Sketches, moves and partitions: counting regions of Catalan deformations of reflection arrangements

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Abstract

The collection of reflecting hyperplanes of a finite Coxeter group is called a reflection arrangement and it appears in many subareas of combinatorics and representation theory. We focus on the problem of counting regions of reflection arrangements and their deformations. Inspired by the recent work of Bernardi, we show that the notion of moves and sketches can be used to provide a uniform and explicit bijection between regions of (the Catalan deformation of) a reflection arrangement and certain non-nesting partitions. We then use the exponential formula to describe a statistic on these partitions such that distribution is given by the coefficients of the characteristic polynomial. Finally, we consider a sub-arrangement of type C arrangement called the threshold arrangement and its Catalan and Shi deformations.

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1 Introduction

A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of affine hyperplanes (i.e., codimension 1 subspaces and their translates) in $\mathbb{R}^n$. A *flat* of $\mathcal{A}$ is a nonempty intersection of some of the hyperplanes in $\mathcal{A}$; the ambient vector space is a flat since it is an intersection of no hyperplanes. Flats are naturally ordered by reverse set inclusion; the resulting poset is called the *intersection poset* and is denoted by $\mathrm{L}(\mathcal{A})$. The *rank* of $\mathcal{A}$ is the dimension of the span of the normal vectors to the hyperplanes. An arrangement in $\mathbb{R}^n$ is called *essential* if its rank is n. A *region* of $\mathcal{A}$ is a connected component of $\mathbb{R}^n \setminus \bigcup \mathcal{A}$. A region is said to be *bounded* if its intersection with the subspace spanned by the normal vectors to the hyperplanes is bounded. Counting the number of regions of arrangements using diverse combinatorial methods is an active area of research.

The *characteristic polynomial* of $\mathcal{A}$ is defined as $\chi_{\mathcal{A}}(t) := \sum \mu(\hat{0}, x)\, t^{\dim(x)}$ where x runs over all flats in $\mathrm{L}(\mathcal{A})$, μ is the Möbius function and $\hat{0}$ corresponds to the flat $\mathbb{R}^n$.

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Using the fact that every interval of the intersection poset of an arrangement is a geometric lattice, we have

$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^{n} (-1)^{n-i} c_i t^i \tag{1}$$

where c_i is a non-negative integer for all $0 \leqslant i \leqslant n$ [19, Corollary 3.4]. The characteristic polynomial is a fundamental combinatorial and topological invariant of the arrangement and plays a significant role throughout the theory of hyperplane arrangements.

In this article, our focus is on the enumerative aspects of (rational) arrangements in $\mathbb{R}^n$. In that direction we have the following seminal result by Zaslavsky.

Theorem 1 ([21]). *Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$. Then the number of regions of $\mathcal{A}$ is given by*

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1) = \sum_{i=0}^{n} c_i$$

and the number of bounded regions is given by

$$b(\mathcal{A}) = (-1)^{\mathrm{rank}(A)} \chi_{\mathcal{A}}(1).$$

The finite field method, developed by Athanasiadis [1], converts the computation of the characteristic polynomial to a point counting problem. A combination of these two results allowed for the computation of the number of regions of several arrangements of interest.

Another way to count the number of regions is to give a bijective proof. This approach involves finding a combinatorially defined set whose elements are in bijection with the regions of the given arrangement and are easier to count. For example, the *braid arrangement* in $\mathbb{R}^n$ is given by

$$\{x_i - x_j = 0 \mid 1 \leqslant i < j \leqslant n\}.$$

It is straightforward to verify that its regions correspond to the permutations of $[n]$. Hence the number of regions of the braid arrangement in $\mathbb{R}^n$ is $n!$.

The Catalan arrangement of type A in $\mathbb{R}^n$ is given by

$$\mathcal{A}_n = \{x_i - x_j = -1, 0, 1 \mid 1 \leqslant i < j \leqslant n\}.$$

This arrangement and its sub-arrangements have been studied in great detail (for example, see [6]). It is well-known that the number of regions of $\mathcal{A}_n$ where $x_1 < x_2 < \cdots < x_n$ (also known as the dominant regions) is given by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Using this, it is easy to see that

$$r(\mathcal{A}_n) = \frac{n!}{n+1}\binom{2n}{n}.$$

Let Φ be a (not necessarily reduced) crystallographic root system and let Φ^+ be a choice of positive roots. The reflection (or Coxeter) arrangement $\mathcal{A}(\Phi)$ corresponding to Φ consists of hyperplanes with the defining equations

$$(\alpha, x) = 0 \quad \text{for } \alpha \in \Phi^+.$$

Note that these are the same hyperplanes that are fixed by the Weyl group of Φ. A deformation of a reflection arrangement is an arrangement each of whose hyperplanes is parallel to some hyperplane in $\mathcal{A}(\Phi)$. Our main focus in the present paper is the Catalan deformation; for brevity we sometimes write Catalan arrangement of type Φ. The defining equations of Catalan arrangements of other types are as follows:

- The Catalan arrangement of type B in $\mathbb{R}^n$ is given by

$$\{x_i = -1, 0, 1 \mid i \in [n]\} \cup \{x_i + x_j = -1, 0, 1 \mid 1 \leqslant i < j \leqslant n\} \cup \mathcal{A}_n.$$

- The Catalan arrangement of type C in $\mathbb{R}^n$ is given by

$$\{2x_i = -1, 0, 1 \mid i \in [n]\} \cup \{x_i + x_j = -1, 0, 1 \mid 1 \leqslant i < j \leqslant n\} \cup \mathcal{A}_n.$$

- The Catalan arrangement of type D in $\mathbb{R}^n$ is given by

$$\{x_i + x_j = -1, 0, 1 \mid 1 \leqslant i < j \leqslant n\} \cup \mathcal{A}_n.$$

- The Catalan arrangement of type BC in $\mathbb{R}^n$ (defined in [3]) is the union of the type B and type C Catalan arrangements in $\mathbb{R}^n$.

In addition to these, we also consider 'type C extended Catalan arrangements'; consisting of hyperplanes of the form $(\alpha, x) = k$ for $k = -m, \dots, m$ for a fixed integer $m \geqslant 1$. The characteristic polynomials, and hence the number of regions, of these arrangements are known (for example, see [3]). We provide bijective proofs for the number of regions as well as bounded regions of these arrangements. Bijective proofs for the number of regions of the type C Catalan arrangement have already been established in [11] and [14]. However, the proofs we present for the other arrangements seem to be new.

The idea used for the bijections is fairly simple but effective. This was used by Bernardi in [6, Section 8] to obtain bijections for the regions of several deformations of the braid arrangement. This idea, that we call 'sketches and moves', is to consider an arrangement $\mathcal{B}$ whose regions we wish to count as a sub-arrangement of an arrangement $\mathcal{A}$. This is done in such a way that the regions of $\mathcal{A}$ are well-understood and are usually total orders on certain symbols. These total orders are what we call *sketches*. Since $\mathcal{B} \subseteq \mathcal{A}$, the regions of $\mathcal{B}$ partition the regions of $\mathcal{A}$ and hence define an equivalence on sketches. We define operations called *moves* on sketches to describe the equivalence classes. In regions of $\mathcal{A}$, moves correspond to crossing hyperplanes in $\mathcal{A} \setminus \mathcal{B}$.

Apart from Bernardi's results, the results in [4] and [16] can also be viewed as applications of the sketches and moves idea to count regions of hyperplane arrangements.

When studying an arrangement, another interesting question is whether the coefficients of its characteristic polynomial can be combinatorially interpreted. By Theorem 1, we know that the sum of the absolute values of the coefficients is the number of regions. Hence, one could ask if there is a statistic on the regions whose distribution is given by the coefficients of the characteristic polynomial. The characteristic polynomial of the braid arrangement in $\mathbb{R}^n$ is $t(t-1)\cdots(t-n+1)$ [19, Corollary 2.2]. Hence, the coefficients are the Stirling numbers of the first kind. Consequently, the distribution of the statistic 'number of cycles' on the set of permutations of $[n]$ (which correspond to the regions of the arrangement) is given by the coefficients of the characteristic polynomial.

The paper is structured as follows: In Section 2, we describe the sketches and moves idea mentioned above and use it to study the regions of some simple arrangements. In Section 3, we reprove the results in [14] about the type C Catalan arrangement with a modification inspired by [2]. We then use the sketches and moves idea in Section 4 to obtain bijections for the regions of the Catalan arrangements of other types. In Section 5, we describe statistics on the regions of the arrangements we have studied whose distribution is given by the corresponding characteristic polynomials. Finally, in Section 6, we use similar techniques to study an interesting arrangement called the threshold arrangement as well as some of its deformations.

2 Sketches, moves and trees: a quick overview of Bernardi's bijection and elementary examples

In his paper [6], Bernardi describes a method to count the regions of any deformation of the braid arrangement using certain objects called *boxed trees*. He also obtains explicit bijections with certain trees for several deformations. The general strategy to establish the bijection is to consider an arrangement $\mathcal{B}$ whose regions we wish to count as a sub-arrangement of an arrangement $\mathcal{A}$ whose regions are well-understood. The regions of $\mathcal{B}$ then define an equivalence on the regions of $\mathcal{A}$. This is done by declaring two regions of $\mathcal{A}$ to be equivalent if they lie inside the same region of $\mathcal{B}$. Now counting the number of regions of $\mathcal{B}$ is the same as counting the number of equivalence classes of this equivalence on the regions of $\mathcal{A}$. This is usually done by choosing a canonical representative for each equivalence class, which also gives a bijection between the regions of $\mathcal{B}$ and certain regions of $\mathcal{A}$.

In particular, a deformation of the braid arrangement is a sub-arrangement of the (extended or) m-Catalan arrangement (for some large m) in $\mathbb{R}^n$, whose hyperplanes are

$$\{x_i - x_j = k \mid 1 \leqslant i < j \leqslant n, k \in [-m, m]\}.$$

The regions of these arrangements are known to correspond to labeled $(m+1)$-ary trees with n nodes (see [6, Section 8.1]). Using the idea mentioned above, Bernardi showed that the regions of a large class of deformations of the braid arrangement, which he calls *transitive* deformations, correspond to certain trees. We should mention that while he obtains direct combinatorial arguments to describe this bijection for some transitive deformations (see [6, Section 8.2]), the proof for the general bijection uses much stronger results (see [6, Section 8.3]).

Coming back to the general strategy, which we aim to generalize in order to apply it to deformations of other types. It is clear that any two equivalent regions of $\mathcal{A}$ have to be on the same side of each hyperplane of $\mathcal{B}$. However, it turns out that this equivalence is the transitive closure of a simpler relation. This follows from the fact that one can reach a region in an arrangement from another by crossing exactly one hyperplane at a time with respect to which the regions lie on opposite sides.

The result mentioned above is well-known and such operations have been used by many authors (for example, see [10]). However, for the benefit of the reader, we reprove it here, for which we require the following definition.

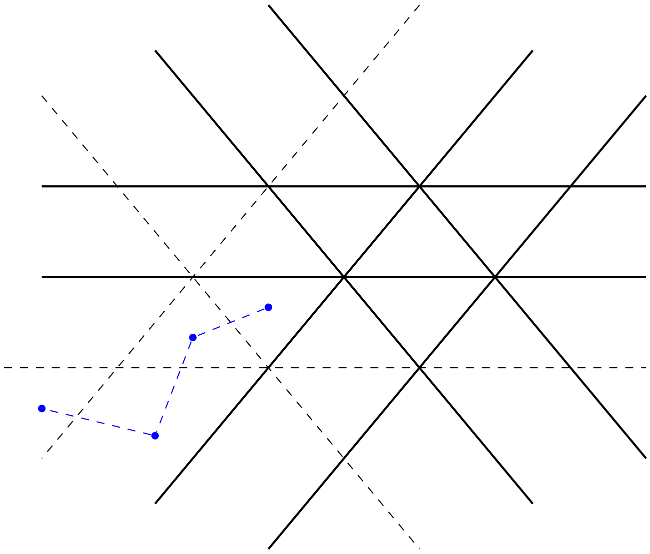


Figure 1: Bold lines form $\mathcal{B}$ and the dotted lines form $\mathcal{A} \setminus \mathcal{B}$. Equivalent $\mathcal{A}$ regions can be connected by changing one $\mathcal{A} \setminus \mathcal{B}$ inequality at a time.

Definition 2. Let R be a region of an arrangement $\mathcal{A}$. A *determining set* of R is a sub-arrangement $\mathcal{D} \subseteq \mathcal{A}$ such that the region of the arrangement $\mathcal{D}$ containing R, denoted $R_{\mathcal{D}}$, is equal to R.

Note that a region of $\mathcal{A}$ always has the entire arrangement $\mathcal{A}$ as a determining set. Also, if a region R' is on the same side as a region R for each hyperplane in a determining set of R, then we must have $R = R'$.

Before going forward, we explicitly describe regions of an arrangement. First note that any hyperplane H in $\mathbb{R}^n$ is a set of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) = 0\}$$

where $P_H(\boldsymbol{x}) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + c$ for some constants $a_1, \ldots, a_n, c \in \mathbb{R}$. Also, the regions of an arrangement $\mathcal{A}$ are precisely the non-empty intersections of sets of the form

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) > 0\} \text{ or } \{\boldsymbol{x} \in \mathbb{R}^n \mid P_H(\boldsymbol{x}) < 0\}$$

where we have one set for each $H \in \mathcal{A}$. Hence, crossing exactly one hyperplane H in an arrangement corresponds to changing the inequality chosen for H in this description of the region.

Theorem 3. *If $\mathcal{D}$ is a minimal determining set of a region R of an arrangement $\mathcal{A}$, then changing the inequality in the definition of R of exactly one $H \in \mathcal{D}$, and keeping all other inequalities of hyperplanes in $\mathcal{A}$ the same, describes a non-empty region of $\mathcal{A}$.*

Before proving this, we will see how it proves the fact mentioned above. Start with two distinct regions R and R' of an arrangement $\mathcal{A}$. We want to get from R to R' by crossing exactly one hyperplane at a time with respect to which the regions lie on opposite sides.

1. Let $\mathcal{D}$ be a minimal determining set of R.
2. Since $R \neq R'$ there is some $H \in \mathcal{D}$ for which R' is on the opposite side as R.
3. Change the inequality corresponding to H in R, call this new region R''.
4. The number of hyperplanes in $\mathcal{A}$ for which R'' and R' lie on opposite sides is less than that for R and R'.
5. Repeat this process to get to R' by changing one inequality at a time.

Proof of Theorem 3. Let $H \in \mathcal{D}$. Since $\mathcal{D}$ is a minimal determining set, $\mathcal{E} = \mathcal{D} \setminus \{H\}$ is not a determining set. So R is strictly contained in $R_{\mathcal{E}}$. This means that the hyperplane H intersects $R_{\mathcal{E}}$ and splits it into two open convex sets, one of which is R.

So we can choose a point $p \in H$ that lies inside $R_{\mathcal{E}}$ and an n-ball centered at p that does not touch any other hyperplanes of $\mathcal{A}$ (since $\mathcal{A}$ is finite). One half of the ball lies in R and the other half lies in a region R' of $\mathcal{A}$. Since R' can be reached from R by just crossing the hyperplane H, we get the required result. ☐

To sum up, we start with an arrangement $\mathcal{B} \subseteq \mathcal{A}$. We know the regions of $\mathcal{A}$ and usually represent them by combinatorial objects we call 'sketches'. We then define 'moves' on these sketches that correspond to changing exactly one inequality of a hyperplane in $\mathcal{A} \setminus \mathcal{B}$. We define sketches to be equivalent if one can be obtained from another through a series of moves. We then count the number of equivalence classes to obtain the number of regions of $\mathcal{B}$.

Before using this method to study the Catalan arrangements of various types, we first look at some simpler arrangements: sub-arrangements of the type C arrangement. These results are well-known and we reprove them here just to exhibit the 'sketches and moves' technique. Hence, in the spirit of Bernardi [6], we will define certain sketches corresponding to the region of the type C arrangement and for any sub-arrangement, we choose a canonical sketch from each region.

2.1 The type C arrangement

This arrangement in $\mathbb{R}^n$ is the set of reflecting hyperplanes of the root system C_n. The defining equations of hyperplanes are

$$\begin{aligned} 2x_i &= 0 \\ x_i + x_j &= 0 \\ x_i - x_j &= 0 \end{aligned}$$

for $1 \leqslant i < j \leqslant n$. Though we could write $x_i = 0$ for the first type of hyperplanes, we think of them as $x_i + x_i = 0$ to define sketches.

We can write the hyperplanes of the type C arrangement as follows:

$$\begin{aligned} x_i &= x_j, & 1 \leqslant i < j \leqslant n \\ x_i &= -x_j, & i, j \in [n]. \end{aligned}$$

Hence, any region of the arrangement is given by a *valid* total order on

$$x_1, \dots, x_n, -x_1, \dots, -x_n.$$

A total order is said to be valid if there is some point in $\mathbb{R}^n$ that satisfies it. We will represent x_i by $\overset{+}{i}$ and $-x_i$ by $\overset{-}{i}$ for all $i \in [n]$.

Example 4. The region $-x_2 < x_3 < x_1 < -x_1 < -x_3 < x_2$ is represented as $\overset{-}{2}\ \overset{+}{3}\ \overset{+}{1}\ \overset{-}{1}\ \overset{-}{3}\ \overset{+}{2}$.

It can be shown that words of the form

$$\overset{w_1}{i_1} \quad \overset{w_2}{i_2} \quad \cdots \quad \overset{w_n}{i_n} \quad \overset{-w_n}{i_n} \quad \cdots \quad \overset{-w_2}{i_2} \quad \overset{-w_1}{i_1}$$

where $\{i_1, \dots, i_n\} = [n]$ and each $w_i \in \{-, +\}$ are the ones that correspond to regions. Such orders are the only ones that can correspond to regions since negatives reverse order. Also, choosing n distinct negative numbers, it is easy to construct a point satisfying the inequalities specified by such a word. Hence the number of regions of the type C arrangement is $2^n n!$. We will call such words *sketches* (which are basically signed permutations). We will draw a line after the first n symbols to denote the reflection and call the part of the sketch before the line its first half and similarly define the second half.

Example 5. $\overset{+}{3}\ \overset{-}{1}\ \overset{-}{2}\ \overset{+}{4} \mid \overset{-}{4}\ \overset{+}{2}\ \overset{+}{1}\ \overset{-}{3}$ is a sketch.

We now study some sub-arrangements of the type C arrangement. For each such arrangement, we will define the moves that we can apply to the sketches (which represent changing exactly one inequality corresponding to a hyperplane not in the arrangement) and then choose a canonical representative from each equivalence class. By Theorem 3, this gives a bijection between these canonical sketches and the regions of the sub-arrangement.

2.2 The Boolean arrangement

One of the first examples one encounters when studying hyperplane arrangements is the Boolean arrangement. The Boolean arrangement in $\mathbb{R}^n$ has hyperplanes $x_i = 0$ for all $i \in [n]$. It is fairly straightforward to see that the number of regions is 2^n. We will do this using the idea of moves on sketches.

The hyperplanes missing from the type C arrangement in the Boolean arrangement are

$$\begin{aligned} x_i + x_j &= 0 \\ x_i - x_j &= 0 \end{aligned}$$

for $1 \leqslant i < j \leqslant n$. Hence, the Boolean moves are as follows:

1. Swapping adjacent $\overset{+}{i}$ and $\overset{-}{j}$ as well as $\overset{+}{j}$ and $\overset{-}{i}$ for distinct $i, j \in [n]$.
2. Swapping adjacent $\overset{+}{i}$ and $\overset{+}{j}$ as well as $\overset{-}{j}$ and $\overset{-}{i}$ for distinct $i, j \in [n]$.

The first kind of move corresponds to changing inequality corresponding to the hyperplane $x_i + x_j = 0$ and keeping all the other inequalities the same. Similarly, the second kind of move corresponds to changing only the inequality corresponding to $x_i - x_j = 0$.

Example 6. We can use a series of Boolean moves on a sketch as follows:

$$\overset{-}{4}\,\overset{+}{1}\,\overset{+}{2}\,\overset{-}{3} \mid \overset{+}{3}\,\overset{-}{2}\,\overset{-}{1}\,\overset{+}{4} \longrightarrow \overset{-}{4}\,\overset{+}{2}\,\overset{+}{1}\,\overset{-}{3} \mid \overset{+}{3}\,\overset{-}{1}\,\overset{-}{2}\,\overset{+}{4} \longrightarrow \overset{-}{4}\,\overset{+}{2}\,\overset{-}{3}\,\overset{+}{1} \mid \overset{-}{1}\,\overset{+}{3}\,\overset{-}{2}\,\overset{+}{4} \longrightarrow \overset{-}{4}\,\overset{-}{3}\,\overset{+}{2}\,\overset{+}{1} \mid \overset{-}{1}\,\overset{-}{2}\,\overset{+}{3}\,\overset{+}{4}$$

It can be shown that for any sketch, we can use Boolean moves to convert it to a sketch where the order of absolute values in the second half is $1, 2, \ldots, n$ (since adjacent transpositions generate the symmetric group). Also, since the signs of the numbers in the second half do not change there is exactly one such sketch in each equivalence class. Hence the number of Boolean regions is the number of ways of assigning signs to the numbers $1, 2, \ldots, n$ which is 2^n.

2.3 The type D arrangement

The type D arrangement in $\mathbb{R}^n$ has the hyperplanes

$$\begin{aligned} x_i + x_j &= 0 \\ x_i - x_j &= 0 \end{aligned}$$

for $1 \leqslant i < j \leqslant n$. The hyperplanes missing from missing from the type C arrangement are

$$2x_i = 0$$

for all $i \in [n]$. Hence a type D move, which we call a D move, is swapping adjacent $\overset{+}{i}$ and $\overset{-}{i}$ for any $i \in [n]$.

Example 7. $\overset{+}{4}\,\overset{+}{1}\,\overset{-}{3}\,\overset{+}{2} \mid \overset{-}{2}\,\overset{+}{3}\,\overset{-}{1}\,\overset{-}{4} \xrightarrow{D\ move} \overset{+}{4}\,\overset{+}{1}\,\overset{-}{3}\,\overset{-}{2} \mid \overset{+}{2}\,\overset{+}{3}\,\overset{-}{1}\,\overset{-}{4}$

In a sketch the only such pair is the last term of the first half and the first term of the second half. Hence D moves actually define an involution on the sketches. Hence the number of regions of the type D arrangement is $2^{n-1}n!$. We could also choose a canonical sketch in each type D region to be the one where the first term of the second half is positive.

2.4 The braid arrangement

The braid arrangement in $\mathbb{R}^n$ has hyperplanes

$$x_i - x_j = 0$$

for $1 \leqslant i < j \leqslant n$. The hyperplanes missing from the type C arrangement are

$$2x_i = 0$$
$$x_i + x_j = 0$$

for all $1 \leqslant i < j \leqslant n$. Hence the braid moves are as follows:

1. (D move) Swapping adjacent $\overset{+}{i}$ and $\overset{-}{i}$ for any $i \in [n]$.
2. Swapping adjacent $\overset{+}{i}$ and $\overset{-}{j}$ as well as $\overset{+}{j}$ and $\overset{-}{i}$ for distinct $i, j \in [n]$.

Example 8. We can use a series of braid moves on a sketch as follows:

$$\overset{-}{4}\,\overset{-}{1}\,\overset{+}{3}\,\overset{+}{2} \mid \overset{-}{2}\,\overset{-}{3}\,\overset{+}{1}\,\overset{+}{4} \longrightarrow \overset{-}{4}\,\overset{-}{1}\,\overset{+}{3}\,\overset{-}{2} \mid \overset{+}{2}\,\overset{-}{3}\,\overset{+}{1}\,\overset{+}{4} \longrightarrow \overset{-}{4}\,\overset{-}{1}\,\overset{-}{2}\,\overset{+}{3} \mid \overset{-}{3}\,\overset{+}{2}\,\overset{+}{1}\,\overset{+}{4} \longrightarrow \overset{-}{4}\,\overset{-}{1}\,\overset{-}{2}\,\overset{-}{3} \mid \overset{+}{3}\,\overset{+}{2}\,\overset{+}{1}\,\overset{+}{4}$$

Any sketch is braid equivalent to one where the signs of all the numbers in the second half are positive. This can be proved using induction on the number of positive terms in the first half of the sketch. Find the rightmost term in the first half that is positive. Moves of the second type can be used to take it to the last position in the first half of the sketch. Then a D move takes it to the second half of the sketch.

It can also be checked that braid moves do not change the order in which the positive terms appear in a sketch. This shows that there is a unique sketch in each braid equivalence class where all the terms in the second half are positive. Hence, the number of braid regions is the number of such sketches, which is $n!$.

Remark 9. The union of the braid and Boolean arrangements in $\mathbb{R}^n$ is just the essentialization of the braid arrangement in $\mathbb{R}^{n+1}$. However, using the idea of moves on sketches, one can show that the regions of this arrangement are in bijection with sketches where the second half is of the form

$$\overset{+}{i_1}\,\overset{+}{i_2}\,\cdots\,\overset{+}{i_k}\,\overset{-}{i_{k+1}}\,\cdots\,\overset{-}{i_n}$$

for some $k \in [0, n]$. This shows that the number of regions is $(n+1)!$.

We also study two other interesting sub-arrangements of the type C arrangement in Section 6.1.

3 Catalan deformation of type C

In this section we reprove, with a modification inspired by [2], the results of [14] about the regions of the type C Catalan arrangements.

Fix $n \geqslant 1$ throughout this section. The type C Catalan arrangement in $\mathbb{R}^n$ is the arrangement with hyperplanes

$$\begin{aligned} 2X_i &= -1, 0, 1 \\ X_i + X_j &= -1, 0, 1 \\ X_i - X_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. In this case, instead of looking at this arrangement directly, we will study the arrangement obtained by performing the translation $X_i = x_i + \frac{1}{2}$ for all $i \in [n]$. It is easy to see that this does not change the combinatorics of the arrangement. The translated arrangement, which we call $\mathcal{C}_n$, has hyperplanes

$$\begin{aligned} 2x_i &= -2, -1, 0 \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned} \tag{2}$$

for all $1 \leqslant i < j \leqslant n$. The arrangement $\mathcal{C}_n$ consists of all hyperplanes of the form $x_i + s = \pm(x_j + t)$ for $i, j \in [n]$ and $s, t \in \{0, 1\}$. This shows that the regions of $\mathcal{C}_n$ are given by valid total orders on

$$\{x_i + s \mid i \in [n],\ s \in \{0, 1\}\} \cup \{-x_i - s \mid i \in [n],\ s \in \{0, 1\}\}.$$

Such orders will be represented using the symbol $\alpha_i^{(s)}$ for $x_i + s$ and $\alpha_{-i}^{(-s)}$ for $-x_i - s$ for all $i \in [n]$ and $s \in \{0, 1\}$. Let $C(n)$ be the set

$$\{\alpha_i^{(s)} \mid i \in [n],\ s \in \{0, 1\}\} \cup \{\alpha_i^{(s)} \mid -i \in [n],\ s \in \{-1, 0\}\}.$$

Hence, we use orders on the letters of $C(n)$ to represent regions of $\mathcal{C}_n$.

Example 10. The total order

$$x_1 < -x_2 - 1 < x_1 + 1 < x_2 < -x_2 < -x_1 - 1 < x_2 + 1 < -x_1$$

is represented as $\alpha_1^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_1^{(1)}\ \alpha_2^{(0)}\ \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_2^{(1)}\ \alpha_{-1}^{(0)}$.

Considering $-x_i$ as x_{-i}, the letter $\alpha_i^{(s)}$ represents $x_i + s$ for any $\alpha_i^{(s)} \in C(n)$. For any $\alpha_i^{(s)} \in C(n)$, we use $\overline{\alpha_i^{(s)}}$ to represent the letter $\alpha_{-i}^{(-s)}$, which we call the *conjugate* of $\alpha_i^{(s)}$.

Definition 11. A *symmetric sketch* is an order on the letters in $C(n)$ such that the following hold for any $\alpha_i^{(s)}, \alpha_j^{(t)} \in C(n)$:

1. If $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$, then $\overline{\alpha_j^{(t)}}$ appears before $\overline{\alpha_i^{(s)}}$.

2. If $\alpha_i^{(s-1)}$ appears before $\alpha_j^{(t-1)}$, then $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$.

3. $\alpha_i^{(s-1)}$ appears before $\alpha_i^{(s)}$.

Proposition 12. *An order on the letters of $C(n)$ corresponds to a region of $\mathcal{C}_n$ if and only if it is a symmetric sketch.*

Proof. The idea of the proof is the same as that of [2, Lemma 5.2]. It is clear that any order that corresponds to a region must satisfy the properties in Definition 11 and hence be a symmetric sketch. For the converse, we show that there is a point in $\mathbb{R}^n$ satisfying the inequalities given by a symmetric sketch.

We prove this using induction on n, the case $n = 1$ being clear. Let $n \geqslant 2$ and w be a symmetric sketch. Without loss of generality, we can assume that the first letter of w is $\alpha_n^{(0)}$. Deleting the letters with subscript n and $-n$ from w gives a symmetric sketch w' in the letters $C(n-1)$. Using the induction hypothesis, we can choose a point $\mathbf{x}' \in \mathbb{R}^{n-1}$ satisfying the inequalities given by w'. Suppose the letter before $\alpha_n^{(1)}$ in w is $\alpha_i^{(s)}$ and the letter after it is $\alpha_j^{(t)}$. We choose $x_n \neq -1$ such that $x'_i + s < x_n + 1 < x'_j + t$ in such a way that $x_n + 1$ is also in the correct position with respect to 0 as specified by w. This is possible since $\mathbf{x}'$ satisfies w'.

We show that $(x'_1, \ldots, x'_{n-1}, x_n)$ satisfies the inequalities given by w. We only have to check that x_n and $(x_n + 1)$ are in the correct relative position with respect to the other letters since property (1) of Definition 11 will then show that $-x_n$ and $-x_n - 1$ are also in the correct relative position. By the choice of x_n, we see that $x_n + 1$ is in the correct position. We have to show that x_n is less than $\pm x'_i$ and $\pm(x'_i + 1)$ for all $i' \in [n-1]$. If $x_n > x'_1$, then $x_n + 1 > x'_1 + 1$ and since $x_n + 1$ satisfies the inequalities specified by w, $\alpha_1^{(1)}$ must be before $\alpha_n^{(1)}$ in w. But by property (2) of Definition 11, this means that $\alpha_1^{(0)}$ must be before $\alpha_n^{(0)}$ in w, which is a contradiction. The same logic can be used to show that $x_n < x'_i$ and $x_n < -x'_i - 1$ for all $i \in [n-1]$, which also gives us that $x_n < x'_i + 1$ and $x_n < -x'_i$ for all $i \in [n-1]$. □

We now derive some properties of symmetric sketches. A symmetric sketch has $4n$ letters, so we call the word made by the first $2n$ letters its first half. Similarly we define its second half.

Lemma 13. *The second half of a symmetric sketch is completely specified by its first half. In fact, it is the 'mirror' of the first half, i.e., it is the reverse of the first half with each letter replaced with its conjugate.*

Proof. For any symmetric sketch, the letter $\alpha_i^{(s)}$ is in the first half if and only if the letter $\overline{\alpha_i^{(s)}}$ is in the second half. This property can be proved as follows: Suppose there is a pair of conjugates in the first half of a symmetric sketch. Since conjugate pairs partition $C(n)$, this means that there is a pair of conjugates in the second half as well. But this would contradict property (1) of a symmetric sketch in Definition 11.

Hence, the set of letters in the second half are the conjugates of the letters in the first half. The order in which they appear is forced by property (1) of Definition 11, that is, the conjugates appear in the opposite order as the corresponding letters in the first half. So if the first half of a symmetric sketch is $a_1 \cdots a_{2n}$ where $a_i \in C(n)$ for all $i \in [2n]$, the sketch is

$$a_1 \quad a_2 \quad \cdots \quad a_{2n} \quad \overline{a_{2n}} \quad \cdots \quad \overline{a_2} \quad \overline{a_1}.$$

□

We draw a vertical line between the $2n^{th}$ and $(2n+1)^{th}$ letter in a symmetric sketch to indicate both the mirroring and the change in sign (note that if the $2n^{th}$ letter is $\alpha_i^{(s)}$, we have $x_i + s < 0 < -x_i - s$ in the corresponding region).

Example 14. $\alpha_{-3}^{(-1)}\ \alpha_{-3}^{(0)}\ \alpha_1^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_1^{(1)}\ \alpha_2^{(0)} \mid \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_2^{(1)}\ \alpha_{-1}^{(0)}\ \alpha_3^{(0)}\ \alpha_3^{(1)}$.

A letter in $C(n)$ is called an *α-letter* if it is of the form $\alpha_i^{(0)}$ or $\alpha_{-i}^{(-1)}$ where $i \in [n]$. The other letters are called *β-letters*. The β-letter 'corresponding' to an α-letter is the one with the same subscript. Hence, in a symmetric sketch, an α-letter always appears before its corresponding β-letter by property (3) in Definition 11. The order in which the subscripts of the α-letters appear is the same as the order in which the subscripts of the β-letters appear by property (2) of Definition 11. The proof of the following lemma is very similar to that of the previous lemma.

Lemma 15. *The order in which the subscripts of the α-letters in a symmetric sketch appear is of the form*

$$i_1 \quad i_2 \quad \cdots \quad i_n \quad -i_n \quad \cdots \quad -i_2 \quad -i_1$$

where $\{|i_1|, \ldots, |i_n|\} = [n]$.

Using Lemmas 13 and 15, to specify the sketch, we only need to specify the following:

1. The α, β-word corresponding to the first half.

2. The signed permutation given by the first n α-letters.

The α, β-word corresponding to the first half is a word of length $2n$ in the letters $\{\alpha, \beta\}$ such that the i^{th} letter is an α if and only if the i^{th} letter of the symmetric sketch is an α-letter.

There is at most one sketch corresponding to a pair of an α, β-word and a signed permutation. This is because the signed permutation tells us, by Lemma 15, the order in which the subscripts of the α-letters (and hence β-letters) appears. Using this and the α, β-word, we can construct the first half and, by Lemma 13, the entire sketch.

Example 16. To the symmetric sketch

$$\alpha_{-3}^{(-1)}\ \alpha_{-3}^{(0)}\ \alpha_1^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_1^{(1)}\ \alpha_2^{(0)} \mid \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_2^{(1)}\ \alpha_{-1}^{(0)}\ \alpha_3^{(0)}\ \alpha_3^{(1)}$$

we associate the pair consisting of the following:

1. α, β-word: $\alpha\beta\alpha\alpha\beta\alpha$.

2. Signed permutation: $-3 \quad 1 \quad -2$.

If we are given the α, β-word and signed permutation above, the unique sketch corresponding to it is the one given above.

The next proposition characterizes the pairs of α, β-words and signed permutations that correspond to symmetric sketches.

Proposition 17. *A pair consisting of*

1. *an α, β-word of length $2n$ such that any prefix of the word has at least as many α-letters as β-letters and*

2. *any signed permutation*

corresponds to a symmetric sketch and all symmetric sketches correspond to such pairs.

Proof. By property (3) of Definition 11, any α, β-word corresponding to the first half of a sketch should have at least as many α-letters as β-letters in any prefix.

We now prove that given such a pair, there is a symmetric sketch corresponding to it. If the given α, β-word is $l_1 l_2 \cdots l_{2n}$ and the given signed permutation is $i_1 i_2 \cdots i_n$, we construct the symmetric sketch as follows:

1. Extend the α, β-word to the one of length $4n$ given by

$$l_1 \quad l_2 \quad \cdots \quad l_{2n} \quad \overline{l_{2n}} \quad \cdots \quad \overline{l_2} \quad \overline{l_1}$$

where $\overline{l_i} = \alpha$ if and only if $l_i = \beta$ for all $i \in [2n]$.

2. Extend the signed permutation to the sequence of length $2n$ given by

$$i_1 \quad i_2 \quad \cdots \quad i_n \quad -i_n \quad \cdots \quad -i_2 \quad -i_1.$$

3. Label the subscripts of the α-letters of the extended α, β-word in the order given by the extended signed permutation and similarly label the β-letters.

If we show that the word constructed is a symmetric sketch, it is clear that it will correspond to the given α, β-word and signed permutation. We have to check that the constructed word satisfies the properties in Definition 11.

The way the word was constructed, we see that it is of the form

$$a_1 \quad a_2 \quad \cdots \quad a_{2n} \quad \overline{a_{2n}} \quad \cdots \quad \overline{a_2} \quad \overline{a_1}$$

where $a_i \in C(n)$ for all $i \in [2n]$. Since the conjugate of the i^{th} α is the $(2n-i+1)^{th}$ β and vice-versa, the first half of the word cannot have a pair of conjugates. Hence the word has all letters of $C(n)$. This shows that property (1) of Definition 11 holds. Property (2) is taken care of since, by construction, the subscripts of the α-letters appear in the same order as those of the β-letters.

To show that property (3) holds, it suffices to show that any prefix of the word has at least as many α-letters as β-letters. This is already true for the first half. To show that this is true for the entire word, we consider α as $+1$ and β as -1. Hence, the condition is that any prefix has a non-negative sum. Since any prefix of size greater than $2n$ is of the form

$$l_1 \quad l_2 \quad \cdots \quad l_{2n} \quad \overline{l_{2n}} \quad \cdots \quad \overline{l_k}$$

for some $k \in [2n]$, the sum is $l_1 + \cdots + l_{k-1} \geqslant 0$. So property (3) holds as well and hence the constructed word is a symmetric sketch. □

We use this description to count symmetric sketches.

Lemma 18. *The number of α, β-words of length $2n$ having at least as many α-letters as β-letters in any prefix is $\binom{2n}{n}$.*

Proof. We consider these α, β-words as lattice paths. Using the step $U = (1, 1)$ for α and the step $D = (1, -1)$ for β, we have to count those lattice paths with each step U or D that start at the origin, have $2n$ steps, and never fall below the x-axis.

Using the reflection principle (for example, see [12]), we get that the number of such lattice paths that end at $(2n, 2k)$ for $k \in [0, n]$ is given by

$$\binom{2n}{n+k} - \binom{2n}{n+k+1}.$$

Here, $\binom{2n}{n+k}$ counts all paths with steps U, D from $(0, 0)$ to $(2n, 2k)$ and $\binom{2n}{n+k+1}$ counts paths from $(0, -2)$ to $(2n, 2k)$. The (telescoping) sum over $k \in [0, n]$ gives the required result. □

The above lemma and Proposition 17 immediately give the following.

Theorem 19. *The number of symmetric sketches and hence regions of $\mathcal{C}_n$ is*

$$2^n n! \binom{2n}{n}.$$

In [2], Athanasiadis obtains bijections between several classes of non-nesting partitions and regions of certain arrangements. We will mention the one for the arrangement $\mathcal{C}_n$, which gives a bijection between the α, β-words associated to symmetric sketches and certain non-nesting partitions.

Definition 20. A *symmetric non-nesting partition* is a partition of $[-2n, 2n] \setminus \{0\}$ such that the following hold:

1. Each block is of size 2.
2. If $B = \{a, b\}$ is a block, so is $-B = \{-a, -b\}$.

3. If $\{a, b\}$ is a block and $c, d \in [-2n, 2n] \setminus \{0\}$ are such that $a < c < d < b$, then $\{c, d\}$ is not a block.

Symmetric non-nesting partitions are usually represented using arc-diagrams. This is done by using $4n$ dots to represent the numbers in $[-2n, 2n] \setminus \{0\}$ in order and joining dots in the same block using an arc. The properties of these partitions imply that there are no nesting arcs and that the diagram is symmetric, which we represent by drawing a line after $2n$ dots.

Example 21. The arc diagram associated to the symmetric non-nesting partition of $[-6, 6] \setminus \{0\}$

$$\{-6, -3\}, \{-5, -1\}, \{-4, 2\}, \{-2, 4\}, \{1, 5\}, \{3, 6\}$$

is given in Figure 2.

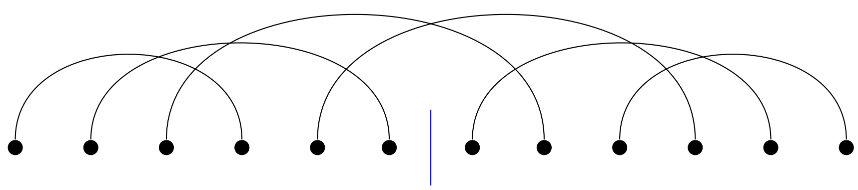


Figure 2: The symmetric non-nesting partition of Example 21.

It can also be seen that there are exactly n pairs of blocks of the form $\{B, -B\}$ with no block containing both a number and its negative. Also, the first n blocks, with blocks being read in order of the smallest element in it, do not have a pair of the form $\{B, -B\}$. Hence, we can label the first n blocks with a signed permutation and label the block $-B$ with the negative of the label of B to obtain a labeling of all blocks. We call such objects *labeled symmetric non-nesting partitions*. In the arc diagram, the labeling is done by replacing the dots representing the elements in a block with its label.

We can obtain a labeled symmetric non-nesting partition from a symmetric sketch by joining the letters $\alpha_i^{(0)}$ and $\alpha_i^{(1)}$ and similarly $\alpha_{-i}^{(-1)}$ and $\alpha_{-i}^{(0)}$ with arcs and replacing each letter in the sketch with its subscript. It can be shown that this construction is a bijection between symmetric sketches and labeled symmetric non-nesting partitions. In particular, the α, β-words associated with symmetric sketches are in bijection with symmetric non-nesting partitions.

Example 22. To the symmetric sketch

$$\alpha_3^{(0)}\alpha_2^{(0)}\alpha_{-1}^{(-1)}\alpha_3^{(1)}\alpha_1^{(0)}\alpha_2^{(1)} | \alpha_{-2}^{(-1)}\alpha_{-1}^{(0)}\alpha_{-3}^{(-1)}\alpha_1^{(1)}\alpha_{-2}^{(0)}\alpha_{-3}^{(0)}$$

we associate the labeled symmetric non-nesting partition in Figure 3.

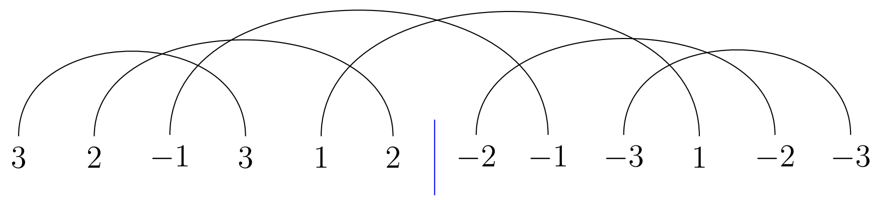


Figure 3: Arc diagram associated to the symmetric sketch in Example 22.

We now describe another way to represent the regions. We have already seen that a sketch corresponds to a pair consisting of an α, β-word and a signed permutation. We represent the α, β-word as a lattice path just as we did in the proof of Lemma 18. We specify the signed permutation by labeling the first n up-steps of the lattice path.

Example 23. The lattice path associated to the symmetric sketch

$$\alpha_{-3}^{(-1)}\ \alpha_{-3}^{(0)}\ \alpha_{1}^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha_{1}^{(1)}\ \alpha_{2}^{(0)}\ \mid\ \alpha_{-2}^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha_{2}^{(1)}\ \alpha_{-1}^{(0)}\ \alpha_{3}^{(0)}\ \alpha_{3}^{(1)}$$

is given in Figure 4.

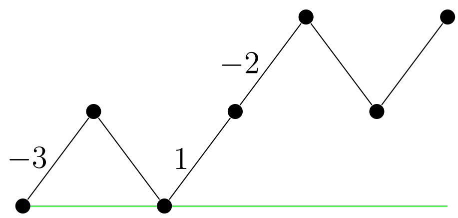


Figure 4: Lattice path associated to the symmetric sketch in Example 23.

These representations for the regions of $\mathcal{C}_n$ also allow us to determine and count which regions are bounded.

Theorem 24. *The number of bounded regions of the arrangement $\mathcal{C}_n$ is*

$$2^n n! \binom{2n-1}{n}.$$

Proof. First note that the arrangement $\mathcal{C}_n$ has rank n and is hence essential. From the bijection defined above, it can be seen that the arc diagram associated to any region R of $\mathcal{C}_n$ can be obtained by plotting a point $(x_1, \ldots, x_n) \in R$ on the real line. This is done by marking x_i and $x_i + 1$ on the real line using i for all $i \in [n]$ and then joining them with an arc and similarly marking $-x_i - 1$ and $-x_i$ using $-i$ and joining them with an arc.

This can be used to show that a region of $\mathcal{C}_n$ is bounded if and only if the arc diagram is 'interlinked'. For example, Figure 3 shows an arc diagram that is interlinked and Figure 5 shows one that is not. In terms of lattice paths, the bounded regions are those whose corresponding lattice path never touches the x-axis except at the origin.

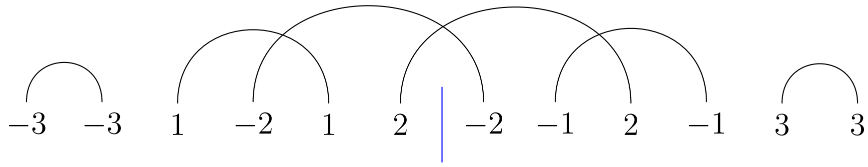


Figure 5: Arc diagram associated to the symmetric sketch of Example 16.

This shows that the number of bounded regions of $\mathcal{C}_n$ is $2^n n!$ times the number of unlabeled lattice paths of length $2n$ that never touch the x-axis except at the origin. Deleting the first step (which is necessarily an up-step) gives a bijection between such paths and those of length $2n-1$ that never fall below the x-axis. Using the same idea as in the proof of Lemma 18, it can be checked that the number of such paths is $\binom{2n-1}{n}$. This proves the required result. $\square$

Remark 25. In [14], the authors study the type C Catalan arrangement directly, i.e., without using the translation $\mathcal{C}_n$ mentioned above. Hence, using the same logic, they use orders on the letters

$$\{\alpha_i^{(s)} \mid i \in [-n,n] \setminus \{0\},\ s \in \{0,1\}\}$$

to represent the regions of the type C Catalan arrangement. They claim that these orders are those such that the following hold for any $i, j \in [-n,n] \setminus \{0\}$ and $s \in \{0,1\}$:

1. If $\alpha_i^{(0)}$ appear before $\alpha_j^{(0)}$, then $\alpha_i^{(1)}$ appears before $\alpha_j^{(1)}$.

2. $\alpha_i^{(0)}$ appears before $\alpha_i^{(1)}$.

3. If $\alpha_i^{(0)}$ appears before $\alpha_j^{(s)}$, then $\alpha_{-j}^{(0)}$ appears before $\alpha_{-i}^{(s)}$.

Though this can be shown to be true, the method used in [14] to construct a point satisfying the inequalities given by such an order does not seem to work in general. We describe their method and then exhibit a case where it does not work.

Let $w = w_1 \cdots w_{4n}$ be an order satisfying the properties given above. Then construct $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as follows: Let $z_0 = 0$ (or pick z_0 arbitrarily). Then define z_p for $p = 1, 2, \ldots, 4n$ in order as follows: If $w_p = \alpha_i^{(0)}$ then set $z_p = z_{p-1} + \frac{1}{2n+1}$ and $x_i = z_p$, and if $w_p = \alpha_i^{(1)}$ then set $z_p = x_i + 1$. Here we consider $x_{-i} = -x_i$ for any $i \in [n]$. Then $\boldsymbol{x}$ satisfies the inequalities given by w.

The following example shows that this method does not always work; in fact $\boldsymbol{x}$ is not always well-defined. Consider the order $w = \alpha_{-2}^{(0)}\alpha_1^{(0)}\alpha_{-2}^{(1)}\alpha_1^{(1)}\alpha_{-1}^{(0)}\alpha_2^{(0)}\alpha_{-1}^{(1)}\alpha_2^{(1)}$. Following the above procedure, we would get that x_1 is both $\frac{2}{5}$ as well as $-1-\frac{3}{5}$.

3.1 Extended type C Catalan

Fix $m, n \geqslant 1$. The type C m-Catalan arrangement in $\mathbb{R}^n$ has hyperplanes

$$\begin{aligned} 2X_i &= 0, \pm 1, \pm 2, \dots, \pm m \\ X_i + X_j &= 0, \pm 1, \pm 2, \dots, \pm m \\ X_i - X_j &= 0, \pm 1, \pm 2, \dots, \pm m \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. We will study the arrangement obtained by performing the translation $X_i = x_i + \frac{m}{2}$ for all $i \in [n]$. The translated arrangement, which we call $\mathcal{C}_n^{(m)}$, has hyperplanes

$$\begin{aligned} 2x_i &= -2m, -2m+1, \dots, 0 \\ x_i + x_j &= -2m, -2m+1, \dots, 0 \\ x_i - x_j &= 0, \pm 1, \pm 2, \dots, \pm m \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. Note that $\mathcal{C}_n = \mathcal{C}_n^{(1)}$. The regions of $\mathcal{C}_n^{(m)}$ are given by valid total orders on

$$\{x_i + s \mid i \in [n],\ s \in [0, m]\} \cup \{-x_i - s \mid i \in [n],\ s \in [0, m]\}.$$

Just as we did for $\mathcal{C}_n$, such orders will be represented by using the symbol $\alpha_i^{(s)}$ for $x_i + s$ and $\alpha_{-i}^{(-s)}$ for $-x_i - s$ for all $i \in [n]$ and $s \in [0, m]$. Let $C^{(m)}(n)$ be the set

$$\{\alpha_i^{(s)} \mid i \in [n],\ s \in [0, m]\} \cup \{\alpha_i^{(s)} \mid -i \in [n],\ s \in [-m, 0]\}.$$

For any $\alpha_i^{(s)} \in C^{(m)}(n)$, $\overline{\alpha_i^{(s)}}$ represents $\alpha_{-i}^{(-s)}$ and is called the conjugate of $\alpha_i^{(s)}$. Letters of the form $\alpha_i^{(0)}$ or $\alpha_{-i}^{(-m)}$ for any $i \in [n]$ are called α-letters. The others are called β-letters.

Definition 26. An order on the letters in $C^{(m)}(n)$ is called a *symmetric m-sketch* if the following hold for all $\alpha_i^{(s)}, \alpha_j^{(t)} \in C^{(m)}(n)$:

1. If $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$, then $\overline{\alpha_j^{(t)}}$ appears before $\overline{\alpha_i^{(s)}}$.
2. If $\alpha_i^{(s-1)}$ appears before $\alpha_j^{(t-1)}$, then $\alpha_i^{(s)}$ appears before $\alpha_j^{(t)}$.
3. $\alpha_i^{(s-1)}$ appears before $\alpha_i^{(s)}$.

The following result can be proved just as Proposition 12.

Proposition 27. *An order on the letters in $C^{(m)}(n)$ corresponds to a region of $\mathcal{C}_n^{(m)}$ if and only if it is a symmetric m-sketch.*

Similar to Lemma 15, it can be shown that the order in which the subscripts of the α-letters appear in a symmetric m-sketch is of the form

$$i_1 \quad i_2 \quad \cdots \quad i_n \quad -i_n \quad \cdots \quad -i_2 \quad -i_1$$

where $\{|i_1|, \dots, |i_n|\} = [n]$. Just as in the case of symmetric sketches, we associate an α, β-word and signed permutation to a symmetric m-sketch which completely determines it.

Example 28. To the symmetric 2-sketch

$$\alpha_2^{(0)}\alpha_{-1}^{(-2)}\alpha_2^{(1)}\alpha_{-1}^{(-1)}\alpha_1^{(0)}\alpha_{-2}^{(-2)} \mid \alpha_2^{(2)}\alpha_{-1}^{(0)}\alpha_1^{(1)}\alpha_{-2}^{(-1)}\alpha_1^{(2)}\alpha_{-2}^{(0)}$$

we associate the pair consisting of the following:

1. α, β-word: $\alpha\alpha\beta\beta\alpha\alpha$.
2. Signed permutation: $2 \quad -1$.

The set of α, β-words associated to symmetric m-sketches for $m > 1$ does not seem to have a simple characterization like those for symmetric sketches (see Proposition 17). However, looking at symmetric m-sketches as labeled non-nesting partitions as done in [2], we see that such objects have already been counted bijectively (refer [11]).

Definition 29. A *symmetric m-non-nesting partition* is a partition of $[-(m+1)n, (m+1)n] \setminus \{0\}$ such that the following hold:

1. Each block is of size $(m+1)$.
2. If B is a block, so is $-B$.
3. If a, b are in some block B, $a < b$ and there is no number $a < c < b$ such that $c \in B$, then if $a < c < d < b$, c and d are not in the same block.

Just as we did for the $m = 1$ case, we can obtain a labeled symmetric m-non-nesting partition from a symmetric m-sketch by joining the letters $\alpha_i^{(0)}, \alpha_i^{(1)}, \ldots, \alpha_i^{(m)}$ and similarly $\alpha_{-i}^{(-m)}, \alpha_{-i}^{(-m+1)}, \ldots, \alpha_{-i}^{(0)}$ with arcs and labeling each such chain with the subscript of the letters being joined.

Example 30. To the symmetric 2-sketch in Example 28, we associate the labeled 2-non-nesting partition of Figure 6.

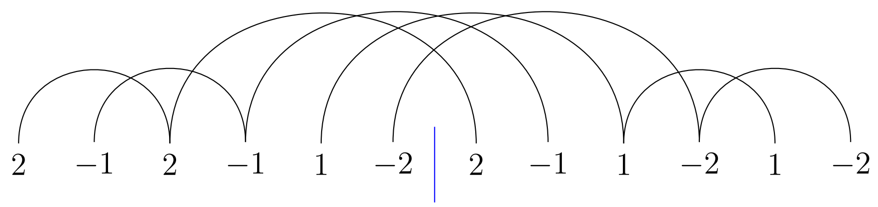


Figure 6: A labeled 2-non-nesting partition

The number of various classes of non-nesting partitions have been counted bijectively. In terms of [11] or [2], the symmetric m-non-nesting partitions defined above are called type C partitions of size $(m+1)n$ of type $(m+1, \ldots, m+1)$ where this is an n-tuple representing the size of the (nonzero) block pairs $\{B, -B\}$. The number of such partitions is

$$\binom{(m+1)n}{n}.$$

Hence we get the following theorem.

Theorem 31. *The number of symmetric m-sketches, which is the number of regions of $\mathcal{C}_n^{(m)}$ is*

$$2^n n!\binom{(m+1)n}{n}.$$

4 Catalan deformations of other types

We will now use 'sketches and moves', as in [6], to count the regions of Catalan arrangements of other types. Depending on the context, we represent the regions of arrangements using sketches, arc diagrams, or lattice paths and frequently make use of the bijections identifying them. We usually use sketches to define moves and use arc diagrams and lattice paths to count regions as well as bounded regions.

4.1 Type D Catalan

Fix $n \geqslant 2$. The type D Catalan arrangement in $\mathbb{R}^n$ has hyperplanes

$$X_i + X_j = -1, 0, 1$$
$$X_i - X_j = -1, 0, 1$$

for $1 \leqslant i < j \leqslant n$. Translating this arrangement by setting $X_i = x_i + \frac{1}{2}$ for all $i \in [n]$, we get the arrangement $\mathcal{D}_n$ with hyperplanes

$$x_i + x_j = -2, -1, 0$$
$$x_i - x_j = -1, 0, 1$$

for $1 \leqslant i < j \leqslant n$. Figure 7 shows $\mathcal{D}_2$ as a sub-arrangement of $\mathcal{C}_2$. It also shows how the regions of $\mathcal{D}_2$ partition the regions of $\mathcal{C}_2$.

We use the idea of moves to count the regions of $\mathcal{D}_n$ by considering it as a sub-arrangement of $\mathcal{C}_n$. The hyperplanes from $\mathcal{C}_n$ that are missing in $\mathcal{D}_n$ are

$$2x_i = -2, -1, 0$$

for all $i \in [n]$. Hence, the type D Catalan moves on symmetric sketches (regions of $\mathcal{C}_n$), which we call $\mathcal{D}$ moves, are as follows:

1. Swapping the $2n^{th}$ and $(2n+1)^{th}$ letter.
2. Swapping the n^{th} and $(n+1)^{th}$ α-letters if they are adjacent, along with the n^{th} and $(n+1)^{th}$ β-letters.

The first move covers the inequalities corresponding to the hyperplanes $x_i+1 = -x_i-1$ and $x_i = -x_i$ for all $i \in [n]$ since the only conjugates that are adjacent, by Lemma 13, are the $2n^{th}$ and $(2n+1)^{th}$ letter.

The second move covers the inequalities corresponding to the hyperplanes $x_i = -x_i-1$ (equivalently, $x_i + 1 = -x_i$) for all $i \in [n]$. This is due to the fact that the only way $\alpha_i^{(0)}$

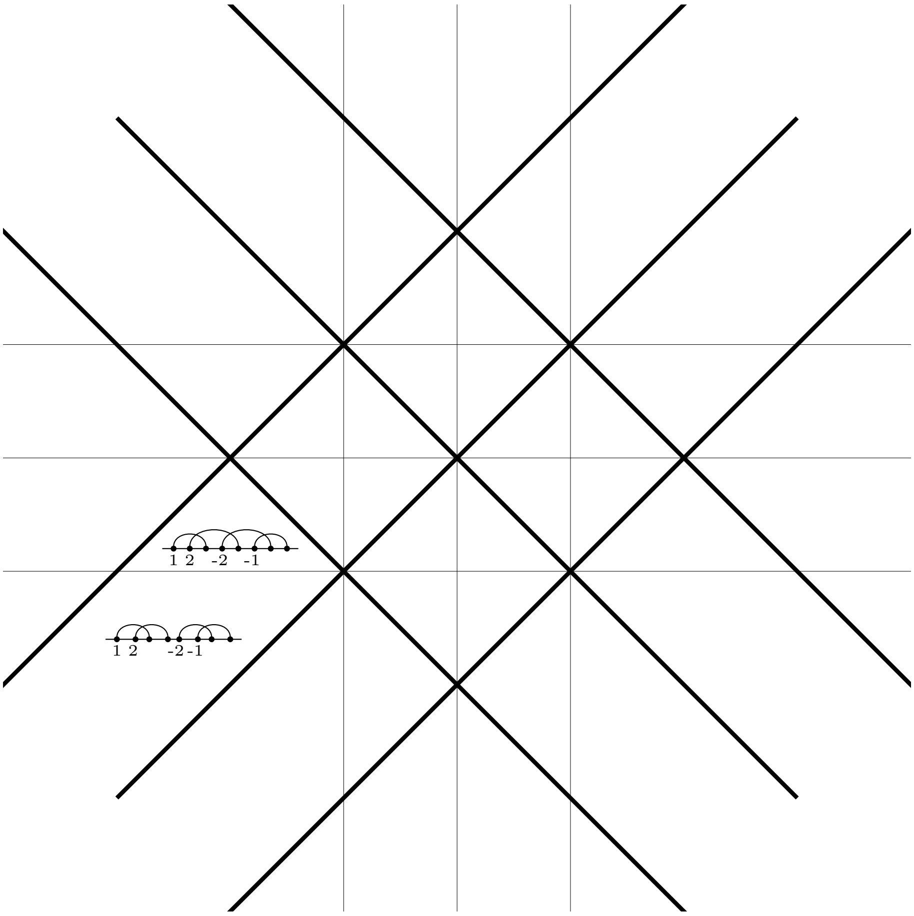


Figure 7: The arrangement $\mathcal{C}_2$ with the hyperplanes in $\mathcal{D}_2$ in bold. Two regions of $\mathcal{C}_2$ are labeled with their symmetric labeled non-nesting partition.

and $\alpha_{-i}^{(-1)}$ as well as $\alpha_i^{(1)}$ and $\alpha_{-i}^{(0)}$ can be adjacent is, by Lemma 15, when the n^{th} and $(n+1)^{th}$ α-letters are adjacent. Also, by Lemma 13, the n^{th} and $(n+1)^{th}$ α-letters are adjacent if and only if the n^{th} and $(n+1)^{th}$ β-letters are adjacent.

Example 32. A series of $\mathcal{D}$ moves applied to a symmetric sketch is given below:

$$\begin{aligned}
&\alpha_{-1}^{(-1)}\alpha_{2}^{(0)}\alpha_{-2}^{(-1)}\alpha_{-1}^{(0)} \mid \alpha_{1}^{(0)}\alpha_{2}^{(1)}\alpha_{-2}^{(0)}\alpha_{1}^{(1)}\\
\xrightarrow{\mathcal{D}\text{ move}}\ &\alpha_{-1}^{(-1)}\alpha_{2}^{(0)}\alpha_{-2}^{(-1)}\alpha_{1}^{(0)} \mid \alpha_{-1}^{(0)}\alpha_{2}^{(1)}\alpha_{-2}^{(0)}\alpha_{1}^{(1)}\\
\xrightarrow{\mathcal{D}\text{ move}}\ &\alpha_{-1}^{(-1)}\alpha_{-2}^{(-1)}\alpha_{2}^{(0)}\alpha_{1}^{(0)} \mid \alpha_{-1}^{(0)}\alpha_{-2}^{(0)}\alpha_{2}^{(1)}\alpha_{1}^{(1)}\\
\xrightarrow{\mathcal{D}\text{ move}}\ &\alpha_{-1}^{(-1)}\alpha_{-2}^{(-1)}\alpha_{2}^{(0)}\alpha_{-1}^{(0)} \mid \alpha_{1}^{(0)}\alpha_{-2}^{(0)}\alpha_{2}^{(1)}\alpha_{1}^{(1)}
\end{aligned}$$

To count the regions of $\mathcal{D}_n$, we have to count the number of equivalence classes of symmetric sketches where two sketches are equivalent if one can be obtained from the other via a series of $\mathcal{D}$ moves. In Figure 7, the two labeled regions of $\mathcal{C}_2$ are adjacent and lie in the same region of $\mathcal{D}_2$. They are related by swapping of the fourth and fifth letters of their sketches, which is a $\mathcal{D}$ move.

The fact about these moves that will help with the count is that a series of $\mathcal{D}$ moves do not change the sketch too much. Hence we can list the sketches that are $\mathcal{D}$ equivalent to a given sketch.

First, consider the case when the n^{th} α-letter of the symmetric sketch is not in the $(2n-1)^{th}$ position. In this case, the n^{th} α-letter is far enough from the $2n^{th}$ letter that a $\mathcal{D}$ move of the first kind (swapping the $2n^{th}$ and $(2n+1)^{th}$ letter) will not affect the letter after the n^{th} α-letter. Hence it does not change whether the n^{th} and $(n+1)^{th}$ α-letters are adjacent.

Let w be a sketch where the n^{th} α-letter is not in the $(2n-1)^{th}$ position. The number of sketches $\mathcal{D}$ equivalent to w is 4 when the n^{th} and $(n+1)^{th}$ α-letters are adjacent. They are illustrated below:

$$\begin{gathered}
\cdots \alpha_{-i}^{(-1)}\alpha_{i}^{(0)} \cdots \alpha_{j}^{(s)} \mid \alpha_{-j}^{(-s)} \cdots \alpha_{-i}^{(0)}\alpha_{i}^{(1)} \cdots \\
\cdots \alpha_{-i}^{(-1)}\alpha_{i}^{(0)} \cdots \alpha_{-j}^{(-s)} \mid \alpha_{j}^{(s)} \cdots \alpha_{-i}^{(0)}\alpha_{i}^{(1)} \cdots \\
\cdots \alpha_{i}^{(0)}\alpha_{-i}^{(-1)} \cdots \alpha_{j}^{(s)} \mid \alpha_{-j}^{(-s)} \cdots \alpha_{i}^{(1)}\alpha_{-i}^{(0)} \cdots \\
\cdots \alpha_{i}^{(0)}\alpha_{-i}^{(-1)} \cdots \alpha_{-j}^{(-s)} \mid \alpha_{j}^{(s)} \cdots \alpha_{i}^{(1)}\alpha_{-i}^{(0)} \cdots
\end{gathered}$$

The number of sketches $\mathcal{D}$ equivalent to w is 2 when the n^{th} and $(n+1)^{th}$ α-letter are not adjacent. They are illustrated below:

$$\cdots \alpha_{j}^{(s)} \mid \alpha_{-j}^{(-s)} \cdots \qquad \cdots \alpha_{-j}^{(-s)} \mid \alpha_{j}^{(s)} \cdots$$

Notice also that the equivalent sketches also satisfy the same properties (n^{th} α-letter not being in the $(2n-1)^{th}$ position and whether the n^{th} and $(n+1)^{th}$ α-letters are adjacent).

In case the n^{th} α-letter is in the $(2n-1)^{th}$ position of the symmetric sketch, it can be checked that it has exactly 4 equivalent sketches all of which also have the n^{th} α-letter in the $(2n-1)^{th}$ position:

$$\begin{gathered}
\cdots \alpha_{i}^{(0)}\alpha_{i}^{(1)} \mid \alpha_{-i}^{(-1)}\alpha_{-i}^{(0)} \cdots \\
\cdots \alpha_{i}^{(0)}\alpha_{-i}^{(-1)} \mid \alpha_{i}^{(1)}\alpha_{-i}^{(0)} \cdots \\
\cdots \alpha_{-i}^{(-1)}\alpha_{i}^{(0)} \mid \alpha_{-i}^{(0)}\alpha_{i}^{(1)} \cdots \\
\cdots \alpha_{-i}^{(-1)}\alpha_{-i}^{(0)} \mid \alpha_{i}^{(0)}\alpha_{i}^{(1)} \cdots
\end{gathered}$$

Figure 7 shows that each region of $\mathcal{D}_2$ contains exactly 2 or 4 regions of $\mathcal{C}_2$, as expected from the above observations.

Theorem 33. *The number of $\mathcal{D}$ equivalence classes on symmetric sketches and hence the number of regions of $\mathcal{D}_n$ is*

$$2^{n-1} \cdot \frac{(2n-2)!}{(n-1)!} \cdot (3n-2).$$

Proof. By the observations made above, the number of sketches equivalent to a given sketch only depends on its α, β-word (see Proposition 17). So, we need to count the number of α, β-words of length $2n$ with any prefix having at least as many α-letters as β-letters that are of the following types:

1. The n^{th} α-letter is not in the $(2n-1)^{th}$ position and
   (a) the letter after the n^{th} α-letter is an α.
   (b) the letter after the n^{th} α-letter is a β.
2. The n^{th} α-letter is in the $(2n-1)^{th}$ position.

We first count the second type of α,β-words. If the n^{th} α-letter is in the $(2n-1)^{th}$ position, the first $(2n-2)$ letters have $(n-1)$ α-letters and $(n-1)$ β-letters and hence form a ballot sequence. This means that there is no restriction on the $2n^{th}$ letter; it can be α or β. So, the total number of such α,β-words is

$$2 \cdot \frac{1}{n}\binom{2n-2}{n-1}.$$

The number of both the types 1(a) and 1(b) of α,β-words mentioned above are the same. This is because changing the letter after the n^{th} α-letter is an involution on the set of α,β-word of length $2n$ with any prefix having at least as many α-letters as β-letters. We have just counted such words that have the n^{th} α-letter in the $(2n-1)^{th}$ position. Hence, using Lemma 18, we get that the number of words of type 1(a) and 1(b) are both equal to

$$\frac{1}{2} \cdot \left[\binom{2n}{n} - \frac{2}{n}\binom{2n-2}{n-1}\right].$$

Combining the observations made above, we get that the number of regions of $\mathcal{D}_n$ is

$$2^n n! \cdot \left(\frac{1}{4} \cdot \left[\frac{2}{n}\binom{2n-2}{n-1} + \frac{1}{2} \cdot \left[\binom{2n}{n} - \frac{2}{n}\binom{2n-2}{n-1}\right]\right] + \frac{1}{2} \cdot \left[\frac{1}{2} \cdot \left[\binom{2n}{n} - \frac{2}{n}\binom{2n-2}{n-1}\right]\right]\right)$$

which simplifies to the required formula. □

Just as we did for $\mathcal{C}_n$, we can describe and count which regions of $\mathcal{D}_n$ are bounded.

Theorem 34. *The number of bounded regions of $\mathcal{D}_n$ is*

$$2^{n-1} \cdot \frac{(2n-3)!}{(n-2)!} \cdot (3n-4).$$

Proof. For $n \geqslant 2$, both $\mathcal{C}_n$ and $\mathcal{D}_n$ have rank n. Hence, a region of $\mathcal{D}_n$ is bounded exactly when all the regions of $\mathcal{C}_n$ it contains are bounded.

We have already seen in Theorem 24 that a region of $\mathcal{C}_n$ is bounded exactly when its corresponding lattice path does not touch the x-axis except at the origin. Such regions are not closed under $\mathcal{D}$ moves. However, if we also include regions whose corresponding lattice paths touch the x-axis only at the origin and $(2n, 0)$, this set of regions, which we call S, is closed under the action of $\mathcal{D}$ moves because such lattice paths are closed under the action of changing the $2n^{th}$ step. Denote by $S_{\mathcal{D}}$ the set of equivalence classes that $\mathcal{D}$ moves partition S into, i.e., $S_{\mathcal{D}}$ is the set of regions of $\mathcal{D}_n$ that contain regions of S.

Just as in the proof of Theorem 33, one can check that the set S is closed under the action of changing the letter after the n^{th} α-letter. Also, note that the lattice paths in S do not touch the x-axis at $(2n-2, 0)$, and hence the n^{th} α-letter cannot be in the $(2n-1)^{th}$ position. Using the above observations and the same method to count regions of $\mathcal{D}_n$ as in the proof of Theorem 33, we get the number of regions in $S_{\mathcal{D}}$ is

$$2^n n! \cdot \frac{3}{8}\left(\binom{2n-1}{n} + \frac{1}{n}\binom{2n-2}{n-1}\right).$$

Here, $\binom{2n-1}{2n}$ counts lattice paths that touch the x-axis only at the origin, $\frac{1}{n}\binom{2n-2}{n-1}$ counts those that touch the x-axis only at the origin and $(2n, 0)$, and the term $\frac{3}{8} = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{2}\right)$ corresponds to choosing a representative from each $\mathcal{D}$ equivalence class.

It can also be checked that each unbounded region in S is $\mathcal{D}$ equivalent to exactly one other region of S, and this region is bounded. This is because the lattice paths corresponding to these unbounded regions touch the x-axis at $(2n, 0)$. Hence, they cannot have the n^{th} and $(n+1)^{th}$ α-letters being adjacent and changing the $2n^{th}$ letter to an α gives a bounded region. Since the unbounded regions in S correspond to Dyck paths of length $(2n-2)$ (by deleting the first and last step), we get that the number of unbounded regions in $S_{\mathcal{D}}$ is

$$2^n n! \cdot \frac{1}{n}\binom{2n-2}{n-1}.$$

Combining the above results, we get that the number of bounded regions of $\mathcal{D}_n$ is

$$2^n n! \left(\frac{3}{8}\left(\binom{2n-1}{n} + \frac{1}{n}\binom{2n-2}{n-1}\right) - \frac{1}{n}\binom{2n-2}{n-1}\right).$$

This simplifies to give our required result. □

As mentioned earlier, we can choose a specific sketch from each $\mathcal{D}$ equivalence class to represent the regions of $\mathcal{D}_n$. It can be checked that symmetric sketches that satisfy the following are in bijection with regions of $\mathcal{D}_n$:

1. The last letter of the α, β-word is a β.
2. The n^{th} α-letter must have a negative label if the letter following it is an α-letter or the n^{th} β-letter.

We will call such sketches type D sketches. They will be used in Section 5 to interpret the coefficients of $\chi_{\mathcal{D}_n}$. Note that the type D sketches that correspond to bounded regions of $\mathcal{D}_n$ are those, when converted to a lattice path, that do not touch the x-axis except at the origin.

4.2 Type BC Catalan

The type B and type BC Catalan arrangements we are going to consider now are not sub-arrangements of the type C Catalan arrangement. While it is possible to consider these arrangements as sub-arrangements of the type C 2-Catalan arrangement (see Section 3.1), this would add many extra hyperplanes. This would make defining moves and counting equivalence classes difficult. Also, we do not have a simple characterization of α, β-words associated to symmetric 2-sketches, as we do for symmetric sketches (see Proposition 17). We instead study the type BC Catalan arrangement directly and define sketches corresponding to its regions. We then consider the type B Catalan arrangement as a sub-arrangement and use moves to study its regions.

The type BC Catalan arrangement in $\mathbb{R}^n$ has hyperplanes

$$\begin{aligned} X_i &= -1, 0, 1 \\ 2X_i &= -1, 0, 1 \\ X_i + X_j &= -1, 0, 1 \\ X_i - X_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. Translating this arrangement by setting $X_i = x_i + \frac{1}{2}$, we get the arrangement $\mathcal{BC}_n$ with hyperplanes

$$\begin{aligned} x_i &= -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2} \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. It can be checked that the regions of $\mathcal{BC}_n$ are given by valid total orders on

$$\{x_i + s \mid i \in [n],\ s \in \{0,1\}\} \cup \{-x_i - s \mid i \in [n],\ s \in \{0,1\}\} \cup \{-\frac{1}{2}, \frac{1}{2}\}.$$

We now define sketches that represent such orders. Just as before, we represent $x_i + s$ as $\alpha_i^{(s)}$ and $-x_i - s$ as $\alpha_{-i}^{(-s)}$ for any $i \in [n]$ and $s \in \{0, 1\}$. The numbers $-\frac{1}{2}$ and $\frac{1}{2}$ will be represented as $\alpha^{(-0.5)}$ and $\alpha^{(0.5)}$ respectively.

Example 35. The total order

$$x_2 < -x_1 - 1 < -\frac{1}{2} < x_1 < x_2 + 1 < -x_2 - 1 < -x_1 < \frac{1}{2} < x_1 + 1 < -x_2$$

is represented as $\alpha_2^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha^{(-0.5)}\ \alpha_1^{(0)}\ \alpha_2^{(1)}\ \alpha_{-2}^{(-1)}\ \alpha_{-1}^{(0)}\ \alpha^{(0.5)}\ \alpha_1^{(1)}\ \alpha_{-2}^{(0)}$.

Set $BC(n)$ to be the set

$$\{\alpha_i^{(s)} \mid i \in [n],\ s \in \{0,1\}\} \cup \{\alpha_i^{(s)} \mid -i \in [n],\ s \in \{-1,0\}\} \cup \{\alpha^{(-0.5)}, \alpha^{(0.5)}\}.$$

We define *type BC sketches* to be the words in $BC(n)$ that correspond to regions of $\mathcal{BC}_n$. Denote by $\overline{\alpha_x^{(s)}}$ the letter $\alpha_{-x}^{(-s)}$ for any $\alpha_x^{(s)} \in BC(n)$. We set $\overline{\alpha^{(-0.5)}} = \alpha^{(0.5)}$ and vice versa. We have the following characterization of type BC sketches:

Proposition 36. *A word in the letters $BC(n)$ is a type BC sketch if and only if the following hold for any $\alpha_x^{(s)}, \alpha_y^{(t)} \in B(n)$:*

1. *If $\alpha_x^{(s)}$ appears before $\alpha_y^{(t)}$ then $\overline{\alpha_y^{(t)}}$ appears before $\overline{\alpha_x^{(s)}}$.*

2. *If $\alpha_x^{(s-1)}$ appears before $\alpha_y^{(t-1)}$ then $\alpha_x^{(s)}$ appears before $\alpha_y^{(t)}$.*

3. *$\alpha_x^{(s-1)}$ appears before $\alpha_x^{(s)}$.*

4. *Each letter of $BC(n)$ appears exactly once.*

Just as was done in the proof of Proposition 12, we can inductively construct a point in $\mathbb{R}^n$ satisfying the inequalities specified by a type BC sketch. Also, just as for type C sketches, it can be shown that these sketches are symmetric about the center. We also represent such sketches using arc diagrams in a similar manner. Note that in this case, we also include an arc between $\alpha^{(-0.5)}$ and $\alpha^{(0.5)}$.

Example 37. To the type BC sketch given below, we associate the arc diagram in Figure 8, where we use $-$ to denote $\alpha^{(-0.5)}$ and $+$ to denote $\alpha^{(0.5)}$.

$$\alpha_2^{(0)}\ \alpha_{-1}^{(-1)}\ \alpha^{(-0.5)}\ \alpha_2^{(1)}\ \alpha_1^{(0)}\ |\ \alpha_{-1}^{(0)}\ \alpha_{-2}^{(-1)}\ \alpha^{(0.5)}\ \alpha_1^{(1)}\ \alpha_{-2}^{(0)}$$

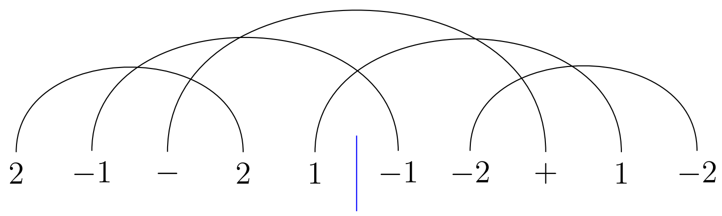


Figure 8: Arc diagram associated to the type BC sketch in Example 37.

To a type BC sketch, we can associate an α, β-word of length $(2n+1)$ and a signed permutation as follows:

1. For the letters in the first half of the type BC sketch of the form $\alpha_i^{(0)}$, $\alpha_{-i}^{(-1)}$ or $\alpha^{(-0.5)}$, we write α and for the others we write β (α corresponds to 'openers' in the arc diagram and β to 'closers').

2. The subscripts of the first n α-letters gives us the signed permutation.

Example 38. To the type BC sketch in Example 37, we associate the following pair:

1. α, β-word: $\alpha\alpha\alpha\beta\alpha$.

2. Signed permutation: $2 \quad -1$.

In the example above $n = 2$, and the α corresponding to $\alpha^{(-0.5)}$ is the 3^{rd}, i.e., the $(n+1)^{th}$ α in the α, β-word. This is true in general. We will now prove this as well as characterize the α, β-words and signed permutations associated to type BC sketches.

Proposition 39. *A pair consisting of*

1. *an α, β-word of length $(2n+1)$ satisfying the property that in any prefix, there are at least as many α-letters as β-letters, and*

2. *any signed permutation*

corresponds to exactly one type BC sketch and all type BC sketches correspond to such pairs.

Proof. Most of the proof is the same as that for type C sketches. The main difference is determining which α-letter corresponds to $\alpha^{(-0.5)}$. The property we have to take care of is that there is no nesting in the arc joining $\alpha^{(-0.5)}$ to $\alpha^{(0.5)}$.

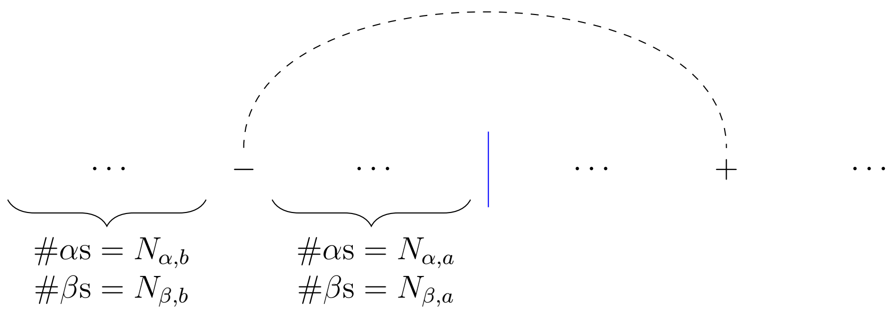


Figure 9: Arc from $\alpha^{(-0.5)}$ to its mirror $\alpha^{(0.5)}$.

Denote by $N_{\alpha,b}$ the number of α-letters before $\alpha^{(-0.5)}$. Define $N_{\alpha,a}$ to be the number of α-letters in the first half after $\alpha^{(-0.5)}$. Similarly define $N_{\beta,b}$ and $N_{\beta,a}$. The condition that we do not want an arc inside the one joining the $\alpha^{(-0.5)}$ to its mirror $\alpha^{(0.5)}$ is given by

$$N_{\alpha,b} \geqslant N_{\beta,b} + N_{\beta,a} + N_{\alpha,a}.$$

This is because we want any β-letter between the $\alpha^{(-0.5)}$ and $\alpha^{(0.5)}$ to have its corresponding α before the $\alpha^{(-0.5)}$. In the inequality above, the right-hand side counts the β-letters before $\alpha^{(0.5)}$. Note that the symmetry of the diagram tells us that the number of β-letters between the center of symmetry and $\alpha^{(0.5)}$ is $N_{\alpha,a}$. Since we need all these β-letters to have their corresponding α-letters before $\alpha^{(-0.5)}$, we get the inequality mentioned above.

Similarly, the condition that we do not want the arc joining $\alpha^{(-0.5)}$ to its mirror to be contained in any arc is given by

$$N_{\alpha,b} \leqslant N_{\beta,b} + N_{\beta,a} + N_{\alpha,a}.$$

This is because of the symmetry of the arc diagram and the fact that we want any α-letter before $\alpha^{(-0.5)}$ to have its corresponding β before $\alpha^{(0.5)}$.

Combining the above observations, we get

$$N_{\alpha,b} = N_{\beta,b} + N_{\beta,a} + N_{\alpha,a}.$$

But this says that the number of α-letters before $\alpha^{(-0.5)}$ should be equal to the number of remaining letters in the first half (other than $\alpha^{(-0.5)}$). Since the total number of letters in the first half is $(2n+1)$, we get the α corresponding to $\alpha^{(-0.5)}$ has to be the $(n+1)^{th}$ α.

The same proof can be used to show that if we start with an α, β-word of the form mentioned in the proposition, we get a type BC sketch precisely when we use the $(n+1)^{th}$ α-letter for $\alpha^{(-0.5)}$. □

Just as we used lattice paths for symmetric sketches, we also do so for type BC sketches. The one corresponding to the sketch in Example 37 is given in Figure 10.

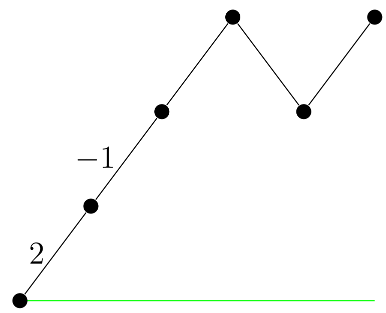


Figure 10: Lattice path corresponding to the type BC sketch in Example 37.

Theorem 40. *The number of type BC sketches, which is the number of regions of $\mathcal{BC}_n$, is*

$$2^n n! \binom{2n+1}{n}.$$

Proof. Since there is no condition on the signed permutations, we just have to count the α, β-words of the form mentioned in Proposition 39. Hence, we have to count lattice paths with steps $U = (1,1)$ and $D = (1,-1)$ of length $2n+1$ that start at the origin and never fall below the x-axis. The number of such paths can be shown to be $\binom{2n+1}{n}$ just as in Lemma 18. □

Theorem 41. *The number of bounded regions of $\mathcal{BC}_n$ is*

$$2^n n! \binom{2n}{n}.$$

Proof. Just as for type C regions, the region corresponding to a type BC sketch is bounded if and only if its arc diagram is interlinked. The signed permutation does not play a role in determining if a region is bounded. Note that in this case, there is an arc joining $\alpha^{(-0.5)}$ to its mirror image $\alpha^{(0.5)}$. If the arc diagram obtained by deleting this arc is interlinked, then clearly so was the initial arc diagram. Also, if the arc diagram consists of two interlinked pieces when this arc is removed (one on either side of the reflecting line), and $\alpha^{(-0.5)}$ is not the last letter of the first half, then the corresponding region would still be bounded.

Examining the bijection between arc diagrams and lattice paths, it can be checked that this means that lattice paths corresponding to bounded regions are those that never touch the x-axis after the origin. Deleting the first step of such paths gives a bijection with the paths counted in Lemma 18. □

4.3 Type B Catalan

Fix $n \geqslant 1$. The type B Catalan arrangement in $\mathbb{R}^n$ has the hyperplanes

$$\begin{aligned} X_i &= -1, 0, 1 \\ X_i + X_j &= -1, 0, 1 \\ X_i - X_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. Translating this arrangement by setting $X_i = x_i + \frac{1}{2}$, we get the arrangement $\mathcal{B}_n$ with hyperplanes

$$\begin{aligned} x_i &= -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2} \\ x_i + x_j &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. We consider $\mathcal{B}_n$ as a sub-arrangement of $\mathcal{BC}_n$. The hyperplanes missing from $\mathcal{BC}_n$ are

$$x_i = -1, 0$$

for all $i \in [n]$. The move on type BC sketches corresponding to changing one of the inequalities associated to these hyperplanes is: Swapping the $(2n+1)^{th}$ and $(2n+2)^{th}$ letters if they are not $\alpha^{(-0.5)}$ and $\alpha^{(0.5)}$.

We can see that such moves correspond to changing the last letter of the α, β-word associated to a type BC sketch. So if we force that the last letter of the sketch has to be an α-letter, we get a canonical sketch in each equivalence class. We will call such sketches type B sketches.

Theorem 42. *The number of type B sketches, which is the number of regions of $\mathcal{B}_n$, is*

$$2^n n! \binom{2n}{n}.$$

Proof. Since there is no condition on the signed permutation, we count the α, β-words associated to type B sketches. Deleting the last letter (which is an α) of such words, gives a bijection with the words counted in Lemma 18. □

Theorem 43. *The number of bounded regions of $\mathcal{B}_n$ is*

$$2^n n!\binom{2n-1}{n}.$$

Proof. Both $\mathcal{B}_n$ and $\mathcal{BC}_n$ have rank n. Hence a region of $\mathcal{B}_n$ is bounded if and only if all regions of $\mathcal{BC}_n$ that it contains are bounded.

In the proof of Theorem 41 we have characterized the α, β-words associated to bounded regions of $\mathcal{BC}_n$. It can be checked that the $(2n+1)^{th}$ letter of a type BC sketch corresponding to a bounded region is never $\alpha^{(-0.5)}$. Also, swapping the $(2n+1)^{th}$ and $(2n+2)^{th}$ letters of such a sketch results in another type BC sketch that corresponds to a bounded region of $\mathcal{BC}_n$. Hence, the number of bounded regions of $\mathcal{B}_n$ is half the number of bounded regions of $\mathcal{BC}_n$, which gives us the required result. □

Note that deleting the last letter (necessarily α) of the α, β-words corresponding to type B sketches gives a boundedness-preserving bijection with type C sketches.

5 Statistics on regions via generating functions

As mentioned in Section 1, the characteristic polynomial of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is of the form

$$\chi_{\mathcal{A}}(t) = \sum_{i=0}^{n} (-1)^{n-i} c_i t^i$$

where c_i is a non-negative integer for all $0 \leqslant i \leqslant n$ and Zaslavsky's theorem tells us that

$$\begin{aligned} r(\mathcal{A}) &= (-1)^n \chi_{\mathcal{A}}(-1) \\ &= \sum_{i=0}^{n} c_i. \end{aligned}$$

In this section, we interpret the coefficients of the characteristic polynomials of the arrangements we have studied. More precisely, for each arrangement we have studied, we first define a statistic on objects which correspond to its regions. We then show that the distribution of this statistic is given by the coefficients of the characteristic polynomial.

We do this by giving combinatorial meaning to the exponential generating functions for the characteristic polynomials of the arrangements we have studied. To obtain these generating functions, we use [19, Exercise 5.10], which we state and prove for convenience.

Definition 44. A sequence of arrangements $(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots)$ is called a *Generalized Exponential Sequence of Arrangements* (GESA) if

- $\mathcal{A}_n$ is an arrangement in $\mathbb{R}^n$ such that every hyperplane is parallel to one of the form $x_i = cx_j$ for some $c \in \mathbb{R}$.

- For any k-subset I of $[n]$, the arrangement

$$\mathcal{A}_n^I = \{H \in \mathcal{A}_n \mid H \text{ is parallel to } x_i = cx_j \text{ for some } i, j \in I \text{ and some } c \in \mathbb{R}\}$$

satisfies $\mathrm{L}(\mathcal{A}_n^I) \cong \mathrm{L}(\mathcal{A}_k)$ (isomorphic as posets).

Note that all the arrangements we have studied are GESAs. In a GESA, $\mathcal{A}_0$ is the empty arrangement in $\mathbb{R}^0$ with $r(\mathcal{A}_0) = b(\mathcal{A}_0) = \chi_{\mathcal{A}_0}(t) = 1$ and $\mathrm{rank}(\mathcal{A}_0) = 0$.

Proposition 45. *Let $(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots)$ be a GESA, and define*

$$F(x) = \sum_{n \geqslant 0} (-1)^n r(\mathcal{A}_n) \frac{x^n}{n!}$$

$$G(x) = \sum_{n \geqslant 0} (-1)^{\mathrm{rank}(\mathcal{A}_n)} b(\mathcal{A}_n) \frac{x^n}{n!}.$$

Then, we have

$$\sum_{n \geqslant 0} \chi_{\mathcal{A}_n}(t) \frac{x^n}{n!} = \frac{G(x)^{(t+1)/2}}{F(x)^{(t-1)/2}}.$$

Proof. The idea of the proof is the same as that of [19, Theorem 5.17]. By Whitney's Theorem [19, Theorem 2.4], we have for all n,

$$\chi_{\mathcal{A}_n}(t) = \sum_{\mathcal{B} \subseteq \mathcal{A}_n,\ \bigcap \mathcal{B} \neq \phi} (-1)^{\#\mathcal{B}} t^{n - \mathrm{rank}(\mathcal{B})}.$$

To each $\mathcal{B} \subseteq \mathcal{A}_n$, such that $\bigcap \mathcal{B} \neq \phi$, we associate a graph $G(\mathcal{B})$ on the vertex set $[n]$ where there is an edge between the vertices i and j if there is a hyperplane in $\mathcal{B}$ parallel to a hyperplane of the form $x_i = cx_j$ for some $c \in \mathbb{R}$.

Using [18, Corollary 5.1.6], we get

$$\sum_{n \geqslant 0} \chi_{\mathcal{A}_n}(t) \frac{x^n}{n!} = \exp \sum_{n \geqslant 1} \tilde{\chi}_{\mathcal{A}_n}(t) \frac{x^n}{n!}$$

where for any n we define

$$\tilde{\chi}_{\mathcal{A}_n}(t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A}_n,\ \bigcap \mathcal{B} \neq \phi \\ G(\mathcal{B}) \text{ connected}}} (-1)^{\#\mathcal{B}} t^{n - \mathrm{rank}(\mathcal{B})}.$$

Note that if $G(\mathcal{B})$ is connected, then any point in $\bigcap \mathcal{B}$ is determined by any one of its coordinates, say x_1. This is because any path from the vertex 1 to a vertex i in $G(\mathcal{B})$ can be used to determine x_i. This shows us that $\mathrm{rank}(\mathcal{B})$ is either n or $n-1$. Hence, $\tilde{\chi}_{\mathcal{A}_n}(t) = c_n t + d_n$ for some $c_n, d_n \in \mathbb{Z}$. Setting

$$\exp \sum_{n \geqslant 1} c_n \frac{x^n}{n!} = \sum_{n \geqslant 0} b_n \frac{x^n}{n!}$$

$$\exp \sum_{n \geqslant 1} d_n \frac{x^n}{n!} = \sum_{n \geqslant 0} a_n \frac{x^n}{n!}$$

we get

$$\sum_{n\geqslant 0}\chi_{\mathcal{A}_n}(t)\frac{x^n}{n!}=\left(\sum_{n\geqslant 0}b_n\frac{x^n}{n!}\right)^t\left(\sum_{n\geqslant 0}a_n\frac{x^n}{n!}\right).$$

Substituting $t=1$ and $t=-1$ and using Theorem 1, we obtain expressions for the exponential generating functions of $\{b_n\}$ and $\{c_n\}$ and this gives us the required result. □

Before looking at the characteristic polynomials of these arrangements, we recall a few results from [18]. Suppose that $c:\mathbb{N}\to\mathbb{N}$ is a function and for each $n,j\in\mathbb{N}$, we define

$$c_j(n)=\sum_{\{B_1,\ldots,B_j\}\in\Pi_n}c(|B_1|)\cdots c(|B_j|)$$

where Π_n is the set of partitions of $[n]$. Define for each $n\in\mathbb{N}$,

$$h(n)=\sum_{j=0}^{n}c_j(n).$$

From [18, Example 5.2.2], we know that in such a situation,

$$\sum_{n,j\geqslant 0}c_j(n)t^j\frac{x^n}{n!}=\left(\sum_{n\geqslant 0}h(n)\frac{x^n}{n!}\right)^t.$$

Informally, we consider $h(n)$ to be the number of "structures" that can be placed on an n-set where each structure can be uniquely broken up into a disjoint union of "connected sub-structures". Here $c(n)$ denotes the number of connected structures on an n-set and $c_j(n)$ denotes the number of structures on an n-set with exactly j connected sub-structures. We will call such structures *exponential structures*.

In fact, in most of the computations below, we will be dealing with generating functions of the form

$$\left(\sum_{n\geqslant 0}h(n)\frac{x^n}{n!}\right)^{\frac{t+1}{2}}. \tag{3}$$

We can interpret such a generating function as follows. Suppose that there are two types of connected structures, say positive and negative connected structures. Also, suppose that the number of positive connected structures on $[n]$ is the same as the number of negative ones, i.e., $c(n)/2$. Then the coefficient of $t^j\frac{x^n}{n!}$ in the generating function given above is the number of structures on $[n]$ that have j positive connected sub-structures.

Also, note that since the coefficients of the characteristic polynomial alternate in sign, the distribution of any appropriate statistic we define would be

$$\sum_{n\geqslant 0}\chi_{\mathcal{A}_n}(-t)\frac{(-x)^n}{n!}.$$

5.1 Reflection arrangements

Before defining statistics for the Catalan arrangements, we first do so for the reflection arrangements we studied in Section 2. As we will see, the same statistic we define for sketches (regions of the type C arrangement) works for the canonical sketches we have chosen for the other arrangements as well.

5.1.1 The type C arrangement

We have seen that the regions of the type C arrangement in $\mathbb{R}^n$ correspond to sketches (Section 2.1) of length $2n$. We use the second half of the sketch to represent the regions, and call them signed permutations on $[n]$.

A statistic on signed permutations whose distribution is given by the coefficients of the characteristic polynomial is given in [9, Section 2]. We define a similar statistic. First break the signed permutation into *compartments* using right-to-left minima as follows: Ignoring the signs, draw a line before the permutation and then repeatedly draw a line immediately following the least number after the last line drawn. This is repeated until a line is drawn at the end of the permutation. It can be checked that compartments give signed permutations an exponential structure. A *positive compartment* of a signed permutations is one where the last term is positive.

Example 46. The signed permutation given by

$$\overset{+}{3}\,\overset{+}{1}\,\overset{-}{6}\,\overset{-}{7}\,\overset{-}{5}\,\overset{+}{2}\,\overset{-}{4}$$

is split into compartments as

$$\mid \overset{+}{3}\,\overset{+}{1} \mid \overset{-}{6}\,\overset{-}{7}\,\overset{-}{5}\,\overset{+}{2} \mid \overset{-}{4} \mid$$

and hence has 3 compartments, 2 of which are positive.

By the combinatorial interpretation of (3), the distribution of the statistic 'number of positive compartments' on signed permutations is given by

$$\left(\frac{1}{1-2x}\right)^{\frac{t+1}{2}}.$$

Note that for the type C arrangement, in terms of Proposition 45, we have

$$F(x) = \left(\frac{1}{1+2x}\right),$$
$$G(x) = 1.$$

Hence, we get that the distribution of the statistic 'number of positive compartments' on signed permutations is given by the coefficients of the characteristic polynomial.

For the arrangements that follow, we have described canonical sketches and hence signed permutations that correspond to regions in Section 2. For each arrangement, we will show that the distribution of the statistic 'number of positive compartments' on these canonical signed permutations is given by the characteristic polynomial.

5.1.2 The Boolean arrangement

The signed permutations that correspond to the Boolean arrangement in $\mathbb{R}^n$ (Section 2.2) are those that have all compartments of size 1, i.e., the underlying permutation is $1\ 2\ \cdots\ n$. Just as before, it can be seen that the distribution of the statistic 'number of positive compartments' on such signed permutations is given by

$$(e^{2x})^{\frac{t+1}{2}}.$$

This agrees with the generating function for the characteristic polynomial we get from Proposition 45 using $F(x) = e^{-2x}$ and $G(x) = 1$.

5.1.3 The type D arrangement

From Section 2.3, we can see that the regions of the type D arrangement in $\mathbb{R}^n$ correspond to signed permutations on $[n]$ where the first sign is positive. Given $i \in [n]$ and a signed permutation σ of $[n] \setminus \{i\}$, the signed permutation of $[n]$ obtained by appending $\bar{i}$ to the start of σ has the same number of positive compartments as σ. This shows that the distribution of the statistic on signed permutations whose first term is positive is

$$(1-x)\left(\frac{1}{1-2x}\right)^{\frac{t+1}{2}}.$$

This agrees with the generating function for the characteristic polynomial we get from Proposition 45 since we have

$$\begin{aligned} F(x) &= \left(\frac{1+x}{1+2x}\right), \\ G(x) &= 1+x. \end{aligned}$$

Note that the expression for $G(x)$ is due to the fact that the type D arrangement in $\mathbb{R}^1$ is empty.

5.1.4 The braid arrangement

From Section 2.4, we get that the regions of the braid arrangement in $\mathbb{R}^n$ corresponds to the signed permutations on $[n]$ where all terms are positive. Hence, the number of positive compartments is just the number of compartments in the underlying permutation. Since compartments give permutations an exponential structure, the distribution of this statistic is

$$\left(\frac{1}{1-x}\right)^t.$$

This agrees with the generating function for the characteristic polynomial we get from Proposition 45 since $F(x) = \frac{1}{1+x}$ and $G(x) = 1 + x$.

We summarize the results of this section as follows. For any reflection arrangement $\mathcal{A}$, we use $\mathcal{A}$-signed permutation to mean those described above to represent the regions of $\mathcal{A}$.

Theorem 47. *For any reflection arrangement $\mathcal{A}$, the absolute value of the coefficient of t^j in $\chi_{\mathcal{A}}(t)$ is the number of $\mathcal{A}$-signed permutations that have j positive compartments.*

5.2 Catalan deformations

We start with defining a statistic for the extended type C Catalan arrangements. Using Proposition 45, we then show that the generating function for the statistic and the characteristic polynomials match.

Fix $m \geqslant 1$. We define a statistic on labeled symmetric non-nesting partitions and show that its distribution is given by the characteristic polynomial. To do this, we first recall some definitions and results about the type A extended Catalan arrangement.

Definition 48. An *m-non-nesting partition of size n* is a partition of $[(m+1)n]$ such that the following hold:

1. Each block is of size $(m+1)$.
2. If a, b are in the same block B and $[a, b] \cap B = \{a, b\}$, then for any c, d such that $a < c < d < b$, c and d are not in the same block.

Just as before, such partitions can be represented using arc diagrams.

Example 49. The arc diagram corresponding to the 2-non-nesting partition of size 3

$$\{1, 2, 4\}, \{3, 5, 6\}, \{7, 8, 9\}$$

is given in Figure 11.

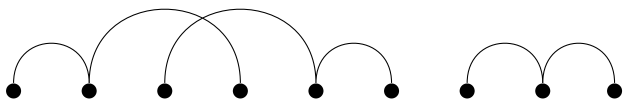


Figure 11: Arc diagram corresponding to the 2-non-nesting partition in Example 49.

It is known (for example, see [2, Theorem 2.2]) that the number of m-non-nesting partitions of size n is

$$\frac{1}{mn+1}\binom{(m+1)n}{n}.$$

These numbers are called the Fuss-Catalan numbers or generalized Catalan numbers. Setting $m = 1$ gives us the usual Catalan numbers. Labeling the n blocks distinctly using $[n]$ gives us labeled m-non-nesting partitions. These objects correspond to the regions of the type A m-Catalan arrangement in $\mathbb{R}^n$ whose hyperplanes are

$$x_i - x_j = 0, \pm 1, \pm 2, \ldots, \pm m$$

for all $1 \leqslant i < j \leqslant n$ (for example, see [6, Section 8.1]).

We now define a statistic on labeled non-nesting partitions similar to the one defined in [7, Section 4]. The statistic defined in [7] is for labeled m-Dyck paths but these objects are in bijection with labeled m-non-nesting partitions.

A labeled non-nesting partition can be broken up into interlinked pieces, say $P_1, \ldots, P_k$. We group these pieces into *compartments* as follows. If the label 1 is in the r^{th} interlinked piece, then the interlinked pieces $P_1, \ldots, P_r$ form the first compartment. Let j be the smallest number in $[n] \setminus A$ where A is the set of labels in first compartment. If j is in the s^{th} interlinked piece then interlinked pieces $P_{r+1}, \ldots, P_s$ form the second compartment. Continuing this way, we break up a labeled non-nesting partition into compartments.

Example 50. The labeled non-nesting partition in Figure 12 has 3 interlinked pieces. The first compartment consists of just the first interlinked piece since it contains the label 1. The smallest label in the rest of the diagram is 3 which is in the last interlinked piece. Hence, this labeled non-nesting partition has 2 compartments.

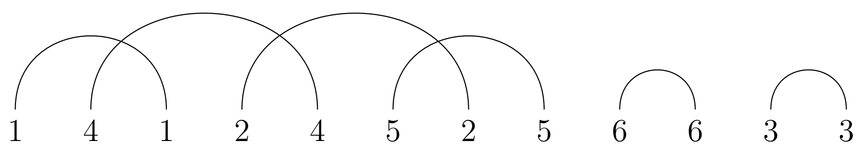


Figure 12: A labeled non-nesting partition with 3 interlinked pieces and 2 compartments.

A non-nesting partition labeled with distinct integers (not necessarily of the form $[n]$) can be broken up into compartments in the same way. Here the first compartment consists of the interlinked pieces up to the one containing the smallest label.

It can be checked that compartments give labeled non-nesting partitions an exponential structure. This is because the order in which they appear can be determined by their labels. A labeled non-nesting partition is said to be *connected* if it has only one compartment.

We now define a similar statistic for labeled symmetric non-nesting partitions. To a symmetric non-nesting partition we can associate a pair consisting of

1. an interlinked symmetric non-nesting partition, which we call the *bounded part* and

2. a non-nesting partition, which we call the *unbounded part*.

This is easy to do using arc diagrams, as illustrated in the following example. The terminology becomes clear when one considers the boundedness of the coordinates in the region corresponding to a labeled symmetric non-nesting partition.

Example 51. To the symmetric 2-non-nesting partition in Figure 13 we associate

1. the interlinked symmetric 2-non-nesting partition marked A and

2. the 2-non-nesting partition marked B.

Here A is the bounded part and B is the unbounded part. We can obtain the original arc diagram back from A and B by placing a copy of B on either side of A.

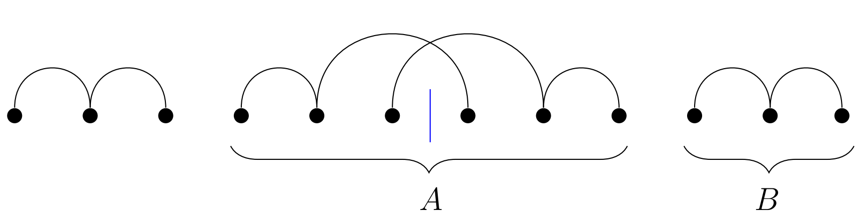


Figure 13: Break up of a symmetric 2-non-nesting partition.

This is a bijection between symmetric non-nesting partitions and such pairs. Given a labeled symmetric non-nesting partition, we define the statistic using just the unbounded part. Ignoring the signs, we break the unbounded part into compartments just as we did for non-nesting partitions. A *positive compartment* is one whose last element has a positive label.

Example 52. Suppose the arc diagram in Figure 14 is the unbounded part of some symmetric non-nesting partition. Notice that ignoring the signs, this arc diagrams breaks up into compartments just as Figure 12. But only the first compartment is positive since its last element has label 6 which is positive.

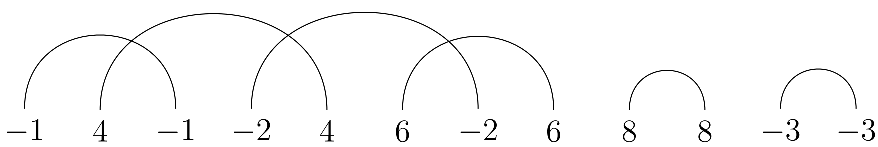


Figure 14: The unbounded part of a symmetric non-nesting partition that has 1 positive compartment.

We claim that the statistic 'number of positive compartments' meets our requirements. To prove that the distribution of this statistic is given by the characteristic polynomial, we apply Proposition 45 to the sequence of arrangements $\{\mathcal{C}_n^{(m)}\}$. Using the bijection between labeled symmetric m-non-nesting partitions and regions of $\mathcal{C}_n^{(m)}$, we note that those arc diagrams that are interlinked are the ones that correspond to bounded regions. Hence, using the notations form Proposition 45, and [18, Proposition 5.1.1], we have

$$F(-x) = G(-x) \cdot \left(\sum_{n \geqslant 0} \frac{2^n n!}{mn+1} \binom{(m+1)n}{n} \frac{x^n}{n!} \right). \tag{4}$$

Note that $\text{rank}(\mathcal{C}_n^{(m)}) = n$. This gives us

$$\sum_{n \geqslant 0} \chi_{\mathcal{A}_n}(-t) \frac{(-x)^n}{n!} = G(-x) \cdot \left(\sum_{n \geqslant 0} \frac{2^n n!}{mn+1} \binom{(m+1)n}{n} \frac{x^n}{n!} \right)^{\frac{t+1}{2}}.$$

Using the combinatorial interpretation of (3), we see that the right hand side of the above equation is the generating function for the distribution of the statistic.

We also obtain corresponding statistics on symmetric sketches using the bijection in Section 3.1. This gives us the following result.

Theorem 53. *The absolute value of the coefficient of t^j in $\chi_{\mathcal{C}_n^{(m)}}(t)$ is the number of symmetric m-sketches of size n that have j positive compartments.*

For the arrangements $\mathcal{D}_n$, $\mathcal{B}_n$, and $\mathcal{BC}_n$ as well, the analogue of (4) holds. That is, for each of these arrangements, using the notation of Proposition 45, we have

$$F(-x) = G(-x) \cdot \left(\sum_{n \geqslant 0} \frac{2^n n!}{n+1} \binom{2n}{n} \frac{x^n}{n!} \right).$$

This can be proved using the definitions of type D, type B, and type BC sketches and the description of which sketches correspond to bounded regions.

There is a slight difference in the proof for the sequence of arrangements $\{\mathcal{D}_n\}$. The arrangement $\mathcal{D}_1$ is empty and hence

$$G(-x) = 1 - x + \sum_{n \geqslant 2} b(\mathcal{D}_n) \frac{x^n}{n!}.$$

However, from the definition of type D sketches, we see that we must not allow those symmetric non-nesting partitions where the bounded part is empty and the first interlinked piece of the unbounded part is of size 1 with negative label. Hence, we still get the required expression for $F(-x)$.

Just as we did for the extended type C Catalan arrangements, we define positive compartments for the arc diagrams corresponding to the regions of these arrangements, which gives corresponding statistics on the sketches.

Example 54. The arc diagram in Figure 15 corresponds to a type BC sketch with 2 positive compartments.

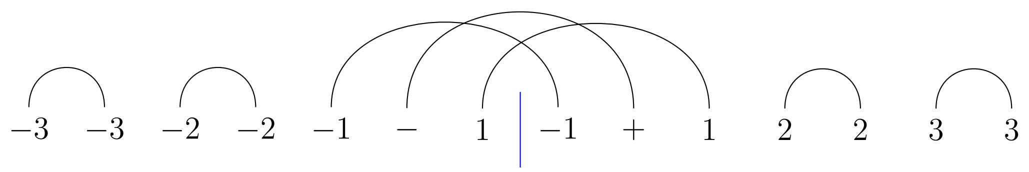


Figure 15: Arc diagram corresponding to a type BC sketch with 2 positive compartments.

The following result can be proved just as before.

Theorem 55. *The absolute value of the coefficient of t^j in $\chi_{\mathcal{A}}(t)$ for $\mathcal{A} = \mathcal{D}_n$ (respectively $\mathcal{B}_n, \mathcal{BC}_n$) is the number of type D (respectively type B, type BC) sketches of size n that have j positive compartments.*

6 Deformations of the threshold arrangement

The threshold arrangement in $\mathbb{R}^n$ consists of the hyperplanes $x_i+x_j = 0$ for $1 \leqslant i < j \leqslant n$. These arrangements are of interest because their regions correspond to certain labeled graphs called *threshold graphs* which have been extensively studied (see [13]). In this section, we study this arrangement and some of its deformations using similar techniques as in previous sections.

6.1 Sketches and moves

We use the sketches and moves idea to study the regions of the threshold arrangement by considering it as a sub-arrangement of the type C arrangement (Section 2.1). Before doing that, we first study the arrangement obtained by adding the coordinate hyperplanes to the threshold arrangement.

6.1.1 Fubini arrangement

We define the Fubini arrangement in $\mathbb{R}^n$ to be the one with hyperplanes

$$\begin{aligned} 2x_i &= 0 \\ x_i + x_j &= 0 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. The hyperplanes missing from the type C arrangement are

$$x_i - x_j = 0$$

for all $1 \leqslant i < j \leqslant n$. Hence a Fubini move, which we call an F move, is swapping adjacent $\overset{+}{i}$ and $\overset{+}{j}$ as well as $\overset{-}{j}$ and $\overset{-}{i}$ for distinct $i, j \in [n]$.

Example 56. We can use a series of F moves on a sketch as follows:

$$\overset{-}{3}\,\overset{-}{6}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{4}\,\overset{-}{5} \mid \overset{+}{5}\,\overset{-}{4}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{6}\,\overset{+}{3} \longrightarrow \overset{-}{6}\,\overset{-}{3}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{4}\,\overset{-}{5} \mid \overset{+}{5}\,\overset{-}{4}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{3}\,\overset{+}{6} \longrightarrow \overset{-}{6}\,\overset{-}{3}\,\overset{-}{2}\,\overset{+}{4}\,\overset{+}{1}\,\overset{-}{5} \mid \overset{+}{5}\,\overset{-}{1}\,\overset{-}{4}\,\overset{+}{2}\,\overset{+}{3}\,\overset{+}{6}$$

We define a *block* to be the set of absolute values in a maximal string of contiguous terms in the second half of a sketch that have the same sign. The blocks of the initial sketch in Example 56 are $\{5\}, \{1, 4\}, \{2, 3, 6\}$ (these blocks appear in this order with the first one being positive). It can be checked that F moves do not change the sequence of signs (above the numbers) and that they can only be used to reorder the elements in a block. Hence, each equivalence class has a unique sketch where the numbers in each block appear in ascending order. The last sketch in Example 56 is the unique such sketch in its equivalence class.

The number of such sketches is equal to the number of ways of choosing an ordered partition of $[n]$ (which correspond to the blocks of the sketch in order) and then choosing a sign for the first block. Hence the number of regions of the Fubini arrangement is $2 \cdot a(n)$ where $a(n)$ is the n^{th} Fubini number, which is the number of ordered partitions of $[n]$ listed as A000670 in the OEIS [17].

6.1.2 Threshold arrangement

The threshold arrangement in $\mathbb{R}^n$ has the hyperplanes

$$x_i + x_j = 0$$

for all $1 \leqslant i < j \leqslant n$. The hyperplanes missing from the type C arrangement are

$$\begin{aligned} 2x_i &= 0 \\ x_i - x_j &= 0 \end{aligned}$$

for all $1 \leqslant i < j \leqslant n$. Hence the threshold moves, which we call T moves, are as follows:

1. (D move) Swapping adjacent $\overset{+}{i}$ and $\overset{-}{i}$ for any $i \in [n]$.
2. (F move) Swapping adjacent $\overset{+}{i}$ and $\overset{+}{j}$ as well as $\overset{-}{j}$ and $\overset{-}{i}$ for distinct $i, j \in [n]$.

For any sketch, there is a T equivalent sketch for which the first block has more than 1 element. This is because, if the sketch has first block of size 1, applying a D move (swapping the n^{th} and $(n+1)^{th}$ term), will result in a sketch where the first block has size greater than 1 (first step in Example 57).

Example 57. We can use a series of T moves on a sketch as follows:

$$\overset{+}{5}\,\overset{-}{4}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{6}\,\overset{-}{3} \mid \overset{+}{3}\,\overset{-}{6}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{4}\,\overset{-}{5} \xrightarrow{D \text{ move}} \overset{+}{5}\,\overset{-}{4}\,\overset{-}{1}\,\overset{+}{2}\,\overset{+}{6}\,\overset{+}{3} \mid \overset{-}{3}\,\overset{-}{6}\,\overset{-}{2}\,\overset{+}{1}\,\overset{+}{4}\,\overset{-}{5} \xrightarrow{F \text{ moves}} \overset{+}{5}\,\overset{-}{4}\,\overset{-}{1}\,\overset{+}{6}\,\overset{+}{3}\,\overset{+}{2} \mid \overset{-}{2}\,\overset{-}{3}\,\overset{-}{6}\,\overset{+}{1}\,\overset{+}{4}\,\overset{-}{5}$$

To obtain a canonical sketch for each threshold region, we will need a small lemma.

Lemma 58. *Two T equivalent sketches that have their first block of size greater than 1 have the same blocks which appear in the same order with the same signs.*

Proof. Looking at what the T moves do to the sequence of signs (above the numbers), we can see that they at most swap the n^{th} and $(n+1)^{th}$ sign (D move). Hence, if we require the first blocks to have size greater than 1, both the sketches have the same number of blocks and the number of elements in the corresponding blocks are the same. An F move can only reorder elements in the same block of a sketch. A D move changes the sign of the first element of the second half. So if there are $k > 1$ elements in the first block of a T equivalent sketch, then the set of absolute values of the first k elements of the second half remains the same in all T equivalent sketches. This gives us the required result. $\square$

Using the above lemma, we can see that for any sketch there is a unique T equivalent sketch where the size of the first block is greater than 1 and the elements of each block are in ascending order. The last sketch in Example 57 is the unique such sketch in its equivalence class. Similar to the count for Fubini regions, we get that the number of regions of the threshold arrangement is

$$2 \cdot (a(n) - n \cdot a(n-1))$$

where, as before, $a(n)$ is the n^{th} Fubini number. The number of regions of the threshold arrangement is listed as A005840 in the OEIS [17].

Remark 59. The regions of the threshold arrangement in $\mathbb{R}^n$ are known to be in bijection with labeled threshold graphs on n vertices (see [19, Exercise 5.25]). Labeled threshold graphs on n vertices are inductively constructed starting from the empty graph. Vertices labeled $1, \ldots, n$ are added in a specified order. At each step, the vertex added is either 'dominant' or 'recessive'. A dominant vertex is one that is adjacent to all vertices added before it and a recessive vertex is one that is isolated from all vertices added before it. It is not difficult to see that the canonical sketches described above are in bijection with threshold graphs.

6.2 Statistics

The characteristic polynomial of the threshold arrangement and a statistic on its regions whose distribution is given by the characteristic polynomial has been studied in [9]. This is done by directly looking at the coefficients of the characteristic polynomial. In fact, even the coefficients of the characteristic polynomial of the Fubini arrangement (Section 6.1.1) have already been combinatorially interpreted in [9, Section 4.1]. This can be used to define an appropriate statistic on the regions of the Fubini arrangement. Here, just as in Section 5, we use Proposition 45 to combinatorially interpret the generating functions of the characteristic polynomials for the Fubini and threshold arrangements. Just as before, we will show that the statistic 'number of positive compartments' works for our purposes.

6.2.1 Fubini arrangement

We will use the second half of the canonical sketches described in Section 6.1.1 to represent the regions. We define blocks for signed permutations just as we did for sketches, *i.e.*, maximal strings of contiguous terms that have the same sign. Hence, the regions of the Fubini arrangement in $\mathbb{R}^n$ correspond to signed permutations on $[n]$ where each block is increasing (ignoring the signs). Recall from Section 5.1.1 that we break a signed permutation into compartments using right-to-left minima and a compartment is positive if its last term is positive.

Example 60. The signed permutation given by

$$\overset{+}{2}\,\overset{+}{3}\,\overset{-}{1}\,\overset{-}{5}\,\overset{-}{7}\,\overset{+}{4}\,\overset{-}{6}$$

has each block increasing and splits into compartments as

$$\mid \overset{+}{2}\,\overset{+}{3}\,\overset{-}{1} \mid \overset{-}{5}\,\overset{-}{7}\,\overset{+}{4} \mid \overset{-}{6} \mid$$

and hence has 3 compartments, 1 of which are positive.

Even for this special class of signed permutations where each block is increasing, compartments give them an exponential structure. This is because there is no condition linking the signs of the last element of a compartment and the first element of the compartment following it. This is because the last element of a compartment is necessarily smaller in absolute value than the element following it. Also, suppose we are given a signed permutation such that each block is increasing. It can be checked that the signed permutation obtained by changing all the signs also satisfies this property.

Using the above observations and the combinatorial interpretation of (3), we get that

$$\left(\frac{e^x}{2-e^x}\right)^{\frac{t+1}{2}}$$

is the exponential generating function for signed permutations where each block is increasing where t keeps track of the number of positive compartments. This is because

$$2\left(\frac{1}{1-(e^x-1)}\right)-1=\left(\frac{e^x}{2-e^x}\right)$$

is the exponential generating function for ordered set partitions where the first part is given a sign. This follows from the combinatorial interpretation of multiplication and addition of exponential generating functions and the fact that $(e^x - 1)$ is the exponential generating function for the 'non-empty set' structure. Note that we want the constant term of this generating function to be 1.

This agrees with the generating function for the characteristic polynomial we get from Proposition 45 since we have

$$\begin{aligned} F(x) &= \left(\frac{1}{2e^x-1}\right), \\ G(x) &= 1. \end{aligned}$$

6.2.2 Threshold arrangement

From Section 6.1.2, we can see that the regions of the threshold arrangement in $\mathbb{R}^n$ correspond to signed permutations on $[n]$ where each block is increasing and the first block has size greater than 1. If such a permutation starts with $\overset{-}{1}$, we instead use the signed permutation obtained by changing $\overset{-}{1}$ to $\overset{+}{1}$ to represent the region. Similar to how we obtained the generating function for the statistic for type D from the one for type C, we obtain our generating function from the one we have for the Fubini arrangement.

Suppose that we are given $i \in [n]$ and a signed permutation σ on $[n] \setminus \{i\}$ whose blocks are increasing. If $i = 1$ we construct the signed permutation on $[n]$ obtained by appending $\overset{-}{1}$ to the front of σ. If $i > 1$, and the first element of σ is $\overset{\pm}{j}$. We construct the signed permutation on $[n]$ obtained by appending $\overset{\mp}{i}$ to the start of σ. In both cases, it can be checked that the number of positive compartment of the new signed permutation constructed is the same as that for σ.

This shows that the distribution of the statistic 'number of positive compartments' on the signed permutations that correspond to regions of the threshold arrangement is

$$(1-x)\left(\frac{e^x}{2-e^x}\right)^{\frac{t+1}{2}}.$$

This agrees with the generating function for the characteristic polynomial we get from Proposition 45 since we have

$$\begin{aligned} F(x) &= \left(\frac{1+x}{2e^x-1}\right), \\ G(x) &= 1+x. \end{aligned}$$

6.3 Some deformations

Deformations of the threshold arrangement have not been as well-studied as those of the braid arrangement. However, the finite field method has been used to compute the characteristic polynomial for some deformations. In [15, 16], Seo computed the characteristic polynomials of the so called Shi and Catalan threshold arrangements. Expressions for the characteristic polynomials of more general deformations have been computed in [5].

In this section, we use the sketches and moves technique to obtain certain non-nesting partitions that are in bijection with the regions of the Catalan and Shi threshold arrangements. We do this by considering these arrangements as sub-arrangements of the type C Catalan arrangement (Section 3). Unfortunately, we were not able to directly count the non-nesting partitions we obtained since their description is not as simple as the ones we have seen before.

Fix $n \geqslant 2$ throughout this section. Recall that we studied the type C Catalan arrangement by considering a translation of it called $\mathcal{C}_n$ whose hyperplane are given by (2) and whose regions correspond to symmetric sketches of size n (see Definition 11). Symmetric sketches can also be viewed as labeled symmetric non-nesting partitions (see Example 22).

6.3.1 Catalan threshold

The Catalan threshold arrangement in $\mathbb{R}^n$ consists of the hyperplanes

$$X_i + X_j = -1, 0, 1$$

for all $1 \leqslant i < j \leqslant n$. The translated arrangement by setting $X_i = x_i + \frac{1}{2}$, which we call $\mathcal{CT}_n$, has hyperplanes

$$x_i + x_j = -2, -1, 0$$

for all $1 \leqslant i < j \leqslant n$. We consider this arrangement as a sub-arrangement of $\mathcal{C}_n$. Using Bernardi's idea of moves, we can define an equivalence on the symmetric sketches such that two sketches are equivalent if they lie in the same region of $\mathcal{CT}_n$.

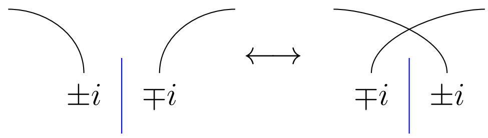
An α_+ letter is an α-letter whose subscript is positive. We similarly define α_-, β_+ and β_- letters. The 'mod-value' of a letter $\alpha_i^{(s)}$ is $|i|$.

The hyperplanes in $\mathcal{C}_n$ that are not in $\mathcal{CT}_n$ are

$$\begin{aligned} 2x_i &= -2, -1, 0 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

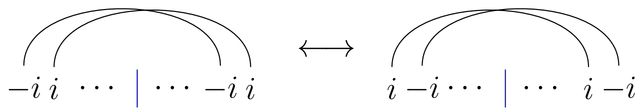
where $1 \leqslant i < j \leqslant n$. Changing the inequality corresponding to exactly one of these hyperplanes is given by the following moves on a sketch, which we call $\mathcal{CT}$ moves.

(a) Swapping the $2n^{th}$ and $(2n+1)^{th}$ letter.



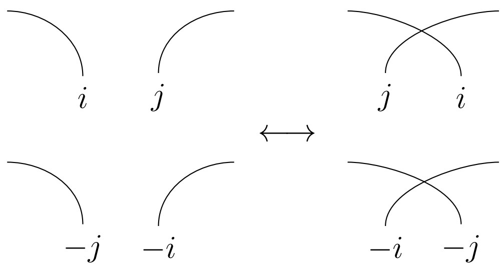
This corresponds to changing the inequality corresponding to a hyperplane of the form $2x_i = -2$ or $2x_i = 0$.

(b) Swapping the n^{th} and $(n+1)^{th}$ α-letter if they are consecutive (along with the n^{th} and $(n+1)^{th}$ β).



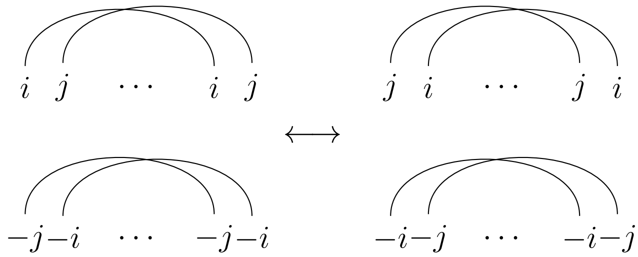
This corresponds to changing the inequality corresponding to a hyperplane of the form $2x_i = -1$.

(c) Swapping consecutive α_+ and β_+ letters (along with their negatives).



This corresponds to changing the inequality corresponding to a hyperplane of the form $x_i - x_j = 1$ for distinct $i, j \in [n]$.

(d) Swapping $\{\alpha_i^{(0)}, \alpha_j^{(0)}\}$ as well as $\{\alpha_i^{(1)}, \alpha_j^{(1)}\}$ if both pairs are consecutive (as well as their negatives) where $i, j \in [n]$ are distinct.



This corresponds to changing the inequality corresponding to the hyperplane $x_i - x_j = 0$.

Two sketches are in the same region of $\mathcal{CT}_n$ if and only if they are related by a series of $\mathcal{CT}$ moves. We call such sketches $\mathcal{CT}$ equivalent.

Consider the sketches to be ordered in the lexicographic order induced by the following order on the letters.

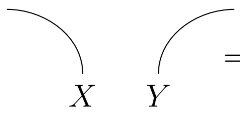
$$\alpha_n^{(0)} \succ \cdots \succ \alpha_1^{(0)} \succ \alpha_{-1}^{(-1)} \succ \cdots \succ \alpha_{-n}^{(-1)} \succ \alpha_n^{(1)} \succ \cdots \succ \alpha_1^{(1)} \succ \alpha_{-1}^{(0)} \succ \cdots \succ \alpha_{-n}^{(0)}$$

In other words, the α-letters are greater than the β-letters and for letters of the same type, the order is given by comparing the subscripts.

A sketch is called $\mathcal{CT}$ maximal if it is greater (in the lexicographic order) than all sketches to which it is $\mathcal{CT}$ equivalent. Hence the regions of $\mathcal{CT}_n$ are in bijection with the $\mathcal{CT}$ maximal sketches.

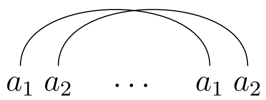
Theorem 61. *A symmetric sketch is $\mathcal{CT}$ maximal if and only if the following hold.*

1. *If a β-letter is followed by an α-letter, they should be of opposite signs and different mod-values.*



$\Longrightarrow$ *X and Y of opposite sign and different mod value.*

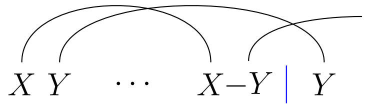
2. *If two α-letters and their corresponding β-letters are both consecutive and of the same sign then the subscript of the first one should be greater.*



and a_1, a_2 same sign $\Longrightarrow a_1 > a_2$.

3. *If the n^{th} and $(n+1)^{th}$ α-letters are consecutive, then so are the $(n-1)^{th}$ and n^{th} with the n^{th} α-letter being positive. In such a situation, if the $(n-1)^{th}$ α-letter is negative and the $(n-1)^{th}$ and n^{th} β-letters are consecutive, the $(n-1)^{th}$ α-letter should have a subscript greater than that of the $(n+1)^{th}$ α.*

4. *If the $(2n-1)^{th}$ and $(2n+1)^{th}$ letters are both β-letters of the same sign and their corresponding α-letters are consecutive, the subscript of the $(2n-1)^{th}$ letter should be greater than that of the $(2n+1)^{th}$.*



and $\quad X, Y$ *same sign* $\implies X > Y.$

Hence the regions of $\mathcal{CT}_n$ are in bijection with sketches of the form described above.

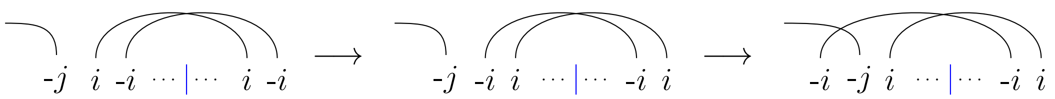
Remark 62. The idea of ordering sketches and choosing the maximal sketch in each region of $\mathcal{CT}_n$ to represent it is the same one used by Bernardi [6] to study certain deformations of the braid arrangement. In fact, [6, Lemma 8.13] shows that in this case, any sketch that is locally maximal (greater than any sketch that can be obtained by applying a single move) is maximal. Note that the sketches described in the above theorem are precisely the *2-locally maximal* sketches. That is, these are the sketches that can neither be converted into a greater sketch by applying a single $\mathcal{CT}$ move nor by applying two $\mathcal{CT}$ moves. It is clear that any $\mathcal{CT}$ maximal sketch is 2-locally maximal. The theorem states the converse is true as well.

Proof of Theorem 61. We first show that these conditions are required for a sketch to be $\mathcal{CT}$ maximal.

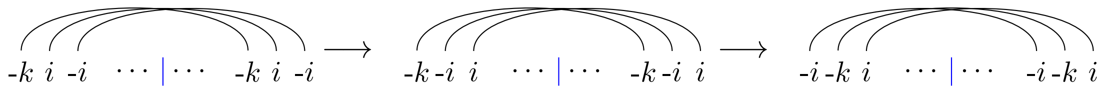
1. The first condition is necessary since the $\mathcal{CT}$ moves of type (a) or (c) would result in a greater sketch if it were false.

2. The second condition corresponds to $\mathcal{CT}$ moves of type (d).

3. The part about the n^{th} α-letter being positive if the n^{th} and $(n+1)^{th}$ α-letters are consecutive is due to $\mathcal{CT}$ moves of type (b). Suppose the letter before the n^{th} α-letter is a β-letter. Then it can't be positive since we have already seen that condition (1) of the theorem statement must be satisfied. But if it is negative, we can do the following to obtain a larger $\mathcal{CT}$ equivalent sketch:

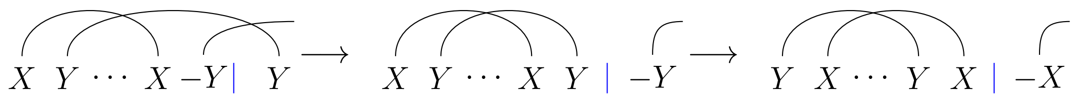


Hence the letter before the n^{th} α-letter has to be an α-letter. Now, suppose that the $(n-1)^{th}$ α-letter is negative and the $(n-1)^{th}$ and n^{th} β-letters are consecutive. Let the subscript of the $(n-1)^{th}$ α-letter be $-k$ and that of the $(n+1)^{th}$ α-letter be $-i$ for some $k, i \in [n]$. If $-k < -i$, we can do the following to obtain a larger $\mathcal{CT}$ equivalent sketch:



Hence we must have $-k > -i$ in this case.

4. Suppose the $(2n-1)^{th}$ and $(2n+1)^{th}$ letters are both β-letters of the same sign and their corresponding α-letters are consecutive but the subscript X of the $(2n-1)^{th}$ letter is less than the subscript Y of the $(2n+1)^{th}$ letter. We can do the following to obtain a larger $\mathcal{CT}$ equivalent sketch:



We now have to prove that these conditions are sufficient for a sketch to be $\mathcal{CT}$ maximal. Suppose w is a symmetric sketch that satisfies the four properties mentioned in the statement of the theorem. Suppose there is a sketch w' which is $\mathcal{CT}$ equivalent to w but larger in the lexicographic order. This means that if $w = w_1 \cdots w_{4n}$ and $w' = w'_1 \cdots w'_{4n}$, there is some $p \in [4n]$ such that

$$w_i = w'_i \text{ for } i \in [p-1] \text{ and } w_p \prec w'_p.$$

The possible ways in which this can happen are listed below.

1. w_p is a β_+ letter and w'_p is an α_+ letter.
2. w_p is a β_- letter and w'_p is an α_- letter.
3. w_p is a β_+ letter and w'_p is an α_- letter.
4. w_p is a β_- letter and w'_p is an α_+ letter.
5. w_p and w'_p are both α_+ letters.
6. w_p and w'_p are both α_- letters.
7. w_p is an α_- letter and w'_p is an α_+ letter.

The case of both w_p and w'_p being β-letters is not possible since, by the non-nesting property of a sketch, this would mean $w_p = w'_p$. Since $\alpha_- \prec \alpha_+$ we cannot have w_p being an α_+ letter and w'_p being an α_- letter. The proof will be complete if we show that none of the cases mentioned above are possible. We only prove that the odd-numbered cases lead to a contradiction. The proofs for the other cases are similar.

Before going forward, we formulate the meaning of w and w' being $\mathcal{CT}$ equivalent in terms of sketches. Since they have to be in the same region of $\mathcal{CT}_n$, the inequalities corresponding to the hyperplanes

$$x_i + x_j = -2, -1, 0$$

for all $1 \leqslant i < j \leqslant n$ are the same in both sketches. This means that the relationship between the pairs of the form

$$\{\alpha_i^{(1)}, \alpha_{-j}^{(-1)}\},\ \{\alpha_i^{(1)}, \alpha_{-j}^{(0)}\},\ \{\alpha_i^{(0)}, \alpha_{-j}^{(-1)}\},\ \text{and } \{\alpha_i^{(0)}, \alpha_{-j}^{(0)}\}$$

for any distinct $i, j \in [n]$ are the same in both w and w'. This can be written as follows:

The relationship between letters of opposite sign and different mod value have to be the same in both w and w'. (5)

<u>**Case 1:** w_p is a β_+ letter and w'_p is an α_+ letter.</u>

In this case w and w' are of the form

$$\begin{aligned} w &= w_1 \cdots w_{p-1} \alpha_k^{(1)} \cdots \\ w' &= w'_1 \cdots w'_{p-1} \alpha_l^{(0)} \cdots \end{aligned}$$

for some $k, l \in [n]$. Hence, $\alpha_l^{(0)}$ appears after $\alpha_k^{(1)}$ in w. By (5), every letter between $\alpha_k^{(1)}$ and $\alpha_l^{(0)}$ in w should be positive or one of $\alpha_{-l}^{(-1)}$ and $\alpha_{-l}^{(0)}$. If all the letters are positive, since $\alpha_k^{(1)}$ is a β_+ letter and $\alpha_l^{(0)}$ is an α_+ letter, there would be a consecutive pair of the form $\beta_+\alpha_+$ in w, which is a contradiction to property (1).

Now suppose $\alpha_{-l}^{(0)}$ is between $\alpha_k^{(1)}$ and $\alpha_l^{(0)}$ in w. It cannot be immediately before $\alpha_l^{(0)}$ since this would contradict property (1). But if it is not immediately before $\alpha_l^{(0)}$, since $\alpha_{-l}^{(0)}$ and $\alpha_l^{(0)}$ are negatives of each other, there should be some negative letter between them. But this letter cannot be $\alpha_{-l}^{(-1)}$ (since this should be before $\alpha_{-l}^{(0)}$). This is a contradiction to (5). Hence $\alpha_{-l}^{(0)}$ cannot be between $\alpha_k^{(1)}$ and $\alpha_l^{(0)}$.

So we must have $\alpha_{-l}^{(-1)}$ between $\alpha_k^{(1)}$ and $\alpha_l^{(0)}$ in w. Again, $\alpha_{-l}^{(-1)}$ cannot be immediately before $\alpha_l^{(0)}$ since this would contradict property (3). This means that there is at least one letter between $\alpha_{-l}^{(-1)}$ and $\alpha_l^{(0)}$ and all such letters are positive. If one of them is a β_+ letter, since $\alpha_l^{(0)}$ is an α_+ letter, there would be a consecutive pair of the form $\beta_+\alpha_+$, which is a contradiction to property (1). Hence all the letters between $\alpha_{-l}^{(-1)}$ and $\alpha_l^{(0)}$ are α_+ letters. But this is impossible by Lemma 15.

Case 3: w_p is a β_+ letter and w'_p is an α_- letter.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1}\alpha_k^{(1)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1}\alpha_{-l}^{(-1)} \cdots$$

for some $k, l \in [n]$. If $k \neq l$, this will contradict (5) since $\alpha_k^{(1)}$ will be before $\alpha_{-l}^{(-1)}$ in w but not in w'. So $\alpha_{-k}^{(-1)}$ appears after $\alpha_k^{(1)}$ in w and all letters between them are negative by (5) (note that $\alpha_k^{(0)}$ is before $\alpha_k^{(1)}$). However, if there were some letters between $\alpha_k^{(1)}$ and $\alpha_{-k}^{(-1)}$, at least one of them would be positive (since $\alpha_k^{(1)}$ and $\alpha_{-k}^{(-1)}$ are negatives of each other). But we also cannot have $\alpha_{-k}^{(-1)}$ immediately after $\alpha_k^{(1)}$ since this would contradict property (1).

Case 5: w_p and w'_p are both α_+ letters.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1}\alpha_k^{(0)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1}\alpha_l^{(0)} \cdots$$

for some $1 \leqslant k < l \leqslant n$. We split this case into two possibilities depending on whether or not $\alpha_l^{(0)}$ is before $\alpha_k^{(1)}$.

Case 5(a): $\alpha_l^{(0)}$ is before $\alpha_k^{(1)}$ in w.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1}\alpha_k^{(0)} \cdots \alpha_l^{(0)} \cdots \alpha_k^{(1)} \cdots \alpha_l^{(1)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1}\alpha_l^{(0)} \cdots .$$

By (5), each letter between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$ in w is positive or one of $\alpha_{-l}^{(-1)}$ or $\alpha_{-l}^{(0)}$. Just as in the **Case 1**, we can prove that $\alpha_{-l}^{(-1)}$ and $\alpha_{-l}^{(0)}$ cannot be between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$. Hence all the letters between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$ are positive. In fact, they all have to be α-letters. Otherwise we would have a consecutive pair of the form $\beta_+\alpha_+$, which contradicts property (1).

Each letter between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$ is positive or one of $\alpha_{-k}^{(-1)}$, $\alpha_{-l}^{(-1)}$, $\alpha_{-k}^{(0)}$ or $\alpha_{-l}^{(0)}$. Neither $\alpha_{-k}^{(0)}$ nor $\alpha_{-l}^{(0)}$ can be between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$, since this would mean that $\alpha_{-k}^{(-1)}$ or $\alpha_{-l}^{(-1)}$ is between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$, which cannot happen since we have already seen that there are only positive α-letters between them.

If $\alpha_{-k}^{(-1)}$ were between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$, it could not be immediately after $\alpha_k^{(1)}$ since this would contradict property (1). If there were some letters between $\alpha_k^{(1)}$ and $\alpha_{-k}^{(-1)}$, at least one of them would be a negative letter other than $\alpha_{-l}^{(-1)}$, which contradicts (5) (since $\alpha_l^{(1)}$ is after $\alpha_{-k}^{(-1)}$).

So the only negative letter that can be between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$ is $\alpha_{-l}^{(-1)}$. First, suppose that all letters between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$ are positive. Then all of them would have to be β_+ letters (otherwise there would be consecutive $\beta_+\alpha_+$ which contradicts property (1)). Then we would have that all letters between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$ are α_+ letters and all letters between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$ are β_+ letters and repeated application of property (2) would give $k > l$, which is a contradiction.

Next, suppose $\alpha_{-l}^{(-1)}$ is between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$. If $\alpha_{-l}^{(-1)}$ is not immediately before $\alpha_l^{(1)}$, there will be some negative letter other than $\alpha_{-l}^{(-1)}$ between $\alpha_k^{(1)}$ and $\alpha_l^{(1)}$, which we have already shown is not possible. So $\alpha_{-l}^{(-1)}$ is immediately before $\alpha_l^{(1)}$ and all the letters between $\alpha_k^{(1)}$ and $\alpha_{-l}^{(-1)}$ are positive and they have to all be β_+ letters (otherwise there would be a consecutive pair of the form $\beta_+\alpha_+$). If $\alpha_{k'}^{(1)}$ is the β_+ letter before $\alpha_{-l}^{(-1)}$ (k' could be k), then $\alpha_{k'}^{(0)}$ is the letter before $\alpha_l^{(0)}$ and hence we get that the letters between $\alpha_k^{(0)}$ and $\alpha_{k'}^{(0)}$ are all α_+ letters and their corresponding β-letters are consecutive and so by property (2), $k \geqslant k'$. But property (4) tells us that $k' > l$. So we get $k > l$, which is a contradiction.

Case 5(b): $\alpha_l^{(0)}$ is after $\alpha_k^{(1)}$ in w.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1}\alpha_k^{(0)} \cdots \alpha_k^{(1)} \cdots \alpha_l^{(0)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1}\alpha_l^{(0)} \cdots .$$

By (5), each letter between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$ in w is positive or one of $\alpha_{-l}^{(-1)}$ or $\alpha_{-l}^{(0)}$. Just as in **Case 1**, we can prove that $\alpha_{-l}^{(-1)}$ and $\alpha_{-l}^{(0)}$ cannot be between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$. Hence all the letters between $\alpha_k^{(0)}$ and $\alpha_l^{(0)}$ are positive. Since $\alpha_k^{(1)}$ is a β_+ letter and $\alpha_l^{(0)}$ is an α_+ letter and all letters in between are positive, there is a consecutive pair of the form $\beta_+\alpha_+$, which is a contradiction to property (1).

Case 7: w_p is a α_- letter and w'_p is an α_+ letter.

In this case w and w' are of the form

$$w = w_1 \cdots w_{p-1}\alpha_{-k}^{(-1)} \cdots$$
$$w' = w'_1 \cdots w'_{p-1}\alpha_l^{(0)} \cdots$$

for some $k, l \in [n]$. If $k \neq l$, we would get a contradiction to (5) since $\alpha_{-k}^{(-1)}$ is before $\alpha_l^{(0)}$ is w but not in w'. So $\alpha_k^{(0)}$ appears after $\alpha_{-k}^{(-1)}$ in w and each letter between them is positive or $\alpha_{-k}^{(0)}$. Just as before $\alpha_{-k}^{(0)}$ being between $\alpha_{-k}^{(-1)}$ and $\alpha_k^{(0)}$ would either contradict property (1) or (5). So all letters between $\alpha_{-k}^{(-1)}$ and $\alpha_k^{(0)}$ are positive. If there is some β_+ letter between them, there will be a consecutive pair of the form $\beta_+\alpha_+$, which would contradict property (1). Hence, all letters between $\alpha_{-k}^{(-1)}$ and $\alpha_k^{(0)}$ are α_+ letters. But this contradicts Lemma 15. ☐

6.3.2 Shi threshold

The Shi threshold arrangement in $\mathbb{R}^n$ consists of the hyperplanes

$$X_i + X_j = 0, 1$$

for all $1 \leqslant i < j \leqslant n$. The translated arrangement by setting $X_i = x_i + \frac{1}{2}$, which we call $\mathcal{ST}_n$, has hyperplanes

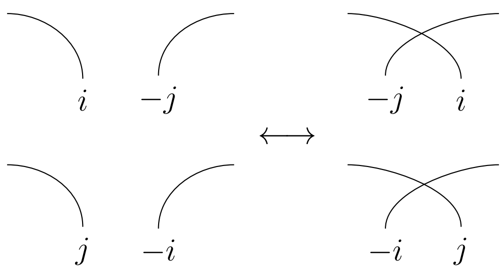
$$x_i + x_j = -1, 0$$

for all $1 \leqslant i < j \leqslant n$. We use the same method as before to study the regions of this arrangement by considering $\mathcal{ST}_n$ as a sub-arrangement of $\mathcal{C}_n$.

The hypeplanes in $\mathcal{C}_n$ that are not in $\mathcal{ST}_n$ are

$$\begin{aligned} 2x_i &= -2, -1, 0 \\ x_i + x_j &= -2 \\ x_i - x_j &= -1, 0, 1 \end{aligned}$$

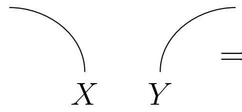
where $1 \leqslant i < j \leqslant n$. Changing the inequality corresponding to exactly one of these hyperplanes are given by the $\mathcal{CT}$ moves as well as the move corresponding to $x_i + x_j = -2$ where $i \neq j$ are in $[n]$: Swapping consecutive β_+ and α_- letters (along with their negatives).



Two sketches are in the same region of $\mathcal{ST}_n$ if and only if they are related by a series of such moves and we call such sketches $\mathcal{ST}$ equivalent. A sketch is called $\mathcal{ST}$ maximal if it is greater (in the lexicographic order) than all sketches to which it is $\mathcal{ST}$ equivalent. Hence the regions of $\mathcal{ST}_n$ are in bijection with the $\mathcal{ST}$ maximal sketches. The following result can be proved just as Theorem 61.

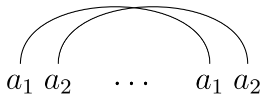
Theorem 63. *A symmetric sketch is $\mathcal{ST}$ maximal if and only if the following hold.*

1. *If a β-letter is followed by an α-letter, the β-letter should be negative and the α-letter should be positive with different mod-values.*



$\implies$ *X negative and Y positive and different mod value.*

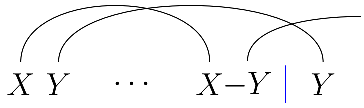
2. *If two α-letters and their corresponding β-letters are both consecutive and of the same sign then the subscript of the first one should be greater.*



and a_1, a_2 *same sign* $\implies a_1 > a_2$.

3. *If the n^{th} and $(n+1)^{th}$ α-letters are consecutive, then so are the $(n-1)^{th}$ and n^{th} with the n^{th} α-letter being positive. In such a situation, if the $(n-1)^{th}$ α-letter is negative and the $(n-1)^{th}$ and n^{th} β-letters are consecutive, the $(n-1)^{th}$ α-letter should have a subscript greater than that of the $(n+1)^{th}$ α-letter.*

4. *If the $(2n-1)^{th}$ and $(2n+1)^{th}$ letters are both negative β-letters and their corresponding α-letters are consecutive, the subscript of the $(2n-1)^{th}$ letter should be greater than that of the $(2n+1)^{th}$.*



and X, Y *negative* $\implies X > Y$.

Hence the regions of $\mathcal{ST}_n$ are in bijection with sketches of the form described above.

7 Concluding remarks

We end the paper with some open questions. Bernardi [6] has dealt with arbitrary deformations of the braid arrangement. The first (ambitious) problem is to generalize all the results in his paper to arbitrary deformations of all reflection arrangements. This is easier said than done! Bernardi proves that the number of regions is equal to the signed sum of certain "boxed trees". So the first step is to generalize the notion of boxed trees to certain decorated forests and then prove the counting formula, this is a work in progress. For certain well-behaved arrangements called "transitive deformations" Bernardi establishes an explicit bijection between the regions and the corresponding trees, via sketches. We don't have trees for all deformations of reflection arrangements but, we do have sketches that are in bijection with regions of (extended) Catalan deformations.

The main motivation behind Bernardi's work is an interesting pattern concerning certain statistic on labeled binary trees. Ira Gessel observed that the multivariate generating function for this statistic specializes to region counts of certain deformations of the braid arrangement. So a new research direction could be to try and define a statistic on non-nesting partitions (of all types) such that the associated generating function specializes to region counts.

Another aspect of Bernardi's work that has not been discussed in the present paper is the coboundary and Tutte polynomials. Using either, the finite field method or the method inspired by statistical mechanics one should get a closed form expression for these polynomials of the deformations we have considered. Moreover, the expression should be in terms of either sketches or non-nesting partitions.

Having a combinatorial model for the coefficients of the characteristic polynomial could be quite useful. Especially to derive various inequalities that they satisfy. For example, denote by $C(m,n,j)$ be the number of symmetric m-non-nesting partitions of size n with j positive compartments. Then following inequalities are not difficult to prove:

1. $C(m,n,j) \leqslant C(m,n+1,j)$
2. $C(m,n,j) \leqslant C(m,n+1,j+1)$
3. $C(m,n,j) \geqslant \sum_{k \geqslant j+1} \binom{k}{j} C(m,n,k)$.

A research direction here is to develop a case-free strategy to obtain more such information. For example, we know that the coefficients are unimodal, so identify the peak in each case.

Recall the Raney numbers that are defined by

$$A_n(m,r) := \frac{r}{n(m+1)+r}\binom{n(m+1)+r}{n}$$

for all positive integers n, m, r. The Catalan numbers are a special case of Raney numbers, obtained by setting $m = r = 1$. It was shown in [8] that the number of regions of the hyperplane arrangement

$$\{x_i = 0 \mid i \in [n]\} \cup \{x_i = 2^k x_j \mid k \in [-m,m], 1 \leqslant i < j \leqslant n\}$$

is equal to $n!A_n(m,2)$. Note that these arrangements define a GESA. Find a family of arrangements which is GESA and the number of regions is $n!A_n(m,r)$. One can use tuples of labeled Dyck paths to enumerate these regions. So one can try and apply techniques from this paper to find a statistic for these objects.

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