Extremal Number of Arborescences

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Abstract

In this paper we study the following extremal graph theoretic problem: Given an undirected Eulerian graph G, which Eulerian orientation minimizes or maximizes the number of arborescences? We solve the minimization for the complete graph K_n, the complete bipartite graph $K_{n,m}$, and for the so-called double graphs, where there are even number of edges between any pair of vertices.

In fact, for K_n we prove the following stronger statement. If T is a tournament on n vertices with out-degree sequence $d_1^+, \dots, d_n^+$, then

$$\mathrm{allarb}(T) \geqslant \frac{1}{n}\left(\prod_{k=1}^{n}(d_k^+ + 1) + \prod_{k=1}^{n} d_k^+\right),$$

where $\mathrm{allarb}(T)$ is the total number of arborescences. Equality holds if and only if T is a locally transitive tournament.

We also give an upper bound for the number of arborescences of an Eulerian orientation for an arbitrary graph G. This upper bound can be achieved on K_n for infinitely many n.

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1 Introduction

Let D be a connected directed graph with multiple edges allowed. An arborescence of D rooted at some vertex v is a spanning tree of D such that each vertex different from v has out-degree 1, and the root vertex v has out-degree 0. In other words, every edge of the spanning tree is oriented towards the root vertex. We denote the number of arborescences rooted at vertex v by $\mathrm{arb}(D, v)$. Given an Eulerian directed graph D the quantity $\mathrm{arb}(D, v)$ does not depend on v (this follows from the BEST theorem [11], see further explanation below) and we will simply denote it by $\mathrm{arb}(D)$. For a not necessarily Eulerian directed graph let

$$\mathrm{allarb}(D) = \sum_{v \in V} \mathrm{arb}(D, v).$$

In this paper we study the following extremal graph theoretical problems.

Problem 1.1. Given an undirected Eulerian graph G which Eulerian orientation O minimizes or maximizes $\mathrm{arb}(O)$?

Problem 1.2. Among all orientations of an undirected graph, which orientation minimizes or maximizes the quantity $\mathrm{allarb}(O)$?

Problem 1.1 is a natural question on its own that fits into a series of research on counting and sampling arborescences [2, 3, 8, 7, 7, 13]. Another, perhaps surprising motivation comes from a geometric problem, namely the study of the symmetric edge polytope of graphs and regular matroids. It turns out that Problem 1.1 is equivalent to finding the facet of the symmetric edge polytope of the cographic matroid of G with minimal or maximal volume. For a more detailed explanation of this connection, see Section 1.2. Note that we will not use this connection in this paper, and every result in this paper can be understood without understanding the geometric motivation.

Problem 1.1 is also closely connected to the celebrated BEST theorem due to de Bruijn, van Aardenne–Ehrenfest [11], Smith and Tutte claiming that the number of Eulerian tours of an Eulerian directed graph D is

$$\mathrm{arb}(D) \prod_{v \in V} (d^+(v) - 1)!,$$

where $d^+(v)$ is the out-degree of vertex v. This theorem shows that the maximizing or minimizing Eulerian orientation also maximizes or minimizes the number of Eulerian tours.

1.1 Results.

Our first result is a general lower bound for allarb on tournaments. A *tournament* is an oriented complete graph. A tournament is called *transitive* if for any vertices u, v, and w, if uv is oriented towards v and vw is oriented towards w, then uw is oriented towards w. To spell out the case of equality of the theorem we need the following definition.

Definition 1.3. A directed graph is called locally transitive if for every vertex v the out-neighbors $N^+(v)$ and the in-neighbors $N^-(v)$ both induce a transitive tournament.

Theorem 1.4. *Let T be a tournament on n vertices with out-degree sequence $d_1^+, \dots, d_n^+$. Then*

$$\operatorname{allarb}(T) \geqslant \frac{1}{n}\left(\prod_{k=1}^{n}(d_k^+ + 1) + \prod_{k=1}^{n} d_k^+\right).$$

Equality holds if and only if T is a locally transitive tournament.

Note that Theorem 1.4 is not true for general directed graphs as it can occur that a directed graph has no arborescence at all.

Definition 1.5. For odd $n = 2d+1$, we call the following tournament the *swirl tournament on n vertices* and we will denote it by SW_n: Let the vertex set be $[n]$, and let the edges be $\{(i, i+k \mod n) : i \in [n], 1 \leqslant k \leqslant d\}$. See Figure 1 for an example.

The following theorem is a simple specialization of Theorem 1.4 for Eulerian tournaments.

Theorem 1.6. *Let T_n be an Eulerian tournament on n vertices. Then*

$$\operatorname{arb}(T_n) \geqslant \frac{1}{n^2}\left(\left(\frac{n+1}{2}\right)^n + \left(\frac{n-1}{2}\right)^n\right)$$

with equality if and only if $T_n \cong SW_n$.

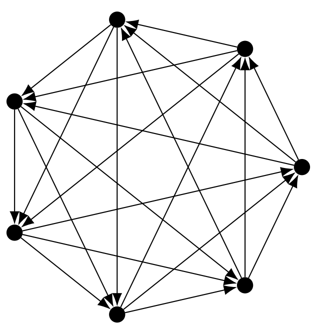

Figure 1: The swirl tournament SW_7, which is the minimizing Eulerian orientation of K_7.

The following theorem is another immediate corollary of Theorem 1.4.

Theorem 1.7. *Let T_n be a tournament on n vertices, and let TR_n be the transitive tournament on n vertices. Then*

$$\operatorname{allarb}(T_n) \geqslant \operatorname{allarb}(TR_n)$$

with equality if and only if $T_n \cong TR_n$.

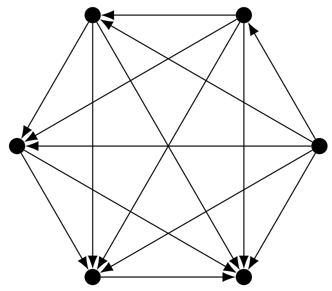

Figure 2: The transitive tournament TR_6.

Theorem 1.4 is surprisingly tight as the following simple statement is true for any directed graph.

Proposition 1.8. *Let D be a directed graph on n vertices with out-degree sequence $d_1^+, \dots, d_n^+$. Then*

$$\operatorname{allarb}(D) \leqslant \sum_{k=1}^{n} \prod_{j \neq k} d_j^+ < \prod_{k=1}^{n} (d_k^+ + 1).$$

Next, we give an upper bound for $\operatorname{arb}(O)$ for Eulerian orientations of K_n (for n odd) that is tight for infinitely many values of n. Let us start with a general upper bound for $\operatorname{allarb}(O)$ for any graph G, which is also tight in several cases.

Theorem 1.9. *Let D be a simple directed graph on n vertices and m edges, with out-degree sequence $d_1^+, \dots, d_n^+$. Then*

$$\operatorname{allarb}(D) \leqslant \left(\frac{1}{n-1}\right)^{\frac{n-1}{2}} \left(\sum_{i=1}^{n} (d_i^+)^2 + m\right)^{\frac{n-1}{2}}.$$

In particular, if G is a simple Eulerian graph with degree sequence $d_1, \dots, d_n$, and O is an Eulerian orientation of G, then we have

$$\operatorname{arb}(O) \leqslant \frac{1}{n} \left(\frac{1}{n-1}\right)^{\frac{n-1}{2}} \left(\frac{1}{4}\sum_{i=1}^{n} d_i^2 + m\right)^{\frac{n-1}{2}}.$$

Next we study tournaments on n vertices, where n is an odd integer. It turns out that for certain n the so-called Hadamard tournaments will be the maximizing orientations.

Definition 1.10. A tournament on n vertices is an Hadamard tournament if its adjacency matrix A satisfies $AA^T = \frac{n+1}{4}I + \frac{n-3}{4}J$ where J is the $n \times n$ matrix with each entry being 1. Note that Hadamard tournaments are sometimes referred to as doubly regular tournaments or homogeneous tournaments in the literature.

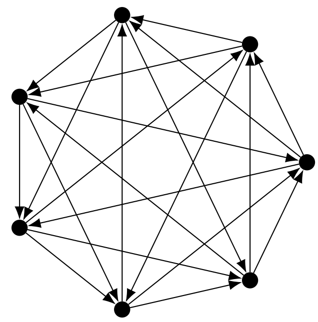

Figure 3: An Hadamard tournament, namely the Paley tournament of order 7.

Recall that a ± 1 matrix H of size $N \times N$ is Hadamard if $H^T H = NI_N$. There is a simple connection between Hadamard matrices and tournaments. There is an Hadamard tournament on n vertices if and only if there is a skew-symmetric Hadamard matrix of size $n+1$ (see [28]). It is known that the size of an Hadamard matrix is either $1, 2$ or divisible by 4 (see Theorem 18.1 of [31]), this means that for $n > 1$ we can only expect an Hadamard tournament on n vertices if $n \equiv 3 \pmod 4$.

Specializing Theorem 1.9 to the complete graph K_n we get an upper bound for the number of arborescences of Eulerian tournaments. In this case we can also characterize the equality case.

Theorem 1.11. *Let T_n be an Eulerian tournament on n vertices. Then*

$$\mathrm{arb}(T_n) \leqslant \frac{1}{n}\left(\frac{n(n+1)}{4}\right)^{\frac{n-1}{2}}$$

with equality if and only if T_n is an Hadamard tournament.

Next, let us study the minimization problem for the complete bipartite graph $K_{n,m}$. It turns out that in this case the minimization problem for $\mathrm{allarb}(O)$ among all orientations is trivial. Indeed, if a graph G has two non-adjacent vertices, then the minimization problem for $\mathrm{allarb}(O)$ is trivial as orienting each incident edge toward these vertices will immediately imply that there are no arborescences in the obtained directed graph as there can be only one root, and any non-root vertex should have at least one out-going edge. So among simple graphs this question is only non-trivial if there are no non-adjacent vertices, that is, when G is a complete graph. In particular, for $K_{n,m}$ the minimal number of arborescences among all orientations is simply 0. This means that only minimization among Eulerian orientations is worth considering.

Theorem 1.12. *Let O be an Eulerian orientation of $K_{n,m}$, where n and m are an even integers. Then*

$$\mathrm{arb}(O) \geqslant \left(\frac{m}{2}\right)^{n-1}\left(\frac{n}{2}\right)^{m-1}.$$

The following orientation of $K_{n,m}$ achieves the lower bound and is the unique minimizer up to isomorphism: take an oriented 4-cycle and blow up every second vertex with $\frac{n}{2}$ vertices and every second vertex with $\frac{m}{2}$ vertices.

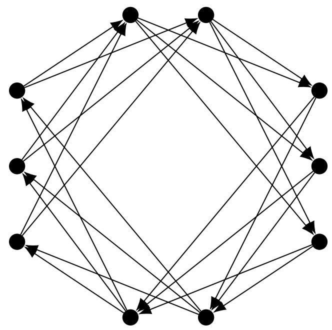

Figure 4: The minimizing Eulerian orientation of $K_{4,6}$.

Our last result is the solution of the Eulerian minimization problem for the so-called double graphs. A graph G is a double graph if there is an even number of edges between any two vertices. In this case we show that the following Eulerian orientation minimizes $\mathrm{arb}(O)$: for each $u, v \in V$, half the edges between u and v are oriented toward u, and half of them toward v. We call this orientation the symmetric orientation of G.

Figure 5: A double graph on the left and its symmetric orientation on the right.

Theorem 1.13. *Let G be a connected double graph. Then the unique Eulerian orientation O of G minimizing $\mathrm{arb}(O)$ is the symmetric orientation.*

1.2 A geometric motivation

We explain a geometric motivation for our investigations. This section is a high-level explanation, and it is not needed for the rest of the paper.

The symmetric edge polytope of a graph $G = (V, E)$ was defined by Matsui, Higashitani, Nagazawa, Ohsugi, and Hibi [25] as the following polytope:

$$\mathcal{Q}_G = \mathrm{Conv}(\{ \mathbf{1}_u - \mathbf{1}_v, \mathbf{1}_v - \mathbf{1}_u \mid uv \in E \}) \subset \mathbb{R}^V$$

Here $\mathbf{1}_v$ denotes the vector where the coordinate corresponding to v is 1, and the rest of the coordinates are 0. This polytope has recently garnered considerable interest [16, 9, 25, 27, 6] due to its nice combinatorial properties. Moreover, it has connections to the Kuramoto synchronization model of physics, where its volume yields an upper bound for the number of steady states [6]. Recently, the symmetric edge polytope was also generalized to regular matroids [29, 10].

Let us call a matroid bipartite if each circuit has even cardinality. It turns out (see [21]), that for a bipartite cographic matroid, the facets of the symmetric edge polytope correspond to Eulerian orientations of the dual (Eulerian) graph, and the volumes of these facets are the arborescence numbers of the corresponding Eulerian orientations.

Hence finding Eulerian orientations of an Eulerian graph with minimal or maximal arborescence number corresponds to finding facets of minimal or maximal volume for the symmetric edge polytope of the cographic matroid.

This paper is organized as follows. In the next section we introduce the necessary tools to study the number of arborescences. In Section 3 we prove the lower bound results, that is, Theorems 1.4, 1.6, 1.12 and 1.13. In Section 4 we prove the upper bound results, Theorem 1.11 and Proposition 1.8. We end the paper with some concluding remarks and open problems.

2 Preliminaries

Notation. Throughout the paper $G = (V, E)$ denotes a graph, D denotes a directed graph, and O is an orientation of some undirected graph G. Our graphs and directed graphs might have multiple edges. K_n denotes the complete graph on n vertices. $K_{n,m}$ denotes the complete bipartite graph with parts of size n and m. The notation $[n]$ stands for $\{1, 2, \dots, n\}$.

The matrix I_k is the $k \times k$ identity matrix, the matrix J_k is the $k \times k$ matrix consisting only of 1's. If the size of the matrix is clear from the context, we drop the subscript. We denote by $\underline{0}$ the all-zero vector, and by $\underline{1}$ the all-one vector (we do not indicate the sizes as it will be clear from the context).

For a matrix M and $S, T \subseteq [n]$, $M_{S,T}$ is the submatrix of M with rows from S and columns from T. We use $M_{\bar{i},\bar{j}}$ for $M_{[n]\setminus\{i\},[n]\setminus\{j\}}$, that is, for the matrix obtained from M by deleting the i^{th} row and j^{th} column. We denote the i^{th} column of M by $M_{*,i}$, and the i^{th} row by $M_{i,*}$.

We denote the characteristic polynomial of matrix M by φ_M. That is, $\varphi_M(x) = \det(xI - M)$. Recall that $\varphi_M(x) = \prod_{i=1}^n (x - \lambda_i)$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of M with multiplicity.

The Frobenius norm of a matrix M is denoted by $||M||_F$. Recall that $||M||_F = \sqrt{\sum_{i,j} M_{i,j}^2}$.

For a directed graph D, we denote by $d^+(v)$ the out-degree of vertex v, by $d^-(v)$ the in-degree of vertex v. We denote by $m(u, v)$ the number of directed edges pointing from u to v. Similarly, for an undirected graph G the degree of a vertex v is denoted by $d(v)$ and $m(u, v)$ denotes the number of edges between u and v.

Given a directed graph D we can associate matrices to D in many different ways. We will need the following three matrices: the adjacency matrix, the skew-symmetric adjacency matrix and the Laplacian matrix.

Definition 2.1. For a directed graph D with vertex set $\{v_1, \dots, v_n\}$, let $A(D)$ refer to the adjacency matrix of D, where $A(D)_{i,j} = m(v_i, v_j)$, the number of directed edges from v_i to v_j.

Definition 2.2. For a directed graph D with vertex set $\{v_1, \dots, v_n\}$, let $L(D)$ refer to the Laplacian matrix of D, defined as

$$L(D)_{i,j} = \begin{cases} d^+(v_i) & \text{if } i = j, \\ -m(v_i, v_j) & \text{if } i \neq j. \end{cases}$$

If G is an undirected graph with vertex set $\{v_1, \dots, v_n\}$, then the Laplacian matrix $L(G)$ is defined by

$$L(G)_{i,j} = \begin{cases} d(v_i) & \text{if } i = j, \\ -m(v_i, v_j) & \text{if } i \neq j. \end{cases}$$

The third matrix is the skew-symmetric adjacency matrix, that is slightly special in that we only associate it to orientations of simple graphs.

Definition 2.3. For a directed graph D with a simple underlying graph G, let $M(D)$ refer to the skew-symmetric adjacency matrix of D, where $M(D)_{i,j} = 1$ if there is a directed edge from v_i to v_j, -1 if there is a directed edge from v_j to v_i, and 0 otherwise. In other words, $M(D) = A(D) - A(D)^T$.

The following result of Tutte is fundamental for us.

Theorem 2.4 (Tutte's matrix-tree theorem [30])**.** *The number of arborescences of a directed graph D rooted at a vertex v_k is equal to* $\det(L(D)_{\overline{k},\overline{k}})$.

This is the counterpart of Kirchhoff's classical matrix tree theorem [22] for directed graphs that can be found in many textbooks [4, 15].

Theorem 2.5 (Kirchhoff's matrix-tree theorem [22])**.** *The number of spanning trees of an undirected graph G is equal to* $\det(L(G)_{\overline{n},\overline{n}})$.

$\det(L(D)_{\overline{k},\overline{k}})$ can be written as the product of the eigenvalues of $L(D)_{\overline{k},\overline{k}}$, but it turns out that it is more convenient to work with the eigenvalues of $L(D)$ instead. Lemma 2.6 below shows that the eigenvalues of $L(D)$ also give us meaningful input for studying the number of arborescences.

Concerning the eigenvalues of $L(D)$ one needs to be a bit careful. Unlike in the case of undirected graphs (where $L(G)$ is a symmetric positive semidefinite matrix) for directed graphs it is not true anymore that the eigenvalues are real or that the matrix is positive semidefinite. It might even occur that $L(D)$ is not diagonalizable, that is, there is no basis consisting of eigenvectors of $L(D)$. Nevertheless, it is true that $L(D)\underline{1} = \underline{0}$ and we will refer to the corresponding 0 eigenvalue as λ_n. Among the eigenvalues $\lambda_1, \dots, \lambda_{n-1}$ there can be another 0, but as the next lemma shows this can only happen if there are no arborescences in the directed graph D.

Lemma 2.6. *For any directed graph D on n vertices we have*

$$\operatorname{allarb}(D) = \prod_{i=1}^{n-1} \lambda_i \tag{2.1}$$

where $\lambda_1, \dots, \lambda_{n-1}$ are the eigenvalues of the Laplacian matrix different from $\lambda_n = 0$.

Proof. Take

$$\varphi_{L(D)}(x) = \prod_{i=1}^{n} (x - \lambda_i) = x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} - \dots + (-1)^{n-1} a_1 x.$$

Then, according to Viéte's formula

$$a_1 = \lambda_2\lambda_3 \cdots \lambda_n + \lambda_1\lambda_3 \cdots \lambda_n + \dots + \lambda_1\lambda_2 \cdots \lambda_{n-1}.$$

As $\lambda_n = 0$ we get $a_1 = \prod_{i=1}^{n-1} \lambda_i$. Furthermore, by taking the cofactor expansion of $\varphi_{L(D)}(x) = \det(xI - L(D))$, we obtain the coefficient

$$a_1 = \sum_{k=1}^{n} \det(L(D)_{\overline{k},\overline{k}}) = \sum_{r \in V(D)} \operatorname{arb}(D, r)$$

due to Tutte's Matrix-Tree theorem. Thus, $\prod\limits_{k=1}^{n-1} \lambda_k = \sum\limits_{r \in V(D)} \operatorname{arb}(D, r)$. ☐

The following corollary of Lemma 2.6 will be a key tool for us.

Corollary 2.7. *Let D be an Eulerian directed graph on n vertices. Since D is Eulerian, we have* $\operatorname{arb}(D, u) = \operatorname{arb}(D, v)$ *for all $u, v \in V(D)$. Thus*

$$\operatorname{arb}(D) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.$$

Lemma 2.8. *Let D be a directed graph on n vertices. Then*

$$\det(L(D) + \alpha J_n) = n\alpha \cdot \operatorname{allarb}(D).$$

Proof. For the all-1 vector $\underline{1}$ we have $L(D)\underline{1} = \underline{0}$ and $(L(D) + \alpha J_n)\underline{1} = n\alpha\underline{1}$. Let $V = \langle \underline{1} \rangle^{\perp}$. For a vector $\underline{v} \in V$ we have $L(D)\underline{v} = (L(D) + \alpha J_n)\underline{v}$ showing that the remaining eigenvalues of $L(D)$ and $L(D) + \alpha J_n$ are the same. By writing up the two linear maps in a basis of V together with $\underline{1}$ they take the form of a block matrix $\begin{pmatrix} c & \underline{u} \\ \underline{0} & M \end{pmatrix}$, where $c = 0$ for $L(D)$ and $n\alpha$ for $L(D) + \alpha J_n$, respectively. So the rest of the eigenvalues are the eigenvalues of M for both matrices. If R denotes the multiset of these eigenvalues, then by Lemma 2.6 we have

$$n\alpha \cdot \mathrm{allarb}(D) = n\alpha \prod_{\lambda \in R} \lambda = \det(L(D) + \alpha J_n). \qquad \square$$

The following lemma is well-known, see Theorem 2.5.3 and its proof in [17].

Lemma 2.9. *Let $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Then*

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqslant \sum_{i,j} |A_{i,j}|^2$$

with equality if and only if A is normal, that is, $A^*A = AA^*$.

Finally, we will use the following basic facts about skew-symmetric matrices.

Lemma 2.10 ([5, 23])**.** *Let $M \in \mathbb{C}^{n\times n}$ be a skew-symmetric matrix, that is, $M^* = -M$.*
(i) Then M is normal, thus, M has an orthonormal basis of eigenvectors.
(ii) The eigenvalues of M are of the form $\alpha_1\mathrm{i}, -\alpha_1\mathrm{i}, \alpha_2\mathrm{i}, -\alpha_2\mathrm{i}, \dots,$ where $\alpha_1, \alpha_2, \dots \in \mathbb{R}$.
(iii) 0 is an eigenvalue of M if n is odd, and so $\det(M) = 0$ in this case.
(iv) The determinant of M is non-negative. In fact, the determinant is the square of the Pfaffian, and so $\det(M)$ is a perfect square if M has only integer values.

We will also use the following formula, which follows directly by the definition of the determinant.

Claim 1. *Let A and D be $n \times n$ matrices, where D is a diagonal matrix. Then,*

$$\det(D - A) = \sum_{S \subseteq [n]} (-1)^{|S|} \det(A_{S,S}) \cdot \prod_{i \in [n]\setminus S} d_{i,i}.$$

3 Lower bounds

In this section we prove Theorems 1.4, 1.6, 1.7 and 1.12.

3.1 Lower bounds for tournaments

Proof of Theorem 1.4. By Lemma 2.8 we know that

$$\det(L(T) + \alpha J_n) = n\alpha \cdot \mathrm{allarb}(T).$$

Let us consider the matrix $2L(T) + J_n$. First observe that

$$\det(2L(T) + J_n) = 2^n \det\left(L(T) + \frac{1}{2}J_n\right) = 2^{n-1} n \cdot \mathrm{allarb}(T).$$

Furthermore, $2L(T) + J_n = D(T) - M(T)$, where $D(T)$ is the diagonal matrix consisting of the elements $2d_i^+ + 1$, and $M(T)$ is the skew-symmetric adjacency matrix of T. Hence

$$\det(2L(T) + J_n) = \det(D(T) - M(T)) = \sum_{S \subseteq [n]} \left(\prod_{k \in S} (2d_k^+ + 1) \right) (-1)^{n-|S|} \det(M(T)_{S^c,S^c}),$$

where $S^c = [n] \setminus S$, and we used Claim 1 in the last equality. Here $M(T)_{S^c,S^c}$ is a skew-symmetric matrix. If it has odd size, then its determinant is 0 according to Lemma 2.10 (iii). If it has even size, then its determinant is non-negative according to Lemma 2.10 (iv). Furthermore, it is an integer since all entries are integer. If $|S^c| = 2k$, then the parity of the determinant of $M(T)_{S^c,S^c}$ is the same as the parity of the determinant of $J_{2k} - I_{2k}$ since the two matrices have the same parity entrywise and the determinant is a polynomial of the entries with integer coefficients. The eigenvalues of the latter matrix are $2k-1$ with multiplicity 1 with eigenvector $\underline{1}$, and -1 with multiplicity $2k-1$ with the eigenspace being all vectors orthogonal to $\underline{1}$, so $\det(J_{2k} - I_{2k}) = (2k-1)(-1)^{2k-1} = -(2k-1)$ is odd. This means that $\det(M(T)_{S^c,S^c})$ is a non-negative odd number, that is,

$$\det(M(T)_{S^c,S^c}) \geqslant 1. \tag{3.1}$$

Hence

$$\sum_{S \subseteq [n]} \left(\prod_{k \in S} (2d_k^+ + 1) \right) (-1)^{n-|S|} \det(M(T)_{S^c,S^c}) \geqslant \sum_{\substack{S \subseteq [n] \\ |S| \equiv n \ (2)}} \prod_{k \in S} (2d_k^+ + 1).$$

Observe that

$$\begin{aligned} \sum_{\substack{S \subseteq [n] \\ |S| \equiv n \ (2)}} \prod_{k \in S} (2d_k^+ + 1) &= \frac{1}{2} \left(\prod_{k=1}^{n} ((2d_k^+ + 1) + 1) + \prod_{k=1}^{n} ((2d_k^+ + 1) - 1) \right) \\ &= 2^{n-1} \left(\prod_{k=1}^{n} (d_k^+ + 1) + \prod_{k=1}^{n} d_k^+ \right). \end{aligned}$$

Putting all these together, we get that

$$2^{n-1} n \cdot \mathrm{allarb}(T) = \det(2L(T) + J) \geqslant 2^{n-1} \left(\prod_{k=1}^{n} (d_k^+ + 1) + \prod_{k=1}^{n} d_k^+ \right).$$

Hence

$$\mathrm{allarb}(T) \geqslant \frac{1}{n} \left(\prod_{k=1}^{n} (d_k^+ + 1) + \prod_{k=1}^{n} d_k^+ \right).$$

This proves the inequality part of the theorem.

To have equality, we need that $\det(M(T)_{S^c,S^c}) = 1$ for each $S \subseteq [n]$ such that $|S^c|$ is even. As Lemma 3.1 shows below, this is equivalent to the tournament being locally transitive. □

Lemma 3.1. *The following are equivalent for a tournament T_n on n vertices.*
(i) T_n is locally transitive.
(ii) T_n does not contain a 4 vertex tournament that contains a cyclically oriented triangle and a vertex with either out-degree or in-degree 0, that is, a 4 vertex tournament that is not locally transitive.
(iii) For the skew-symmetric matrix $M(T_n)$ we have

$$\varphi_{M(T_n)}(x) = \frac{1}{2}((x+1)^n + (x-1)^n).$$

(iv) For the skew-symmetric matrix $M(T_n)$ we have $\det(M(T)_{S,S}) = 1$ *for each $S \subseteq [n]$ with $|S|$ even.*

Proof. The equivalence of (i) and (ii) is trivial as a non-transitive tournament always contains a cyclically oriented triangle.

First note that

$$\frac{1}{2}((x+1)^n + (x-1)^n) = \sum_{k=0}^{n/2} \binom{n}{2k} x^{n-2k}.$$

To prove the equivalence of (iii) and (iv), observe that (by Claim 1)

$$\varphi_{M(T_n)}(x) = \sum_{S \subseteq [n]} x^{n-|S|}(-1)^{|S|} \det(M(T)_{S,S}).$$

By Lemma 2.10 (iii) and (iv) and inequality (3.1), $\det(M(T)_{S,S}) = 0$ if $|S|$ is odd, and $\det(M(T)_{S,S}) \geqslant 1$ is $|S|$ is even. So the coefficient of x^{n-2k} is at least $\binom{n}{2k}$ and equality holds if and only if $\det(M(T)_{S,S}) = 1$ for all set S of size $2k$.

Finally, let us prove the equivalence of (ii) and (iv). It turns out that for the tournaments on 4 vertices that are not locally transitive, the determinant of the skew-symmetric adjacency matrix is 9 and for all other tournaments on 4 vertices, the determinant is 1. This shows that for sets with $|S| = 4$, $\det(M(T_n)_{S,S}) = 1$ if and only if (ii) holds.

Next we show that if for all $|S| = 4$ we have $\det(M(T_n)_{S,S}) = 1$, then we also have $\det(M(T_n)_{S,S}) = 1$ whenever $|S|$ is even. We prove this statement by induction on the size of $|S| = 2r$. Suppose that $|S| \geqslant 6$ and we already know the statement for all $R \subset S$. Then

$$\varphi_{M(T)_{S,S}}(x) = x^{2r} + \binom{2r}{2} x^{2r-2} + \cdots + \binom{2r}{2r-4} x^4 + \binom{2r}{2r-2} x^2 + \det(M(T)_{S,S}).$$

Note that

$$\varphi_{M(T)_{S,S}}(x) = \prod_{j=1}^{r} (x - i\alpha_j)(x + i\alpha_j) = \prod_{j=1}^{r} (x^2 + \alpha_j^2)$$

as $M(T)_{S,S}$ is a skew-symmetric matrix itself. This means that

$$y^r + \binom{2r}{2} y^{r-1} + \cdots + \binom{2r}{2r-4} y^2 + \binom{2r}{2r-2} y + \det(M(T)_{S,S}) = \prod_{j=1}^{r} (y + \alpha_j^2),$$

that is, it is a real-rooted polynomial. For a real-rooted polynomial $\sum_{j=0}^{d} a_j y^j$ with non-negative coefficients Newton's inequality says (see for instance Lemma 3.2 of [14])) that for $1 \leqslant t \leqslant d-1$ we have

$$\frac{a_{t-1}}{\binom{d}{t-1}} \cdot \frac{a_{t+1}}{\binom{d}{t+1}} \leqslant \left(\frac{a_t}{\binom{d}{t}}\right)^2.$$

Let us apply this inequality for the above polynomial and $t = 1$:

$$\frac{\binom{2r}{4}}{\binom{r}{2}} \cdot \det(M(T)_{S,S}) \leqslant \left(\frac{\binom{2r}{2}}{\binom{r}{1}}\right)^2.$$

This gives that

$$\det(M(T)_{S,S}) \leqslant 3 \cdot \frac{2r-1}{2r-3}.$$

Since $r \geqslant 3$ we get that $\det(M(T)_{S,S}) \leqslant 5$. On the other hand, $\det(M(T)_{S,S})$ is an odd number, moreover, it is the square of the Pfaffian (by Lemma 2.10 (*iv*)) so if $\det(M(T)_{S,S}) > 1$, then it is at least 9. Thus, $\det(M(T)_{S,S}) = 1$ for every even-size S. This completes the proof of the equivalence of the four conditions. □

Next we prove Theorem 1.6. We will need the following result of Huang [18] characterizing locally transitive directed graphs (see also the paper [1]).

Theorem 3.2 (Huang [18]). *If $D = (V, E)$ is a simple connected directed graph, then the following two conditions are equivalent:*
(i) D is locally transitive.
(ii) There exists a cyclic ordering of the vertices (say clockwise) such that if $(u, v) \in E$, then for every vertex w between u and v we have $(u, w) \in E$ and $(w, v) \in E$.

Proof of Theorem 1.6. If T is an Eulerian tournament, then all vertices have degree $\frac{n-1}{2}$. From Theorem 1.4 we immediately get that

$$\mathrm{arb}(T) \geqslant \frac{1}{n^2}\left(\left(\frac{n+1}{2}\right)^n + \left(\frac{n-1}{2}\right)^n\right).$$

It is also clear that the swirl tournament is locally transitive, and so achieves the lower bound.

The uniqueness follows from Huang's theorem (Theorem 3.2). From that theorem we immediately see that given the out-degree sequence of locally transitive tournament, the vertices have a cyclic ordering such that for every vertex v the out-neighbor set $N^+(v)$ is simply the next $d^+(v)$ vertices in the cyclic order. In particular, this shows the uniqueness of the minimizing Eulerian tournament. □

Below we give another proof that the swirl tournament achieves the minimum number of arborescences. This proof is more direct and has the advantage that it connects the eigenvalues of $M(T)$ with the number of arborescences.

First we prove a lemma that connects the number of arborescences with the characteristic polynomial of $M(T)$.

Lemma 3.3. *Let T be a d-regular tournament on $n = 2d + 1$ vertices. Let $M(T)$ be the skew-symmetric adjacency matrix of T. If the eigenvalues of $M(T)$ are $\alpha_1, \dots, \alpha_{n-1}, \alpha_n = 0$, then the eigenvalues of $L(T)$ are $\frac{n-\alpha_1}{2}, \dots, \frac{n-\alpha_{n-1}}{2}$ and 0. Furthermore,*

$$\operatorname{arb}(T) = \frac{1}{n^2 2^{n-1}} \varphi_{M(T)}(n).$$

Proof. Note that we have $L(T) = \frac{1}{2}(nI - J - M(T))$. We have $M(T)\underline{1} = \underline{0}$ and $L(T)\underline{1} = \underline{0}$. Real skew-symmetric matrices are diagonalizable (since they are normal), that is, there are eigenvectors $\underline{v}_1, \dots, \underline{v}_{n-1}, \underline{v}_n = \underline{1}$ of $M(T)$ belonging to the eigenvalues $\alpha_1, \dots, \alpha_n = 0$ that form a basis. Note that $\underline{1}^T M(T) = \underline{0}$, which implies that $\underline{1}$ and $\underline{v}_k$ are orthogonal whenever $\lambda_k \neq 0$. (Indeed, $\underline{0} = (\underline{1}^T M(T))\underline{v}_k = \underline{1}^T(M(T)\underline{v}_k) = \lambda_k \underline{1}^T \underline{v}_k$.) If $\lambda_k = 0$, then we can choose an orthogonal eigenbasis from the eigensubspace belonging to the eigenvalue 0, so we can assume that $\underline{1}$ and $\underline{v}_k$ are orthogonal in this case, too.

Then for $1 \leqslant k \leqslant n-1$, we have

$$L(T)\underline{v}_k = \frac{1}{2}(nI - J - M(T))\underline{v}_k = \frac{n - \alpha_k}{2}\underline{v}_k$$

since $J\underline{v}_k = \underline{0}$ as $\underline{1}$ and $\underline{v}_k$ are orthogonal. This proves the first part of the claim.

To prove the second part, observe that

$$\operatorname{arb}(T) = \frac{1}{n}\prod_{i=1}^{n-1}\frac{n-\alpha_i}{2} = \frac{1}{n^2 2^{n-1}} \cdot n \prod_{i=1}^{n-1}(n - \alpha_i) = \frac{1}{n^2 2^{n-1}}\varphi_{M(T)}(n),$$

where in the first equality we used Corollary 2.7. This proves the second part of the claim. □

Next, we compute the characteristic polynomial of $M(T)$ for the swirl tournament T. We claim that

$$\varphi_{M(T)}(x) = \frac{1}{2}((x+1)^n + (x-1)^n).$$

$M(T)$ is a circulant matrix (see for example [17, 0.9.6] for the definition) so its eigenvectors are of the form $(1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1})$, where $\varepsilon^n = 1$. (See for example [17, 2.2.P10].)

When $\varepsilon = 1$ we get the usual $\underline{1}$ eigenvector of $M(T)$ with eigenvalue 0. Otherwise we get the eigenvalue

$$\sum_{k=1}^{d}\varepsilon^k - \sum_{k=d+1}^{n-1}\varepsilon^k = (1-\varepsilon^d)\varepsilon\frac{1-\varepsilon^d}{1-\varepsilon} = \frac{\varepsilon - 2\varepsilon^{d+1} + 1}{1-\varepsilon}.$$

Since n is odd, there is a unique n-th root of unity $\eta \neq 1$ for which $\eta^2 = \varepsilon$. Then

$$\frac{\varepsilon - 2\varepsilon^{d+1} + 1}{1-\varepsilon} = \frac{\eta^2 - 2\eta^{2(d+1)} + 1}{1-\eta^2} = \frac{\eta^2 - 2\eta + 1}{1-\eta^2} = \frac{1-\eta}{1+\eta}$$

whence

$$\varphi_{M(T)}(x) = x \prod_{\substack{\eta^n=1\\ \eta\neq 1}} \left(x - \frac{1-\eta}{1+\eta} \right).$$

We claim that for odd n this is nothing else than

$$\frac{1}{2}((x+1)^n + (x-1)^n).$$

It is clear that both polynomials are monic and that 0 is a root of both polynomials. If $\alpha \neq 0$ is a root of the latter polynomial, then $\alpha \neq -1$, and so

$$\frac{1}{2}((\alpha+1)^n + (\alpha-1)^n) = \frac{1}{2}(\alpha+1)^n \left(1 + \left(\frac{\alpha-1}{\alpha+1} \right)^n \right).$$

Since n is odd we get that $\frac{\alpha-1}{\alpha+1} = -\eta$ for some $\eta \neq 1$ for which $\eta^n = 1$. We get $\alpha = \frac{1-\eta}{1+\eta}$, proving that the two polynomials are equal. This completes the proof that the swirl tournament achieves the lower bound for the number of arborescences of Eulerian tournaments.

Next we prove Theorem 1.7.

Proof of Theorem 1.7. For the transitive tournament, $L(TR_n)$ is an upper triangular matrix, so $\det(L(TR_n)_{n,n}) = (n-1)!$ (the product of the values in the main diagonal).

We show first that for any tournament with outdegree sequence $d_1^+, \ldots, d_k^+$, we have

$$\prod_{k=1}^{n} (d_k^+ + 1) \geqslant n!.$$

The out-degree sequence of the transitive tournament is $0, 1, \ldots, n-1$, which achieves the bound, and this degree sequence determines the transitive tournament. Suppose for contradiction that some tournament T_n with a different degree sequence $d_1^+, \ldots, d_n^+$ minimizes $\prod_{k=1}^{n}(d_k^+ + 1)$. Then T_n must have two vertices u and v with the same out-degree, say $d^+(u) = d^+(v) = d$. Now flip the orientation of the edge between u and v. Suddenly, one of them has out-degree $d+1$, the other one has $d-1$. No other out-degree changed. This implies that $\prod_{k=1}^{n}(d_k^+ + 1)$ decreased as $(d-1+1)(d+1+1) < (d+1)^2$, contradicting the assumption that T_n minimizes the quantity $\prod_{k=1}^{n}(d_k^+ + 1)$. So (using Theorem 1.4)

$$\mathrm{allarb}(T_n) \geqslant \frac{1}{n} \left(\prod_{k=1}^{n} (d_k^+ + 1) + \prod_{k=1}^{n} d_k^+ \right) \geqslant \frac{1}{n} \prod_{k=1}^{n} (d_k^+ + 1) \geqslant \frac{1}{n} n! = (n-1)! = \mathrm{allarb}(TR_n).$$

It is also clear from the proof that equality only holds if T_n is isomorphic to TR_n, otherwise one can strictly decrease $\prod_{k=1}^{n}(d_k^+ + 1)$. □

3.2 Minimizing orientation for complete bipartite graphs

In this section we prove Theorem 1.12. The proof of Theorem 1.12 is very similar to the proof of Theorem 1.4. In fact, it is a bit simpler.

Proof of Theorem 1.12. Let D be an Eulerian orientation of $K_{n,m}$. Then the Laplacian matrix of D looks as follows:

$$L(D) = \begin{pmatrix} \frac{m}{2}I_n & -A_1 \\ -A_2 & \frac{n}{2}I_m \end{pmatrix},$$

where A_1 and A_2 are matrices of size $n \times m$ and $m \times n$, respectively. Now let us consider the matrix

$$S(D) = \begin{pmatrix} \frac{m}{2}I_n & \frac{1}{2}J_{n,m} - A_1 \\ \frac{1}{2}J_{m,n} - A_2 & \frac{n}{2}I_m \end{pmatrix},$$

where $J_{n,m}$ and $J_{m,n}$ are the matrices of all 1's of size $n \times m$ and $m \times n$, respectively. Let us consider the following 4 vectors in $\mathbb{C}^{n+m}$:

$$\underline{v}_1 = (1, 1, \ldots, 1) \text{ and } \underline{v}_2 = \left(1, \ldots, 1, -\frac{n}{m}, \ldots, -\frac{n}{m}\right)$$

$$\underline{v}_3 = (1, \ldots, 1, 0, \ldots, 0) \text{ and } \underline{v}_4 = (0, \ldots, 0, 1, \ldots, 1),$$

where the first n coordinates and the last m coordinates are equal. Observe that

$$L(D)\underline{v}_1 = \underline{0}, \quad L(D)\underline{v}_2 = \frac{n+m}{2}\underline{v}_2, \quad S(D)\underline{v}_3 = \frac{m}{2}\underline{v}_3, \quad S(D)\underline{v}_4 = \frac{n}{2}\underline{v}_4.$$

Clearly, $\langle \underline{v}_1, \underline{v}_2 \rangle = \langle \underline{v}_3, \underline{v}_4 \rangle$, let us denote this 2-dimensional vector space by V_2. This is an invariant subspace for both $L(D)$ and $S(D)$. If $\underline{v} \in V_2^{\perp}$, then the sum of the first n coordinates of $\underline{v}$ and the sum of the last m coordinates of $\underline{v}$ are both 0 implying that $L(D)\underline{v} = S(D)\underline{v}$. This shows that the remaining $n+m-2$ eigenvalues of $L(D)$ and $S(D)$ are the same, let us denote the multiset of these eigenvalues by R. Indeed, if we take the vectors $\underline{v}_3$ and $\underline{v}_4$ and we extend it to a basis of $\mathbb{C}^{n+m}$ by taking a basis of $V_2^{\perp}$, then in this basis both $L(D)$ and $S(D)$ will have the form $\begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$, where M_{11} is a 2×2 matrix describing the action of $L(D)$ and $S(D)$ on V_2 (this block is potentially different for $L(D)$ and $S(D)$) and M_{12} and M_{22} are the same for $L(D)$ and $S(D)$ as for $\underline{v} \in V_2^{\perp}$ we have $L(D)\underline{v} = S(D)\underline{v}$. From this it follows that R is the multiset of eigenvalues of M_{22}.

From Corollary 2.7 and the known eigenvalues corresponding to v_1, v_2, v_3 and v_4 we get that

$$\operatorname{arb}(D) = \frac{1}{n+m} \cdot \frac{n+m}{2} \cdot \prod_{\lambda \in R} \lambda = \frac{1}{n+m} \cdot \frac{n+m}{2} \cdot \frac{2}{n} \cdot \frac{2}{m} \cdot \det(S(D)) = \frac{2}{nm} \det(S(D)).$$

Next observe that $\left(\frac{1}{2}J_{n,m} - A_1\right)^T = -\left(\frac{1}{2}J_{n,m} - A_2\right)$, so for $B = \frac{1}{2}J_{n,m} - A_1$,

$$S(D) = \begin{pmatrix} \frac{m}{2}I_n & B \\ -B^T & \frac{n}{2}I_m \end{pmatrix}.$$

Thus

$$\det(S(D)) = \sum_{k=0}^{\min(n,m)} \left(\frac{m}{2}\right)^{n-k} \left(\frac{n}{2}\right)^{m-k} \sum_{|S|=|T|=k} \det(B_{S,T})^2.$$

For $k = 1$ we have $\sum_{|S|=|T|=k} \det(B_{S,T})^2 = \frac{nm}{4}$ as all elements of B are $\pm\frac{1}{2}$. Since all terms are non-negative, we get a lower bound by taking only the terms $k = 0$ and $k = 1$, that is

$$\det(S(D)) \geqslant \left(\frac{m}{2}\right)^{n} \left(\frac{n}{2}\right)^{m} + \frac{nm}{4}\left(\frac{m}{2}\right)^{n-1} \left(\frac{n}{2}\right)^{m-1} = 2\left(\frac{m}{2}\right)^{n} \left(\frac{n}{2}\right)^{m},$$

that is

$$\mathrm{arb}(D) \geqslant \left(\frac{m}{2}\right)^{n-1} \left(\frac{n}{2}\right)^{m-1}.$$

Note that for $B = \frac{1}{2}\begin{pmatrix} J_{n/2,m/2} & -J_{n/2,m/2} \\ -J_{n/2,m/2} & J_{n/2,m/2} \end{pmatrix}$ we have $\det(B_{S,T}) = 0$ whenever $|S| = |T| \geqslant 2$ since this is a rank 1 matrix. So we have equality for this matrix. This matrix corresponds exactly to the directed graph described in the theorem.

Next we show that the minimizing orientation is unique up to isomorphism. Indeed, B must be a rank 1 matrix as otherwise there would be sets S and T such that $|S| = |T| = 2$ and $\det(B_{S,T}) > 0$ showing that $\mathrm{arb}(D) > \left(\frac{m}{2}\right)^{n-1}\left(\frac{n}{2}\right)^{m-1}$. Since B is rank 1 there are vectors $\underline{u}$ and $\underline{v}$ such that $B = \frac{1}{2}\underline{u}\underline{v}^T$. By replacing $\underline{u}$ and $\underline{v}$ with $\alpha\underline{u}$ and $\frac{1}{\alpha}\underline{v}$ for some $\alpha \neq 0$ we can assume that $\underline{u}_1 = 1$. Since all entries of $2B_{ij}$ are ± 1, we immediately get that all entries of $\underline{u}$ and $\underline{v}$ are ± 1. Since each row of $2B$ contains $\frac{m}{2}$ entries of 1 and $\frac{m}{2}$ entries of -1, and each column of $2B$ contains $\frac{n}{2}$ entries of 1 and $\frac{n}{2}$ entries of -1, we get that also for $\underline{u}$ and $\underline{v}$, half their coordinates are 1 and half of them are -1. This means that $2B$ is isomorphic to $\begin{pmatrix} J_{n/2,m/2} & -J_{n/2,m/2} \\ -J_{n/2,m/2} & J_{n/2,m/2} \end{pmatrix}$ up to the permutations of rows and columns. In other words, any minimizing orientation is isomorphic to the one described in the theorem. □

Let us mention that if $n = m$ we can also describe the eigenvalues of $L(D)$. We do not detail the proof as it is practically the same as the first part of the above proof.

Lemma 3.4. *Let D be an orientation of $K_{n,n}$, where n is even. Let $M(D)$ be its skew-symmetric adjacency matrix, and let $L(D)$ be its Laplacian matrix. Let $\underline{1}$ be the all-1 vector of length $2n$, and let $\underline{j}$ be the vector of length $2n$ whose first n coordinates are 1, and last n coordinates are -1. Then*

$$L(D)\underline{1} = \underline{0}, \quad L(D)\underline{j} = n\underline{j}, \quad M(D)\underline{1} = \underline{0}, \quad M(D)\underline{j} = \underline{0}.$$

Furthermore, if $\underline{v}$ is an eigenvector of $M(D)$ corresponding to eigenvalue α, that is orthogonal to the vectors $\underline{1}$ and $\underline{j}$, then v is an eigenvector of $L(D)$ corresponding to eigenvalue $\frac{n+\alpha}{2}$. In particular,

$$\mathrm{arb}(D) = \frac{1}{2^{2n-1}n^2}\varphi_{M(D)}(n).$$

3.3 Proof of Theorem 1.13

In this section we prove Theorem 1.13. The proof is based on the following result of Ostrowski and Taussky. This inequality can be found in [17] as Theorem 7.8.19.

Lemma 3.5 (Ostrowski and Taussky)**.** *Let A be a real square matrix. Assume its symmetric part $\frac{A+A^T}{2}$ is positive-definite. Then,*

$$\det\left(\frac{A+A^T}{2}\right) \leqslant \det(A)$$

with equality if and only if $\frac{A+A^T}{2} = A$.

The following theorem implies Theorem 1.13.

Theorem 3.6. *Let G be an undirected connected graph with an Eulerian orientation O, and let* $\mathrm{sp}(G)$ *denote the number of spanning trees of the graph G. Then*

$$\mathrm{arb}(O) \geqslant \frac{1}{2^{n-1}}\mathrm{sp}(G)$$

with equality if and only if G is a double graph and O is its symmetric orientation.

Proof. Since O is Eulerian, we can see that $L(O) + L(O)^T = L(G)$, where $L(G)$ is the Laplacian matrix of the undirected graph, and $L(O)$ is the Laplacian matrix of the directed graph O. This is because (using the notation $D(O)$ for the diagonal matrix containing the out-degrees of O, and $D(G)$ for the diagonal matrix containing the degrees of G)

$$L(O) + L(O)^T = (D(O) + D(O)^T) - (A(O) + A(O)^T) = D(G) - A(G) = L(G)$$

where we use the fact that O is Eulerian to deduce $D(O) + D(O)^T = D(G)$. As usual let $L(O)_{\overline{n},\overline{n}}$ be the matrix obtained from $L(O)$ by deleting the n^{th} row and column. Then we have $L(O)_{\overline{n},\overline{n}} + L(O)^T_{\overline{n},\overline{n}} = L(G)_{\overline{n},\overline{n}}$. Note that $L(G)$ is a symmetric matrix because G is undirected. Furthermore $L(G)$ is positive-definite since G is connected. This means that $L(G)_{\overline{n},\overline{n}}$ is positive-definite. This means $\frac{L(O)_{\overline{n},\overline{n}}+L(O)^T_{\overline{n},\overline{n}}}{2} = \frac{1}{2}L(G)_{\overline{n},\overline{n}}$ is also positive-definite. From Lemma 3.5 this implies that

$$\det\left(\frac{L(O)_{\overline{n},\overline{n}} + L(O)^T_{\overline{n},\overline{n}}}{2}\right) \leqslant \det(L(O)_{\overline{n},\overline{n}}).$$

Therefore we have,

$$\begin{aligned}
\frac{1}{2^{n-1}}\mathrm{sp}(G) &= \frac{1}{2^{n-1}}\det(L(G)_{\overline{n},\overline{n}}) \\
&= \det\left(\frac{1}{2}L(G)_{\overline{n},\overline{n}}\right) \\
&= \det\left(\frac{L(O)_{\overline{n},\overline{n}} + L(O)^T_{\overline{n},\overline{n}}}{2}\right) \\
&\leqslant \det(L(O)_{\overline{n},\overline{n}}) \\
&= \mathrm{arb}(O).
\end{aligned}$$

We have equality in the above if and only if $\det\left(\frac{L(O)_{\overline{n},\overline{n}}+L(O)^T_{\overline{n},\overline{n}}}{2}\right) = \det(L(O)_{\overline{n},\overline{n}})$. From Lemma 3.5 this happens if and only if $\frac{L(O)_{\overline{n},\overline{n}}+L(O)^T_{\overline{n},\overline{n}}}{2} = L(O)_{\overline{n},\overline{n}}$. Because $\frac{L(O)_{\overline{n},\overline{n}}+L(O)^T_{\overline{n},\overline{n}}}{2} = \frac{1}{2}L(G)_{\overline{n},\overline{n}}$ this happens if and only if $\frac{1}{2}L(G)_{\overline{n},\overline{n}} = L(O)_{\overline{n},\overline{n}}$. Finally, since the rows and columns must sum to zero, this is further equivalent to $\frac{1}{2}L(G) = L(O)$. Which means we have equality if and only if O is the symmetric orientation. □

Proof of Theorem 1.13. Immediate from Theorem 3.6. □

Remark 3.7. There is another connection between the number of arborescences and the number of spanning trees. A simple double counting argument shows that if G is a graph on n vertices and m edges, then

$$\frac{1}{2^m}\sum_O \operatorname{allarb}(O) = \frac{n}{2^{n-1}}\operatorname{sp}(G),$$

where the summation is for all orientations, not just the Eulerian ones.

4 Upper bounds

In this section we prove Theorems 1.9 and 1.11.

4.1 General upper bound for directed graphs.

Proof of Theorem 1.9. Let $L = L(D)$ be the Laplacian, and $A = A(D)$ be the adjacency matrix of D. Let the eigenvalues of L be $\lambda_1, \dots, \lambda_n = 0$. Then applying the geometric-quadratic mean inequality we have,

$$\prod_{i=1}^{n-1}\lambda_i = \prod_{i=1}^{n-1}|\lambda_i| \leqslant \left(\frac{1}{n-1}\sum_{i=1}^{n-1}|\lambda_i|^2\right)^{(n-1)/2}.$$

Using the fact that $\sum_{i=1}^n |\lambda_i|^2 \leqslant ||L||_F^2$ (Lemma 2.9) we have,

$$\left(\frac{1}{n-1}\right)^{(n-1)/2}\left(\sum_{i=1}^{n}|\lambda_i|^2\right)^{(n-1)/2} \leqslant \left(\frac{1}{n-1}\right)^{\frac{n-1}{2}}\left(||L||_F^2\right)^{(n-1)/2}.$$

On the other hand,

$$||L||_F^2 = \sum_{i,j} L_{ij}^2 = \sum_i L_{ii}^2 + \sum_{i,j} A_{ij}^2 = \sum_{i=1}^n (d_i^+)^2 + m$$

where we use the fact that D is simple. Altogether,

$$\operatorname{allarb}(D) = \prod_{i=1}^{n-1}\lambda_i \leqslant \left(\frac{1}{n-1}\right)^{\frac{n-1}{2}}(||L||_F^2)^{(n-1)/2} = \left(\frac{1}{n-1}\right)^{\frac{n-1}{2}}\left(\sum_{i=1}^n (d_i^+)^2 + m\right)^{(n-1)/2}$$

which is the desired inequality.

If O is an Eulerian orientation of the simple graph G with degree sequence $d_1, \ldots, d_n$, then $d_i^+ = \frac{d_i}{2}$ and $\operatorname{allarb}(O) = n \cdot \operatorname{arb}(O)$, whence the second inequality follows. $\square$

Corollary 4.1. *Let D be a simple directed graph on n vertices and m edges. If Δ^+ is the maximum out-degree of vertices in D, then we have*

$$\operatorname{allarb}(D) \leqslant n^{\frac{n-1}{2}} \left(\frac{1}{n-1}\right)^{\frac{n-1}{2}} \left((\Delta^+)^2 + \Delta^+\right)^{\frac{n-1}{2}}.$$

In particular if G is a simple Eulerian graph, Δ is the maximum degree of vertices in G, and O is an Eulerian orientation of G, then we have

$$\operatorname{arb}(O) \leqslant n^{\frac{n-3}{2}} \left(\frac{1}{n-1}\right)^{\frac{n-1}{2}} \left(\frac{1}{2}\right)^{n-1} \left(\Delta^2 + 2\Delta\right)^{\frac{n-1}{2}}.$$

Proof. This is immediate from Theorem 1.9. For the first inequality we have $d_i^+ \leqslant \Delta^+$ which means $\sum_{i=1}^n (d_i^+)^2 \leqslant n(\Delta^+)^2$ and $m \leqslant n\Delta^+$, then the upper bound follows from Theorem 1.9. For the second inequality we have $d_i \leqslant \Delta$ which means $\sum_{i=1}^n d_i^2 \leqslant n\Delta^2$ and $m \leqslant \frac{1}{2}n\Delta$, then the upper bound follows from Theorem 1.9. $\square$

Proof of Theorem 1.11. Since $d_i = n-1$ and $m = \binom{n}{2}$, the bound in Theorem 1.9 implies

$$\begin{aligned}
\operatorname{arb}(O) &\leqslant \frac{1}{n}\left(\frac{1}{n-1}\right)^{\frac{n-1}{2}} \left(\frac{1}{4}\sum_{i=1}^n d_i^2 + m\right)^{\frac{n-1}{2}} \\
&= \frac{1}{n}\left(\frac{1}{n-1}\right)^{\frac{n-1}{2}} \left(\frac{n(n-1)^2}{4} + n\frac{n-1}{2}\right)^{\frac{n-1}{2}} \\
&= \frac{1}{n}\left(\frac{n(n+1)}{4}\right)^{\frac{n-1}{2}}
\end{aligned}$$

which is the desired inequality.

Next, we show that Hadamard tournaments attain the upper bound. Suppose that O is an Hadamard tournament. The definition of Hadamard tournament implies that $A_{i,i} = \frac{n-1}{2}$. Thus, O is Eulerian. By [12, Proposition 3.1], the adjacency matrix $A := A(O)$ has an eigenvalue n with multiplicity 1 and eigenvalues $\frac{-1}{2} + i\frac{\sqrt{n}}{2}$ and $\frac{-1}{2} - i\frac{\sqrt{n}}{2}$, each with multiplicity $\frac{n-1}{2}$. Therefore its Laplacian $L := L(O)$ has an eigenvalue 0 with multiplicity 1 and eigenvalue $\frac{n}{2} \pm i\frac{\sqrt{n}}{2}$, each with multiplicity $\frac{n-1}{2}$. By Corollary 2.7,

$$\operatorname{arb}(O) = \frac{1}{n}\prod_{\lambda_i \neq 0} \lambda_i = \frac{1}{n}\left(\frac{n(n+1)}{4}\right)^{\frac{n-1}{2}}$$

as required.

Now suppose $\operatorname{arb}(O) = \frac{1}{n}\left(\frac{n(n+1)}{4}\right)^{\frac{n-1}{2}}$. Then in the proof of Theorem 1.9, we need equality in the geometric mean- quadratic mean inequality for $|\lambda_1|, \ldots, |\lambda_{n-1}|$, which implies $|\lambda_1| = \cdots = |\lambda_{n-1}|$, thus $|\lambda_i| = \sqrt{\frac{n(n+1)}{4}}$ for $i = 1, \ldots, n-1$.

By Lemma 3.3 we also know that the real part of all non-zero eigenvalues are $\frac{n}{2}$ as the eigenvalues of the skew-symmetric adjacency matrix $M(T)$ are purely complex according to Lemma 2.10. This implies that all eigenvalues are $\frac{n \pm i\sqrt{n}}{2}$ and since they are the eigenvalues of a real matrix, the multiplicities of both numbers are $\frac{n-1}{2}$. So T has exactly three eigenvalues. Because $\operatorname{arb}(O) > 0$, we know O has an Eulerian tour due to the BEST theorem, and thus is strongly connected. Since O is a strongly connected tournament and has exactly 3 distinct eigenvalues, it follows that O is an Hadamard tournament by [12, Theorem 3.2]. □

Remark 4.2. The bound in Theorem 1.11 is not tight for all n, for example when $n = 5$ the bound gives an upper bound of 11.25. But when n is prime and $n \equiv 3 \pmod{4}$, then the bound is attained by the Paley tournament on n vertices. So the bound is tight for an infinite family of orientations.

Remark 4.3. In general, it is not true that an Eulerian orientation maximizes $\operatorname{allarb}(O)$ when there is such an orientation. For instance, for the graph $K_{2,4}$ the Eulerian orientation O –which is unique up to isomorphism– gives $\operatorname{allarb}(O) = 12$, whereas there is an orientation O' for which $\operatorname{allarb}(O') = 16$. Nevertheless, we conjecture that allarb is maximized by an Eulerian orientation for K_n when n is odd, see Conjecture 5.4.

4.2 Trivial upper bound.

Proof of Proposition 1.8. Once we fix the root to v_k, any arborescence rooted at v_k uses exactly one of the d_j^+ edges at vertex j. Hence the number of such arborescences is at most $\prod_{j \neq k} d_j^+$, and the number of all arborescences is at most

$$\operatorname{allarb}(T) \leqslant \sum_{k=1}^{n} \prod_{j \neq k} d_j^+ < \prod_{k=1}^{n} (d_k^+ + 1). \qquad \square$$

Remark 4.4. The bounds in Theorem 1.4 and Proposition 1.8 have ratio less than n. Probably an even stronger upper bound is true since for Eulerian tournaments the ratio of the upper and lower bounds provided by Theorem 1.11 and 1.6 are within constant factor:

$$\frac{\left(\frac{n(n+1)}{4}\right)^{(n-1)/2}}{\frac{1}{n}\left(\left(\frac{n+1}{2}\right)^n + \left(\frac{n-1}{2}\right)^n\right)} \leqslant \frac{2e^{3/2}}{e^2+1} \approx 1.06846.$$

5 Concluding remarks and questions

There is a general intuition that guides in the solution of many Eulerian minimization problems: the orientation minimizing the number of arborescences is the one that has many short directed cycles. Similar intuition appears at many different problems: For the number of spanning trees this is justified by McKay [26]. In case of Eulerian orientations, the situation is opposite: short cycles tend to increase their number [19]. In case of Eulerian tours this phenomenon was also observed by Creed (see page 154 of [8]: "a strong connection between the number of short cycles of different lengths and the number of Eulerian tours of graphs").

In our paper the short cycle phenomenon is less apparent for the complete graph K_n, but both for complete bipartite graphs $K_{n,m}$ and double graphs, the minimizing Eulerian orientations are the ones that contain the most directed cycles among cycles of minimal length. The following conjecture is also related to the intuition on many short directed cycles. As we explain below, this conjecture is the special case of [20, Conjecture 5.6].

Conjecture 5.1. Let G be an Eulerian planar graph. The orientation minimizing the number of arborescences is the one that is alternatingly in- and outward oriented at each vertex, that is, where the cycle around each face is oriented.

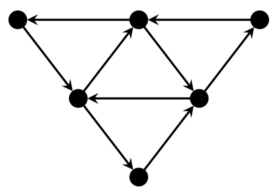

Figure 6: An Eulerian plane graph and its alternating orientation.

Remark 5.2. As we remarked in Section 1.2, the problems considered in this paper are related to a problem concerning volumes of facets of symmetric edge polytopes. Conjecture 5.1 can be translated to the symmetric edge polytope language as follows: take the planar dual G^* of the planar Eulerian graph G. Since G was Eulerian, G^* is bipartite (say, with partite classes A and B). The planar dual of the orientation mentioned in Conjecture 5.1 is the orientation of G^* where each edge points from A to B. In [20], this is called a standard orientation of G^*.

The facets of the symmetric edge polytope of G^* correspond to planar duals of Eulerian orientations of G, and their normalized volume is the arborescence number of these Eulerian orientations. The duals of the Eulerian orientations are exactly those orientations of G^*, where for each cycle, the number of edges in the two cyclic directions agree. (In [20], these are called semi-balanced orientations.) Conjecture 5.1 says that among these orientations, the standard orientation corresponds to a facet of minimal volume.

In [20, Conjecture 5.6], it is further conjectured that the standard orientation corresponds to a facet of minimal volume not only for planar bipartite graphs, but for arbitrary bipartite graphs. Furthermore, it is conjectured that the standard orientation not only minimizes the volume of the corresponding facet of the symmetric edge polytope, but it coefficientwise minimizes the h^*-polynomial of the corresponding facet. (The latter would imply the minimization of the volume as well.)

It seems that the structures of the maximizing orientations are much more intricate than that of minimizing orientations. Even in the case of complete graph K_n we do not know the answer if there is no Hadamard tournament on n vertices. In particular, the following problem is open.

Problem 5.3. Which Eulerian tournament T maximizes $\text{arb}(T)$ on n vertices if $n \equiv 1 \pmod 4$?

The following conjecture seems natural, too.

Conjecture 5.4. Let n be odd. Then the tournament maximizing $\text{allarb}(T_n)$ is Eulerian.

Seemingly Conjecture 5.4 does not follow the short cycle heuristic since the number of directed triangles in a tournament is

$$\binom{n}{3} - \sum_{k=1}^{n} \binom{d_k^+}{2},$$

and this is maximized for Eulerian orientations. The reason is simple: the short cycle intuition is valid only after fixing the out-degree sequence. Among Eulerian tournaments it is indeed the swirl tournament that maximizes the number of directed 4-cycles, see [24].

The same questions for $K_{n,m}$ are also open.

Problem 5.5. Which Eulerian orientation O maximizes $\text{arb}(O)$ on $K_{n,m}$? Which orientation O maximizes $\text{allarb}(O)$ on $K_{n,m}$?

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