Infinite Families of Congruences Modulo 81 for Overpartitions with Restricted Odd Differences

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Abstract

Let $\overline{t}(n)$ denote the number of overpartitions of weight n in which (i) the difference between the successive parts may be odd only if the larger part is overlined and (ii) if the smallest part is odd then it is overlined. In this paper, by employing fundamental generating function dissection techniques and various q-series identities, we prove several new infinite families of congruences modulo 81 for $\overline{t}(n)$. For example, for $\alpha \geqslant 0$ and $n \geqslant 0$,

$$\overline{t}\left(4^{2\alpha+1}(8n+5)\right) \equiv 0 \pmod{81}.$$

Consequently, we extend a result of Hirschhorn and Sellers to an infinite family, namely, for $\alpha \geqslant 0$ and $n \geqslant 0$,

$$\overline{t}\left(4^{\alpha}(8n+5)\right) \equiv 0 \pmod{9}.$$

Mathematics Subject Classifications: 05A17, 11P83

1 Introduction

An overpartition of a nonnegative integer n is a partition of n in which the last (or first) occurrence of each distinct part may be overlined. Denote the number of overpartitions of n by $\overline{p}(n)$. For instance, there are fourteen overpartitions of 4:

$$4,\ \overline{4},\ 3+1,\ \overline{3}+1,\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2,\ 2+\overline{2},\ 2+1+1,$$
$$\overline{2}+1+1,\ 2+1+\overline{1},\ \overline{2}+1+\overline{1},\ 1+1+1+1,\ 1+1+1+\overline{1}.$$

Arithmetic properties of $\overline{p}(n)$ have been widely studied (see, for example, [3, 5, 10, 12]).

Recently, Bringmann, Dousse, Lovejoy and Mahlburg [2] defined the function $\overline{t}(n)$ to be the number of overpartitions of n, in which (i) the difference between two successive

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parts may be odd only if the larger part is overlined and (ii) if the smallest part is odd then it is overlined. For example, there are eight such overpartitions of 4:

$$4,\ \overline{4},\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2,\ 2+\overline{2},\ \overline{2}+1+\overline{1},\ 1+1+1+\overline{1}.$$

Using certain q-difference equations, Bringmann et al. [2] established the generating function of $\overline{t}(n)$ that is given by

$$\sum_{n=0}^{\infty} \overline{t}(n)q^n = \frac{f_3}{f_1 f_2}, \tag{1}$$

where here and throughout this paper, f_k is defined by

$$f_k := (q^k;q^k)_\infty := \prod_{n=1}^{\infty}(1-q^{kn}).$$

Subsequently, Hirschhorn and Sellers [6] demonstrated several congruences modulo 2, 4, 5, 9 and 12 for $\overline{t}(n)$ by applying the elementary dissection techniques of generating functions. For example, for $n \geqslant 0$,

$$\overline{t}(8n+5) \equiv 0 \pmod{9}. \tag{2}$$

By means of the theory of modular forms, Chern and Hao [4] obtained some congruences modulo 2, 3, 6, 8, 9 and 12 for $\overline{t}(n)$, such as, for $n \geqslant 0$,

$$\overline{t}(9n+6) \equiv 0 \pmod{9}.$$

Lin, Liu, Wang and Xiao [9] proved two infinite families of congruences modulo 5 and 27, namely, for $\alpha \geqslant 0$ and $n \geqslant 0$,

$$\overline{t}\left(9^\alpha(45n+30)\right) \equiv 0 \pmod{5},$$
$$\overline{t}\left(9^\alpha(72n+69)\right) \equiv 0 \pmod{27}.$$

Naika and Gireesh [11] showed an infinite family of congruences modulo 6 for $\overline{t}(n)$ by using dissections of theta function identities, for $\alpha \geqslant 0$ and $n \geqslant 0$,

$$\overline{t}\left(8p^{2\alpha+1}(pn+j)+3p^{2\alpha+2}\right) \equiv 0 \pmod{6},$$

where $p \geqslant 5$ is any prime such that $\left(\frac{-2}{p}\right) = -1$ and $1 \leqslant j \leqslant p-1$.

In this paper, we aims to construct several infinite families of congruences modulo 81 for $\overline{t}(n)$. Using generating function dissection techniques and various q-series identities, we establish the following two congruence relations modulo 81.

Theorem 1. *For $n \geqslant 0$,*

$$\overline{t}(4^4n) - \overline{t}(4^2n) \equiv \overline{t}(4^3n) - \overline{t}(4n) \pmod{81}. \tag{3}$$

Theorem 2. *For $\alpha \geqslant 0$ and $n \geqslant 0$,*

$$\overline{t}\left(4^{\alpha+3}(8n+5)\right) \equiv \overline{t}\left(4^{\alpha+1}(8n+5)\right) \pmod{81}. \tag{4}$$

Combining the congruence relation (3) with two specific congruences $\alpha = 0$ and $\alpha = 1$ in (5), we arrive at a family of congruences modulo 81 for $\overline{t}(n)$.

Theorem 3. *For $\alpha \geqslant 0$ and $n \geqslant 0$,*

$$\overline{t}\left(4^{2\alpha+1}(8n+5)\right) \equiv 0 \pmod{81}. \tag{5}$$

Based on the congruence relation (4) and the generating function of $\overline{t}\,(4^2(8n+5))$ modulo 81, we obtain a family of generating functions modulo 81 for $\overline{t}(n)$.

Theorem 4. *For $\alpha \geqslant 1$,*

$$\sum_{n=0}^{\infty} \overline{t}\left(4^{2\alpha}(8n+5)\right) q^n \equiv 72\frac{f_1 f_2^4 f_6^2}{f_3^2} + 54q\frac{f_2^{26}}{f_1^{13}} \pmod{81}. \tag{6}$$

In addition, combining the above two theorems with congruence (2), we find a generalisation of (2) to an infinite family of congruences.

Corollary 5. *For $\alpha \geqslant 0$ and $n \geqslant 0$,*

$$\overline{t}\left(4^{\alpha}(8n+5)\right) \equiv 0 \pmod{9}.$$

2 Preliminaries

Recall Ramanujan's bilateral theta function $f(a,b)$ is given by:

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1. \tag{7}$$

In particular,

$$\psi(q) := f(q,q^3) = \frac{f_2^2}{f_1}, \tag{8}$$

$$\varphi(q) := f(q,q) = \frac{f_2^5}{f_1^2 f_4^2}, \tag{9}$$

$$\varphi(-q) := f(-q,-q) = \frac{f_1^2}{f_2}. \tag{10}$$

Lemma 6. *For positive integers k and r,*

$$f_k^{3^r} \equiv f_{3k}^{3^{r-1}} \pmod{3^r}.$$

Proof. See [8, Lemma 2.3]. □

Lemma 7. *The following 2-dissections hold.*

$$\frac{1}{f_1^2}=\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}, \tag{11}$$

$$\frac{1}{f_1^4}=\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}, \tag{12}$$

$$\frac{f_1^3}{f_3}=\frac{f_4^3}{f_{12}}-3q\frac{f_2^2f_{12}^3}{f_4f_6^2}, \tag{13}$$

$$\frac{f_3}{f_1^3}=\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}, \tag{14}$$

$$\frac{f_1}{f_3^3}=\frac{f_2f_4^2f_{12}^2}{f_6^7}-q\frac{f_2^3f_{12}^6}{f_4^2f_6^9}, \tag{15}$$

$$\frac{f_1}{f_3}=\frac{f_2f_{16}f_{24}^2}{f_6^2f_8f_{48}}-q\frac{f_2f_8^2f_{12}f_{48}}{f_4f_6^2f_{16}f_{24}}, \tag{16}$$

$$\frac{f_3}{f_1}=\frac{f_4f_6f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}}+q\frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}}, \tag{17}$$

$$f_1f_3=\frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2}-q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}, \tag{18}$$

$$\frac{1}{f_1f_3}=\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}+q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}. \tag{19}$$

Proof. For (11)–(17), one may refer to [12, Lemma 2.2–Lemma 2.6]. For (18), see [7, eq. (30.12.1)]. And (19) follows from (18) by replacing q by $-q$ and using the fact $(-q;-q)_\infty=\frac{f_2^3}{f_1f_4}$. □

Lemma 8. *We have*

$$\frac{f_4^3}{f_{12}}=\frac{f_1^3}{f_3}+3q\frac{f_2^2f_{12}^3}{f_4f_6^2}, \tag{20}$$

$$\frac{f_{12}}{f_4^3}=\frac{f_3}{f_1^3}-3q\frac{f_2^2f_3f_{12}^4}{f_1^3f_4^4f_6^2}, \tag{21}$$

$$\frac{f_2^6f_3}{f_1^3f_6^2}-\frac{f_2f_3^6}{f_1^2f_6^3}=q\frac{f_1f_6^6}{f_2^2f_3^3}, \tag{22}$$

$$q\frac{f_4^8}{f_2^4}\equiv 1-\frac{f_1f_6^7}{f_2f_3^3f_4^8} \pmod 3, \tag{23}$$

$$2q\frac{f_4^8}{f_2^4}\equiv\frac{f_2^{20}}{f_1^8f_4^8}-\frac{f_1^3f_2^3}{f_3f_6} \pmod 9, \tag{24}$$

$$\frac{f_1^9 f_{12}^3}{f_3^3 f_4^9} \equiv 1 - 9q\frac{f_2^2 f_{12}^4}{f_4^4 f_6^2} \pmod{27}, \tag{25}$$

$$\frac{f_2^{24} f_3^{12}}{f_1^{12} f_6^{12}} \equiv 37\frac{f_1^{24}}{f_2^{12}} + 45\frac{f_1^{16} f_3^8}{f_2^8 f_6^4} \pmod{27}. \tag{26}$$

Proof. (20) follows directly from (13). Multiplying both sides of (13) by $f_3 f_{12}/f_1^3 f_4^3$, we get (21).

From Jin and Zhang [8, Lemma 3.3], we have

$$\frac{\psi^3(q)}{\psi(q^3)} - \frac{\varphi(-q^3)^3}{\varphi(-q)} = q\frac{\psi(q^3)^3}{\psi(q)}. \tag{27}$$

Substituting (8)–(10) into (27) yields (22).

From Berndt [1, p.150, eq.(9.15)], we see

$$\frac{\psi(q)^6}{\psi(q^3)^2} + 18q\psi(q)^2\psi(q^3)^2 - 27q^2\frac{\psi(q^3)^6}{\psi(q)^2} = \varphi(q)^4 + 16q\psi(q^2)^4. \tag{28}$$

Taking modulo 3 on both sides of (28), and using (8), (9) and Lemma 6 deduce

$$q\frac{f_4^8}{f_2^4} \equiv 1 - \frac{f_2^{20}}{f_1^8 f_4^8} \equiv 1 - \frac{f_1 f_6^7}{f_2 f_3^3 f_4^8} \pmod{3}.$$

Thus (23) is proved. Similarly, (24) follows by reducing modulo 9 on both sides of (28), and applying (8), (9) and Lemma 6.

Cube both sides of (21) and take modulo 27, and then multiply both sides by f_1^9/f_3^3 to yield (25).

From Berndt [1, p.357, eq.(4.3)], we find

$$9\frac{\varphi(-q^3)^3}{\varphi(-q)} - \frac{\varphi(-q)^3}{\varphi(-q^3)} = 8\frac{\psi(q)^3}{\psi(q^3)}. \tag{29}$$

Putting (8) and (10) into (29) deduces that

$$9\frac{f_2 f_3^6}{f_1^2 f_6^3} - \frac{f_1^6 f_6}{f_2^3 f_3^2} = 8\frac{f_2^6 f_3}{f_1^3 f_6^2}. \tag{30}$$

Multiplying both sides of (30) by $10f_3^2/f_6$ and calculating modulo 81 yield

$$\frac{f_2^6 f_3^3}{f_1^3 f_6^3} \equiv 10\frac{f_1^6}{f_2^3} - 9\frac{f_2 f_3^8}{f_1^2 f_6^4} \pmod{81}. \tag{31}$$

Then raising both sides of (31) to the fourth power and reducing modulo 81, we arrive at (26). $\square$

Lemma 9. *We have*

$$\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} = \frac{f_3}{f_1^3} - 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{32}$$

$$\frac{f_2^9 f_{12}^2}{f_4^6 f_6^3} = \frac{f_1^3}{f_3} + 3q\frac{f_1^3 f_2^2 f_{12}^4}{f_3 f_4^4 f_6^2}, \tag{33}$$

$$\frac{f_1^3 f_6^2}{f_2^6 f_3} \equiv \frac{f_2^3 f_3^2}{f_1^6 f_6} - 9q\frac{f_2^{16}}{f_1^8} \pmod{27}, \tag{34}$$

$$\frac{f_1^6 f_6^4}{f_2^{12} f_3^2} \equiv \frac{f_2^6 f_3^4}{f_1^{12} f_6^2} + 9q\frac{f_2^{19} f_3^2}{f_1^{14} f_6} \pmod{27}. \tag{35}$$

Proof. (32) follows from (14). Multiplying both sides of (32) by $f_1^3 f_2^9 f_{12}^2 / f_3 f_4^6 f_6^3$ yields (33) immediately.

Multiplying both sides of (30) by $f_3 f_6 / f_1^3 f_2^3$ and then simplifying imply that

$$\begin{aligned}
\frac{f_1^3 f_6^2}{f_2^6 f_3} &= \frac{f_2^3 f_3^2}{f_1^6 f_3} - 9\left(\frac{f_2^3 f_3^2}{f_1^6 f_3} - \frac{f_3^7}{f_1^5 f_2^2 f_6^2}\right) \\
&= \frac{f_2^3 f_3^2}{f_1^6 f_3} - 9\frac{f_3 f_6}{f_1^3 f_2^3}\left(\frac{f_2^6 f_3}{f_1^3 f_6^2} - \frac{f_2 f_3^6}{f_1^2 f_6^3}\right) \\
&= \frac{f_2^3 f_3^2}{f_1^6 f_3} - 9q\frac{f_6^7}{f_1^2 f_2^5 f_3^2} && \text{(using (22))} \\
&\equiv \frac{f_2^3 f_3^2}{f_1^6 f_3} - 9q\frac{f_2^{16}}{f_1^8} \pmod{27}, && \text{(using Lemma 6)}
\end{aligned}$$

so (34) is proved. Squaring both sides of (34) and reducing modulo 27, (35) follows. □

Lemma 10. *We have*

$$\frac{f_2^{16}}{f_1^8} + 2\frac{f_4^8}{f_2^4} + q\frac{f_4^{16}}{f_2^8} \equiv 0 \pmod{3}, \tag{36}$$

$$\frac{f_2^{40}}{f_1^{16} f_4^{16}} \equiv 1 + q\frac{f_4^8}{f_2^4} + q^2\frac{f_4^{16}}{f_2^8} \pmod{3}, \tag{37}$$

$$\frac{f_{12}^2}{f_4^6} \equiv \frac{f_3^2}{f_1^6} - 6q\frac{f_2^2 f_4^8}{f_6^2} \pmod{27}, \tag{38}$$

$$\frac{f_4^6}{f_{12}^2} \equiv \frac{f_1^6}{f_3^2} + 6q\frac{f_1^3 f_2^2 f_4^8}{f_3 f_6^2} + 9q^2\frac{f_4^{16}}{f_2^8} \pmod{27}. \tag{39}$$

Proof. Multiplying both sides of (24) by f_4^8/f_2^4 and using Lemma 6, we get (36).

And multiplying both sides of (36) by f_2^4/f_4^8, then squaring it and taking modulo 3, we conclude (37).

Squaring both sides of (21), reducing modulo 27 and applying (21), we derive that

$$
\begin{aligned}
\frac{f_{12}^2}{f_4^6} &= \frac{f_3^2}{f_1^6} - 6q\frac{f_2^2f_3^2f_{12}^4}{f_1^6f_4^4f_6^2} + 9q^2\frac{f_2^4f_3^2f_{12}^8}{f_1^6f_4^8f_6^4} \\
&\equiv \frac{f_3^2}{f_1^6} - 6q\frac{f_2^2f_3^2f_4^8}{f_1^6f_6^2}\left(\frac{f_{12}}{f_4^3}\right) + 9q^2\frac{f_4^{16}}{f_2^8} \qquad \text{(using Lemma 6)} \\
&\equiv \frac{f_3^2}{f_1^6} - 6q\frac{f_2^2f_4^8}{f_6^2} \pmod{27},
\end{aligned}
$$

so (38) follows.

Squaring both sides of (20), and reducing modulo 27, we deduce that

$$
\begin{aligned}
\frac{f_4^6}{f_{12}^2} &= \frac{f_1^6}{f_3^2} + 6q\frac{f_1^3f_2^2f_{12}^3}{f_3f_4f_6^2} + 9q^2\frac{f_2^4f_{12}^6}{f_4^2f_6^4} \\
&\equiv \frac{f_1^6}{f_3^2} + 6q\frac{f_1^3f_2^2f_4^8}{f_3f_6^2} + 9q^2\frac{f_4^{16}}{f_2^8} \pmod{27}. \qquad \text{(using Lemma 6)}
\end{aligned}
$$

Hence (39) is proved. ☐

Lemma 11. *We have*

$$
\frac{f_1^{37}f_2^{41}f_4}{f_3^{13}f_6^{12}f_{12}} - \frac{f_2^5f_3^2f_4}{f_1^8f_{12}} \equiv -27q\frac{f_6^{14}}{f_2f_3^6f_4^{10}} + 63q\frac{f_6^8}{f_1f_2^3f_3^3f_4^2} \pmod{81}. \tag{40}
$$

Proof. Utilising (20), (25) and (26), we find that

$$
\begin{aligned}
&\frac{f_1^{37}f_2^{41}f_4}{f_3^{13}f_6^{12}f_{12}} - \frac{f_2^5f_3^2f_4}{f_1^8f_{12}} \\
&= \frac{f_1^{37}f_2^{41}}{f_3^{13}f_4^2f_6^{12}}\left(\frac{f_4^3}{f_{12}}\right) - \frac{f_2^5f_3^2}{f_1^8f_4^2}\left(\frac{f_4^3}{f_{12}}\right) \\
&\equiv \frac{f_1^{52}f_2^{17}}{f_3^{26}f_4^2}\left(\frac{f_2^{24}f_3^{12}}{f_1^{12}f_6^{12}}\right) - \frac{f_2^5f_3}{f_1^5f_4^2} - 3q\frac{f_2^7f_3^5f_4^6}{f_1^{17}f_6^2}\left(\frac{f_1^9f_{12}^3}{f_3^3f_4^9}\right) + 3q\frac{f_1f_4^6f_6^4}{f_2^{11}f_3}\left(\frac{f_1^9f_{12}^3}{f_3^3f_4^9}\right) \\
&\equiv \left\{37\frac{f_1^{76}f_2^5}{f_3^{26}f_4^2} - \frac{f_2^5f_3}{f_1^5f_4^2}\right\} - 36\frac{f_1^{68}f_2^9}{f_3^{18}f_4^2f_6^4} - 3q\frac{f_2^7f_3^5f_4^6}{f_1^{17}f_6^2} + 3q\frac{f_1f_4^6f_6^4}{f_2^{11}f_3} \\
&\quad + \left\{27q^2\frac{f_2^9f_3^5f_4^2f_{12}^4}{f_1^{17}f_6^4} - 27q^2\frac{f_1f_4^2f_6^2f_{12}^4}{f_2^9f_3}\right\} \pmod{81}.
\end{aligned} \tag{41}
$$

In view of Lemma 6, it implies that

$$
\frac{f_1^{76}f_2^5}{f_3^{26}f_4^2} \equiv \frac{f_2^5f_3}{f_1^5f_4^2} \pmod{81}, \qquad \frac{f_1^{68}f_2^9}{f_3^{18}f_4^2f_6^4} \equiv \frac{f_3^6}{f_1^4f_4^2f_6} \pmod{9}, \tag{42}
$$

$$\frac{f_2^9 f_3^5 f_4^2 f_{12}^4}{f_1^{17} f_6^4} \equiv \frac{f_1 f_4^2 f_6^2 f_{12}^4}{f_2^9 f_3} \equiv \frac{f_1 f_4^2 f_{12}^4}{f_3 f_6} \pmod{3}. \tag{43}$$

Plugging (42) and (43) into (41), then using (22) and (35), we obtain

$$\begin{aligned}
&\frac{f_1^{37} f_2^{41} f_4}{f_3^{13} f_6^{12} f_{12}} - \frac{f_2^5 f_3^2 f_4}{f_1^8 f_{12}} \\
&\equiv 36\frac{f_2^5 f_3}{f_1^5 f_4^2} - 36\frac{f_3^6}{f_1^4 f_4^2 f_6} - 3q\frac{f_2^7 f_3^5 f_4^6}{f_1^{17} f_6^2} + 3q\frac{f_1 f_4^6 f_6^4}{f_2^{11} f_3} \\
&\equiv 36\frac{f_6^2}{f_1^2 f_2 f_4^2}\left\{\frac{f_2^6 f_3}{f_1^3 f_6^2} - \frac{f_2 f_3^6}{f_1^2 f_3^3}\right\} - 3q\frac{f_2^7 f_3^5 f_4^6}{f_1^{17} f_6^2} + 3q\frac{f_2 f_3 f_4^6}{f_1^5}\left(\frac{f_1^6 f_6^4}{f_2^{12} f_3^2}\right) \\
&\equiv 36q\frac{f_6^8}{f_1 f_2^3 f_4^2 f_3^3} + 27q^2\frac{f_2^{20} f_3^3 f_4^6}{f_1^{19} f_6} \pmod{81}.
\end{aligned} \tag{44}$$

Applying Lemma 6 and (23), we derive that

$$\begin{aligned}
27q^2\frac{f_2^{20} f_3^3 f_4^6}{f_1^{19} f_6} &\equiv 27q^2\frac{f_2^{17} f_4^6}{f_1^{10}} \equiv 27q\frac{f_2^{21}}{f_1^{10} f_4^2}\left(q\frac{f_4^8}{f_2^4}\right) \\
&\equiv 27q\frac{f_2^{21}}{f_1^{10} f_4^2} - 27q\frac{f_2^{20} f_6^7}{f_1^9 f_4^{10} f_3^3} \\
&\equiv 27q\frac{f_6^8}{f_1 f_2^3 f_4^2 f_3^3} - 27q\frac{f_6^{14}}{f_2 f_3^6 f_4^{10}} \pmod{81}.
\end{aligned} \tag{45}$$

We thus get (40) by substituting (45) into (44). Hence the proof is finished. ☐

Lemma 12. *We have*

$$12q\frac{f_1 f_2^{16} f_4^6}{f_3 f_6^5} + 36q\frac{f_1^2 f_6^8 f_{12}}{f_2^3 f_3^4 f_4^5} + 51q\frac{f_1^4 f_4^3 f_6^4 f_{12}}{f_2^{11} f_3^2} \equiv 27q\frac{f_6^8}{f_1 f_2^3 f_3^3 f_4^2} + 72q\frac{f_1 f_4^6 f_6}{f_2^2 f_3} \pmod{81}. \tag{46}$$

Proof. By (21) and Lemma 6, we obtain

$$\begin{aligned}
&12q\frac{f_1 f_2^{16} f_4^6}{f_3 f_6^5} + 36q\frac{f_1^2 f_6^8 f_{12}}{f_2^3 f_3^4 f_4^5} + 51q\frac{f_1^4 f_4^3 f_6^4 f_{12}}{f_2^{11} f_3^2} \\
&\equiv 12q\frac{f_1 f_2^{16} f_4^6}{f_3 f_6^5} + 36q\frac{f_1^2 f_6^8}{f_2^3 f_3^4 f_4^2}\left(\frac{f_{12}}{f_4^3}\right) + 51q\frac{f_1^4 f_4^6 f_6^4}{f_2^{11} f_3^2}\left(\frac{f_{12}}{f_4^3}\right) \\
&\equiv \left\{51q\frac{f_1 f_4^6 f_6^4}{f_2^{11} f_3} + 12q\frac{f_1 f_2^{16} f_4^6}{f_3 f_6^5}\right\} + 36q\frac{f_6^8}{f_1 f_2^3 f_3^3 f_4^2} + 9q^2\frac{f_1 f_4^2 f_6^2 f_{12}^4}{f_2^9 f_3} + 54q^2\frac{f_6^6 f_{12}^4}{f_1 f_2 f_3^3 f_4^6} \\
&\equiv 63q\frac{f_1 f_4^6 f_6}{f_2^2 f_3} + 36q\frac{f_6^8}{f_1 f_2^3 f_3^3 f_4^2} + 9q^2\frac{f_1 f_4^{11} f_{12}}{f_3 f_6} + 54q^2\frac{f_4^6 f_6^6}{f_1 f_2 f_3^3} \pmod{81}.
\end{aligned} \tag{47}$$

Using (21) and (23), then reducing modulo 81, we find that

$$\begin{aligned}
9q^2\frac{f_1f_4^{11}f_{12}}{f_3f_6}+54q^2\frac{f_4^6f_6^6}{f_1f_2f_3^3} &\equiv 9q^2\frac{f_1f_4^{14}}{f_3f_6}\left(\frac{f_{12}}{f_4^3}\right)+54q^2\frac{f_4^{14}}{f_1^2f_6}\left(\frac{f_1f_6^7}{f_2f_3^3f_4^8}\right)\\
&\equiv -18q^2\frac{f_4^{14}}{f_1^2f_6}+\left\{54q^3\frac{f_2^2f_4^{10}f_{12}^4}{f_1^2f_6^3}+27q^3\frac{f_4^{22}}{f_1^2f_2^4f_6}\right\}\\
&\equiv -18q^2\frac{f_4^{14}}{f_1^2f_6} \pmod{81}, \qquad (48)
\end{aligned}$$

where we use the fact from Lemma 6

$$54q^3\frac{f_4^{10}f_2^2f_{12}^4}{f_1^2f_6^3}+27q^3\frac{f_4^{22}}{f_1^2f_2^4f_6}\equiv 81q^3\frac{f_4^{22}}{f_1^2f_2^7}\equiv 0 \pmod{81}.$$

Applying (24) and Lemma 6, we see

$$\begin{aligned}
-18q^2\frac{f_4^{14}}{f_1^2f_6} &= -9q\frac{f_2^4f_4^6}{f_1^2f_6}\left(2q\frac{f_4^8}{f_2^4}\right)\\
&\equiv 9q\frac{f_1f_2^7f_4^6}{f_3f_6^2}-9q\frac{f_2^{24}}{f_1^{10}f_4^2f_6}\\
&\equiv 9q\frac{f_1f_4^6f_6}{f_2^2f_3}-9q\frac{f_6^8}{f_1f_2^3f_3^3f_4^2} \pmod{81}. \qquad (49)
\end{aligned}$$

Combining (47)–(49), we arrive at (46) to complete the proof. □

Lemma 13. *We have*

$$30\frac{f_4^6f_6^5}{f_1^5f_2^5f_{12}^2}+48\frac{f_2^{13}f_3^2f_{12}^2}{f_1^{11}f_4^6f_6}-3\frac{f_1^{10}f_2^{13}}{f_3^5f_6}+6\frac{f_1f_6^5}{f_2^5f_3^2}\equiv 54q\frac{f_2^6f_4^8}{f_1^5}+27q^2\frac{f_2^2f_4^{16}}{f_1^5} \pmod{81}. \qquad (50)$$

Proof. Substitute (32)–(34) into the left side of (50) and reduce modulo 81 to deduce that

$$\begin{aligned}
&30\frac{f_4^6f_6^5}{f_1^5f_2^5f_{12}^2}+48\frac{f_2^{13}f_3^2f_{12}^2}{f_1^{11}f_4^6f_6}-3\frac{f_1^{10}f_2^{13}}{f_3^5f_6}+6\frac{f_1f_6^5}{f_2^5f_3^2}\\
&=30\frac{f_2^4f_6^2}{f_1^5}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2}\right)+48\frac{f_2^4f_3^2f_6^2}{f_1^{11}}\left(\frac{f_2^9f_{12}^2}{f_4^6f_6^3}\right)-3\frac{f_1^{16}f_2^{10}}{f_3^7}\left(\frac{f_2^3f_3^2}{f_1^6f_6}\right)+6\frac{f_2f_6^3}{f_1^2f_3}\left(\frac{f_1^3f_6^2}{f_2^6f_3}\right)\\
&\equiv\left\{3\frac{f_2^4f_3f_6^2}{f_1^8}-3\frac{f_1^{19}f_2^4f_6^2}{f_3^8}\right\}+72q\frac{f_4^2f_6^3f_{12}^2}{f_1^5f_2^3}+63q\frac{f_2^6f_3f_{12}^4}{f_1^8f_4^4}+\left\{27q\frac{f_2^{17}f_6^3}{f_1^{10}f_3}+54q\frac{f_1^8f_2^{26}}{f_3^7}\right\}.
\end{aligned} \qquad (51)$$

In view of Lemma 6, it is easy to verify

$$\frac{f_1^{19}f_2^4f_6^2}{f_3^8}\equiv\frac{f_2^4f_3f_6^2}{f_1^8} \pmod{27}, \qquad \frac{f_4^2f_6^3f_{12}^2}{f_1^5f_2^3}\equiv\frac{f_2^{15}f_4^2f_{12}^2}{f_1^5f_6^3} \pmod{9}, \qquad (52)$$

$$\frac{f_2^6 f_3 f_{12}^4}{f_1^8 f_4^4} \equiv \frac{f_3 f_4^{14} f_6^3}{f_1^8 f_2^3 f_{12}^2} \pmod 9, \qquad \frac{f_2^{17} f_6^3}{f_1^{10} f_3} \equiv \frac{f_1^8 f_2^{26}}{f_3^7} \equiv \frac{f_6^9}{f_1 f_2 f_3^4} \pmod 3. \tag{53}$$

Substituting (52) and (53) into (51), then using (32), (33) and Lemma 6, we derive that

$$\begin{aligned}
&30\frac{f_4^6 f_6^5}{f_1^5 f_2^5 f_{12}^2} + 48\frac{f_2^{13} f_3^2 f_{12}^2}{f_1^{11} f_4^6 f_6} - 3\frac{f_1^{10} f_2^{13}}{f_3^5 f_6} + 6\frac{f_1 f_6^5}{f_2^5 f_3^2} \\
&\equiv 72q\frac{f_2^{15} f_4^2 f_{12}^2}{f_1^5 f_6^3} + 63q\frac{f_3 f_4^{14} f_6^3}{f_1^8 f_2^3 f_{12}^2} \\
&= 72q\frac{f_2^6 f_4^8}{f_1^5}\left(\frac{f_2^9 f_{12}^2}{f_4^6 f_6^3}\right) + 63q\frac{f_2^6 f_3 f_4^8}{f_1^8}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2}\right) \\
&\equiv \left\{72q\frac{f_2^6 f_4^8}{f_1^2 f_3} + 63q\frac{f_2^6 f_3^2 f_4^8}{f_1^{11}}\right\} + \left\{54q^2\frac{f_2^8 f_4^4 f_{12}^4}{f_1^2 f_3 f_6^2} + 54q^2\frac{f_3 f_4^{10} f_6 f_{12}^2}{f_1^8 f_2}\right\} \\
&\equiv 54q\frac{f_2^6 f_4^8}{f_1^2 f_3} + 27q^2\frac{f_2^2 f_4^{16}}{f_1^5} \pmod{81}.
\end{aligned}$$

Thus (50) is proved. ∎

Lemma 14. *We have*

$$\begin{aligned}
&69\frac{f_1^{10} f_2^{13}}{f_3^5 f_6} + 24\frac{f_4^6 f_6^5}{f_1^5 f_2^5 f_{12}^2} + 6\frac{f_2^{13} f_3^2 f_{12}^2}{f_1^{11} f_4^6 f_6} + 54\frac{f_2^{50}}{f_1^{21} f_4^{16}} + 54q^2\frac{f_2^2 f_4^{16}}{f_1^5} \\
&\equiv 72\frac{f_1 f_2^4 f_6^2}{f_3^2} + 54q\frac{f_2^{26}}{f_1^{13}} \pmod{81}.
\end{aligned} \tag{54}$$

Proof. Using (37)–(39), we find that

$$\begin{aligned}
&24\frac{f_4^6 f_6^5}{f_1^5 f_2^5 f_{12}^2} + 6\frac{f_2^{13} f_3^2 f_{12}^2}{f_1^{11} f_4^6 f_6} + 54\frac{f_2^{50}}{f_1^{21} f_4^{16}} \\
&= 24\frac{f_6^5}{f_1^5 f_2^5}\left(\frac{f_4^6}{f_{12}^2}\right) + 6\frac{f_2^{13} f_3^2}{f_1^{11} f_6}\left(\frac{f_{12}^2}{f_4^6}\right) + 54\frac{f_2^{10}}{f_1^5}\left(\frac{f_2^{40}}{f_1^{16} f_4^{16}}\right) \\
&\equiv 24\frac{f_1 f_6^5}{f_2^5 f_3^2} + 6\frac{f_2^{13} f_3^4}{f_1^{17} f_6} + 54\frac{f_2^{10}}{f_1^5} + \left\{63q\frac{f_4^8 f_6^3}{f_1^2 f_2^3 f_3} + 45q\frac{f_2^{15} f_3^2 f_4^8}{f_1^{11} f_6^3} + 54q\frac{f_2^6 f_4^8}{f_1^5}\right\} \\
&\quad + \left\{54q^2\frac{f_4^{16} f_6^5}{f_1^5 f_2^{13}} + 54q^2\frac{f_2^2 f_4^{16}}{f_1^5}\right\} \pmod{81}.
\end{aligned} \tag{55}$$

In light of Lemma 6, it is readily verifiable that

$$\frac{f_2^{13} f_3^4}{f_1^{17} f_6} \equiv \frac{f_1^{10} f_2^{13}}{f_3^5 f_6} \pmod{27}, \qquad \frac{f_4^8 f_6^3}{f_1^2 f_2^3 f_3} \equiv \frac{f_2^{15} f_3^2 f_4^8}{f_1^{11} f_6^3} \equiv \frac{f_2^6 f_4^8}{f_1^2 f_3} \pmod 9, \tag{56}$$

$$\frac{f_2^6 f_4^8}{f_1^5} \equiv \frac{f_2^6 f_4^8}{f_1^2 f_3} \pmod 3, \qquad \frac{f_4^{16} f_6^5}{f_1^5 f_2^{13}} \equiv \frac{f_2^2 f_4^{16}}{f_1^5} \pmod 3. \tag{57}$$

Plugging (56) and (57) into (55), then reducing modulo 81, we obtain

$$\begin{aligned}
&24\frac{f_4^6 f_6^5}{f_1^5 f_2^5 f_{12}^2}+6\frac{f_2^{13} f_3^2 f_{12}^2}{f_1^{11} f_4^6 f_6}+54\frac{f_2^{50}}{f_1^{21} f_4^{16}}\\
&\equiv 24\frac{f_1 f_6^5}{f_2^5 f_3^2}+6\frac{f_1^{10} f_2^{13}}{f_3^5 f_6}+54\frac{f_2^{10}}{f_1^5}+27q^2\frac{f_2^2 f_4^{16}}{f_1^5} \pmod{81}. \qquad (58)
\end{aligned}$$

Applying (35) and Lemma 6, we are led to

$$\begin{aligned}
24\frac{f_1 f_6^5}{f_2^5 f_3^2} &= 24\frac{f_2^7 f_6}{f_1^5}\left(\frac{f_1^6 f_6^4}{f_2^{12} f_3^2}\right)\\
&\equiv 24\frac{f_3^4 f_2^{13}}{f_1^{17} f_6}+54q\frac{f_2^{26} f_3^2}{f_1^{19}}\\
&\equiv 24\frac{f_1^{10} f_2^{13}}{f_3^5 f_6}+54q\frac{f_2^{26}}{f_1^{13}} \pmod{81}. \qquad (59)
\end{aligned}$$

By (58) and (59), the left side of (54) becomes into

$$\begin{aligned}
&69\frac{f_1^{10} f_2^{13}}{f_3^5 f_6}+24\frac{f_4^6 f_6^5}{f_1^5 f_2^5 f_{12}^2}+6\frac{f_2^{13} f_3^2 f_{12}^2}{f_1^{11} f_4^6 f_6}+54\frac{f_2^{50}}{f_1^{21} f_4^{16}}+54q^2\frac{f_2^2 f_4^{16}}{f_1^5}\\
&\equiv \left\{18\frac{f_1^{10} f_2^{13}}{f_3^5 f_6}+54\frac{f_2^{10}}{f_1^5}\right\}+54q\frac{f_2^{26}}{f_1^{13}}\\
&\equiv 72\frac{f_1 f_2^4 f_6^2}{f_3^2}+54q\frac{f_2^{26}}{f_1^{13}} \pmod{81}, \qquad (60)
\end{aligned}$$

where we use the facts from Lemma 6

$$\frac{f_1^{10} f_2^{13}}{f_3^5 f_6}\equiv\frac{f_1 f_2^4 f_6^2}{f_3^2} \pmod 9, \qquad \frac{f_2^{10}}{f_1^5}\equiv\frac{f_1 f_2^4 f_6^2}{f_3^2} \pmod 3.$$

So (54) follows. □

3 Proof of Theorem 1

In this section, we give the proof of Theorem 1 based on the following two specific generating functions.

Theorem 15. *We have*

$$\sum_{n=0}^{\infty}\left(\overline{t}(4^2n)-\overline{t}(4n)\right)q^n \equiv 27q\frac{f_6^{14}}{f_2 f_3^6 f_4^{10}}+63q\frac{f_6^8}{f_1 f_2^3 f_3^3 f_4^2}+72q\frac{f_1 f_4^6 f_6}{f_2^2 f_3} \pmod{81}. \qquad (61)$$

Proof. We begin by rewriting (1) as

$$\sum_{n=0}^{\infty} \overline{t}(n)q^n = \frac{1}{f_2}\Big(\frac{f_3}{f_1}\Big). \tag{62}$$

Substituting (17) into (62), then extracting all terms of q^{2n} and setting q^2 to q, we obtain

$$\sum_{n=0}^{\infty} \overline{t}(2n)q^n = \frac{f_2 f_8 f_{12}^2}{f_4 f_6 f_{24}}\Big(\frac{f_3}{f_1^3}\Big). \tag{63}$$

Plugging (14) into (63), then extracting all terms of q^{2n} and letting q^2 to q, we find that

$$\sum_{n=0}^{\infty} \overline{t}(4n)q^n = \frac{f_2^5 f_3^2 f_4}{f_1^8 f_{12}} = \frac{f_2^5 f_4}{f_{12}}\Big(\frac{f_3}{f_1^3}\Big)^3\Big(\frac{f_1}{f_3}\Big). \tag{64}$$

Putting (14) and (16) into (64) and reducing modulo 81, then extracting all terms of q^{2n} and setting q^2 to q, we deduce that

$$\sum_{n=0}^{\infty} \overline{t}(8n)q^n \equiv \frac{f_2^{19} f_3^7 f_8 f_{12}^2}{f_1^{21} f_4 f_6^7 f_{24}} + 72q\frac{f_2^{14} f_3^5 f_4^2 f_{24}}{f_1^{19} f_6^2 f_8 f_{12}} + 27q\frac{f_2^{11} f_3^3 f_6 f_8 f_{12}^2}{f_1^{17} f_4 f_{24}} + 54q^2\frac{f_2^6 f_3 f_4^2 f_6^6 f_{24}}{f_1^{15} f_8 f_{12}}. \tag{65}$$

By Lemma 6, it is easy to verify that

$$\frac{f_2^{14} f_3^5 f_4^2 f_{24}}{f_1^{19} f_6^2 f_8 f_{12}} \equiv \frac{f_2^5 f_4^2 f_6 f_{24}}{f_1 f_3 f_8 f_{12}} \pmod 9, \qquad \frac{f_2^{11} f_3^3 f_6 f_8 f_{12}^2}{f_1^{17} f_4 f_{24}} \equiv \frac{f_1 f_4^5 f_6^5}{f_3^3 f_2 f_8^2} \pmod 3, \tag{66}$$

$$\frac{f_2^6 f_3 f_4^2 f_6^6 f_{24}}{f_1^{15} f_8 f_{12}} \equiv \frac{f_6^8 f_8^2}{f_1^{12} f_4} \pmod 3. \tag{67}$$

Substituting (66) and (67) into (65) and reducing modulo 81, we arrive at

$$\sum_{n=0}^{\infty} \overline{t}(8n)q^n \equiv \frac{f_2^{19} f_8 f_{12}^2}{f_4 f_6^7 f_{24}}\Big(\frac{f_3}{f_1^3}\Big)^7 + 72q\frac{f_2^5 f_4^2 f_6 f_{24}}{f_8 f_{12}}\Big(\frac{1}{f_1 f_3}\Big) + 27q\frac{f_4^5 f_6^5}{f_2 f_8^2}\Big(\frac{f_1}{f_3^3}\Big) + 54q^2\frac{f_6^8 f_8^2}{f_4}\Big(\frac{1}{f_1^4}\Big)^3. \tag{68}$$

Plugging (12), (14), (15) and (19) into (68), then extracting all terms of q^{2n} and setting q^2 to q, we get

$$\sum_{n=0}^{\infty} \overline{t}(16n)q^n \equiv \frac{f_2^{41} f_3^{14} f_4}{f_1^{44} f_6^{12} f_{12}} + 72q\frac{f_1 f_2^7 f_{12}^3}{f_3 f_4^3 f_6^2} + 54q\frac{f_2^{41} f_3^8}{f_1^{42} f_4^{10}} + \left\{27q\frac{f_2^{33} f_3^{10} f_4}{f_1^{40} f_6^4 f_{12}} + 54q\frac{f_1^2 f_2^3 f_6^6}{f_3^4 f_4^2}\right\}. \tag{69}$$

Thanks to Lemma 6, it leads to

$$\frac{f_2^{41} f_3^{14} f_4}{f_1^{44} f_6^{12} f_{12}} \equiv \frac{f_1^{37} f_2^{41} f_4}{f_3^{13} f_6^{12} f_{12}} \pmod{81}, \qquad \frac{f_1 f_2^7 f_{12}^3}{f_3 f_4^3 f_6^2} \equiv \frac{f_1 f_4^6 f_6}{f_2^2 f_3} \pmod 9, \tag{70}$$

$$\frac{f_2^{41}f_3^8}{f_1^{42}f_4^{10}} \equiv \frac{f_6^{14}}{f_2f_3^6f_4^{10}} \pmod 3, \qquad \frac{f_2^{33}f_3^{10}f_4}{f_1^{40}f_6^4f_{12}} \equiv \frac{f_1^2f_2^3f_6^6}{f_3^4f_4^2} \pmod 3. \tag{71}$$

Substituting (70) and (71) into (69), then reducing modulo 81, we deduce that

$$\sum_{n=0}^{\infty} \bar{t}(16n)q^n \equiv \frac{f_1^{37}f_2^{41}f_4}{f_3^{13}f_6^{12}f_{12}} + 72q\frac{f_1f_4^6f_6}{f_2^2f_3} + 54q\frac{f_6^{14}}{f_2f_3^6f_4^{10}}. \tag{72}$$

Combining (64) with (72), we obtain

$$\sum_{n=0}^{\infty} \left(\bar{t}(4^2n) - \bar{t}(4n)\right) q^n \equiv \frac{f_1^{37}f_2^{41}f_4}{f_3^{13}f_6^{12}f_{12}} - \frac{f_2^5f_3^2f_4}{f_1^8f_{12}} + 54q\frac{f_6^{14}}{f_2f_3^6f_4^{10}} + 72q\frac{f_1f_4^6f_6}{f_2^2f_3}. \tag{73}$$

Applying Lemma 11, we thus get (61) to complete the proof. □

Theorem 16. *We have*

$$\sum_{n=0}^{\infty} \left(\bar{t}(4^4n) - \bar{t}(4^3n)\right) q^n \equiv 27q\frac{f_6^{14}}{f_2f_3^6f_4^{10}} + 63q\frac{f_6^8}{f_1f_2^3f_3^3f_4^2} + 72q\frac{f_1f_4^6f_6}{f_2^2f_3} \pmod{81}. \tag{74}$$

Proof. Rewriting (72) as

$$\sum_{n=0}^{\infty} \bar{t}(16n)q^n \equiv \frac{f_2^{41}f_4}{f_6^{12}f_{12}}\Big(\frac{f_1^3}{f_3}\Big)^{13}\Big(\frac{1}{f_1^2}\Big) + 72q\frac{f_4^6f_6}{f_2^2}\Big(\frac{f_1}{f_3}\Big) + 54q\frac{f_6^{14}}{f_2f_4^{10}}\Big(\frac{f_1}{f_3^3}\Big)^2\Big(\frac{1}{f_1^2}\Big). \tag{75}$$

Substituting (11), (13), (15) and (16) into (75), then isolating the terms of q^{2n} and setting q^2 to q, we derive that

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{t}(32n)q^n \equiv{}& \frac{f_1^{36}f_2^{40}f_4^5}{f_3^{12}f_6^{14}f_8^2} + 3q\frac{f_1^{38}f_2^{38}f_8^2}{f_3^{14}f_4f_6^{10}} + \left\{54q\frac{f_4^5f_6^8}{f_1^2f_2^{10}f_3^2f_8^2} + 54q\frac{f_1^{40}f_2^{32}f_4^5}{f_3^{16}f_6^6f_8^2}\right\} \\ &+ \left\{27q\frac{f_6^4f_8^2}{f_1^4f_2^4f_4} + 9q\frac{f_2^5f_4^2f_6f_{24}}{f_1f_3f_8f_{12}}\right\} + \left\{27q^2\frac{f_6^{12}f_8^2}{f_2^{12}f_3^4f_4} + 27q^2\frac{f_1^{42}f_2^{30}f_8^2}{f_3^{18}f_4f_6^2}\right\}.\end{aligned} \tag{76}$$

By Lemma 6, it is straightforward to verify that

$$\frac{f_1^{38}f_2^{38}f_8^2}{f_3^{14}f_4f_6^{10}} \equiv \frac{f_1^{11}f_2^{11}f_8^2}{f_3^5f_4f_6} \pmod{27}, \quad \frac{f_4^5f_6^8}{f_1^2f_2^{10}f_3^2f_8^2} \equiv \frac{f_1^{40}f_2^{32}f_4^5}{f_3^{16}f_6^6f_8^2} \equiv \frac{f_1f_4^5f_6^5}{f_3^3f_2f_8^2} \pmod 3, \tag{77}$$

$$\frac{f_6^4f_8^2}{f_1^4f_2^4f_4} \equiv \frac{f_2^5f_4^2f_6f_{24}}{f_1f_3f_8f_{12}} \pmod 3, \qquad \frac{f_6^{12}f_8^2}{f_2^{12}f_3^4f_4} \equiv \frac{f_1^{42}f_2^{30}f_8^2}{f_3^{18}f_4f_6^2} \equiv \frac{f_6^8f_8^2}{f_1^{12}f_4} \pmod 3. \tag{78}$$

Putting (77) and (78) into (76) and reducing modulo 81 yield

$$\sum_{n=0}^{\infty} \bar{t}(32n)q^n \equiv \frac{f_2^{40}f_4^5}{f_6^{14}f_8^2}\Big(\frac{f_1^3}{f_3}\Big)^{12} + 3q\frac{f_2^{11}f_8^2}{f_4f_6}\Big(\frac{f_1^3}{f_3}\Big)^5\Big(\frac{1}{f_1^4}\Big) + 27q\frac{f_4^5f_6^5}{f_2f_8^2}\Big(\frac{f_1}{f_3^3}\Big)$$

$$+36q\frac{f_2^5f_4^2f_6f_{24}}{f_8f_{12}}\Big(\frac{1}{f_1f_3}\Big)+54q^2\frac{f_6^8f_8^2}{f_4}\Big(\frac{1}{f_1^4}\Big)^3. \tag{79}$$

Replacing (12), (13), (15) and (19) into (79) and extracting all terms of q^{2n}, then setting q^2 to q, we find that

$$\sum_{n=0}^{\infty}\bar{t}(64n)q^n \equiv \frac{f_1^{40}f_2^{41}}{f_3^{14}f_4^2f_6^{12}}+\Big\{12q\frac{f_1f_2^{16}f_4^6}{f_3f_6^5}+36q\frac{f_1f_2^7f_{12}^3}{f_3f_4^3f_6^2}\Big\}+\Big\{27q\frac{f_1^{44}f_2^{33}}{f_3^{18}f_4^2f_6^4}+54q\frac{f_1^2f_2^3f_6^6}{f_3^4f_4^2}\Big\}$$
$$+36q\frac{f_2^{24}}{f_1f_3^3f_4^2f_6}+54q\frac{f_2^{41}f_3^8}{f_1^{42}f_4^{10}}+27q^2\frac{f_1^5f_2^8f_4^6f_6^3}{f_3^5}. \tag{80}$$

In light of Lemma 6, it is easy to verify that

$$\frac{f_1f_2^7f_{12}^3}{f_3f_4^3f_6^2}\equiv\frac{f_1f_2^{16}f_4^6}{f_3f_6^5} \pmod 9, \qquad \frac{f_1^{44}f_2^{33}}{f_3^{18}f_4^2f_6^4}\equiv\frac{f_1^2f_2^3f_6^6}{f_3^4f_4^2} \pmod 3, \tag{81}$$

$$\frac{f_2^{24}}{f_1f_3^3f_4^2f_6}\equiv\frac{f_6^8}{f_1f_2^3f_3^3f_4^2} \pmod 9, \qquad \frac{f_2^{41}f_3^8}{f_1^{42}f_4^{10}}\equiv\frac{f_6^{14}}{f_2f_3^6f_4^{10}} \pmod 3, \tag{82}$$

$$\frac{f_1^5f_2^8f_4^6f_6^3}{f_3^5}\equiv\frac{f_4^6f_6^6}{f_1f_2f_3^3} \pmod 3. \tag{83}$$

Substituting (81)–(83) into (80) and reducing modulo 81, we see

$$\sum_{n=0}^{\infty}\bar{t}(64n)q^n \equiv \frac{f_1^{40}f_2^{41}}{f_3^{14}f_4^2f_6^{12}}+48q\frac{f_1f_2^{16}f_4^6}{f_3f_6^5}+36q\frac{f_6^8}{f_1f_2^3f_3^3f_4^2}+54q\frac{f_6^{14}}{f_2f_3^6f_4^{10}}+27q^2\frac{f_4^6f_6^6}{f_1f_2f_3^3}. \tag{84}$$

Rewriting (84) as

$$\sum_{n=0}^{\infty}\bar{t}(64n)q^n \equiv \frac{f_2^{41}}{f_4^2f_6^{12}}\Big(\frac{f_1^3}{f_3}\Big)^{14}\Big(\frac{1}{f_1^2}\Big)+48q\frac{f_2^{16}f_4^6}{f_6^5}\Big(\frac{f_1}{f_3}\Big)+36q\frac{f_6^8}{f_2^3f_4^2}\Big(\frac{f_1}{f_3^3}\Big)\Big(\frac{1}{f_1^2}\Big)$$
$$+54q\frac{f_6^{14}}{f_2f_4^{10}}\Big(\frac{f_1}{f_3^3}\Big)^2\Big(\frac{1}{f_1^2}\Big)+27q^2\frac{f_4^6f_6^6}{f_2}\Big(\frac{f_1}{f_3^3}\Big)\Big(\frac{1}{f_1^2}\Big). \tag{85}$$

Plugging (11), (13), (15) and (16) into (85), extracting all terms of q^{2n} and setting q^2 to q, we deduce that

$$\sum_{n=0}^{\infty}\bar{t}(128n)q^n \equiv \frac{f_1^{36}f_2^{40}f_4^5}{f_3^{12}f_6^{14}f_8^2}+78q\frac{f_1^{38}f_2^{38}f_8^2}{f_3^{14}f_4f_6^{10}}+33q\frac{f_1^{17}f_2^5f_4^2f_6f_{24}}{f_3^7f_8f_{12}}+72q\frac{f_2^2f_3f_6^2f_8^2}{f_1^7f_4}+27q\frac{f_6^4f_8^2}{f_1^4f_2^4f_4}$$
$$+\Big\{9q\frac{f_1^{40}f_2^{32}f_4^5}{f_3^{16}f_6^6f_8^2}+45q\frac{f_4^5f_6^6}{f_1^5f_2^4f_3f_8^2}\Big\}+\Big\{27q\frac{f_2^8f_4^5f_6^2}{f_1^5f_3f_8^2}+54q\frac{f_4^5f_6^8}{f_1^2f_2^{10}f_3^2f_8^2}\Big\}$$
$$+\Big\{27q^2\frac{f_2^6f_6^6f_8^2}{f_1^3f_3^3f_4}+27q^2\frac{f_6^{12}f_8^2}{f_2^{12}f_3^4f_4}+27q^2\frac{f_1^{42}f_2^{30}f_8^2}{f_3^{18}f_4f_6^2}\Big\}. \tag{86}$$

By Lemma 6, it follows that

$$\frac{f_1^{38}f_2^{38}f_8^2}{f_3^{14}f_4f_6^{10}}\equiv\frac{f_1^{11}f_2^{11}f_8^2}{f_3^5f_4f_6}\pmod{27},\qquad \frac{f_1^{17}f_2^5f_4^2f_6f_{24}}{f_3^7f_8f_{12}}\equiv\frac{f_3^2f_2^5f_4^2f_6f_{24}}{f_1^{10}f_8f_{12}}\pmod{27},\tag{87}$$

$$\frac{f_6^4f_8^2}{f_1^4f_2^4f_4}\equiv\frac{f_6^3f_8^2}{f_1^4f_2f_4}\pmod{3},\qquad \frac{f_1^{40}f_2^{32}f_4^5}{f_3^{16}f_6^6f_8^2}\equiv\frac{f_4^5f_6^6}{f_1^5f_2^4f_3f_8^2}\pmod{9},\tag{88}$$

$$\frac{f_4^5f_6^6}{f_1^5f_2^4f_3f_8^2}\equiv\frac{f_1f_4^5f_6^5}{f_2f_3^3f_8^2}\pmod{3},\qquad \frac{f_2^8f_4^5f_6^2}{f_1^5f_3f_8^2}\equiv\frac{f_4^5f_6^8}{f_1^2f_2^{10}f_3^2f_8^2}\equiv\frac{f_1f_4^5f_6^5}{f_2f_3^3f_8^2}\pmod{3},\tag{89}$$

$$\frac{f_2^6f_6^6f_8^2}{f_1^3f_3^3f_4}\equiv\frac{f_6^{12}f_8^2}{f_2^{12}f_3^4f_4}\equiv\frac{f_1^{42}f_2^{30}f_8^2}{f_3^{18}f_4f_6^2}\equiv\frac{f_6^8f_8^2}{f_3^4f_4}\pmod{3}.\tag{90}$$

Substituting (87)–(90) into (86) and reducing modulo 81 follow that

$$\sum_{n=0}^{\infty}\overline{t}(128n)q^n\equiv\frac{f_2^{40}f_4^5}{f_6^{14}f_8^2}\left(\frac{f_1^3}{f_3}\right)^{12}+78q\frac{f_2^{11}f_8^2}{f_4f_6}\left(\frac{f_1^3}{f_3}\right)^5\left(\frac{1}{f_1^4}\right)+33q\frac{f_2^5f_4^2f_6f_{24}}{f_8f_{12}}\left(\frac{f_3}{f_1^3}\right)^2\left(\frac{1}{f_1^4}\right)$$
$$+72q\frac{f_2^2f_6^2f_8^2}{f_4}\left(\frac{f_3}{f_1^3}\right)\left(\frac{1}{f_1^4}\right)+27q\frac{f_6^3f_8^2}{f_2f_4}\left(\frac{1}{f_1^4}\right)+54q\frac{f_4^5f_6^5}{f_2f_8^2}\left(\frac{f_1}{f_3^3}\right).\tag{91}$$

Putting (12)–(15) into (91), extracting all terms of q^{2n}, and setting q^2 to q, we obtain

$$\sum_{n=0}^{\infty}\overline{t}(256n)q^n\equiv\frac{f_1^{40}f_2^{41}}{f_3^{14}f_4^2f_6^{12}}+51q\frac{f_2^{16}f_3^7f_4^3f_{12}}{f_1^{23}f_6^5}+36q\frac{f_2^{24}f_3^5f_{12}}{f_1^{25}f_4^5f_6}$$
$$+\left\{69q\frac{f_1f_2^{16}f_4^6}{f_3f_6^5}+45q\frac{f_2^7f_3^5f_4^6}{f_1^{17}f_6^2}+27q\frac{f_2f_3^3f_4^6}{f_1^{11}}\right\}$$
$$+\left\{45q\frac{f_2^{24}}{f_1f_3^3f_4^2f_6}+27q\frac{f_1^2f_2^3f_6^6}{f_3^4f_4^2}+54q\frac{f_2^{15}f_3^3f_6^2}{f_1^{19}f_4^2}+27q\frac{f_1^{44}f_2^{33}}{f_3^{18}f_4^2f_6^4}\right\}$$
$$+\left\{54q^2\frac{f_1^5f_2^8f_4^6f_6^3}{f_3^5}+54q^2\frac{f_2^8f_3^3f_4^3f_6^3f_{12}}{f_1^{19}}\right\}.\tag{92}$$

In light of Lemma 6, it is readily verifiable that

$$\frac{f_2^{16}f_3^7f_4^3f_{12}}{f_1^{23}f_6^5}\equiv\frac{f_1^4f_4^3f_6^4f_{12}}{f_2^{11}f_3^2}\pmod{27},\qquad \frac{f_2^{24}f_3^5f_{12}}{f_1^{25}f_4^5f_6}\equiv\frac{f_1^2f_6^8f_{12}}{f_2^3f_3^4f_4^5}\pmod{9},\tag{93}$$

$$\frac{f_2^7f_3^5f_4^6}{f_1^{17}f_6^2}\equiv\frac{f_1f_2^{16}f_4^6}{f_3f_6^5}\pmod{9},\qquad \frac{f_2f_3^3f_4^6}{f_1^{11}}\equiv\frac{f_1f_2^{16}f_4^6}{f_3f_6^5}\pmod{3},\tag{94}$$

$$\frac{f_2^{24}}{f_1f_3^3f_4^2f_6}\equiv\frac{f_6^8}{f_1f_2^3f_3^3f_4^2}\pmod{9},\qquad \frac{f_1^2f_2^3f_6^6}{f_3^4f_4^2}\equiv\frac{f_2^{15}f_3^3f_6^2}{f_1^{19}f_4^2}\equiv\frac{f_6^8}{f_1f_2^3f_3^3f_4^2}\pmod{3},\tag{95}$$

$$\frac{f_1^{44}f_2^{33}}{f_3^{18}f_4^2f_6^4}\equiv\frac{f_6^8}{f_1f_2^3f_3^3f_4^2}\pmod{3},\qquad \frac{f_1^5f_2^8f_4^6f_6^3}{f_3^5}\equiv\frac{f_2^8f_3^3f_4^3f_6^3f_{12}}{f_1^{19}}\equiv\frac{f_4^6f_6^6}{f_1f_2f_3^3}\pmod{3}.\tag{96}$$

Substituting (93)–(96) into (92) and reducing modulo 81, we are led to

$$\sum_{n=0}^{\infty} \bar{t}(256n)q^n \equiv \frac{f_1^{40} f_2^{41}}{f_3^{14} f_4^2 f_6^{12}} + 51q \frac{f_1^4 f_4^3 f_6^4 f_{12}}{f_2^{11} f_3^2} + 36q \frac{f_1^2 f_6^8 f_{12}}{f_2^3 f_3^4 f_4^5} + 60q \frac{f_1 f_2^{16} f_4^6}{f_3 f_6^5} + 72q \frac{f_6^8}{f_1 f_2^3 f_3^3 f_4^2} + 27q^2 \frac{f_4^6 f_6^6}{f_1 f_2 f_3^3}. \tag{97}$$

Combining (84) with (97), we conclude that

$$\sum_{n=0}^{\infty} \left(\bar{t}(4^4 n) - \bar{t}(4^3 n)\right) q^n \equiv 12q \frac{f_1 f_2^{16} f_4^6}{f_3 f_6^5} + 36q \frac{f_1^2 f_6^8 f_{12}}{f_2^3 f_3^4 f_4^5} + 51q \frac{f_1^4 f_4^3 f_6^4 f_{12}}{f_2^{11} f_3^2} + 36q \frac{f_6^8}{f_1 f_2^3 f_3^3 f_4^2} + 27q \frac{f_6^{14}}{f_2 f_3^6 f_4^{10}} \pmod{81}. \tag{98}$$

Thanks to Lemma 12 and (98), we imply (74) to finish the proof. □

Proof of Theorem 1. From (61) and (74), we obtain that

$$\bar{t}(4^4 n) - \bar{t}(4^3 n) \equiv \bar{t}(4^2 n) - \bar{t}(4n) \pmod{81}. \tag{99}$$

Hence (3) follows from (99). And the proof of Theorem 1 is complete. □

4 Proofs of Theorem 2 and Theorem 3

In order to prove Theorem 2 and Theorem 3, we need the following two theorems.

Theorem 17. *For $n \geqslant 0$,*

$$\bar{t}(4(8n+5)) \equiv 0 \pmod{81}. \tag{100}$$

Proof. In (64), substituting (14) and (16) into it and extracting all terms of q^{2n+1}, then dividing through by q and setting q^2 to q, we find that

$$\sum_{n=0}^{\infty} \bar{t}(4(2n+1))q^n \equiv -\frac{f_2^{18} f_3^7 f_4^2 f_{24}}{f_1^{21} f_6^6 f_8 f_{12}} + 9 \frac{f_2^{15} f_3^5 f_8 f_{12}^2}{f_1^{19} f_4 f_6^3 f_{24}} + 27q \frac{f_2^7 f_3 f_6^5 f_8 f_{12}^2}{f_1^{15} f_4 f_{24}} + 54q \frac{f_2^{10} f_3^3 f_4^2 f_6^2 f_{24}}{f_1^{17} f_8 f_{12}}. \tag{101}$$

By Lemma 6, it is easy to verify that

$$\frac{f_2^{15} f_3^5 f_8 f_{12}^2}{f_1^{19} f_4 f_6^3 f_{24}} \equiv \frac{f_6^3 f_8 f_{12}^2}{f_1 f_3 f_2^3 f_4 f_{24}} \pmod{9}, \qquad \frac{f_2^7 f_3 f_6^5 f_8 f_{12}^2}{f_1^{15} f_4 f_{24}} \equiv \frac{f_2 f_4^5 f_6^7}{f_1^{12} f_8^2} \pmod{3}, \tag{102}$$

$$\frac{f_2^{10} f_3^3 f_4^2 f_6^2 f_{24}}{f_1^{17} f_8 f_{12}} \equiv \frac{f_1 f_2 f_6^5 f_8^2}{f_3^3 f_4} \pmod{3}. \tag{103}$$

Plugging (102) and (103) into (101), we derive that

$$\sum_{n=0}^{\infty}\overline{t}(4(2n+1))q^n \equiv -\frac{f_2^{18}f_4^2f_{24}}{f_6^6f_8f_{12}}\left(\frac{f_3}{f_1^3}\right)^7+9\frac{f_6^3f_8f_{12}^2}{f_2^3f_4f_{24}}\left(\frac{1}{f_1f_3}\right)+27q\frac{f_2f_4^5f_6^7}{f_8^2}\left(\frac{1}{f_1^4}\right)^3$$
$$+54q\frac{f_2f_6^5f_8^2}{f_4}\left(\frac{f_1}{f_3^3}\right). \tag{104}$$

Putting (12), (14), (15) and (19) into (104), then extracting all terms of q^{2n} and replacing q^2 by q, we deduce that

$$\sum_{n=0}^{\infty}\overline{t}(4(4n+1))q^n \equiv -\frac{f_2^{44}f_3^{15}f_{12}}{f_1^{45}f_4f_6^{15}}+9\frac{f_4^3f_6^7}{f_1^5f_2^2f_3f_{12}^3}+\left\{27q\frac{f_1^4f_4^2f_6^6}{f_2^3f_3^4}+54q\frac{f_2^{36}f_3^{11}f_{12}}{f_1^{41}f_4f_6^7}\right\}$$
$$+27q^2\frac{f_2^{11}f_3^7f_4^{10}}{f_1^{29}}. \tag{105}$$

Thanks to Lemma 6, it is readily verifiable that

$$\frac{f_4^3f_6^7}{f_1^5f_2^2f_3f_{12}^3}\equiv\frac{f_1^4f_4^3f_6^7}{f_3^4f_2^2f_{12}^3} \pmod 9, \qquad \frac{f_2^{11}f_3^7f_4^{10}}{f_1^{29}}\equiv\frac{f_1f_4^{10}f_6^4}{f_2f_3^3} \pmod 3, \tag{106}$$

$$\frac{f_1^4f_4^2f_6^6}{f_2^3f_3^4}\equiv\frac{f_2^{36}f_3^{11}f_{12}}{f_1^{41}f_4f_6^7}\equiv\frac{f_1f_4^2f_6^5}{f_3^3} \pmod 3. \tag{107}$$

Substituting (106) and (107) into (105) and reducing modulo 81, we see

$$\sum_{n=0}^{\infty}\overline{t}(4(4n+1))q^n \equiv -\frac{f_2^{44}f_{12}}{f_4f_6^{15}}\left(\frac{f_3}{f_1^3}\right)^{15}+9\frac{f_4^3f_6^7}{f_2^2f_{12}^3}\left(\frac{f_1^3}{f_3}\right)\left(\frac{f_1}{f_3^3}\right)+27q^2\frac{f_4^{10}f_6^4}{f_2}\left(\frac{f_1}{f_3^3}\right). \tag{108}$$

Putting (13)–(15) into (108) and extracting all terms of q^{2n+1}, then dividing both sides by q and letting q^2 to q, we obtain

$$\sum_{n=0}^{\infty}\overline{t}(4(8n+5))q^n \equiv \left\{36\frac{f_2^{85}f_3^{28}}{f_1^{89}f_6^{25}}+45\frac{f_1f_2^4f_6^2}{f_3^2}\right\}+\left\{27q\frac{f_2^{77}f_3^{24}}{f_1^{85}f_6^{17}}+54q\frac{f_1^2f_2^8f_6^6}{f_3^5}\right\}. \tag{109}$$

Following from Lemma 6, it is straightforward to verify

$$36\frac{f_2^{85}f_3^{28}}{f_1^{89}f_6^{25}}+45\frac{f_1f_2^4f_6^2}{f_3^2}\equiv 81\frac{f_1f_2^4f_6^2}{f_3^2}\equiv 0 \pmod{81}, \tag{110}$$

$$27q\frac{f_2^{77}f_3^{24}}{f_1^{85}f_6^{17}}+54q\frac{f_1^2f_2^8f_6^6}{f_3^5}\equiv 81q\frac{f_2^2f_6^8}{f_1f_3^4}\equiv 0 \pmod{81}. \tag{111}$$

Combining (109)–(111), we arrive at (100). This completes the proof. □

Theorem 18. *For $n \geqslant 0$,*

$$\overline{t}(4^3(8n+5))\equiv 0 \pmod{81}. \tag{112}$$

Proof. In (85), plugging (11), (13), (15) and (16) into it and extracting all terms of q^{2n+1}, then dividing both sides by q and setting q^2 to q, we find that

$$\sum_{n=0}^{\infty}\bar{t}(64(2n+1))q^n \equiv 2\frac{f_1^{36}f_2^{42}f_8^2}{f_3^{12}f_4f_6^{14}}+48\frac{f_1^{17}f_2^6f_8f_{12}^2}{f_3^7f_4f_{24}}+\left\{39\frac{f_1^{38}f_2^{36}f_4^5}{f_3^{14}f_6^{10}f_8^2}+36\frac{f_3f_4^5f_6^2}{f_1^7f_8^2}+54\frac{f_4^5f_6^4}{f_1^4f_2^6f_8^2}\right\}$$
$$+\left\{18q\frac{f_1^{40}f_2^{34}f_8^2}{f_3^{16}f_4f_6^6}+9q\frac{f_6^6f_8^2}{f_1^5f_2^2f_3f_4}\right\}+\left\{27q\frac{f_6^8f_8^2}{f_1^2f_2^8f_3^2f_4}+54q\frac{f_2^{10}f_6^2f_8^2}{f_1^5f_3f_4}\right\}$$
$$+\left\{54q\frac{f_1^{42}f_2^{28}f_4^5}{f_3^{18}f_6^2f_8^2}+54q\frac{f_4^5f_6^{12}}{f_2^{14}f_3^4f_8^2}+54q\frac{f_2^4f_4^5f_6^6}{f_1^3f_3^3f_8^2}\right\}. \tag{113}$$

By Lemma 6, it is easy to verify

$$\frac{f_1^{17}f_2^6f_8f_{12}^2}{f_3^7f_4f_{24}}\equiv\frac{f_3^2f_2^6f_8f_{12}^2}{f_1^{10}f_4f_{24}}\pmod{27},\qquad \frac{f_1^{38}f_2^{36}f_4^5}{f_3^{14}f_6^{10}f_8^2}\equiv\frac{f_1^{11}f_2^9f_4^5}{f_3^5f_6f_8^2}\pmod{27}, \tag{114}$$

$$\frac{f_3f_4^5f_6^2}{f_1^7f_8^2}\equiv\frac{f_1^{11}f_2^9f_4^5}{f_3^5f_6f_8^2}\pmod 9,\qquad \frac{f_4^5f_6^4}{f_1^4f_2^6f_8^2}\equiv\frac{f_1^{11}f_2^9f_4^5}{f_3^5f_6f_8^2}\pmod 3, \tag{115}$$

$$\frac{f_1^{40}f_2^{34}f_8^2}{f_3^{16}f_4f_6^6}\equiv\frac{f_6^6f_8^2}{f_1^5f_2^2f_3f_4}\pmod 9,\qquad \frac{f_6^6f_8^2}{f_1^5f_2^2f_3f_4}\equiv\frac{f_1f_6^6f_8^2}{f_3^3f_2^2f_4}\pmod 3, \tag{116}$$

$$\frac{f_6^8f_8^2}{f_1^2f_2^8f_3^2f_4}\equiv\frac{f_2^{10}f_6^2f_8^2}{f_1^5f_3f_4}\equiv\frac{f_1f_2f_6^5f_8^2}{f_3^3f_4}\pmod 3, \tag{117}$$

$$\frac{f_1^{42}f_2^{28}f_4^5}{f_3^{18}f_6^2f_8^2}\equiv\frac{f_4^5f_6^{12}}{f_2^{14}f_3^4f_8^2}\equiv\frac{f_2^4f_4^5f_6^6}{f_1^3f_3^3f_8^2}\equiv\frac{f_2f_4^5f_6^7}{f_3^4f_8^2}\pmod 3. \tag{118}$$

Substituting (114)–(118) into (113) and reducing modulo 81 lead to

$$\sum_{n=0}^{\infty}\bar{t}(64(2n+1))q^n \equiv 2\frac{f_2^{42}f_8^2}{f_4f_6^{14}}\left(\frac{f_1^3}{f_3}\right)^{12}+48\frac{f_2^6f_8f_{12}^2}{f_4f_{24}}\left(\frac{f_3}{f_1^3}\right)^2\left(\frac{1}{f_1^4}\right)$$
$$+48\frac{f_2^9f_4^5}{f_6f_8^2}\left(\frac{f_1^3}{f_3}\right)^5\left(\frac{1}{f_1^4}\right)+27q\frac{f_6^6f_8^2}{f_2^2f_4}\left(\frac{f_1}{f_3^3}\right). \tag{119}$$

Putting (12)–(15) into (119) and extracting all terms of q^{2n}, then replacing q^2 by q, we derive that

$$\sum_{n=0}^{\infty}\bar{t}(64(4n+1))q^n \equiv 2\frac{f_1^{42}f_2^{35}f_4^2}{f_3^{14}f_6^{12}}+48\frac{f_2^{34}}{f_1^5f_3f_4^6f_6^5}+48\frac{f_2^{25}f_3^6}{f_1^{26}f_4^3f_6^2f_{12}}+18q\frac{f_2^9f_3^4f_4^5f_6^2}{f_1^{20}f_{12}}$$
$$+\left\{54q\frac{f_1f_4^2f_6^6}{f_2^3f_3^3}+54q\frac{f_1^{46}f_2^{27}f_4^2}{f_3^{18}f_6^4}+36q\frac{f_1f_2^{18}f_4^2}{f_3^3f_6}\right\}$$
$$+\left\{27q\frac{f_2^{26}f_6^3}{f_1f_3^5f_4^6}+27q\frac{f_2^{17}f_3^2f_6^6}{f_1^{22}f_4^3f_{12}}\right\}. \tag{120}$$

In view of Lemma 6, it is straightforward to verify that

$$\frac{f_2^{34}}{f_1^5 f_3 f_4^6 f_6^5} \equiv \frac{f_2^7 f_6^4}{f_1^5 f_3 f_4^6} \pmod{27}, \qquad \frac{f_2^{25} f_3^6}{f_1^{26} f_4^3 f_6^2 f_{12}} \equiv \frac{f_1 f_6^7}{f_3^3 f_2^2 f_4^3 f_{12}} \pmod{27}, \tag{121}$$

$$\frac{f_2^9 f_3^4 f_4^5 f_6^2}{f_1^{20} f_{12}} \equiv \frac{f_4^5 f_6^5}{f_1^2 f_3^2 f_{12}} \pmod{9}, \qquad \frac{f_1 f_4^2 f_6^6}{f_2^3 f_3^3} \equiv \frac{f_1^{46} f_2^{27} f_4^2}{f_3^{18} f_6^4} \equiv \frac{f_1 f_4^2 f_6^5}{f_3^3} \pmod{3}, \tag{122}$$

$$\frac{f_1 f_2^{18} f_4^2}{f_3^3 f_6} \equiv \frac{f_1 f_4^2 f_6^5}{f_3^3} \pmod{9}, \qquad \frac{f_2^{26} f_6^3}{f_1 f_3^5 f_4^6} \equiv \frac{f_2^{17} f_3^2 f_6^6}{f_1^{22} f_4^3 f_{12}} \equiv \frac{f_6^{12}}{f_1 f_3^5 f_2 f_4^6} \pmod{3}. \tag{123}$$

Substituting (121)–(123) into (120) and reducing modulo 81, we deduce that

$$\sum_{n=0}^{\infty} \bar{t}(64(4n+1))q^n \equiv 2\frac{f_2^{35} f_4^2}{f_6^{12}}\Big(\frac{f_1^3}{f_3}\Big)^{14} + 48\frac{f_2^7 f_6^4}{f_4^6}\Big(\frac{1}{f_1 f_3}\Big)\Big(\frac{1}{f_1^4}\Big) + 48\frac{f_6^7}{f_2^2 f_4^3 f_{12}}\Big(\frac{f_1}{f_3^3}\Big)$$
$$+ 18q\frac{f_4^5 f_6^5}{f_{12}}\Big(\frac{f_1}{f_3^3}\Big)\Big(\frac{f_3}{f_1^3}\Big) + 63q f_4^2 f_6^5\Big(\frac{f_1}{f_3^3}\Big) + 54q\frac{f_6^{12}}{f_2 f_4^6}\Big(\frac{f_1}{f_3^3}\Big)^2\Big(\frac{f_3}{f_1^3}\Big). \tag{124}$$

Putting (12)–(15) and (19) into (124) and extracting all terms of q^{2n+1}, then dividing through by q and setting q^2 to q, we arrive at

$$\sum_{n=0}^{\infty} \bar{t}(64(8n+5))q^n \equiv 30\frac{f_4^6 f_6^5}{f_1^5 f_2^5 f_{12}^2} + 48\frac{f_2^{13} f_3^2 f_{12}^2}{f_1^{11} f_4^6 f_6} + 78\frac{f_1^{37} f_2^{40}}{f_3^{14} f_6^{10}} + \left\{33\frac{f_1 f_6^5}{f_2^5 f_3^2} + 54\frac{f_2^4 f_3 f_6^2}{f_1^8}\right\}$$
$$+ \left\{63\frac{f_1 f_2^4 f_6^2}{f_3^2} + 18\frac{f_2^{13} f_3}{f_1^8 f_6}\right\} + \left\{27q\frac{f_2^5 f_6^7}{f_1^4 f_3^3} + 54q\frac{f_6^{10}}{f_1^4 f_2^4 f_3^3} + 27q\frac{f_1^{41} f_2^{32}}{f_3^{18} f_6^2}\right\}. \tag{125}$$

Thanks to Lemma 6, it is easy to check that

$$\frac{f_1^{37} f_2^{40}}{f_3^{14} f_6^{10}} \equiv \frac{f_1^{10} f_2^{13}}{f_3^5 f_6} \pmod{27}, \qquad \frac{f_2^4 f_3 f_6^2}{f_1^8} \equiv \frac{f_1 f_6^5}{f_2^5 f_3^2} \pmod{3}, \tag{126}$$

$$\frac{f_1 f_2^4 f_6^2}{f_3^2} \equiv \frac{f_2^{13} f_3}{f_1^8 f_6} \pmod{9}, \qquad \frac{f_2^5 f_6^7}{f_1^4 f_3^3} \equiv \frac{f_6^{10}}{f_1^4 f_2^4 f_3^3} \equiv \frac{f_1^{41} f_2^{32}}{f_3^{18} f_6^2} \equiv \frac{f_2^{26}}{f_1^{13}} \pmod{3}. \tag{127}$$

Putting (126) and (127) into (125) and reducing modulo 81, we are led to

$$\sum_{n=0}^{\infty} \bar{t}(64(8n+5))q^n \equiv 30\frac{f_4^6 f_6^5}{f_1^5 f_2^5 f_{12}^2} + 48\frac{f_2^{13} f_3^2 f_{12}^2}{f_1^{11} f_4^6 f_6} - 3\frac{f_1^{10} f_2^{13}}{f_3^5 f_6} + 6\frac{f_1 f_6^5}{f_2^5 f_3^2} + 27q\frac{f_2^{26}}{f_1^{13}}. \tag{128}$$

By Lemma 13 and (128), we obtain

$$\sum_{n=0}^{\infty} \bar{t}(64(8n+5))q^n \equiv 27q\frac{f_2^{26}}{f_1^{13}} + 54q\frac{f_2^6 f_4^8}{f_1^5} + 27q^2\frac{f_2^2 f_4^{16}}{f_1^5}$$
$$\equiv 27q\frac{f_2^{10}}{f_1^5}\left\{\frac{f_2^{16}}{f_1^8} + 2\frac{f_4^8}{f_2^4} + q\frac{f_4^{16}}{f_2^8}\right\} \pmod{81}. \tag{129}$$

Substituting (36) into (129), we get (112). So the proof is finished. □

Proof of Theorem 2. Define

$$b(n) := \bar{t}(4^3n) - \bar{t}(4n). \tag{130}$$

From (3), we see

$$b(4n) \equiv b(n) \pmod{81}. \tag{131}$$

And by (100) and (112), we imply

$$b(8n+5) \equiv 0 \pmod{81}. \tag{132}$$

For any nonnegative integer α, setting $n \to 4^\alpha(8n+5)$ in (131) leads to

$$b(4^{\alpha+1}(8n+5)) \equiv b(4^\alpha(8n+5)) \pmod{81}. \tag{133}$$

Combining (132) with (133), we arrive at for $\alpha \geqslant 0$ and $n \geqslant 0$,

$$b(4^\alpha(8n+5)) \equiv 0 \pmod{81}. \tag{134}$$

Substituting (134) back to (130), we derive that for $\alpha \geqslant 0$ and $n \geqslant 0$,

$$\bar{t}(4^{\alpha+3}(8n+5)) \equiv \bar{t}(4^{\alpha+1}(8n+5)) \pmod{81}.$$

So (4) is proved. □

Proof of Theorem 3. Combining (4) with (100), we find for $\alpha \geqslant 0$ and $n \geqslant 0$,

$$\bar{t}(4^{2\alpha+1}(8n+5)) \equiv \bar{t}(4(8n+5)) \equiv 0 \pmod{81}.$$

Thus we arrive at (5). □

5 Proof of Theorem 4

Proof of Theorem 4. By Theorem 2, it suffices to show the case $\alpha = 1$, that is

$$\sum_{n=0}^{\infty} \bar{t}\left(16(8n+5)\right) q^n \equiv 72\frac{f_1 f_2^4 f_6^2}{f_3^2} + 54q\frac{f_2^{26}}{f_1^{13}} \pmod{81}. \tag{135}$$

In (75), substituting (11), (13), (15) and (16) into it, isolating all terms of q^{2n+1}, then dividing both sides by q and setting q^2 to q, we derive that

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{t}\left(16(2n+1)\right) q^n \equiv{}& 2\frac{f_1^{36} f_2^{42} f_8^2}{f_3^{12} f_4 f_6^{14}} + 42\frac{f_1^{38} f_2^{36} f_4^5}{f_3^{14} f_6^{10} f_8^2} + \left\{72\frac{f_2^6 f_8 f_{12}^2}{f_1 f_3 f_4 f_{24}} + 54\frac{f_4^5 f_6^4}{f_1^4 f_2^6 f_8^2}\right\}\\ &+ 27q\left\{\frac{f_6^8 f_8^2}{f_1^2 f_2^8 f_3^2 f_4} + \frac{f_1^{40} f_2^{34} f_8^2}{f_3^{16} f_4 f_6^6}\right\} + 54q\left\{\frac{f_4^5 f_6^{12}}{f_2^{14} f_3^4 f_8^2} + \frac{f_1^{42} f_2^{28} f_4^5}{f_3^{18} f_6^2 f_8^2}\right\}.\end{aligned} \tag{136}$$

In light of Lemma 6, it is easy to verify that

$$\frac{f_1^{38}f_2^{36}f_4^5}{f_3^{14}f_6^{10}f_8^2} \equiv \frac{f_1^{11}f_2^9f_4^5}{f_3^5f_6f_8^2} \pmod{27}, \qquad \frac{f_4^5f_6^4}{f_1^4f_2^6f_8^2} \equiv \frac{f_2^6f_8f_{12}^2}{f_1f_3f_4f_{24}} \pmod{3}, \tag{137}$$

$$\frac{f_6^8f_8^2}{f_1^2f_2^8f_3^2f_4} \equiv \frac{f_1^{40}f_2^{34}f_8^2}{f_3^{16}f_4f_6^6} \equiv \frac{f_1f_2f_6^5f_8^2}{f_3^3f_4} \pmod{3}, \tag{138}$$

$$\frac{f_4^5f_6^{12}}{f_2^{14}f_3^4f_8^2} \equiv \frac{f_1^{42}f_2^{28}f_4^5}{f_3^{18}f_6^2f_8^2} \equiv \frac{f_2f_4^5f_6^7}{f_1^{12}f_8^2} \pmod{3}. \tag{139}$$

Substituting (137)–(139) into (136) and applying modulo 81, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{t}\left(16(2n+1)\right)q^n \equiv{}& 2\frac{f_2^{42}f_8^2}{f_4f_6^{14}}\left(\frac{f_1^3}{f_3}\right)^{12} + 42\frac{f_2^9f_4^5}{f_6f_8^2}\left(\frac{f_1^3}{f_3}\right)^5\left(\frac{1}{f_1^4}\right) + 45\frac{f_2^6f_8f_{12}^2}{f_4f_{24}}\left(\frac{1}{f_1f_3}\right)\\ &+ 54q\frac{f_2f_6^5f_8^2}{f_4}\left(\frac{f_1}{f_3^3}\right) + 27q\frac{f_2f_4^5f_6^7}{f_8^2}\left(\frac{1}{f_1^4}\right)^3.\end{aligned} \tag{140}$$

Putting (12), (13), (15) and (19) into (140) and extracting all terms of q^{2n}, then replacing q^2 by q yield that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{t}\left(16(4n+1)\right)q^n \equiv{}& 2\frac{f_1^{42}f_2^{35}f_4^2}{f_3^{14}f_6^{12}} + \left\{42\frac{f_2^{34}}{f_1^5f_3f_4^6f_6^5} + 45\frac{f_1^4f_4^3f_6^7}{f_2^2f_3^4f_{12}^3}\right\} + 72q\frac{f_1f_2^{18}f_4^2}{f_3^3f_6}\\ &+ \left\{27q\frac{f_1^4f_4^2f_6^6}{f_2^3f_3^4} + 54q\frac{f_1^{46}f_4^2f_2^{27}}{f_3^{18}f_6^4}\right\} + 54q\frac{f_2^{26}f_6^3}{f_1f_3^5f_4^6} + 27q^2\frac{f_2^{11}f_3^7f_4^{10}}{f_1^{29}}.\end{aligned} \tag{141}$$

From Lemma 6, it is readily verifiable that

$$\frac{f_2^{34}}{f_1^5f_3f_4^6f_6^5} \equiv \frac{f_2^7f_6^4}{f_1^5f_3f_4^6} \pmod{27}, \qquad \frac{f_1^4f_4^3f_6^7}{f_2^2f_3^4f_{12}^3} \equiv \frac{f_2^7f_6^4}{f_1^5f_3f_4^6} \pmod{9}, \tag{142}$$

$$\frac{f_1f_2^{18}f_4^2}{f_3^3f_6} \equiv \frac{f_1f_4^2f_6^5}{f_3^3} \pmod{9}, \qquad \frac{f_1^4f_4^2f_6^6}{f_2^3f_3^4} \equiv \frac{f_1^{46}f_4^2f_2^{27}}{f_3^{18}f_6^4} \equiv \frac{f_1f_4^2f_6^5}{f_3^3} \pmod{3}, \tag{143}$$

$$\frac{f_2^{26}f_6^3}{f_1f_3^5f_4^6} \equiv \frac{f_6^{12}}{f_1^{16}f_2f_4^6} \pmod{3}, \qquad \frac{f_2^{11}f_3^7f_4^{10}}{f_1^{29}} \equiv \frac{f_4^{10}f_6^4}{f_1^8f_2} \pmod{3}. \tag{144}$$

Putting (142)–(144) into (141) and reducing modulo 81, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{t}\left(16(4n+1)\right)q^n \equiv{}& 2\frac{f_2^{35}f_4^2}{f_6^{12}}\left(\frac{f_1^3}{f_3}\right)^{14} + 6\frac{f_2^7f_6^4}{f_4^6}\left(\frac{1}{f_1^4}\right)\left(\frac{1}{f_1f_3}\right) + 72qf_4^2f_6^5\left(\frac{f_1}{f_3^3}\right)\\ &+ 54q\frac{f_6^{12}}{f_2f_4^6}\left(\frac{1}{f_1^4}\right)^4 + 27q^2\frac{f_4^{10}f_6^4}{f_2}\left(\frac{1}{f_1^4}\right)^2.\end{aligned} \tag{145}$$

Plugging (12), (13), (15) and (19) into (145), and isolating all terms of q^{2n+1}, then dividing through by q and replacing q^2 by q, we produce that

$$\sum_{n=0}^{\infty}\overline{t}\left(16(8n+5)\right)q^n \equiv \left\{78\frac{f_1^{37}f_2^{40}}{f_3^{14}f_6^{10}}+72\frac{f_1f_2^4f_6^2}{f_3^2}\right\}+24\frac{f_4^6f_6^5}{f_1^5f_2^5f_{12}^2}+6\frac{f_2^{13}f_3^2f_{12}^2}{f_1^{11}f_4^6f_6}+54\frac{f_2^{50}f_3^{12}}{f_1^{57}f_4^{16}}$$
$$+\left\{54q\frac{f_2^{26}f_3^4}{f_1^{25}}+27q\frac{f_1^{41}f_2^{32}}{f_3^{18}f_6^2}\right\}+54q^2\frac{f_2^2f_3^{12}f_4^{16}}{f_1^{41}}. \tag{146}$$

In view of Lemma 6, it implies that

$$\frac{f_1^{37}f_2^{40}}{f_3^{14}f_6^{10}} \equiv \frac{f_1^{10}f_2^{13}}{f_3^5f_6} \pmod{27}, \qquad \frac{f_1f_2^4f_6^2}{f_3^2} \equiv \frac{f_1^{10}f_2^{13}}{f_3^5f_6} \pmod 9, \tag{147}$$

$$\frac{f_2^{50}f_3^{12}}{f_1^{57}f_4^{16}} \equiv \frac{f_2^{50}}{f_1^{21}f_4^{16}} \pmod 3, \qquad \frac{f_2^{26}f_3^4}{f_1^{25}} \equiv \frac{f_1^{41}f_2^{32}}{f_3^{18}f_6^2} \equiv \frac{f_6^9}{f_1f_2f_3^4} \pmod 3, \tag{148}$$

$$\frac{f_2^2f_3^{12}f_4^{16}}{f_1^{41}} \equiv \frac{f_2^2f_4^{16}}{f_1^5} \pmod 3. \tag{149}$$

Substituting (147)–(149) into (146) and reducing modulo 81, we conclude that

$$\sum_{n=0}^{\infty}\overline{t}\left(16(8n+5)\right)q^n \equiv 69\frac{f_1^{10}f_2^{13}}{f_3^5f_6}+24\frac{f_4^6f_6^5}{f_1^5f_2^5f_{12}^2}+6\frac{f_2^{13}f_3^2f_{12}^2}{f_1^{11}f_4^6f_6}+54\frac{f_2^{50}}{f_1^{21}f_4^{16}}+54q^2\frac{f_2^2f_4^{16}}{f_1^5}. \tag{150}$$

So (135) follows from Lemma 14 and (150). This finishes the proof. □

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