Improved Two-Colour Rado Numbers for Linear Equations with Certain Coefficients

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Abstract

Let $a_1, \ldots, a_m$ be nonzero integers, $c \in \mathbb{Z}$ and $r \geqslant 2$. The `Rado number` for the equation

$$\sum_{i=1}^{m} a_i x_i = c$$

in r colours is the least positive integer N such that any r-colouring of the integers in the interval $[1, N]$ admits a monochromatic solution to the given equation. We introduce the concept of t-distributability of sets of positive integers, and determine exact values whenever possible, and upper and lower bounds otherwise, for the Rado numbers when the set $\{a_1, \ldots, a_{m-1}\}$ is 2-distributable or 3-distributable, $a_m = -1$, and $r = 2$. This generalizes previous works by several authors.

Mathematics Subject Classifications: 05C55, 05D10

1 Introduction

I. Schur [29] discovered his celebrated result that bears his name in 1916 while attempting to prove Fermat's "last" theorem. Schur's theorem states that for every positive integer r, there exists a positive integer $n = n(r)$ such that for every r-colouring of the integers in the interval $[1, n]$, there exists a monochromatic solution to the equation $x + y = z$. In other words, for each positive integer r, and for every function $\chi : \{1, \ldots, n\} \to \{1, \ldots, r\}$, there exist integers $x, y \in \{1, \ldots, n\}$ with $x + y \in \{1, \ldots, n\}$ such that $\chi(x) = \chi(y) = \chi(x + y)$. The least such $n = n(r)$ is denoted by $\mathtt{s}(r)$ in honour of Schur, and these are called the Schur numbers. The only exact values of $\mathtt{s}(r)$ known are $\mathtt{s}(1) = 2$, $\mathtt{s}(2) = 5$, $\mathtt{s}(3) = 14$, $\mathtt{s}(4) = 45$, and $\mathtt{s}(5) = 160$.

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Schur's theorem was generalized in a series of results in the 1930's by Rado [21, 22, 23] leading to a complete resolution to the following problem: characterize systems of linear homogeneous equations with integral coefficients $\mathcal{L}$ such that for a given positive integer r, there exists a least positive integer $n = \mathtt{Rad}(\mathcal{L};\mathtt{r})$ for which every r-colouring of the integers in the interval $[1, n]$ yields a monochromatic solution to the system $\mathcal{L}$. There has been a growing interest in the determination of the Rado numbers $\mathtt{Rad}(\mathcal{L};\mathtt{r})$, particularly when $\mathcal{L}$ is a single equation and $r = 2$.

Beutelspacher & Brestovansky [4] proved that the 2-colour Rado number for the equation $x_1+\cdots+x_{m-1}-x_m = 0$ equals m^2-m-1 for each $m \geqslant 3$. The 2-colour Rado numbers for the equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$ has been completely resolved. In view of Rado's result on single linear homogeneous regular equations, the only equations to consider are when $a_1 + a_2 = 0$ and when $a_1 + a_2 + a_3 = 0$. The first case was completely resolved by Burr & Loo [5], and the second case by Gupta, Thulasi Rangan & Tripathi [9]. Jones & Schaal [13] determined the 2-colour Rado number for $a_1x_1 + \cdots + a_{m-1}x_{m-1} - x_m = 0$ when $a_1, \ldots, a_{m-1}$ are positive integers with $\min\{a_1, \ldots, a_{m-1}\} = 1$. Hopkins & Schaal [12] resolved the problem in the case $\min\{a_1, \ldots, a_{m-1}\} = 2$ and gave bounds in the general case which they conjectured to hold; this was proved by Guo & Sun in [8]. They showed that the 2-colour Rado number for the general case equals $as^2 + s - a$, where $a = \min\{a_1, \ldots, a_{m-1}\}$ and $s = a_1 + \cdots + a_{m-1}$. The 2-colour Rado number for the non-homogeneous equation $x_1 + \cdots + x_{m-1} - x_m = c$ was determined by Schaal [27] for (i) $c \leqslant 0$ and even, (ii) m is odd, and by Kosek & Schaal [16] when (i) $c < m - 2$, (ii) $c > (m - 1)(m - 2)$. In all other cases in [16], they have given at least upper and lower bounds.

Let $a_1, \ldots, a_m$ be nonzero integers, $c \in \mathbb{Z}$ and $r \in \mathbb{N}$. The $\mathtt{Rado\ number}$ for the equation

$$\sum_{i=1}^{m} a_i x_i = c \tag{1}$$

in r colours, denoted by $\mathtt{Rad}_r\big(a_1, \ldots, a_m; c\big)$, is the least positive integer N such that any r-colouring of the integers in the interval $[1, N]$ admits a monochromatic solution to Eq.(1). This means that for *any* $\chi : [1, N] \to \{1, \ldots, r\}$ there exists $x_1, \ldots, x_m \in [1, N]$ satisfying Eq.(1) and such that $\chi(x_1) = \cdots = \chi(x_m)$.

The case $r = 1$ is trivial. For most of the remainder of this paper, we deal with the case $r = 2$. It is convenient to denote $\mathtt{Rad}_2\big(a_1, \ldots, a_m; c\big)$ more simply by $\mathtt{Rad}\big(a_1, \ldots, a_m; c\big)$. We write $\mathrm{S} = \sum_{i=1}^{m-1} a_i$ throughout this paper. In this paper, we provide exact results whenever possible and upper and lower bounds otherwise for these Rado numbers, mostly in the case $r = 2$. These results cover all integral values of c but hold for a restricted collection of coefficients $\{a_1, \ldots, a_{m-1}\}$; we always assume $a_m = -1$. We call this restricted family "distributed", and introduce the basic concepts of "r-distributability" in Section 2. We give the main results in Section 3. A summary of our results is provided in Table 1.

range for c	r	restriction on the set	Rado number
$(-\infty, \mathrm{S}-1)$	2	no restriction 2-distributable	$\geqslant (\mathrm{S}-1-c)(\mathrm{S}+2)+1$ $(\mathrm{S}-1-c)(\mathrm{S}+2)+1$ (Theorem 13)
$\mathrm{S}-1$	all	no restriction	1 (Theorem 13)
$[\mathrm{S}, 2\mathrm{S}-3]$	2	2-distributable	$\leqslant \mathrm{S}+1$ (Theorem 15)
$2\mathrm{S}-2$	all	no restriction	2 (Theorem 15)
$2\mathrm{S}-1$	2	2-distributable	3 (Theorem 15)
$\big[(\lambda-1)\mathrm{S}, \lambda\mathrm{S}-\lambda\big)$ $(3 \leqslant \lambda \leqslant \mathrm{S})$ $c = \lambda(\mathrm{S}-1)-\mu$ $(1 \leqslant \mu \leqslant \mathrm{S}-\lambda)$	2	no restriction 2-distributable 3-distributable	$\geqslant \lambda+\mu$ $\lambda+1$ if $\mu=1$ $\lambda+\mu$ if $2 \leqslant \mu \leqslant \mathrm{S}-\lambda$ (Theorem 17)
$\lambda(\mathrm{S}-1)$	all	no restriction	λ (Theorem 17)
$\bigcup_{3\leqslant\lambda\leqslant\left\lceil\frac{\mathrm{S}+1}{2}\right\rceil} \big(\lambda\mathrm{S}-\lambda, \lambda\mathrm{S}\big)$ $c = \lambda\mathrm{S}-\mu$ $(1 \leqslant \mu \leqslant \lambda-1)$	2	3-distributable	$\leqslant 2\lambda-\mu$ (Theorem 19)
$\big(\mathrm{S}-1, \mathrm{S}(\mathrm{S}-1)\big)$	2	no restriction 3-distributable	$\geqslant \left\lceil\frac{1+c(\mathrm{S}+2)}{\mathrm{S}^2+\mathrm{S}-1}\right\rceil$ if $c > \mathrm{S}-1$ (Theorem 21) $\leqslant \mathrm{S}+1$ (Theorem 23)
$\big(\mathrm{S}(\mathrm{S}-1), \infty\big)$	2	2-distributable	$\left\lceil\frac{1+c(\mathrm{S}+2)}{\mathrm{S}^2+\mathrm{S}-1}\right\rceil$ if $c > \mathrm{S}(\mathrm{S}-1)$ (Theorem 21)

Table 1: Summary of results on $\mathtt{Rad}_r\big(a_1, \ldots, a_{m-1}, -1; c\big)$, $c\,\mathrm{S}$ even, $\mathrm{S} = \sum_{i=1}^{m-1} a_i$

2 Distributable Sets

The results of our paper hold whenever the collection of positive integers $a_1, \dots, a_{m-1}$ is distributable. In this section, we introduce the notion of t-distributability, for each positive integer t. The case $t = 2$ corresponds to finite “complete” sequences, which are well known.

Definition 1. A sequence $\{u_n\}_{n\geqslant 1}$ of positive integers is said to be `complete` if every positive integer can be expressed as a sum of distinct terms of some finite subsequence $\{u_{n_k}\}_{k\geqslant 1}$.

The following result of Brown [6] characterizes complete sequences.

Proposition 2. (Brown, 1961)
A sequence $\{u_n\}_{n\geqslant 1}$ of positive integers is `complete` *if and only if*

$$u_1 = 1, \quad u_n \leqslant s_{n-1} + 1 \ \text{for} \ n \geqslant 2, \tag{2}$$

where $s_k = u_1 + \cdots + u_k$ for $k \geqslant 1$.

The notion of complete sequences has been generalized in several ways. For instance, Erdős & Lewin [7] extended the notion to d-complete sequences which are infinite sequences of positive integers where every sufficiently large integer can be written as a sum of distinct integers from the sequence, with the added constraint that no term in the sum can divide any other term in the sum. The results in [7] were extended by Park [19] and by Ma & Chen [18], among others; see also the results of Andrews, Beck & Hopkins [3]. We extend the notion of complete sequences in a different manner, as given in the following definition.

Definition 3. Let $\{a_1, \dots, a_k\}$ be a multiset of positive integers and $\{\sigma_1, \dots, \sigma_t\}$ be a multiset of nonnegative integers such that $\sum_{i=1}^t \sigma_i = \sum_{i=1}^k a_i$. The number of partitions of $\{1, \dots, k\}$ into sets $S_1, \dots, S_t$ (some possibly empty) such that $\sum_{i\in S_j} a_i = \sigma_j$ for $j \in \{1, \dots, t\}$ is called a `Set Distribution Coefficient`, and denoted by $\begin{bmatrix} a_1, \dots, a_k \\ \sigma_1, \dots, \sigma_t \end{bmatrix}$.

Suppose the sets $S_1, \dots, S_t$ partition $\{1, \dots, k\}$ such that $\sum_{i\in S_j} a_i = \sigma_j$ for $1 \leqslant j \leqslant t$. Then k belongs to exactly one of $S_1, \dots, S_t$, say $k \in S_1$ after renumbering. Then $\sum_{i\in S_j} a_i = \sigma_j$ for $2 \leqslant j \leqslant t$ and $\sum_{i\in S_1\setminus\{k\}} a_i = \sigma_1 - a_k$. Thus we have the identity

$$\begin{bmatrix} a_1, \dots, a_k \\ \sigma_1, \dots, \sigma_t \end{bmatrix} = \begin{bmatrix} a_1, \dots, a_{k-1} \\ \sigma_1 - a_k, \sigma_2, \dots, \sigma_t \end{bmatrix} + \begin{bmatrix} a_1, \dots, a_{k-1} \\ \sigma_1, \sigma_2 - a_k, \dots, \sigma_t \end{bmatrix} + \cdots + \begin{bmatrix} a_1, \dots, a_{k-1} \\ \sigma_1, \sigma_2, \dots, \sigma_t - a_k \end{bmatrix}. \tag{3}$$

Definition 4. We say a multiset $\{a_1, \dots, a_k\}$ of positive integers is t-`distributable` if

$$\begin{bmatrix} a_1, \dots, a_k \\ \sigma_1, \dots, \sigma_t \end{bmatrix} > 0$$

for all choices of multisets $\{\sigma_1, \dots, \sigma_t\}$ of nonnegative integers for which $\sum_{i=1}^t \sigma_i = \sum_{i=1}^k a_i$.

Remark 5. Any multiset of positive integers is 1-distributable, and any t-distributable multiset is j-distributable for $j \in \{1, \ldots, t-1\}$.

A note about the usage of the term "t-distributable". Fix a positive integer t, and fix a sequence $a_1, \ldots, a_{m-1}$. For every partition $\sigma_1, \ldots, \sigma_t$ of $S = \sum_{i=1}^{m-1} a_i$ into nonnegative integers, we partition the sequence $a_1, \ldots, a_{m-1}$ into t sets such that the set sums are $\sigma_1, \ldots, \sigma_t$. Thus, we are "distributing" the coefficient set over all possible partitions of the sum S.

Proposition 6. *Let $A = \{a_1, \ldots, a_k\}$ be a multiset of positive integers. With $a_i \leqslant a_{i+1}$ for $i \in \{1, \ldots, k-1\}$, A is t-distributable if and only if*

$$a_i \leqslant \left\lceil \frac{s_i}{t} \right\rceil \quad \textit{for} \;\; i \in \{1, \ldots, k\} \tag{4}$$

where $s_i = a_1 + a_2 + \cdots + a_i$ for $i \in \{1, \ldots, k\}$.

Proof. Suppose $A = \{a_1, \ldots, a_k\}$ is t-distributable, with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$, and let $\sum_{i=1}^{k} a_i = \sigma$. Therefore, for each $i \in \{1, \ldots, k\}$,

$$\begin{bmatrix} a_1, \ldots, a_k \\ a_i - 1, \ldots, a_i - 1, \sigma - (t-1)(a_i - 1) \end{bmatrix} > 0.$$

Since the terms $a_i, \ldots, a_k$ cannot contribute to the sum $a_i - 1$, it follows that

$$\sigma - (t-1)(a_i - 1) \geqslant a_i + a_{i+1} + \cdots + a_k. \tag{5}$$

Now Eq.(5) is equivalent to $a_1 + \cdots + a_{i-1} \geqslant (t-1)(a_i - 1)$. Adding a_i to both sides, we have

$$s_i \geqslant t a_i - (t-1) > t(a_i - 1).$$

Hence $a_i \leqslant \left\lceil \frac{s_i}{t} \right\rceil$, proving the necessity of the condition.

Conversely, we must show that any sequence $\{a_i\}_{i=1}^{k}$ satisfying the condition in Eq.(4) is t-distributable. We induct on k. The base case $k = 1$ is obvious, and we assume that all sequences satisfying the condition in Eq.(4) is t-distributable whenever the sequence has fewer than k terms.

Consider any nondecreasing sequence $\{a_i\}_{i=1}^{k}$ of positive integers satisfying the condition in Eq.(4), and let $\sigma_1, \ldots, \sigma_t$ be any nondecreasing sequence of nonnegative integers with sum $\sigma = \sum_{i=1}^{k} a_i$. Since the subsequence $\{a_i\}_{i=1}^{k-1}$ also satisfies the condition in Eq.(4) and $\sigma_1, \ldots, \sigma_{t-1}, \sigma_t - a_k$ has sum $\sigma - a_k = \sum_{i=1}^{k-1} a_i$, by the induction hypothesis we have

$$\begin{bmatrix} a_1, \ldots, a_{k-1} \\ \sigma_1, \ldots, \sigma_{t-1}, \sigma_t - a_k \end{bmatrix} > 0.$$

From Eq.(3) it follows that

$$\begin{bmatrix} a_1, \ldots, a_{k-1}, a_k \\ \sigma_1, \ldots, \sigma_{t-1}, \sigma_t \end{bmatrix} > 0,$$

proving the sufficiency of the condition. □

Remark 7. For any t-distributable sequence $a_1, \ldots, a_k$, $a_p = 1$ implies $a_i = 1$ for $i \in \{1, \ldots, p-1\}$, as a consequence of Eq.(4).

Example 8. There are several instances of 2-distributable and 3-distributable sequences for which $\texttt{Rad}_2(a_1, \ldots, a_m; c)$ has not been determined. We provide two examples each for 2-distributable and 3-distributable sequences which are covered by results in the following sections.

- $1, 2, 3, \ldots, n$ (2-distributable)
- $2^0, 2^1, 2^2, \ldots, 2^n$ (2-distributable)
- $1_k, 2, 3, \ldots, n$, $k > 1$ (3-distributable), where 1_k denotes k-occurrences of 1
- $1_k, 2_k, 3, \ldots, n$, $k > 1$ (3-distributable), where 1_k denotes k-occurrences of 1 and 2_k denotes k-occurrences of 2

3 Main Results

For any collection of nonzero integers $a_1, \ldots, a_m$, and for $c \in \mathbb{Z}$ and $r > 1$, the Rado number $\texttt{Rad}_r\big(a_1, \ldots, a_m; c\big)$ is the smallest positive integer R for which every r-colouring of $[1, R]$ contains a monochromatic solution to Eq.(1). By assigning the colour of x_i in the solution to Eq.(1) to $x_i - 1$, we note that this is equivalent to determining the smallest positive integer R for which every r-colouring of $[0, R-1]$ contains a monochromatic solution to

$$\sum_{i=1}^{m} a_i x_i = c - \sigma, \tag{6}$$

where $\sigma = \sum_{i=1}^{m} a_i$.

Theorem 9. *Let $a_1, \ldots, a_m$ be nonzero integers, $c \in \mathbb{Z}$ and $r > 1$. If $\sum_{i=1}^{m} a_i = \sigma$, lcm$(2, 3, \ldots, r) = L$, and $\gcd(\sigma, L) \nmid c$, then $\texttt{Rad}_r(a_1, \ldots, a_m; c)$ does not exist.*

Proof. Let $R \geqslant 1$. We exhibit an r-colouring of $[1, R]$ which contains no monochromatic solution to Eq.(6). Since $\gcd(\sigma, L) \nmid c$, there exists a prime p that divides both σ and L but does not divide c. Observe that $p \leqslant r$.

Define the colouring $\chi : \mathbb{N} \to \{1, \ldots, p\}$ by $\chi(i) \equiv i \bmod p$. We show that this colouring does not admit a monochromatic solution to Eq.(1). By way of contradiction, assume there exists a monochromatic solution $x_1, \ldots, x_m$. Since $\chi(x_i)$ is constant, $x_i \equiv x_j \pmod{p}$ for $i \neq j$ by definition of χ. Write $x_i \equiv x_0 \pmod{p}$. Then $c = \sum_{i}^{m} a_i x_i \equiv x_0 \sum_{i=1}^{m} a_i = \sigma x_0 \equiv 0 \pmod{p}$, which is a contradiction to our assumption. □

Corollary 10. *Let $a_1, \ldots, a_m$ be nonzero integers, $c \in \mathbb{Z}$ and $r = 2$. If $\sum_{i=1}^{m} a_i = \sigma$ is even and c is odd, then $\texttt{Rad}_2(a_1, \ldots, a_m; c)$ does not exist.*

Remark 11. This generalizes a remark in [16, p. 806] on the non-existence of the Rado number $\texttt{Rad}_2(a_1, \ldots, a_m; c)$ when $a_i = 1$ for each $i \in \{1, \ldots, m\}$.

Theorem 12. *Let $a_1, \ldots, a_m$ be nonzero integers, $c \in \mathbb{Z}$, $r > 1$ and $\lambda \geqslant 1$. If $\sum_{i=1}^{m} a_i = \sigma$, then*

$$\mathtt{Rad}_r(a_1, \ldots, a_m; \lambda c + \sigma) \leqslant 1 + \lambda\big(\mathtt{Rad}_r(a_1, \ldots, a_m; c + \sigma) - 1\big).$$

Proof. Let $N = \mathtt{Rad}_r(a_1, \ldots, a_m; c+\sigma)$. Then any r-colouring of the integers in $[1, N]$ must contain a monochromatic solution to $\sum_{i=1}^{m} a_i x_i = c + \sigma$. We extend any such r-colouring χ to $[0, \lambda(N-1)]$ by first assigning the colour of x in $[1, N]$ to $x - 1$. Thus we have an r-colouring of the integers in $[0, N-1]$. Then we assign x, y in $[0, \lambda(N-1)]$ the same colour if and only if $x \equiv y \pmod{N-1}$. Since solutions $(x_1, \ldots, x_m)$ to $\sum_{i=1}^{m} a_i x_i = c + \sigma$ give rise to solutions $\big(\lambda(x_1 - 1), \ldots, \lambda(x_m - 1)\big)$ to $\sum_{i=1}^{m} a_i\big(\lambda(x_i - 1) + 1\big) = \lambda c + \sigma$, the result follows from the argument at the beginning of this section. □

In this paper, we study the Rado numbers for the equation

$$\sum_{i=1}^{m-1} a_i x_i - x_m = c \tag{7}$$

in the special case where $\{a_1, \ldots, a_{m-1}\}$ is 2-distributable or 3-distributable and $r = 2$, for any $c \in \mathbb{Z}$. With $\mathrm{S} = \sum_{i=1}^{m-1} a_i$, by Corollary 10, $\mathtt{Rad}_2\big(a_1, \ldots, a_{m-1}, -1; c\big)$ exists if and only if either c is even or S is even. Throughout the rest of this paper, we write $\mathrm{S} = \sum_{i=1}^{m-1} a_i$, and assume that at least one of c, S is even.

By assigning the colour of x_i in the solution of Eq.(7) to $x_i - 1$, we note that this is equivalent to determining the smallest positive integer R for which every r-colouring of $[0, R-1]$ contains a monochromatic solution to

$$\sum_{i=1}^{m-1} a_i x_i - x_m = c - (\mathrm{S} - 1). \tag{8}$$

For brevity, we write

$$\mathtt{Rad}_r\big(c\big)$$

for the r-colour Rado number $\mathtt{Rad}_r\big(a_1, \ldots, a_{m-1}, -1; c\big)$, and

$$\mathtt{Rad}_{r;t}\big(c\big)$$

for the r-colour Rado number $\mathtt{Rad}_r\big(a_1, \ldots, a_{m-1}, -1; c\big)$ for t-distributable sets $\{a_1, \ldots, a_{m-1}\}$.

This paper is an extension of the works given by Schaal [27], and by Kosek & Schaal [16]. We extend the results by including a large class of coefficients and summarize the results in Table 1. These results cover all values of c, and are successively presented as Theorems 13, 15, 17, 19, 21, 23.

Theorem 13. *Let $a_1, \ldots, a_{m-1}$ be a set of positive integers, with $\sum_{i=1}^{m-1} a_i = \mathrm{S}$.*

(i) *For any $r > 1$,*

$$\mathtt{Rad}_r\big(\mathrm{S} - 1\big) = 1.$$

(ii) *If* $c\mathrm{S}$ *is even and* $c < \mathrm{S} - 1$*, then*

$$\mathtt{Rad}_2\big(c\big) \geqslant (\mathrm{S} - 1 - c)(\mathrm{S} + 2) + 1.$$

(iii) *If* $c\mathrm{S}$ *is even,* $c < \mathrm{S} - 1$*, and if the set* $\{a_1, \ldots, a_{m-1}\}$ *is* 2*-distributable, then*

$$\mathtt{Rad}_{2;2}\big(c\big) = (\mathrm{S} - 1 - c)(\mathrm{S} + 2) + 1.$$

Proof. (i) If $c = \mathrm{S} - 1$, then Eq.(7) admits the solution $x_1 = \cdots = x_m = 1$. Hence every r-colouring of the set $\{1\}$ admits a monochromatic solution of Eq.(7). Thus $\mathtt{Rad}_r\big(\mathrm{S} - 1\big) = 1$ for each $r > 1$.

(ii) By Corollary 10, $\mathtt{Rad}_2\big(c\big)$ exists if and only if $c\mathrm{S}$ is even. For the rest of the proof, we also suppose $c < \mathrm{S} - 1$. Let $\Delta : [0, (\mathrm{S} - 1 - c)(\mathrm{S} + 2) - 1] \to \{0, 1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in [0, \mathrm{S} - 2 - c] \cup [(\mathrm{S} - 1 - c)(\mathrm{S} + 1), (\mathrm{S} - 1 - c)(\mathrm{S} + 2) - 1]; \\ 1 & \text{if } x \in [\mathrm{S} - 1 - c, (\mathrm{S} - 1 - c)(\mathrm{S} + 1) - 1]. \end{cases}$$

We claim that Δ provides a valid 2-colouring of $[0, (\mathrm{S} - 1 - c)(\mathrm{S} + 2) - 1]$ with respect to Eq.(8).

Suppose $\Delta(x_i) = 1$ for $i \in \{1, \ldots, m - 1\}$. Then

$$x_m = \sum_{i=1}^{m-1} a_i x_i + \mathrm{S} - 1 - c \geqslant \mathrm{S}(\mathrm{S} - 1 - c) + (\mathrm{S} - 1 - c) = (\mathrm{S} + 1)(\mathrm{S} - 1 - c).$$

Hence $\Delta(x_m) = 0$. Therefore, we must have $\Delta(x_i) = 0$ for $i \in \{1, \ldots, m\}$.

If $x_i \in [0, \mathrm{S} - 2 - c]$ for $i \in \{1, \ldots, m - 1\}$, then

$$x_m = \sum_{i=1}^{m-1} a_i x_i + \mathrm{S} - 1 - c \geqslant \mathrm{S} - 1 - c,$$

and

$$x_m = \sum_{i=1}^{m-1} a_i x_i + \mathrm{S} - 1 - c \leqslant \mathrm{S}(\mathrm{S} - 2 - c) + (\mathrm{S} - 1 - c) < (\mathrm{S} - 1 - c)(\mathrm{S} + 1).$$

Therefore, $x_i \in [(\mathrm{S} - 1 - c)(\mathrm{S} + 1), (\mathrm{S} - 1 - c)(\mathrm{S} + 2) - 1]$ for at least one $i \in \{1, \ldots, m - 1\}$. Now

$$x_m = \sum_{i=1}^{m-1} a_i x_i + \mathrm{S} - 1 - c \geqslant (\mathrm{S} - 1 - c)(\mathrm{S} + 1) + (\mathrm{S} - 1 - c) = (\mathrm{S} - 1 - c)(\mathrm{S} + 2),$$

so that x_m is outside the domain of Δ. Hence $\mathtt{Rad}_2\big(c\big) \geqslant (\mathrm{S} - 1 - c)(\mathrm{S} + 2) + 1$.

(iii) Suppose $\{a_1, \ldots, a_{m-1}\}$ is 2-distributable. For any $\chi : [0, (\mathrm{S}-1-c)(\mathrm{S}+2)] \to \{0,1\}$, we show that there exists a monochromatic solution to Eq.(8) under χ. Together with the result in part (ii), this would complete the proof of part (iii). We first resolve the case where $c = \mathrm{S} - 2$; under our assumption, both c and S are even.

Let $c = \mathrm{S} - 2$, and let $\chi : [0, \mathrm{S} + 2] \to \{0, 1\}$. Without loss of generality, let $\chi(0) = 0$. If $\chi(1) = 0$, then $x_1 = \cdots = x_{m-1} = 0$, $x_m = 1$ provides a monochromatic solution to Eq.(8). Hence $\chi(1) = 1$.

If $\chi(\mathrm{S}+1) = 1$, then $x_1 = \cdots = x_{m-1} = 1$, $x_m = \mathrm{S}+1$ provides a monochromatic solution to Eq.(8). Hence $\chi(\mathrm{S} + 1) = 0$.

Since $\{a_1, \ldots, a_{m-1}\}$ is 2-distributable with $\sum_{i=1}^{m-1} a_i = \mathrm{S}$, distribute the sets according to $\sigma_1 = 1$ and $\sigma_2 = \mathrm{S} - 1$. We may assume that $a_1 = 1$, after relabelling if necessary. If $\chi(\mathrm{S} + 2) = 0$, then $x_1 = \mathrm{S} + 1$, $x_2 = \cdots = x_{m-1} = 0$ provides a monochromatic solution to Eq.(8). Hence $\chi(\mathrm{S} + 2) = 1$.

We now distribute the set $\{a_1, \ldots, a_{m-1}\}$ according to $\sigma_1 = \sigma_2 = \mathrm{S}/2$. If $\chi(2) = 0$, then $x_i = 0$ for the a_i's in the first collection and $x_j = 2$ for the a_j's in the second collection provides a monochromatic solution to Eq.(8) since $\chi(\mathrm{S} + 1) = 0$. Hence $\chi(2) = 1$.

Since $\{a_1, \ldots, a_{m-1}\}$ is 2-distributable, we may assume that at least one $a_i = 1$; after relabelling we may assume $a_1 = 1$. Now $x_1 = 2$, $x_2 = \cdots = x_m = 1$ provides a monochromatic solution to Eq.(8) since $\chi(1) = \chi(2) = \chi(\mathrm{S} + 2) = 1$.

Therefore any 2-colouring of the integers in $[0, \mathrm{S} + 2]$ admits a monochromatic solution to $\sum_{i=1}^{m-1} a_i x_i + 1 = x_m$, which is Eq.(8) for $c = \mathrm{S}-2$. Hence $\mathtt{Rad}_{2;2}\big(\mathrm{S}-2\big) \leqslant \mathrm{S} + 3$.

Now let $c < \mathrm{S} - 1$. With $c = -1$ and $\lambda = (\mathrm{S} - 1) - c$, Theorem 12 gives

$$\mathtt{Rad}_{2;2}\big(\sigma - (\mathrm{S} - 1 - c)\big) \leqslant 1 + (\mathrm{S} - 1 - c)\big(\mathtt{Rad}_{2;2}(\sigma - 1) - 1\big),$$

or that

$$\mathtt{Rad}_{2;2}\big(c\big) \leqslant 1 + (\mathrm{S} - 1 - c)(\mathrm{S} + 2).$$

Together with part (ii), this completes the proof of part (iii).

□

Remark 14. Theorem 13 generalizes the results in Theorems 1.1 and 2.1 in [16].

Theorem 15. *Let* $a_1, \ldots, a_{m-1}$ *be a set of positive integers, with* $\sum_{i=1}^{m-1} a_i = \mathrm{S}$. *Let* $\mathrm{S} \leqslant c \leqslant 2\mathrm{S} - 1$ *and let* $c\mathrm{S}$ *be even.*

(i)

$$\mathtt{Rad}_r\big(2\mathrm{S} - 2\big) = 2.$$

(ii) *Suppose* $\{a_1, \ldots, a_{m-1}\}$ *is 2-distributable. If* $c \neq 2\mathrm{S} - 2$, *then*

$$\mathtt{Rad}_{2;2}\big(c\big) \leqslant \mathrm{S} + 1 \ \ \textit{and} \ \ \mathtt{Rad}_{2;2}\big(2\mathrm{S} - 1\big) = 3.$$

Proof. (i) If $c = 2\mathrm{S} - 2$, then Eq.(7) admits the solution $x_1 = \cdots = x_m = 2$. Hence every r-colouring of the set $\{1, 2\}$ admits a monochromatic solution of Eq.(7). Since $x_1 = \cdots = x_m = 1$ is not a solution to Eq.(7), it follows that $\mathtt{Rad}_r\big(2\mathrm{S} - 2\big) = 2$ for each $r > 1$.

(ii) Suppose $\{a_1, \ldots, a_{m-1}\}$ is 2-distributable, and let $r = 2$.

Suppose $c = 2\mathrm{S} - 1$; since c is odd, S must be even. For any $\chi : [1, 3] \to \{0, 1\}$, we show that there exists a monochromatic solution to Eq.(7) under χ.

If $\chi(1) = \chi(2)$, then Eq.(7) admits the solution $x_1 = \cdots = x_{m-1} = 2$, $x_m = 1$. Henceforth suppose $\chi(1) \neq \chi(2)$.

If $\chi(3) = \chi(1)$, distribute the set $\{a_1, \ldots, a_{m-1}\}$ according to $\sigma_1 = \sigma_2 = \mathrm{S}/2$. Then $x_i = 1$ for a_i's in the first collection, $x_i = 3$ for a_i's in the second collection, and $x_m = 1$ provides a monochromatic solution to Eq.(7).

If $\chi(3) = \chi(2)$, distribute the set $\{a_1, \ldots, a_{m-1}\}$ according to $\sigma_1 = \mathrm{S} - 1$ and $\sigma_2 = 1$. Then $x_i = 2$ for a_i's in the first collection, $x_i = 3$ for a_i $(= 1)$ in the second collection, and $x_m = 2$ provides a monochromatic solution to Eq.(7). This proves our claim, and shows that $\mathtt{Rad}_{2;2}\big(c\big) \leqslant 3$ in this case.

To prove that $\mathtt{Rad}_{2;2}\big(c\big) > 2$ in this case, observe that the 2-colouring of $[1, 2]$ with $\chi(1) \neq \chi(2)$ is a valid colouring. This completes the proof in the special case $c = 2\mathrm{S} - 1$.

It remains to show that $\mathtt{Rad}_{2;2}\big(c\big) \leqslant \mathrm{S} + 1$ for $c \in [\mathrm{S}, 2\mathrm{S} - 3]$. Write $c = 2(\mathrm{S} - 1) - k$, with $k \in [1, \mathrm{S} - 2]$. By assigning the colour of x_i in the solution of Eq.(7) to $x_i - 1$, we show that any 2-colouring of $[0, \mathrm{S}]$ admits a monochromatic solution to

$$\sum_{i=1}^{m-1} a_i x_i - x_m = \mathrm{S} - 1 - k, \tag{9}$$

for $k \in [1, \mathrm{S} - 2]$.

Let $\chi : [0, \mathrm{S}] \to \{0, 1\}$, and assume $\chi(1) = 0$ without loss of generality. If $\chi(k + 1) = 0$, then $x_1 = \cdots = x_{m-1} = 1$, $x_m = k + 1$ provides a monochromatic solution to Eq.(9). Therefore we may assume $\chi(k + 1) = 1$.

Suppose $\chi(0) = 0$. Distribute the set $\{a_1, \ldots, a_{m-1}\}$ according to $\sigma_1 = k + 1$ and $\sigma_2 = \mathrm{S} - (k + 1)$. Then $x_i = 0$ for the a_i's in the first collection, $x_i = 1$ for the a_i's in the second collection, and $x_m = 1$ provides a monochromatic solution to Eq.(9). Therefore we may assume $\chi(0) = 1$.

Suppose $\chi(\mathrm{S} - 1 - k) = 1$. Distribute the set $\{a_1, \ldots, a_{m-1}\}$ according to $\sigma_1 = \mathrm{S} - 1$ and $\sigma_2 = 1$. Then $x_i = 0$ for the a_i's in the first collection, $x_i = \mathrm{S} - 1 - k$ for the a_i $(= 1)$ in the second collection, and $x_m = 0$ provides a monochromatic solution to Eq.(9). Therefore we may assume $\chi(\mathrm{S} - 1 - k) = 0$.

Suppose $\chi(\mathrm{S}) = 1$. Distribute the set $\{a_1, \ldots, a_{m-1}\}$ according to $\sigma_1 = \mathrm{S} - 1$ and $\sigma_2 = 1$. Then $x_i = 0$ for the a_i's in the first collection, $x_i = \mathrm{S}$ for the a_i $(= 1)$ in the second collection, and $x_m = k+1$ provides a monochromatic solution to Eq.(9). Therefore we may assume $\chi(\mathrm{S}) = 0$.

Suppose $\chi(\mathrm{S}-1) = 0$. Distribute the set $\{a_1, \dots, a_{m-1}\}$ according to $\sigma_1 = \mathrm{S}-1$ and $\sigma_2 = 1$. Then $x_i = 1$ for the a_i's in the first collection, $x_i = \mathrm{S}-1-k$ for the a_i $(=1)$ in the second collection, and $x_m = \mathrm{S}-1$ provides a monochromatic solution to Eq.(9). Therefore we may assume $\chi(\mathrm{S}-1) = 1$.

Suppose c is even; then k is even. Distribute the set $\{a_1, \dots, a_{m-1}\}$ according to $\sigma_1 = \frac{k}{2}+1$ and $\sigma_2 = \mathrm{S} - \left(\frac{k}{2}+1\right)$. Then $x_i = 0$ for the a_i's in the first collection, $x_i = 2$ for the a_i's in the second collection, and $x_m = \mathrm{S}-1$ provides a monochromatic solution to Eq.(9). Therefore we may assume $\chi(2) = 0$ in this case.

Now suppose c is odd; then k is odd and S is even. Distribute the set $\{a_1, \dots, a_{m-1}\}$ according to $\sigma_1 = (\mathrm{S}+1+k)/2$ and $\sigma_2 = (\mathrm{S}-1-k)/2$. Then $x_i = 0$ for the a_i's in the first collection, $x_i = 2$ for the a_i's in the second collection, and $x_m = 0$ provides a monochromatic solution to Eq.(9). Therefore we may also assume $\chi(2) = 0$ in this case.

In both cases, we must have $\chi(2) = 0$. Distribute the set $\{a_1, \dots, a_{m-1}\}$ according to $\sigma_1 = k+1$ and $\sigma_2 = \mathrm{S} - (k+1)$. Then $x_i = 1$ for the a_i's in the first collection, $x_i = 2$ for the a_i's in the second collection, and $x_m = \mathrm{S}$ provides a monochromatic solution to Eq.(9).

□

Remark 16. Theorem 15 generalizes the results in Lemmas 3.1 and 3.2 in [16].

Theorem 17. *Let* $a_1, \dots, a_{m-1}$ *be a set of positive integers, with* $\sum_{i=1}^{m-1} a_i = \mathrm{S}$. *Let* $c \in \bigcup_{3 \leqslant \lambda \leqslant \mathrm{S}} \left[(\lambda-1)\mathrm{S}, \lambda\mathrm{S}-\lambda\right)$, *and let* $c\,\mathrm{S}$ *be even. Then* $c = \lambda(\mathrm{S}-1) - \mu$, *with* $3 \leqslant \lambda \leqslant \mathrm{S}$, $0 \leqslant \mu \leqslant \mathrm{S}-\lambda$, *and*

(i)
$$\mathtt{Rad}_2\big(c\big) \geqslant \lambda + \mu.$$

(ii) *For any* $r > 1$,
$$\mathtt{Rad}_r\big(\lambda(\mathrm{S}-1)\big) = \lambda.$$

(iii) *If the set* $\{a_1, \dots, a_{m-1}\}$ *is* 2*-distributable, then*
$$\mathtt{Rad}_{2;2}\big(\lambda(\mathrm{S}-1)-1\big) = \lambda + 1.$$

(iv) *If the set* $\{a_1, \dots, a_{m-1}\}$ *is* 3*-distributable and* $\mu \neq 0, 1$, *then*
$$\mathtt{Rad}_{2;3}\big(c\big) = \lambda + \mu.$$

Proof. By Corollary 10, $\mathtt{Rad}_2\big(c\big)$ exists if and only if $c\,\mathrm{S}$ is even. Suppose $c = \lambda(\mathrm{S}-1) - \mu$, with $3 \leqslant \lambda \leqslant \mathrm{S}$, $0 \leqslant \mu \leqslant \mathrm{S}-\lambda$.

(i) Let $\Delta : [0, \lambda + \mu - 2] \to \{0, 1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in [0, \lambda - 2]; \\ 1 & \text{if } x \in [\lambda - 1, \lambda + \mu - 2]. \end{cases}$$

We claim that Δ provides a valid 2-colouring of $[0, \lambda + \mu - 2]$ with respect to Eq.(8).

Suppose $\Delta(x_i) = 0$ for $i \in \{1, \dots, m-1\}$.

$$x_m = \sum_{i=1}^{m-1} a_i x_i - (\lambda - 1)(\mathrm{S} - 1) + \mu \leqslant \mathrm{S}(\lambda - 2) - (\lambda - 1)(\mathrm{S} - 1) + \mu \leqslant -1,$$

so that x_m is outside the domain of Δ. Therefore, we must have $\Delta(x_i) = 1$ for $i \in \{1, \dots, m\}$. But then

$$x_m = \sum_{i=1}^{m-1} a_i x_i - (\lambda - 1)(\mathrm{S} - 1) + \mu \geqslant \mathrm{S}(\lambda - 1) - (\lambda - 1)(\mathrm{S} - 1) + \mu \geqslant \lambda + \mu - 1.$$

Hence $\mathtt{Rad}_2\big(c\big) \geqslant \lambda + \mu$.

(ii) Let $r > 1$. By part (i),

$$\mathtt{Rad}_r\big(\lambda(\mathrm{S} - 1)\big) \geqslant \mathtt{Rad}_2\big(\lambda(\mathrm{S} - 1)\big) \geqslant \lambda.$$

Since Eq.(7) admits the solution $x_1 = \cdots = x_m = \lambda$, every r-colouring of $[1, \lambda]$ admits a monochromatic solution of Eq.(7). Therefore $\mathtt{Rad}_r\big(\lambda(\mathrm{S} - 1)\big) \leqslant \lambda$ for each $r > 1$.

(iii) Let $c = \lambda(\mathrm{S} - 1) - \mu$ with $3 \leqslant \lambda \leqslant \mathrm{S}$, $1 \leqslant \mu \leqslant \mathrm{S} - \lambda$. By part (i), it suffices to show that

$$\mathtt{Rad}_{2;2}\big(c\big) \leqslant \lambda + \mu$$

when the set $\{a_1, \dots, a_{m-1}\}$ is 2-distributable for $\mu = 1$, and 3-distributable for $\mu > 1$.

By assigning the colour of x_i in the solution of Eq.(7) to $x_i - 1$, we show that any 2-colouring of $[0, \lambda + \mu - 1]$ admits a monochromatic solution to

$$\sum_{i=1}^{m-1} a_i x_i - x_m = (\lambda - 1)(\mathrm{S} - 1) - \mu. \tag{10}$$

Let $\chi : [0, \lambda + \mu - 1] \to \{0, 1\}$, and assume $\chi(\lambda - 1) = 0$ without loss of generality. Assume that the set $\{a_1, \dots, a_{m-1}\}$ is 2-distributable.

If $\chi(\lambda-1+\mu)=0$, then $x_1=\cdots=x_{m-1}=\lambda-1$, $x_m=\lambda-1+\mu$ provides a monochromatic solution to Eq.(10). Therefore we may assume $\chi(\lambda-1+\mu)=1$.

Suppose $\chi(\lambda-2)=0$. Distribute the set $\{a_1,\ldots,a_{m-1}\}$ according to $\sigma_1=\mu$ and $\sigma_2=\mathrm{S}-\mu$. Then $x_i=\lambda-2$ for the a_i's in the first collection, $x_i=\lambda-1$ for the a_i's in the second collection, and $x_m=\lambda-1$ provides a monochromatic solution to Eq.(10). Therefore we may assume $\chi(\lambda-2)=1$.

Suppose c is odd; then μ is odd and S is even. Distribute the set $\{a_1,\ldots,a_{m-1}\}$ according to $\sigma_1=\sigma_2=\mathrm{S}/2$. Then $x_i=\lambda$ for the a_i's in the first collection, $x_i=\lambda-2$ for the a_i's in the second collection, and $x_m=\lambda-1+\mu$ provides a monochromatic solution to Eq.(10). Therefore we may also assume $\chi(\lambda)=0$ in this case.

Then $x_1=\cdots=x_{m-1}=\lambda-1$ and $x_m=\lambda-1+\mu$ provides a solution to Eq.(10). However, $\chi(\lambda-1)=0$, whereas $\chi(\lambda-1+\mu)=1$ for $\mu>1$ and $\chi(\lambda)=0$. Hence we have a monochromatic solution to Eq.(10) for $\mu=1$.

(iv) For the rest of this proof, assume that $\mu>1$. If c is odd, again distribute the set $\{a_1,\ldots,a_{m-1}\}$ according to $\sigma_1=(\mu-1)/2$ and $\sigma_2=\mathrm{S}-(\mu-1)/2$. Then $x_i=\lambda-3$ for the a_i's in the first collection, $x_i=\lambda-1$ for the a_i's in the second collection, and $x_m=\lambda$ provides a monochromatic solution to Eq.(10). Therefore we may also assume $\chi(\lambda-3)=1$ in this case.

Next suppose c is even; then μ is even. Distribute the set $\{a_1,\ldots,a_{m-1}\}$ according to $\sigma_1=\mu/2$ and $\sigma_2=\mathrm{S}-\mu/2$. Then $x_i=\lambda-3$ for the a_i's in the first collection, $x_i=\lambda-1$ for the a_i's in the second collection, and $x_m=\lambda-1$ provides a monochromatic solution to Eq.(10). Therefore we may assume $\chi(\lambda-3)=1$ in this case as well.

Henceforth assume that the set $\{a_1,\ldots,a_{m-1}\}$ is 3-distributable. Distribute the set $\{a_1,\ldots,a_{m-1}\}$ according to $\sigma_1=\lceil\frac{\mathrm{S}}{\mu+1}\rceil$, $\sigma_2=\lceil\frac{\mathrm{S}}{\mu+1}\rceil(\mu+1)-\mathrm{S}$, and $\sigma_3=2\mathrm{S}-\lceil\frac{\mathrm{S}}{\mu+1}\rceil(\mu+2)$. Then $x_i=\lambda-1+\mu$ for the a_i's in the first collection, $x_i=\lambda-3$ for the a_i's in the second collection, $x_i=\lambda-2$ for the a_i's in the third collection, together with

$$\begin{aligned} x_m &= \sum_{i=1}^{m-1} a_i x_i-(\lambda-1)(\mathrm{S}-1)+\mu \\ &= (\lambda-1+\mu)\left\lceil\tfrac{\mathrm{S}}{\mu+1}\right\rceil+(\lambda-3)\left(\left\lceil\tfrac{\mathrm{S}}{\mu+1}\right\rceil(\mu+1)-\mathrm{S}\right) \\ &\quad +(\lambda-2)\left(2\mathrm{S}-\left\lceil\tfrac{\mathrm{S}}{\mu+1}\right\rceil(\mu+2)\right)-(\lambda-1)(\mathrm{S}-1)+\mu \\ &= \lambda-1+\mu \end{aligned}$$

provides a monochromatic solution to Eq.(10).

□

Remark 18. Theorem 17 generalizes the results in Lemma 3.3 in [16].

Theorem 19. *Let $a_1, \ldots, a_{m-1}$ be a 3-distributable set of positive integers, with $\sum_{i=1}^{m-1} a_i = \mathrm{S}$. Let $c \in \bigcup_{3 \leqslant \lambda \leqslant \lceil \frac{\mathrm{S}+1}{2} \rceil} \big(\lambda(\mathrm{S}-1), \lambda \mathrm{S}\big)$, and let $c\mathrm{S}$ be even. If $c = \lambda\mathrm{S} - \mu$, with $3 \leqslant \lambda \leqslant \lceil \frac{\mathrm{S}+1}{2} \rceil$, $1 \leqslant \mu \leqslant \lambda - 1$, then*

$$\mathtt{Rad}_{2;3}\big(c\big) \leqslant 2\lambda - \mu.$$

Proof. Let $c = \lambda\mathrm{S} - \mu$, with $3 \leqslant \lambda \leqslant \lceil \frac{\mathrm{S}+1}{2} \rceil$ and $1 \leqslant \mu \leqslant \lambda - 1$. By assigning the colour of x_i in the solution of Eq.(7) to $x_i - 1$, we show that any 2-colouring of $[0, 2\lambda - \mu - 1]$ admits a monochromatic solution to

$$\sum_{i=1}^{m-1} a_i x_i - x_m = (\lambda - 1)\mathrm{S} - (\mu - 1). \tag{11}$$

Let $\chi : [0, 2\lambda - \mu - 1] \to \{0, 1\}$, and assume $\chi(\lambda - 1) = 0$ without loss of generality. Assume that the set $\{a_1, \ldots, a_{m-1}\}$ is 3-distributable.

If $\chi(\mu - 1) = 0$, then $x_1 = \cdots = x_{m-1} = \lambda - 1$, $x_m = \mu - 1$ provides a monochromatic solution to Eq.(11). Therefore we may assume $\chi(\mu - 1) = 1$.

Suppose $\chi(2\lambda - \mu - 1) = 0$. Distribute the set $\{a_1, \ldots, a_{m-1}\}$ according to $\sigma_1 = \mathrm{S} - 1$ and $\sigma_2 = 1$. Then $x_i = \lambda - 1$ for the a_i's in the first collection, $x_i = 2\lambda - \mu - 1$ for the a_i (=1) in the second collection, and $x_m = \lambda - 1$, provides a monochromatic solution to Eq.(11). Therefore we may assume $\chi(2\lambda - \mu - 1) = 1$.

Suppose S is even. Distribute the set $\{a_1, \ldots, a_{m-1}\}$ according to $\sigma_1 = \mathrm{S}/2$ and $\sigma_2 = \mathrm{S}/2$. Then $x_i = 2\lambda - \mu - 1$ for the a_i's in the first collection, $x_i = \mu - 1$ for the a_i's in the second collection, and $x_m = \mu - 1$, provides a monochromatic solution to Eq.(11).

Now suppose S is odd, so that $c = \lambda\mathrm{S} - \mu$ is even. Hence $\lambda - \mu$ is even. Distribute the set $\{a_1, \ldots, a_{m-1}\}$ according to $\sigma_1 = \lambda - \mu$ and $\sigma_2 = \mathrm{S} - (\lambda - \mu)$. Then $x_i = \lambda$ for the a_i's in the first collection, $x_i = \lambda - 1$ for the a_i's in the second collection, and $x_m = \lambda - 1$, provides a solution to Eq.(11). Therefore we may assume $\chi(\lambda) = 1$.

Distribute the set $\{a_1, \ldots, a_{m-1}\}$ according to $\sigma_1 = \lambda - \mu$, $\sigma_2 = (\mathrm{S} - \lambda + \mu + 1)/2$ and $\sigma_3 = (\mathrm{S} - \lambda + \mu - 1)/2$. Then $x_i = \lambda$ for the a_i's in the first collection, $x_i = \mu - 1$ for the a_i's in the second collection, $x_i = 2\lambda - \mu - 1$ for the a_i's in the third collection, and $x_m = \mu - 1$, provides a solution to Eq.(11). □

Remark 20. Theorem 19 generalizes the results in Lemma 3.4 in [16].

Theorem 21. *Let $\{a_1, \ldots, a_{m-1}\}$ be a set of positive integers, with $\sum_{i=1}^{m-1} a_i = \mathrm{S}$. Let $c > \mathrm{S} - 1$, and let $c\mathrm{S}$ be even.*

(i)

$$\mathtt{Rad}_2\big(c\big) \geqslant \left\lceil \frac{1 + c(\mathrm{S}+2)}{\mathrm{S}^2 + \mathrm{S} - 1} \right\rceil.$$

(ii) *If the set $\{a_1, \ldots, a_{m-1}\}$ is 2-distributable and $c > \mathrm{S}(\mathrm{S}-1)$, then*

$$\mathtt{Rad}_{2;2}\big(c\big) = \left\lceil \frac{1 + c(\mathrm{S}+2)}{\mathrm{S}^2 + \mathrm{S} - 1} \right\rceil.$$

Proof. For convenience, we set

$$\mathrm{T} = \left\lceil \frac{1 + c(\mathrm{S}+2)}{\mathrm{S}^2 + \mathrm{S} - 1} \right\rceil - 1,$$

and show that

$$\mathrm{S}^2\mathrm{T} - c(\mathrm{S}+1) < \mathrm{ST} - c < \mathrm{T}. \tag{12}$$

Both inequalities are both equivalent to

$$\mathrm{T} < \frac{c}{\mathrm{S}-1}.$$

From the definition of T, since $c > \mathrm{S} - 1$, we have

$$\mathrm{T} \leqslant \frac{1 + c(\mathrm{S}+2)}{\mathrm{S}^2 + \mathrm{S} - 1} = \frac{1 + c(\mathrm{S}+2)}{1 + (\mathrm{S}-1)(\mathrm{S}+2)} < \frac{c}{\mathrm{S}-1}. \tag{13}$$

Thus both inequalities in Eq.(12) hold.

We have

$$\begin{aligned}
\mathrm{ST} - c &\geqslant \mathrm{S}\left(\frac{1 + c(\mathrm{S}+2)}{\mathrm{S}^2 + \mathrm{S} - 1} - 1\right) - c \\
&= \mathrm{S}\left(\frac{1 + c(\mathrm{S}+2)}{1 + (\mathrm{S}-1)(\mathrm{S}+2)} - 1\right) - c \\
&= \frac{\big(c - (\mathrm{S}-1)\big)\mathrm{S}(\mathrm{S}+2)}{1 + (\mathrm{S}-1)(\mathrm{S}+2)} - c \\
&= \frac{c(\mathrm{S}+1) - \mathrm{S}(\mathrm{S}-1)(\mathrm{S}+2)}{1 + (\mathrm{S}-1)(\mathrm{S}+2)} \\
&> 0 \text{ if } c > (\mathrm{S}+1)(\mathrm{S}-1).
\end{aligned}$$

(i) Let $\Delta : [1, \mathrm{T}] \to \{0, 1\}$ be defined by

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in \big(\max\{0, \mathrm{S}^2\mathrm{T} - c(\mathrm{S}+1)\}, \mathrm{ST} - c\big]; \\ 1 & \text{otherwise.} \end{cases}$$

We claim that Δ provides a valid 2-colouring of $[1, \mathrm{T}]$ with respect to Eq.(7). Suppose $\Delta(x_i) = 0$ for $i \in \{1, \ldots, m-1\}$. Then

$$x_m = \sum_{i=1}^{m-1} a_i x_i - c \leqslant \mathrm{S}(\mathrm{ST} - c) - c = \mathrm{S}^2\mathrm{T} - c(\mathrm{S}+1).$$

Hence $\Delta(x_m) = 1$. Therefore, we must have $\Delta(x_i) = 1$ for $i \in \{1, \ldots, m\}$.

If $x_i > \mathrm{ST} - c$ for $i \in \{1, \ldots, m-1\}$, then

$$x_m = \sum_{i=1}^{m-1} a_i x_i - c > \mathrm{S}(\mathrm{ST} - c) - c = \mathrm{S}^2\mathrm{T} - c(\mathrm{S}+1),$$

and

$$x_m = \sum_{i=1}^{m-1} a_i x_i - c \leqslant \mathrm{ST} - c.$$

Therefore, $x_i \in \big[1, \mathrm{S}^2\mathrm{T} - c(\mathrm{S}+1)\big]$ for at least one $i \in \{1, \ldots, m-1\}$. Now

$$\begin{array}{rcl} x_m & = & \displaystyle\sum_{i=1}^{m-1} a_i x_i - c \\ & \leqslant & \mathrm{S}^2\mathrm{T} - c(\mathrm{S}+1) + (\mathrm{S}-1)\mathrm{T} - c \\ & = & (\mathrm{S}^2 + \mathrm{S} - 1)\mathrm{T} - c(\mathrm{S}+2) \\ & < & (\mathrm{S}^2 + \mathrm{S} - 1) \cdot \dfrac{1 + c(\mathrm{S}+2)}{\mathrm{S}^2 + \mathrm{S} - 1} - c(\mathrm{S}+2) \\ & = & 1, \end{array}$$

so that x_m is outside the domain of Δ.

We note that $\mathrm{S}^2\mathrm{T} - c(\mathrm{S}+1) < \mathrm{ST} - c$ for $c > \mathrm{S} - 1$, and further that $\mathrm{ST} - c > 0$ if $c > (\mathrm{S}+1)(\mathrm{S}-1)$. Thus Δ provides a valid 2-colouring for $c > (\mathrm{S}+1)(\mathrm{S}-1)$. For $c \in [\mathrm{S}, \mathrm{S}^2)$, it may be the case that $\mathrm{ST} - c < 1$, in which case all integers in the interval $[1, \mathrm{T}]$ are coloured 1. Since $x_m = \sum_{i=1}^{m-1} a_i x_i - c \leqslant \mathrm{ST} - c$, Δ provides a valid 1-colouring if $\mathrm{ST} - c < 1$. Therefore $\mathtt{Rad}_1\big(c\big) > \mathrm{T}$ in such cases, and $\mathtt{Rad}_2\big(c\big) > \mathrm{T}$ in any case.

(ii) Suppose that the set $\{a_1, \ldots, a_{m-1}\}$ is 2-distributable, and $c > \mathrm{S}(\mathrm{S}-1)$. By part (i), it suffices to prove that

$$\mathtt{Rad}_{2;2}\big(c\big) \leqslant \left\lceil \frac{1 + c(\mathrm{S}+2)}{\mathrm{S}^2 + \mathrm{S} - 1} \right\rceil = \mathrm{T} + 1.$$

We have

$$\mathrm{T} \geqslant \frac{\big(c - (\mathrm{S}-1)\big)(\mathrm{S}+2)}{\mathrm{S}^2 + \mathrm{S} - 1} \geqslant \frac{\big(1 + \mathrm{S}(\mathrm{S}-1) - (\mathrm{S}-1)\big)(\mathrm{S}+2)}{\mathrm{S}^2 + \mathrm{S} - 1} = \mathrm{S} - 1 + \frac{3}{\mathrm{S}^2 + \mathrm{S} - 1}.$$

Hence $T \geqslant \mathrm{S}$ when $c > \mathrm{S}(\mathrm{S}-1)$.

Let $\chi : [1, \mathrm{T}+1] \to \{0, 1\}$ be any 2-colouring of the integers in the interval $[1, \mathrm{T}+1]$. Consider the complimentary colouring $\overline{\chi} : [1, \mathrm{T}+1] \to \{0, 1\}$ given by

$$\overline{\chi}(x) = \chi(\mathrm{T} + 2 - x).$$

Then monochromatic solutions to Eq.(7) under χ corresponding to monochromatic solutions to

$$\sum_{i=1}^{m-1} a_i x_i - x_m = (\mathrm{S}-1)(\mathrm{T}+2) - c \tag{14}$$

under $\overline{\chi}$.

From Eq.(13),

$$(\mathrm{S}-1)(\mathrm{T}+2) - c < (\mathrm{S}-1)\left(\frac{c}{\mathrm{S}-1} + 2\right) - c = 2(\mathrm{S}-1).$$

Thus we have $c' = (\mathrm{S}-1)(\mathrm{T}+2) - c < 2(\mathrm{S}-1)$ for $c > \mathrm{S}(\mathrm{S}-1)$. If $c' < \mathrm{S}-1$, every 2-colouring of $[1, (\mathrm{S}-1-c')(\mathrm{S}+2)+1]$ admits a monochromatic solution to Eq.(15) by Theorem 13. Now

$$\begin{aligned}
(\mathrm{S}-1-c')(\mathrm{S}+2)+1 &= \big(c - (\mathrm{S}-1)(\mathrm{T}+1)\big)(\mathrm{S}+2)+1 \\
&= 1 + c(\mathrm{S}+2) - \big(1 + (\mathrm{S}-1)(\mathrm{S}+2)\big)(\mathrm{T}+1) + \mathrm{T}+1 \\
&\leqslant \mathrm{T}+1.
\end{aligned}$$

Hence every 2-colouring of $[1, \mathrm{T}+1]$ also admits a monochromatic solution to Eq.(15) in this case. If $c' \in \big[\mathrm{S}-1, 2(\mathrm{S}-1)\big)$, then every 2-colouring of $[1, \mathrm{S}+1]$ admits a monochromatic solution to Eq.(15) by Theorem 15. Since $\mathrm{S} \leqslant \mathrm{T}$, every 2-colouring of $[1, \mathrm{T}+1]$ also admits a monochromatic solution to Eq.(15).

□

Remark 22. Theorem 21 generalizes the results in Theorem 4.1 in [16].

Theorem 23. *Let* $\{a_1, \ldots, a_{m-1}\}$ *be a* 3*-distributable set of positive integers, with* $\sum_{i=1}^{m-1} a_i = \mathrm{S}$. *If* $c \in \big(\mathrm{S}-1, \mathrm{S}(\mathrm{S}-1)\big)$ *and* $c\mathrm{S}$ *is even, then*

$$\mathtt{Rad}_{2;3}\big(c\big) \leqslant \mathrm{S}+1.$$

Proof. The result holds for 2-distributable sets $\{a_1, \ldots, a_{m-1}\}$ when $c \in (\mathrm{S}-1, 2\mathrm{S}-1]$ by Theorem 15.

The range of c in Theorems 17 and 19 together cover $\bigcup_{3 \leqslant \lambda \leqslant \lceil \frac{\mathrm{S}+1}{2} \rceil} [(\lambda-1)\mathrm{S}, \lambda\mathrm{S}) = [2\mathrm{S}, \lceil \frac{\mathrm{S}+1}{2} \rceil \mathrm{S})$. In the cases covered by Theorem 17, the Rado number equals $\lambda + \mu$, which is at most S. In the cases covered by Theorem 19, the Rado number is at most $2\lambda - \mu$, and this is at most $2\frac{\mathrm{S}+2}{2} - 1 = \mathrm{S}+1$. Therefore for 3-distributable sets of positive integers $a_1, \ldots, a_{m-1}$ and for $c \in [2\mathrm{S}, \lceil \frac{\mathrm{S}+1}{2} \rceil \mathrm{S})$, we have

$$\mathtt{Rad}_{2;3}\big(c\big) \leqslant \mathrm{S}+1.$$

Let $\chi : [1, \mathrm{S}+1] \to \{0,1\}$ be any 2-colouring of the integers in the interval $[1, \mathrm{S}+1]$. Consider the complimentary colouring $\overline{\chi} : [1, \mathrm{S}+1] \to \{0,1\}$ given by

$$\overline{\chi}(x) = \chi(\mathrm{S}+2-x).$$

Then monochromatic solutions to Eq.(7) under χ corresponds to monochromatic solutions to

$$\sum_{i=1}^{m-1} a_i x_i - x_m = (\mathrm{S}-1)(\mathrm{S}+2) - c \tag{15}$$

under $\overline{\chi}$.

Note that $\left\lceil \frac{\mathrm{S}+1}{2} \right\rceil \mathrm{S} \leqslant c < \mathrm{S}(\mathrm{S}-1)$ implies

$$2(\mathrm{S}-1) < (\mathrm{S}-1)(\mathrm{S}+2) - c \leqslant (\mathrm{S}-1)(\mathrm{S}+2) - \left\lceil \frac{\mathrm{S}+1}{2} \right\rceil \mathrm{S} < \left\lceil \frac{\mathrm{S}+1}{2} \right\rceil \mathrm{S}.$$

Therefore Eq.(15) translates monochromatic solutions to Eq.(7) corresponding to $c \in \big(\mathrm{S}-1, \lceil \frac{\mathrm{S}+1}{2} \rceil \mathrm{S}\big)$ to monochromatic solutions to Eq.(7) corresponding to $c \in \big[\lceil \frac{\mathrm{S}+1}{2} \rceil \mathrm{S}, \mathrm{S}(\mathrm{S}-1)\big)$. □

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