In [4], Grünbaum proved that every tournament is anti-traceable unless it is isomorphic to one of PT_3, RT_5 and PT_7. Rosenfeld in [14] showed that for any tournament on $n \geqslant 9$ vertices, every vertex is the initial vertex of a Hamiltonian anti-directed path, strengthening the previous result. In general digraphs, Ghouila-Houri [3] proved that every strong digraph D on n vertices with $\delta^+(D) + \delta^-(D) \geqslant n$ is Hamiltonian. Particularly, any digraph D of $\delta^0(D) \geqslant n/2$ is Hamiltonian. Woodall [19] improved Ghouila-Houri's result by verifying that every strong digraph on $n \geqslant 2$ vertices with $d_D^+(u) + d_D^-(v) \geqslant n$ for every pair of nonadjacent vertices u and v in D is Hamiltonian.

Recall that an oriented graph is a digraph without directed cycles of length two. In oriented graphs, there exist a number of beautiful and significant results concerning Hamiltonian oriented cycles. A problem proposed by Thomassen [18] is to determine $\delta^0(D)$ that guarantees an oriented graph to be Hamiltonian. Häggkvist [5] constructed a family of non-Hamiltonian oriented graphs D with n vertices and $\delta^0(D) = (3n - 5)/8$. Furthermore, Häggkvist [5] conjectured that any oriented graph D with n vertices and $\delta^0(D) \geqslant (3n - 4)/8$ is Hamiltonian. This conjecture was confirmed by Keevash, Kühn and Osthus [8] for sufficiently large n. Häggkvist [5] also conjectured that any oriented graph with n vertices and $\delta(D) + \delta^+(D) + \delta^-(D) > (3n - 3)/2$ is Hamiltonian. This conjecture was resolved by Kelly, Kühn and Osthus [10] approximately. Häggkvist and Thomason [6] proposed a conjecture stating that for any positive ϵ, the condition $\delta^0(D) \geqslant (3/8 + \epsilon)n$ is sufficient to guarantee the existence of any orientation of a Hamiltonian cycle in any sufficiently large oriented graph D. The same authors subsequently demonstrated that the stronger condition $\delta^0(D) \geqslant (5/12 + \epsilon)n$ is also adequate. In 2011, Kelly [9] completely resolved this conjecture. Notably, this minimum semi-degree condition ensures that D contains a cycle of every possible orientation and length.

Several interesting results have also been obtained for long oriented paths and long oriented cycles with specific lengths. Jackson [7] showed the following classical result.

Theorem 1 ([7]). *For any positive integer k, every oriented graph D of $\delta^0(D) \geqslant k$ admits a directed path of length $2k$ when $|V(D)| > 2k + 2$. Furthermore, if $|V(D)| \leqslant 2k + 2$, then D is Hamiltonian.*

Zhang [20] refined this result by showing the following result.

Theorem 2 ([20]). *Every oriented graph D with $\delta^0(D) \geqslant k$ admits either a directed path of length at least $2k + 2$ or a directed cycle of length at least $2k + 1$.*

Building upon these earlier results, Stein [16] proposed the following conjecture.

Conjecture 3. ([16]) For any positive integer k, every oriented graph D of $\delta^0(D) > k/2$ contains every oriented path of length k.

Conjecture 3 is, in a sense, best possible due to the existence of disjoint union of RT_k. Theorems 1 and 2 showed that Conjecture 3 holds for directed paths. The following is a list of known results concerning Conjecture 3.

(1) Kelly [9] proved that Conjecture 3 is true for $k \geqslant 3n/4 + o(n)$.

(2) Jackson [7] proved that Conjecture 3 holds for directed paths.

(3) Stein and Zárate-Guerén [17] proved that for all $\eta \in (0,1)$, there is n_0 such that for all $n \geqslant n_0$ and $k \geqslant \eta n$, every oriented graph D on n vertices with $\delta^0(D) > (1+\eta)k/2$ contains every antidirected path with k edges.

(4) Klimošová and Stein [11] proved that for all $k \geqslant 3$, every oriented graph D of $\delta^0(D) \geqslant (3k-2)/4$ contains each anti-directed path of length k.

(5) Chen, Hou, and Zhou [2] improved this lower bound to $(2k+1)/3$.

(6) Skokan and Tyomkyn [15] further improved the lower bound to $5k/8$.

For integers s and t, define $P(s,t)$ to be an oriented path consisting of s consecutive forward arcs followed by t consecutive backward arcs. Up to isomorphism, we have $P(s,t) = P(t,s)$. Jackson's result (see (2)) can be rephrased as Conjecture 3 holds true for $P(k,0)$. The main result of this paper is the following.

Theorem 4. *For any integer k, every oriented graph D with $\delta^0(D) > k/2$ contains a copy of $P(s,t)$ for all $s + t = k$.*

Similarly, we define $Q(s,t)$ to be an oriented path of length $s + t$, consisting of s backward arcs and followed by t forward arcs. Also, $Q(s,t)$ is isomorphic to $Q(t,s)$. Notably, Theorem 4 implies that every oriented graph D with $\delta^0(D) > k/2$ also contains every $Q(s,t)$ for $s+t = k$. Here is the proof: consider the oriented graph $\overleftarrow{D}$, obtained from D by reversing the orientation of each arc. Then we have $\delta^0(\overleftarrow{D}) > k/2$. By Theorem 4, $\overleftarrow{D}$ contains a $P(s,t)$, which is a $Q(s,t)$ included in D.

A *block* of an oriented path P is a maximal directed subpath in P. Consequently, an oriented path with two blocks is either isomorphic to a $P(s,t)$ or to a $Q(s,t)$ for some integers s and t. As a corollary, we conclude that Conjecture 3 is true for every oriented path with two blocks.

Theorem 5. *For any integer k, every oriented graph D of $\delta^0(D) > k/2$ contains every oriented path of length k with two blocks.*

The rest of this paper is arranged as follows. In Section 2, we give the proof of Theorem 4. We conclude this paper with a remark in the last section.

2 The proof of Theorem 4

We begin by introducing additional notation. Let $D = (V, A)$ be a digraph. The notation $u \to v$ stands for $(u, v) \in A(D)$, while, $u \nrightarrow v$ indicates that $(u, v) \notin A(D)$. The *out-neighborhood* of a vertex v, denoted as $N_D^+(v)$, consists of all vertices $u \in V(D)$ such that $v \to u$, and the *out-degree* of v, denoted as $d_D^+(v)$, is the cardinality of $N_D^+(v)$. Analogously, the *in-neighborhood* of a vertex v is defined as $N_D^-(v) = \{u \in V(D) | u \to v\}$, and the *in-degree* of v is given by $d_D^-(v) = |N_D^-(v)|$. The vertices within $N_D^+(v)$ are referred to as *out-neighbors*, while those in $N_D^-(v)$ are considered *in-neighbors* of v. For integers a and b, we use the notation $[a, b] = \{a, a+1, \ldots, b\}$, and for convenience, we further denote $[k] = [1, k]$.

The following observation is straightforward yet highly valuable.

Observation 6. *Let ℓ be a positive integer and let D be an oriented graph. If $P = v_1 v_2 \ldots v_\ell$ is a longest directed path in D, then $N_D^-(v_1) \cup N_D^+(v_\ell) \subseteq V(P)$.*

Now we are ready to give the proof of Theorem 4.

Proof of Theorem 4: Let D be an oriented graph with $\delta^0(D) > k/2$. For $k \leqslant 4$, the statement can be directly verified. Hence, we assume $k \geqslant 5$ in the following. Suppose to the contrary that D contains no path $P(s, t)$ for some s and t with $s + t = k$. By symmetry, we can assume that $s \geqslant t$. Therefore, $s \geqslant \lceil k/2 \rceil$ and $t \leqslant \lfloor k/2 \rfloor$. Moreover, by Theorem 1, we may assume $t \geqslant 1$.

Let $P = v_1 v_2 \ldots v_\alpha$ be a longest directed path in D. By Observation 6, we have $N_D^-(v_1) \cup N_D^+(v_\alpha) \subseteq V(P)$. By Theorem 1, $\alpha \geqslant 2\delta^0(D) + 1 \geqslant k + 2$.

Let $R = \{v_{s+1}, v_{s+2}, \ldots, v_{\alpha-t-1}\}$ and $B = \{v_{t+1}, v_{t+2}, \ldots, v_{\alpha-s-1}\}$. Observe that $|R| = |B| = \alpha - s - t - 1 \geqslant 1$ since $\alpha \geqslant k + 2$ and $s + t = k$.

Claim 7. (i) *No vertex in R is adjacent to v_α;* (ii) *No vertex in B is adjacent to v_α.*

Proof. We only prove (i) as (ii) can be proved with similar discussion. Suppose the assertion is false. Then there must exist $i \in [s+1, \alpha - t - 1]$ such that v_i and v_α are adjacent. If $v_i \to v_\alpha$, then

$$v_{i-s+1} \to v_{i-s+2} \to \ldots \to v_i \to v_\alpha \leftarrow v_{\alpha-1} \leftarrow \ldots \leftarrow v_{\alpha-t}$$

is a path $P(s, t)$, a contradiction. If $v_\alpha \to v_i$, then

$$v_{i-s} \to v_{i-s+1} \to \ldots \to v_i \leftarrow v_\alpha \leftarrow v_{\alpha-1} \leftarrow \ldots \leftarrow v_{\alpha-t+1}$$

is a $P(s, t)$, a contradiction too. □

From Claim 7, we have the following technical statement.

Claim 8. *There exists no vertex v in D such that v serves as the initial vertex of a longest directed path P_1 and also as the terminal vertex of another longest directed path P_2 with $V(P_1) = V(P_2)$.*

Proof. Without loss of generality, assume $V(P_1) = V(P_2) = V(P)$. By Observation 6, we have $N_D^+(v) \cup N_D^-(v) \subseteq V(P)$. From Claim 7, it is clear that v is not adjacent to at least $\alpha - k - 1$ vertices of P. Since D is an oriented graph, it follows that $d_D^+(v) + d_D^-(v) \leqslant k$. On the other hand, we observe that $d_D^+(v) + d_D^-(v) \geqslant 2\delta^0(D) > k$. Consequently, we deduce that $k < d_D^+(v) + d_D^-(v) \leqslant k$, which is a contradiction. $\qquad\square$

Claim 9. $v_\alpha \nrightarrow v_{\alpha-t}$ *and* $v_\alpha \nrightarrow v_{\alpha-s}$.

Proof. If $v_\alpha \rightarrow v_{\alpha-t}$, then

$$v_{\alpha-k} \rightarrow v_{\alpha-k+1} \rightarrow \ldots \rightarrow v_{\alpha-t} \leftarrow v_\alpha \leftarrow v_{\alpha-1} \leftarrow \ldots \leftarrow v_{\alpha-t+1}$$

is a $P(s,t)$, a contradiction. If $v_\alpha \rightarrow v_{\alpha-s}$, then

$$v_{\alpha-s+1} \rightarrow v_{\alpha-s+2} \rightarrow \ldots \rightarrow v_\alpha \rightarrow v_{\alpha-s} \leftarrow v_{\alpha-s-1} \leftarrow \ldots \leftarrow v_{\alpha-k}$$

is a $P(s,t)$, also a contradiction. $\qquad\square$

Let $\gamma = \min\{i \,|\, v_\alpha \rightarrow v_i\}$ and $\xi = \max\{i \,|\, v_i \rightarrow v_1\}$. Since $\delta^0(D) > k/2 \geqslant t$ and $N_D^+(v_\alpha) \subseteq V(P)$, by Claim 7, it is easily seen that $\gamma \leqslant s$.

Claim 10. $N_D^-(v_1) \cap (R \cup \{v_{\alpha-t}\}) = \emptyset$.

Proof. Suppose not, there exists $v_i \in R \cup \{v_{\alpha-t}\}$ such that $v_i \rightarrow v_1$. Then

$$v_{i-s+\gamma} \rightarrow v_{i-s+\gamma+1} \rightarrow \ldots \rightarrow v_i \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_\gamma \leftarrow v_\alpha \leftarrow v_{\alpha-1} \leftarrow \ldots \leftarrow v_{\alpha-t+1}$$

is a $P(s,t)$, a contradiction. $\qquad\square$

Claim 11. $\gamma \geqslant 2$, *i.e.,* $\xi \leqslant \alpha - 1$.

Proof. If not, then $\gamma = 1$ and $P + (v_\alpha, v_1)$ is a directed cycle of length α. By the maximality of P and Observation 6, we have $N_D^+(v) \cup N_D^-(v) \subseteq V(P)$ for any $v \in V(P)$. By Claim 7, we have $N_D^+(v_\alpha) \cup N_D^-(v_\alpha) \subseteq V(P)\backslash(R \cup \{v_\alpha\})$. As D is an oriented graph, $N_D^+(v_\alpha) \cap N_D^-(v_\alpha) = \emptyset$. Therefore, we have

$$k < 2\delta^0(D) \leqslant d_D^+(v_\alpha) + d_D^-(v_\alpha) \leqslant \alpha - |R| - 1 = s + t = k,$$

this is impossible. The claim follows. $\qquad\square$

For convenience, denote $U = \{v_1, v_2, \ldots, v_\gamma\}$ and $W = \{v_\xi, v_{\xi+1}, \ldots, v_\alpha\}$. In the subsequent proof, we divide the discussion into two cases: when $2 \leqslant \gamma \leqslant t$ and when $\gamma > t$.

Case A. $2 \leqslant \gamma \leqslant t$

Recall that $N_D^-(v_1) \subseteq V(P)$ and $d_D^-(v_1) \geqslant \delta^0(D) > k/2 \geqslant t$. It is obvious that $\xi \geqslant t + 3$.

Claim 12. $N_D^-(v_1) \cap (B \cup \{v_{\alpha-s}\}) = \emptyset$.

Proof. On the contrary, suppose that there exists $v_i \in B \cup \{v_{\alpha-s}\}$ such that $v_i \to v_1$. It is straightforward to check that

$$v_{\alpha-s+1} \to v_{\alpha-s+2} \to \ldots \to v_\alpha \to v_\gamma \leftarrow v_{\gamma-1} \leftarrow \ldots \leftarrow v_1 \leftarrow v_i \leftarrow v_{i-1} \leftarrow \ldots \leftarrow v_{i-t+\gamma}$$

is a $P(s,t)$, a contradiction. $\square$

By Claims 12 and 10, we have $\xi \in [\alpha - s + 1, s] \cup [\alpha - t + 1, \alpha - 1]$. Let $F = \{v_{\alpha-s}, v_{\alpha-s+1}, \ldots, v_s\}$. Set $X_1 = \{v_{\gamma+1}, v_{\gamma+2}, \ldots, v_t\}$ and $Y_1 = \{v_{\alpha-t}, v_{\alpha-t+1}, \ldots, v_{\xi-1}\}$. Then

$$V(P) = U \cup X_1 \cup B \cup F \cup R \cup Y_1 \cup W.$$

We divide the proof of Case A into two subcases: when $F = \emptyset$ and when $F \neq \emptyset$.

Case 1. $F = \emptyset$

Then $\xi \in [\alpha - t + 1, \alpha - 1]$ and, clearly, $\alpha \geqslant 2s + 1$. See Figure 1 for an illustration.
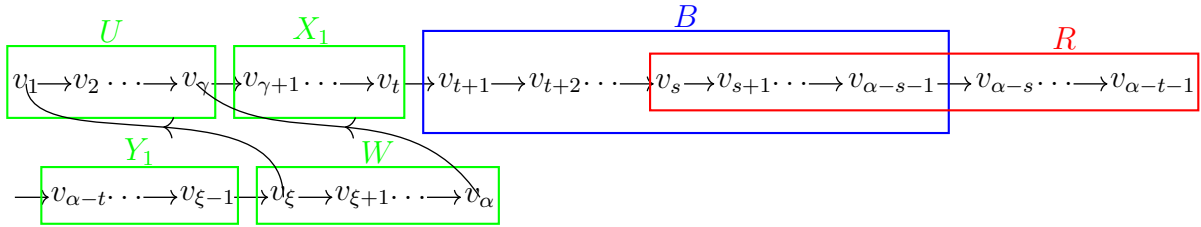
Figure 1

Claim 13. *One of the following must occur:*
  (i). There are vertices $v_\ell \in X_1$ and $v_{\ell-2} \in X_1 \cup \{v_\gamma\}$ such that $v_\alpha \to v_\ell$ and $v_{\ell-2} \to v_1$.
  (ii). There are vertices $v_m \in Y_1$ and $v_{m-2} \in Y_1$ such that $v_\alpha \to v_m$ and $v_{m-2} \to v_1$.

Proof. On the contrary, neither (i) nor (ii) occurs. Let $X_1^* = N_D^+(v_\alpha) \cap X_1$ and $Y_1^* = N_D^+(v_\alpha) \cap Y_1$. Denote by $x_1^* = |X_1^*|$ and $y_1^* = |Y_1^*|$. Since (i) does not occur, we have $|N_D^-(v_1) \cap (X_1 \cup \{v_\gamma\})| \leqslant |X_1 \cup \{v_\gamma\}| - (x_1^* - 1)$. By Claim 9, $v_\alpha \nrightarrow v_{\alpha-t}$. Since (ii) does not occur, we have $|N_D^-(v_1) \cap Y_1| \leqslant |Y_1| - (y_1^* - 1)$. By the maximality of ξ, Claims 10 and 12, we obtain that

$$\begin{aligned}
|N_D^-(v_1)| &= |N_D^-(v_1) \cap (U \backslash \{v_\gamma\})| + |N_D^-(v_1) \cap (X_1 \cup \{v_\gamma\})| + |N_D^-(v_1) \cap (Y_1 \cup \{v_\xi\})| \\
&\leqslant \gamma - 3 + |X_1| + 1 - (x_1^* - 1) + |Y_1| - (y_1^* - 1) + 1 \\
&= \gamma - 3 + t - \gamma + 1 - (x_1^* - 1) + \xi - \alpha + t - (y_1^* - 1) + 1 \\
&= 2t - \alpha - x_1^* - y_1^* + \xi + 1, \tag{1}
\end{aligned}$$

and, by the minimality of γ and Claim 7, we have

$$|N_D^+(v_\alpha)| = |X_1^* \cup \{v_\gamma\}| + |Y_1^*| + |N_D^+(v_\alpha) \cap W|$$

$$\leqslant x_1^* + 1 + y_1^* + \alpha - \xi - 1$$
$$= \alpha + x_1^* + y_1^* - \xi. \tag{2}$$

Combining (1) and (2), we have

$$d_D^-(v_1) + d_D^+(v_\alpha) \leqslant 2t + 1 \leqslant 2\lfloor k/2 \rfloor + 1.$$

On the other hand, $d_D^-(v_1) + d_D^+(v_\alpha) \geqslant 2\delta^0(D) > k$. Therefore, we conclude that

$$k < d_D^-(v_1) + d_D^+(v_\alpha) \leqslant 2\lfloor k/2 \rfloor + 1.$$

If k is odd, then we have $k < 2\lfloor k/2 \rfloor + 1 = k$, a contradiction. Now assume k is even. Then we have $\delta^0(D) \geqslant k/2 + 1$. Thus we have

$$k + 2 \leqslant 2\delta^0(D) \leqslant d_D^-(v_1) + d_D^+(v_\alpha) \leqslant 2\lfloor k/2 \rfloor + 1 = k + 1,$$

a contradiction too. As a consequence, the claim follows. $\square$

If (i) of Claim 13 occurs, i.e., there exist $v_\ell \in X_1$ and $v_{\ell-2} \in X_1 \cup \{v_\gamma\}$ such that $v_\alpha \to v_\ell$ and $v_{\ell-2} \to v_1$, then both

$$v_{\ell-1} \to v_\ell \to \ldots \to v_\alpha \to v_\gamma \to v_{\gamma+1} \to \ldots \to v_{\ell-2} \to v_1 \to v_2 \to \ldots \to v_{\gamma-1}$$

and

$$v_{\xi+1} \to v_{\xi+2} \to \ldots \to v_\alpha \to v_\ell \to v_{\ell+1} \to \ldots \to v_\xi \to v_1 \to v_2 \to \ldots \to v_{\ell-1}$$

are longest directed paths in D. However, this contradicts Claim 8.

If (ii) of Claim 13 occurs, i.e., there exist $v_{m-2}, v_m \in Y_1$ such that $v_\alpha \to v_m$ and $v_{m-2} \to v_1$, then both

$$v_{m-1} \to v_m \to \ldots \to v_\alpha \to v_\gamma \to v_{\gamma+1} \to \ldots \to v_{m-2} \to v_1 \to v_2 \to \ldots \to v_{\gamma-1}$$

and

$$v_{\xi+1} \to v_{\xi+2} \to \ldots \to v_\alpha \to v_m \to v_{m+1} \to \ldots \to v_\xi \to v_1 \to v_2 \to \ldots \to v_{m-1}$$

are longest directed paths in D, a contradiction too.

Case 2. $F \neq \emptyset$

It is obvious that $\alpha \leqslant 2s$. See Figure 2 for an illustration.

Claim 14. *One of the following must occur:*
   (i). There are vertices $v_\ell \in X_1$ and $v_{\ell-2} \in X_1 \cup \{v_\gamma\}$ with $v_\alpha \to v_\ell$ and $v_{\ell-2} \to v_1$.
   (ii). There are vertices $v_m, v_{m-2} \in F$ with $v_\alpha \to v_m$ and $v_{m-2} \to v_1$.
   (iii). There are vertices $v_h, v_{h-2} \in Y_1$ with $v_\alpha \to v_h$ and $v_{h-2} \to v_1$.
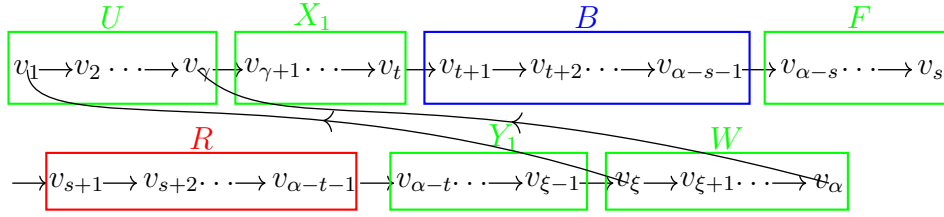
Figure 2

Proof. Suppose that none of (i), (ii) and (iii) occurs. Let $X_1^* = N_D^+(v_\alpha) \cap X_1$, $F^* = N_D^+(v_\alpha) \cap F$ and $Y_1^* = N_D^+(v_\alpha) \cap Y_1$. Denote by $x_1^* = |X_1^*|$, $f^* = |F^*|$ and $y_1^* = |Y_1^*|$. Since (i) does not occur, we have $|N_D^-(v_1) \cap (X_1 \cup \{v_\gamma\})| \leqslant |X_1 \cup \{v_\gamma\}| - (x_1^* - 1)$. By Claim 9, we get $v_\alpha \nrightarrow v_{\alpha-t}$ and $v_\alpha \nrightarrow v_{\alpha-s}$. Since (ii) and (iii) do not occur, we have $|N_D^-(v_1) \cap Y_1| \leqslant |Y_1| - (y_1^* - 1)$ and $|N_D^-(v_1) \cap F| \leqslant |Y_1| - (f^* - 1)$. Therefore, we have

$$
\begin{aligned}
|N_D^-(v_1)| &= |N_D^-(v_1) \cap (U \setminus \{v_\gamma\})| + |N_D^-(v_1) \cap (X_1 \cup \{v_\gamma\})| + |N_D^-(v_1) \cap F| \\
&\quad + |N_D^-(v_1) \cap (Y_1 \cup \{v_\xi\})| \\
&\leqslant \gamma - 3 + |X_1| + 1 - (x_1^* - 1) + |F| - (f^* - 1) + |Y_1| - (y_1^* - 1) + 1 \\
&= \gamma - 3 + t - \gamma + 1 - (x_1^* - 1) + 2s - \alpha + 1 - (f^* - 1) \\
&\quad + \xi - \alpha + t - (y_1^* - 1) + 1 \\
&= 2s + 2t - 2\alpha - x_1^* - f^* - y_1^* + \xi + 3 \\
&= 2k - 2\alpha - x_1^* - f^* - y_1^* + \xi + 3,
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
|N_D^+(v_\alpha)| &= |X_1^* \cup \{v_\gamma\}| + |F^*| + |Y_1^*| + |N_D^+(v_\alpha) \cap W| \\
&\leqslant x_1^* + 1 + f^* + y_1^* + \alpha - \xi - 1 \\
&= \alpha + x_1^* + f^* + y_1^* - \xi.
\end{aligned}
\tag{4}
$$

Combining (3) and (4), we have

$$
d_D^-(v_1) + d_D^+(v_\alpha) \leqslant 2k - \alpha + 3.
\tag{5}
$$

By $d_D^-(v_1) + d_D^+(v_\alpha) \geqslant 2\delta^0(D) > k$, we obtain that

$$
k < d_D^-(v_1) + d_D^+(v_\alpha) \leqslant 2k - \alpha + 3.
\tag{6}
$$

Recall that $\alpha \geqslant k + 2$. Thus we have $\alpha = k + 2$ by (6), and $d_D^-(v_1) + d_D^+(v_\alpha) = k + 1$. If k is even, then we have $k + 2 \leqslant 2\delta^0(D) \leqslant d_D^-(v_1) + d_D^+(v_\alpha) = k + 1$, a contradiction. Now assume that k is odd. Then $\delta^0(D) \geqslant (k+1)/2$. Note that the longest directed path P has length $\alpha - 1 = k + 1 \leqslant 2\delta^0(D)$. By Theorem 2, D contains a directed cycle C of length $2\delta^0(D) + 1 = k + 2$. Choose P as the longest directed path in C. This implies that $v_\alpha \to v_1$. However, this contradicts Claim 11.

Therefore, the claim follows. $\qquad\square$

If (i) of Claim 14 occurs, i.e., there exist $v_\ell \in X_1$ and $v_{\ell-2} \in X_1 \cup \{v_\gamma\}$ such that $v_\alpha \to v_\ell$ and $v_{\ell-2} \to v_1$, then both

$$v_{\ell-1} \to v_\ell \to \ldots \to v_\alpha \to v_\gamma \to v_{\gamma+1} \to \ldots \to v_{\ell-2} \to v_1 \to v_2 \to \ldots \to v_{\gamma-1}$$

and

$$v_{\xi+1} \to v_{\xi+2} \to \ldots \to v_\alpha \to v_\ell \to v_{\ell+1} \to \ldots \to v_\xi \to v_1 \to v_2 \to \ldots \to v_{\ell-1}$$

are longest directed paths in D. We have the desired contradiction to Claim 8.

If (ii) of Claim 14 occurs, i.e., there exist $v_{m-2}, v_m \in F$ such that $v_\alpha \to v_m$ and $v_{m-2} \to v_1$, then both

$$v_{m-1} \to v_m \to \ldots \to v_\alpha \to v_\gamma \to v_{\gamma+1} \to \ldots \to v_{m-2} \to v_1 \to v_2 \to \ldots \to v_{\gamma-1}$$

and

$$v_{\xi+1} \to v_{\xi+2} \to \ldots \to v_\alpha \to v_m \to v_{m+1} \to \ldots \to v_\xi \to v_1 \to v_2 \to \ldots \to v_{m-1}$$

are longest directed paths in D, a contradiction to Claim 8 too.

If (iii) of Claim 14 occurs, i.e., there exist $v_{h-2}, v_h \in Y_1$ such that $v_\alpha \to v_h$ and $v_{h-2} \to v_1$, then

$$v_{h-1} \to v_h \to \ldots \to v_\alpha \to v_\gamma \to v_{\gamma+1} \to \ldots \to v_{h-2} \to v_1 \to v_2 \to \ldots \to v_{\gamma-1}$$

and

$$v_{\xi+1} \to v_{\xi+2} \to \ldots \to v_\alpha \to v_h \to v_{h+1} \to \ldots \to v_\xi \to v_1 \to v_2 \to \ldots \to v_{h-1}$$

are longest directed paths in D, the same contradiction as above.

Case B. $\gamma > t$

Then $t < \gamma \leqslant s$. Recall that $(N_D^+(v_\alpha) \cup N_D^-(v_\alpha)) \cap R = \emptyset$, $N_D^-(v_1) \cap (R \cup \{v_{\alpha-t}\}) = \emptyset$ and $v_\alpha \nrightarrow v_{\alpha-t}$.

Claim 15. $\xi \geqslant \gamma + 4$.

Proof. By the minimality of γ and since D is oriented, we have

$$d_D^+(v_\alpha) \subseteq \{v_\gamma, v_{\gamma+1}, \ldots, v_{\alpha-2}\}.$$

Further, since $N_D^+(v_\alpha) \cap (R \cup \{v_{\alpha-t}\}) = \emptyset$, we deduce that

$$d_D^+(v_\alpha) \leqslant \alpha - 2 - \gamma + 1 - (\alpha - s - t) = k - \gamma - 1.$$

By $d_D^-(v_1) \geqslant 2\delta^0(D) - d_D^+(v_\alpha) > \gamma + 1$, we have $\xi \geqslant \gamma + 4$. $\square$
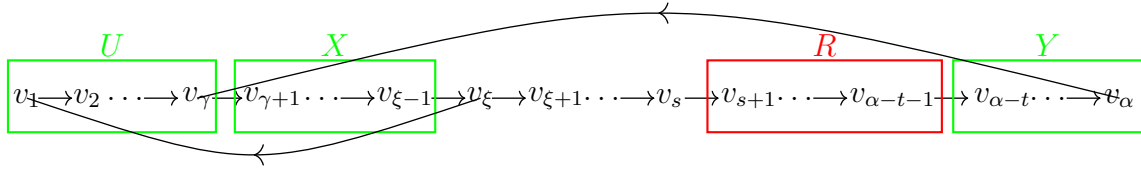
Figure 3

Together with Claim 11 and $N_D^-(v_1) \cap (R \cup \{v_{\alpha-t}\}) = \emptyset$, we obtain $\xi \in [\gamma+4, s] \cup [\alpha - t + 1, \alpha - 1]$. Here to proceed with the proof, we will consider the following two cases.

Case 1. $\gamma + 4 \leqslant \xi \leqslant s$

Let us denote $X = \{v_{\gamma+1}, v_{\gamma+2}, \ldots, v_{\xi-1}\}$ and $Y = \{v_{\alpha-t}, v_{\alpha-t+1}, \ldots, v_\alpha\}$. Write X^* for $N_D^+(v_\alpha) \cap X$ and let $x^* = |X^*|$. See Figure 3 for an illustration.

Claim 16. *There are vertices $v_\ell \in X$ and $v_{\ell-2} \in X \cup \{v_\gamma\}$ with $v_\alpha \to v_\ell$ and $v_{\ell-2} \to v_1$.*

Proof. Suppose to the contrary that the claim is false. Then there are at least $x^* - 1$ vertices in $X \cup \{v_\gamma\}$ not belonging to $N_D^-(v_1)$. Thus we have

$$
\begin{aligned}
|N_D^-(v_1)| &= |N_D^-(v_1) \cap (U \backslash \{v_\gamma\})| + |N_D^-(v_1) \cap (X \cup \{v_\gamma, v_\xi\})| \\
&\leqslant \gamma - 3 + |X| + 1 - (x^* - 1) + 1 \\
&= \gamma - 3 + \xi - \gamma - (x^* - 1) + 1 \\
&= \xi - x^* - 1,
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
|N_D^+(v_\alpha)| &= |X^* \cup \{v_\gamma\}| + |N_D^+(v_\alpha) \cap Y| + |N_D^+(v_\alpha) \cap \{v_\xi, v_{\xi+1}, \ldots, v_s\}| \\
&\leqslant x^* + 1 + t - 2 + s - \xi + 1 \\
&= x^* + k - \xi.
\end{aligned}
\tag{8}
$$

By (7) and (8), we deduce that $d_D^-(v_1) + d_D^+(v_\alpha) \leqslant k - 1$. Additionally, considering that $d_D^-(v_1) + d_D^+(v_\alpha) \geqslant 2\delta^0(D) > k$. We conclude that $k < d_D^-(v_1) + d_D^+(v_\alpha) \leqslant k - 1$, a contradiction. $\square$

By Claim 16, there exist $v_\ell \in X$ and $v_{\ell-2} \in X \cup \{v_\gamma\}$ such that $v_\alpha \to v_\ell$ and $v_{\ell-2} \to v_1$, then both

$$
v_{\ell-1} \to v_\ell \to \ldots \to v_\alpha \to v_\gamma \to v_{\gamma+1} \to \ldots \to v_{\ell-2} \to v_1 \to v_2 \to \ldots \to v_{\gamma-1}
$$

and

$$
v_{\xi+1} \to v_{\xi+2} \to \ldots \to v_\alpha \to v_\ell \to v_{\ell+1} \to \ldots \to v_\xi \to v_1 \to v_2 \to \ldots \to v_{\ell-1}
$$

are longest directed paths in D satisfying that $v_{\ell-1}$ is the initial vertex of one path and the terminal vertex of the other. This contradicts Claim 8.

Case 2. $\alpha - t + 1 \leqslant \xi \leqslant \alpha - 1$.

Let $X_2 = \{v_{\gamma+1}, v_{\gamma+2}, \ldots, v_s\}$ and $Y_2 = \{v_{\alpha-t}, v_{\alpha-t+1}, \ldots, v_{\xi-1}\}$. Then $V(P)$ has the partition $U \cup X_2 \cup R \cup Y_2 \cup W$. See Figure 4 for an illustration.
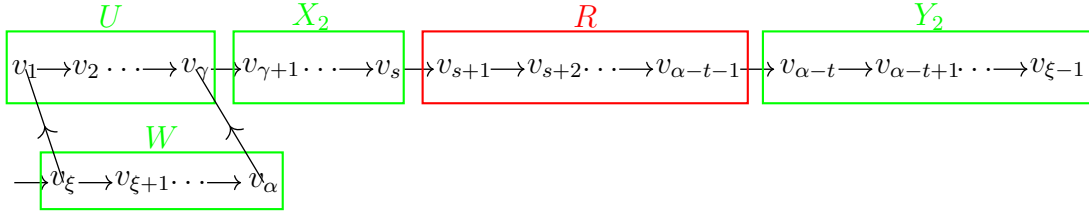

Figure 4

Claim 17. *One of the following must occur:*
   (i). There are vertices $v_\ell \in X_2$ and $v_{\ell-2} \in X_2 \cup \{v_\gamma\}$ with $v_\alpha \to v_\ell$ and $v_{\ell-2} \to v_1$.
   (ii). There are vertices $v_m \in Y_2$ and $v_{m-2} \in Y_2$ with $v_\alpha \to v_m$ and $v_{m-2} \to v_1$.

Proof. Suppose, on the contrary, that neither (i) nor (ii) occurs. Let $X_2^* = N_D^+(v_\alpha) \cap X_2$ and $Y_2^* = N_D^+(v_\alpha) \cap Y_2$. Denote by $x_2^* = |X_2^*|$ and $y_2^* = |Y_2^*|$. Then there are at least $x_2^* - 1$ vertices in $X_2 \cup \{v_\gamma\}$ not belonging to $N_D^-(v_1)$, and there are at least $y_2^* - 1$ vertices in Y_2 not belonging to $N_D^-(v_1)$ because $v_\alpha \nrightarrow v_{\alpha-t}$. Consequently, we obtain that

$$
\begin{aligned}
|N_D^-(v_1)| &= |N_D^-(v_1) \cap (U \backslash \{v_\gamma\})| + |N_D^-(v_1) \cap (X_2 \cup \{v_\gamma\})| + |N_D^-(v_1) \cap (Y_2 \cup \{v_\xi\})| \\
&\leqslant \gamma - 3 + |X_2| + 1 - (x_2^* - 1) + |Y_2| - (y_2^* - 1) + 1 \\
&= \gamma - 3 + s - \gamma + 1 - (x_2^* - 1) + \xi - \alpha + t - (y_2^* - 1) + 1 \\
&= k - \alpha - x_1^* - y_1^* + \xi + 1,
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
|N_D^+(v_\alpha)| &= |X_2^* \cup \{v_\gamma\}| + |Y_2^*| + |N_D^+(v_\alpha) \cap W| \\
&\leqslant x_2^* + 1 + y_2^* + \alpha - \xi - 1 \\
&= \alpha + x_2^* + y_2^* - \xi.
\end{aligned} \tag{10}
$$

Combining (9) and (10), we have

$$
d_D^-(v_1) + d_D^+(v_\alpha) \leqslant k + 1.
$$

However,

$$
k < 2\delta^0(D) \leqslant d_D^-(v_1) + d_D^+(v_\alpha) \leqslant k + 1
$$

forces that $d_D^-(v_1) + d_D^+(v_\alpha) = k + 1$ and thus $d_D^-(v_1) = k - \alpha - x_1^* - y_1^* + \xi + 1$ and $d_D^+(v_\alpha) = \alpha + x_2^* + y_2^* - \xi$. Particularly, we have $|N_D^-(v_1) \cap Y_2| = |Y_2| - (y_2^* - 1)$ and

$|N_D^+(v_\alpha) \cap W| = \alpha - \xi - 1$. $|N_D^-(v_1) \cap Y_2| = |Y_2| - (y_2^* - 1)$ implies that $v_{\xi-2} \to v_1$, and $|N_D^+(v_\alpha) \cap W| = \alpha - \xi - 1$ implies that $v_\alpha \to v_\xi$. Then

$$v_{\xi-1} \to v_\xi \to \ldots \to v_\alpha \to v_\gamma \to v_{\gamma+1} \to \ldots \to v_{\xi-2} \to v_1 \to v_2 \to \ldots \to v_{\gamma-1}$$

and

$$v_{\xi+1} \to v_{\xi+2} \to \ldots \to v_\alpha \to v_\xi \to v_1 \to v_2 \to \ldots \to v_{\xi-1}$$

are longest directed paths in D satisfying that $v_{\xi-1}$ is the initial vertex of one path and the terminal vertex of the other. This contradicts Claim 8.

Therefore, the claim follows. $\square$

With the same discussion as the above, if either (i) or (ii) of Claim 17 occurs, we have a vertex $v \in V(P)$ such that v is the initial vertex of a longest directed path P_1 and the terminal vertex of another longest directed path P_2 with $V(P_1) = V(P_2) = V(P)$. This contradicts Claim 8.

This completes our proof of the theorem. $\square$

3 Remarks

Jackson [7] has demonstrated that Conjecture 3 holds true for every oriented path with one block. In this paper, we show that Conjecture 3 remains valid for every oriented path with two blocks. The natural question arises: Can we extend the confirmation of Conjecture 3 to oriented paths with three or more blocks using similar techniques.

References

[1] P. Camion, Chemins et circuits hamiltoniens des graphes complets, C. R. Acad. Sci. Paris. 249 (1959) 2151-2152

[2] B. Chen, X. Hou and X. Zhou, Long antipaths and anticycles in oriented graphs, Discrete Math. 348 (2025) 114412

[3] A. Ghouila-Houri, Une condition suffisante d'existence d'un circuit hamiltonien, C. R. Math. Acad. Sci. Paris. 251 (1960) 495-497

[4] B. Grünbaum, Antidirected Hamiltonian paths in tournaments, J. Comb. Theory Ser. B. 11 (1971) 249-257

[5] R. Häggkvist, Hamilton cycles in oriented graphs, Comb. Probab. Comput. 2 (1993) 25-32

[6] R. Häggkvist and A. Thomason, Oriented Hamilton cycles in oriented graphs, in: Combinatorics, Geometry and Probability, Cambridge University Press. (1997) 339-353

[7] B. Jackson, Long paths and cycles in oriented graphs, J. Graph Theory. 5(2) (1981) 145-157

[8] P. Keevash, D. Kühn and D. Osthus, An exact minimum degree condition for Hamilton cycles in oriented graphs, J. Lond. Math. Soc. 79 (2009) 144-166

[9] L. Kelly, Arbitrary orientations of Hamilton cycles in oriented graphs, Electron. J. Comb. 18(1) (2011) #P186

[10] L. Kelly, D. Kühn and D. Osthus, A Dirac type result on Hamilton cycles in oriented graphs, Comb. Probab. Comput. 17 (2008) 689-709

[11] T. Klimošová and M. Stein, Antipaths in oriented graphs, Discrete Math. 346 (2023) 113515

[12] J.W. Moon. On subtournaments of a tournament, Can. Math. Bull. 9 (1966) 297-301

[13] L. Rédei, Ein kombinatorischer Satz. Acta Litt. Szeged. 7 (1934) 39-43

[14] M. Rosenfeld, Antidirected Hamiltonian paths in tournaments, J. Comb. Theory Ser. B. 12 (1971) 93-99

[15] J. Skokan and M. Tyomkyn, Alternating paths in oriented graphs with large semidegree, arXiv:2406.03166.

[16] M. Stein, Tree containment and degree conditions, in: A.M. Raigorodskii, M.T. Rassias (Eds.), Discrete Mathematics and Applications, in: Springer Optimization and Its Applications, Springer. 165 (2020) 459-486

[17] M. Stein and C. Zárate-Guerén, Antidirected subgraphs of oriented graphs, Comb. Probab. Comput. 33 (2024) 446-466

[18] C. Thomassen, Long cycles in digraphs with constraints on the degrees, in: Surveys in Combinatorics, in: London Math. Soc. Lecture Notes, Cambridge University Press 38 (1979) 211-228

[19] D. Woodall, Sufficient conditions for cycles in digraphs, Proc. Lond. Math. Soc. 24 (1972) 739-755

[20] C. Zhang, Paths and cycles in oriented graphs, Kexue Tongbao. (English Ed.) 26(10) (1981) 865-868