On Faces and Hilbert Bases of Kostka Cones

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Abstract

The r-Kostka cone is the real polyhedral cone generated by pairs of partitions with at most r parts such that the corresponding Kostka coefficient is nonzero. We provide several results showing that its faces have interesting structural and enumerative properties. We show that, for fixed d, the number of d-faces of the r-Kostka cone is a polynomial in r with a positive integer expansion in the binomial basis, and we provide exact formulas for $d \leqslant 4$. We prove that the maximum number of extremal rays in a d-face stabilizes to some explicit constant as r increases. We then work towards a generalization of the Gao-Kiers-Orelowitz-Yong Width Bound.

Mathematics Subject Classifications: 05E10, 52B05

1 Introduction

1.1 Background

The r-Kostka cone, denoted by Kostka_r, is the real polyhedral cone generated by pairs $(\lambda, \mu) \in \mathbb{R}^{2r}$ of non-increasing r-tuples of equal sum such that, for all $1 \leqslant i < r$, the sum of the first i parts of λ is at least the sum of the first i parts of μ. The integral points of the r-Kostka cone are precisely the pairs (λ, μ) of integer partitions with at most r parts such that the *Kostka number* $K_{\lambda,\mu}$ is positive, or equivalently, such that $\lambda - \mu$ is a sum of positive weights in $\mathfrak{sl}_r$. Kostka numbers have connections to Young tableaux [7], representation theory [2], symmetric functions [6], dimer configurations [5], and supergravity theories [17].

Carl Kostka introduced Kostka numbers in 1882 while studying symmetric function expansions [6]. Kostka numbers are hard to compute in general, as their computation is #P-complete [9]. Kostka numbers also appear in the representation theory of the general linear group. By Young's Rule, the Kostka number $K_{\lambda,\mu}$ is the multiplicity with which the weight μ appears in the irreducible representation of $\mathrm{GL}_r(\mathbb{C})$ with highest weight λ. It is also the coefficient of the monomial symmetric function corresponding to μ in the expansion of the Schur polynomial corresponding to λ. See, [15, Chapter 7] for a more

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thorough history of Kostka numbers and [2] for details on the representation-theoretic perspective.

Slicing the r-Kostka cone by the affine hyperplane $\{x \in \mathbb{R}^{2r} : (1,1,\dots,1)\cdot x = 1\}$ yields a $(2r-2)$-dimensional polytope, which we call the *Kostka polytope* and denote by $\mathrm{P}_r^{\mathsf{Kostka}}$. There are numerous other polytopes defined in terms of partitions, the faces of which have previously been shown to have interesting enumerative properties. The *Fibonacci polytopes*, or *ordered partition polytopes*, have vertex sets satisfying a Fibonacci-like recurrence [11] and are related to alternating permutations [16]. For the family of *unordered partition polytopes*, Shlyk gave a description of the dynamic behavior of the vertices and a characterization of the facets [13]. Each unordered partition polytope is combinatorially equivalent to a face of $\mathrm{P}_r^{\mathsf{Kostka}}$.

Several recent works on the Kostka cone have focused on its Hilbert basis and extremal rays. In 2021, Gao, Kiers, Orelowitz, and Yong [3] gave a criterion for Hilbert basis membership, though they show that this decision problem is NP-complete in general. They use this criterion to give a simple description of the extremal rays and a "Width Bound" on the integer pairs (λ_1, μ_1) that can be the first parts of partitions λ, μ forming a Hilbert basis element (λ, μ) of the r-Kostka cone for $r \leqslant \lambda_1$. Kim has since provided a strengthening of this Width Bound via a study of generalized Dyck paths [4]. Similar studies have also been carried out in other Lie types. Besson, Jeralds, and Kiers [1] took a representation-theoretic approach to enumerate the rays of the *generalized Kostka cones* of types D_r and E_r, where type A_r is the classical case handled in [3].

1.2 Results

Our work focuses on studying the faces and Hilbert basis of the r-Kostka cone Kostka_r, with a focus on enumerative and structural properties. We typically refer to r-Kostka polytope $\mathrm{P}_r^{\mathsf{Kostka}}$ instead of the Kostka cone when discussing the face structure, as d-faces of $\mathrm{P}_r^{\mathsf{Kostka}}$ are naturally identified with $(d+1)$-faces of Kostka_r.

We begin by computing the maximum number of vertices contained in a face of fixed dimension (see Theorem 21).

Theorem 1. *For $r > d+1$, the maximum number of vertices contained in a d-face of the polytope P_r^{Kostka} is $\prod_{i=1}^{3} \left\lfloor \frac{d+2+i}{3} \right\rfloor$, which is the maximum product of three positive integers summing to $d+3$.*

We then study the set of d-faces of $\mathrm{P}_r^{\mathsf{Kostka}}$ using a connection to cells of the braid arrangement. By reducing the d-face structure of $\mathrm{P}_r^{\mathsf{Kostka}}$ to that of $\mathrm{P}_{3d+3}^{\mathsf{Kostka}}$ (Theorem 32), we can provide exact formulas for the number of d-faces of $\mathrm{P}_r^{\mathsf{Kostka}}$ for $d = 1, 2, 3$. We also show that in general, the face-counting functions $f_d(r)$ are polynomials of a rather nice form.

Theorem 2. *The number of edges of P_r^{Kostka} is*

$$f_1(r) = \binom{r}{6} + 2\binom{r}{5} + 6\binom{r}{4} + 7\binom{r}{3} + 3\binom{r}{2},$$

the number of two-dimensional faces of P_r^{Kostka} *is*

$$f_2(r) = \binom{r}{9} + 3\binom{r}{8} + 12\binom{r}{7} + 23\binom{r}{6} + 33\binom{r}{5} + 31\binom{r}{4} + 13\binom{r}{3} + \binom{r}{2},$$

and the number of three-dimensional faces of P_r^{Kostka} *is*

$$f_3(r) = \binom{r}{12} + 4\binom{r}{11} + 19\binom{r}{10} + 49\binom{r}{9} + 105\binom{r}{8} + 163\binom{r}{7} + 177\binom{r}{6} + 131\binom{r}{5} + 53\binom{r}{4} + 7\binom{r}{3}.$$

For any $d \geqslant 1$*, the function* $f_d(r)$ *has a positive integer expansion in the basis* $\binom{r}{k}_{0 \leqslant k \leqslant 3d+3}$*.*

We also determine that the coefficient of the top degree term $\binom{r}{3d+3}$ is always 1, yielding precise asymptotics for the number of d-faces.

The main result of the last section concerns the Hilbert basis of Kostka_r. We say that an integer pair (λ_1, μ_1) is *r-initial* if there is an element (λ, μ) in the Hilbert basis of Kostka_r such that λ has first element λ_1 and μ has first element μ_1. The Width Bound of Gao-Kiers-Orelowitz-Yong [3, Theorem 1.4] implies that (λ_1, μ_1) is λ_1-initial if and only if λ_1 and μ_1 are coprime. We provide several sufficient conditions for a pair (λ_1, μ_1) to be $(\lambda_1 + 1)$-initial.

Theorem 3. *If any of the following conditions hold:*

- λ_1 *and* μ_1 *are coprime [3, Theorem 1.4], or*
- $\lambda_1 + 1$ *and* μ_1 *are coprime, or*
- $\lambda_1 + 1$ *and* $\mu_1 + 1$ *are coprime with* $2\mu_1 \geqslant \lambda_1$*,*

then the pair (λ_1, μ_1) *is* $(\lambda_1 + 1)$*-initial. Moreover, this holds even if we consider only those Hilbert basis elements contained in the* 2*-skeleton of* Kostka_r*.*

The first criterion follows directly from the work of Gao-Kiers-Orelowitz-Yong, while the latter two conditions are the result of new constructions of Hilbert basis elements. The proof of the last claim in Theorem 3 relies on our characterization of the 2-faces of the Kostka cone (Theorem 29).

1.3 Outline

We begin by providing some preliminaries on the Kostka cone and Kostka polytope in Section 2. We study the maximum number of vertices contained in a face of the Kostka polytope in Section 3. The edge characterization of the Kostka polytope is in Section 4, and the enumerative results on the faces of fixed dimension are in Section 5. The construction of Hilbert basis elements is discussed in Section 6.

2 Preliminaries

2.1 The Kostka Cone

For positive integers r and n, we denote the set of integer partitions of n into at most r parts by $\mathsf{Par}_r(n)$, where such partitions are written as non-increasing r-tuples. Each partition can be viewed as a Young diagram, where the length of the i^{th} row is the i^{th} entry of the r-tuple.

Consider two partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$ in $\mathsf{Par}_r(n)$. A *semistandard tableau of shape* λ *and content* μ is a filling of the Young diagram corresponding to λ with integer entries such that the rows are non-decreasing to the right, the columns strictly increase downward, and there are precisely μ_i boxes with entry i for all $1 \leqslant i \leqslant r$. These are counted by the *Kostka coefficient* $K_{\lambda,\mu}$.

Example 4. The Kostka coefficient $K_{(4,2),(2,2,1,1)}$ is equal to 4, as shown by the following four tableaux of shape $(4, 2)$ and content $(2, 2, 1, 1)$.

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&4\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&4\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline 2&3\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 2&2\\\cline{1-2}\end{array}$$

There is a well-known condition for when a Kostka coefficient is nonzero. This occurs precisely when λ *dominates* μ, i.e.,

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{j=1}^{k} \mu_j \text{ for all } k \leqslant r\,.$$

This is denoted by $\lambda \geqslant_{\mathsf{Dom}} \mu$, and this ordering on partitions is called the *dominance order* (also known as the *majorization order* or *natural order*) [15, Section 7.10].

Definition 5. The r-Kostka cone is the $(2r-1)$-dimensional polyhedral cone formed by taking the convex hull in $\mathbb{R}^{2r}$ of the points $(\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r) \in \mathbb{Z}^{2r}_{\geqslant 0}$ where $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$ are both elements of $\mathsf{Par}_r(n)$ for some n and where λ dominates μ.

Note that the Kostka cone is pointed, i.e., contains no nontrivial linear subspace. The Kostka cones can be viewed as nested via the following observation.

Observation 6. *The cone* Kostka_r *is combinatorially equivalent to the codimension-*2 *face of* Kostka_{r+1} *obtained by intersecting with the hyperplane given by the equation* $\mu_r = 0$.

2.1.1 Facets

The bounding hyperplanes of Kostka_r are simple to describe by examining the required inequalities satisfied by individual entries of each element.

Lemma 7. *The Kostka cone is bounded by the following hyperplanes for $1 \leqslant i < r$:*

$$\begin{aligned} H_i &= \{(\lambda, \mu) \in \mathbb{R}^{2r} : \lambda_i = \lambda_{i+1}\}, \\ H_r &= \{(\lambda, \mu) \in \mathbb{R}^{2r} : \lambda_r = 0\}, \\ \widehat{H}_i &= \{(\lambda, \mu) \in \mathbb{R}^{2r} : \mu_i = \mu_{i+1}\}, \ \textit{and} \\ J_i &= \left\{(\lambda, \mu) \in \mathbb{R}^{2r} : \sum_{j=1}^{i} \lambda_j = \sum_{k=1}^{i} \mu_k\right\}. \end{aligned}$$

Hence for $r > 2$, Kostka_r has $3r - 2$ facets.

Proof. It is straightforward to check that each of these hyperplanes intersects Kostka_r along a facet. Moreover, for $r > 2$ the defining equations for these hyperplanes are all distinct, and hence the resulting facets are distinct. □

2.1.2 The Kostka polytope

Since a large portion of this work concerns the face structure of Kostka_r, it is often more convenient to work with a polytopal slice of this cone.

Definition 8. Let $\mathrm{P}_r^{\mathsf{Kostka}}$ be the $(2r-2)$-dimensional polytope obtained by intersecting Kostka_r with the affine hyperplane $\{\sum_{i=1}^{r}(\lambda_i + \mu_i) = 1\}$.

In other words, $\mathrm{P}_r^{\mathsf{Kostka}}$ is the set of points (λ, μ) in Kostka_r such that λ and μ each have entries summing to $\frac{1}{2}$. Since we are interested only in the combinatorial type of $\mathrm{P}_r^{\mathsf{Kostka}}$, we could have equivalently intersected Kostka_r with any affine hyperplane nontrivially intersecting all faces of Kostka_r except the origin.

Observation 9. *The d-faces of P_r^{Kostka} are in bijection with the $(d+1)$-faces of Kostka_r. In particular, each $(d+1)$-face of Kostka_r is obtained by taking all points along any ray emanating from the origin and passing through some fixed d-face of P_r^{Kostka}. Thus, the vertices of P_r^{Kostka} correspond to the extremal rays of Kostka_r.*

2.1.3 Extremal Rays

The extremal rays of Kostka_r were described in [3]. In particular, we have

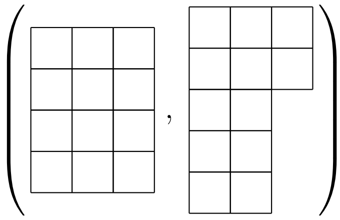
Proposition 10. *[3, Proposition 4.1, Corollary 1.7] Let a, b, ℓ satisfy $0 \leqslant \ell < b \leqslant a \leqslant r$. Then*

$$\begin{aligned} (\lambda, \mu) &= \Bigg(\underbrace{a-\ell, \dots, a-\ell}_{b}, 0 \dots, 0; \underbrace{a-\ell, \dots, a-\ell}_{\ell}, \underbrace{b-\ell, \dots, b-\ell}_{a-\ell}, 0, \dots, 0\Bigg) \\ &= ((a-\ell)^b, 0^{r-b});\ ((a-\ell)^\ell, (b-\ell)^{a-\ell}, 0^{r-a})), \end{aligned}$$

generates an extremal ray of Kostka_r, and all extremal rays are generated by such an element. In particular, the number of extremal rays of Kostka_r is $\binom{r}{3} + \binom{r}{2} + \binom{r}{1}$.

Example 11. Let $r = a = 5$, $b = 4$, and $\ell = 2$. Then

$$(\lambda, \mu) = ((3,3,3,3,0),(3,3,2,2,2)) =$$



generates an extremal ray of Kostka_5.

Definition 12. We say that the extremal ray in Theorem 10 is *labeled* by the triple (a, b, ℓ) whenever $a \neq b$. Whenever $a = b$, the extremal ray in Theorem 10 is not dependent on the choice of ℓ, and we say it is *labeled* by the triple (a, a, a). We also say that the corresponding vertex of $\mathrm{P}_r^{\mathsf{Kostka}}$ (using Theorem 9) is *labeled* by the same triple.

Example 13. The seven extremal rays of Kostka_3 are labeled by the triples $(1, 1, 1)$, $(2, 1, 0)$, $(2, 2, 2)$, $(3, 1, 0)$, $(3, 2, 0)$, $(3, 2, 1)$, and $(3, 3, 3)$.

Remark 14. Note that our usage of the parameters a, b, ℓ differs from the convention in [3]; in particular, we relabel their parameter $a + \ell$ by a and $b + \ell$ by b. While the choice of label (a, a, a) may seem arbitrary for the case when $a = b$, this choice simplifies the statement of Theorem 27.

2.2 Hilbert Bases

Let $C \subseteq \mathbb{R}^d$ be a rational convex polyhedral cone. By Gordan's Lemma [12, Theorem 16.4], there exists a finite set $\mathcal{H}(C) \subseteq C \cap \mathbb{Z}^d$, such that

- every integral point of C can be expressed as a nonnegative integer combination of points in $\mathcal{H}(C)$, and
- $\mathcal{H}(C)$ has minimal cardinality with respect to the first property.

In the case that C is pointed, the set $\mathcal{H}(C)$ is unique and is known as the *Hilbert basis* of C. Moreover, an element of $C \cap \mathbb{Z}^d$ is in the Hilbert basis if and only if it is *irreducible*, i.e., cannot be expressed as a nonnegative integer combination of any other integral points of C; otherwise it is called *reducible*. See [12, Section 16.4] for further background.

Remark 15. Since Kostka_r is pointed and has integral points corresponding to pairs in $\mathsf{Par}_r(n)$, we can express Hilbert basis membership in terms of the partitions. Namely, an element $(\lambda, \mu) \in \mathsf{Kostka}_r \cap \mathbb{Z}^{2r}$ is a Hilbert basis element if and only if no nontrivial subset of the columns of λ has total size equal to the total size of a subset of the columns of μ.

3 The Maximum Number of Vertices of a Face

In this section, we look at the maximum number of vertices contained in a d-face of the polytope $\mathrm{P}_r^{\mathsf{Kostka}}$. Equivalently (see Theorem 9), we look at the maximum number of extremal rays contained in a $(d+1)$-face of the cone Kostka_r. We give a uniform upper bound on this quantity for fixed d, and furthermore show that this upper bound is exact for $r > d+1$.

Definition 16. For integers $r \geqslant 1$ and $0 \leqslant d \leqslant 2r-2$, let $m(r,d)$ denote the maximum number of vertices in a d-dimensional face of the polytope $\mathrm{P}_r^{\mathsf{Kostka}}$. Let $m(d)$ denote the maximum number of vertices of a d-face in any polytope $\mathrm{P}_j^{\mathsf{Kostka}}$ over all choices of $j \geqslant 1$.

By Theorem 6, we have that $m(r,d)$ is non-decreasing as a function in r. Moreover, since any proper face can be extended to a face of higher dimension, the function $m(r,d)$ is strictly increasing in d. Table 1 depicts some values of $m(r,d)$.

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3														
3	4	6	7												
4	4	7	10	13	14										
5	4	8	11	15	19	24	25								
6	4	8	12	17	23	28	34	40	41						
7	4	8	12	18	25	32	40	48	55	62	63				
8	4	8	12	18	27	34	45	53	64	75	83	91	92		
9	4	8	12	18	27	36	46	58	69	82	95	110	119	128	129

Table 1: Some values of $m(r,d)$, the maximum number of vertices in a d-face of $\mathrm{P}_r^{\mathsf{Kostka}}$, are shown, appearing in the row labeled by r and the column labeled by d.

Remark 17. Note that $m(d)$ is a priori not guaranteed to exist, but Theorem 21 shows that it is well-defined.

Our main result is an exact calculation of $m(d)$ in Theorem 21, which in turn gives an upper bound on $m(r,d)$.

From Theorem 7 and Theorem 10 the following is clear.

Proposition 18. *Let v be a vertex of P_r^{Kostka} labeled by the triple (a,b,ℓ). Then*

- $v \in H_i$ *if and only if* $b \neq i$,
- $v \in \widehat{H}_k$ *if and only if* $a \neq k$ *and* $\ell \neq k$.
- $v \in J_j$ *if and only if* $j \leqslant \ell$, $j \geqslant a$, *or* $a = b$.

Theorem 19. *A d-dimensional face of P_r^{Kostka} has at most $\prod_{i=1}^{3} \left\lfloor \frac{d+2+i}{3} \right\rfloor$ vertices.*

Proof. Let

$$F = \mathrm{P}_r^{\mathsf{Kostka}} \cap \left(\bigcap_{i \in I} H_i \right) \cap \left(\bigcap_{j \in J} J_j \right) \cap \left(\bigcap_{k \in K} \widehat{H}_k \right)$$

be a d-dimensional face of $\mathrm{P}_r^{\mathsf{Kostka}}$, where $I \subseteq \{1, 2, \ldots, r\}$ and $J, K \subseteq \{1, 2, \ldots, r-1\}$ are (possibly empty) index sets. We can furthermore assume that the set of hyperplanes is chosen minimally to have this intersection, i.e., $|I| + |J| + |K| = 2r - 1 - d$.

We are interested in bounding the possible triples $(a, b, \ell) \in \mathbb{Z}_{\geqslant 0}$ labeling the vertices of F. According to Theorem 18, such a triple must satisfy that $b \notin I$, $a, \ell \notin K$, and an element of J is weakly between a and ℓ only if $a = b = \ell$. Let F_1 be the set of triples (a, b, ℓ) meeting these conditions. The minimality condition implies that, for any elements $j < j' < j''$ of $J \cup \{0, r\}$, there must be some $(a, b, \ell) \in F_1$ such that $j \leqslant \ell < j' < a \leqslant j''$. That is, the sets $\{j, j+1, \ldots, j'-1\} \setminus K$ and $\{j-1, j, \ldots, j'\} \setminus K$ are nonempty for any elements $j < j'$ in $J \cup \{0, r\}$.

Fix $b \notin I$. We then set

$$z_1(b) = |\{a : (a, b, \ell) \in F_1 \text{ for some } \ell\}| \text{ and } z_2(b) = |\{\ell : (a, b, \ell) \in F_1 \text{ for some } a\}| \,.$$

If $b \in J$, then we must have $a = b = \ell$, so $z_1(b) + z_2(b) \leqslant 2$. If $b \notin J$, since each $j \in J$ has an element of $\{0, \ldots, r\} \setminus K$ on either side of it, we have $z_1(b) + z_2(b) \leqslant r + 1 - |J| - |K|$. Thus, summing over our choices for b, we have

$$\begin{aligned} |F_1| &\leqslant \sum_{b \in [r] \setminus I} z_1(b) \cdot z_2(b) \\ &\leqslant \sum_{b \in [r] \setminus I} \left\lfloor \frac{r + 1 - |J| - |K|}{2} \right\rfloor \cdot \left\lfloor \frac{r + 2 - |J| - |K|}{2} \right\rfloor \\ &\leqslant (r - |I|) \cdot \left\lfloor \frac{r + 1 - |J| - |K|}{2} \right\rfloor \cdot \left\lfloor \frac{r + 2 - |J| - |K|}{2} \right\rfloor \,, \end{aligned}$$

where, in the second step, we replace the summand by the maximum value of the product of two numbers summing to $r + 1 - |J| - |K|$. The sum of the three factors in the final expression is

$$2r + 1 - |I| - |J| - |K| = d + 3 \,,$$

so their product is at most $\prod_{i=1}^{3} \left\lfloor \frac{d + 2 + i}{3} \right\rfloor$. This yields the desired upper bound. □

Via a construction, we can prove a lower bound on $m(r, d)$.

Theorem 20. *Suppose $r > d + 1$. Given any positive integers z_1, z_2, z_3 summing to $d + 3$, the intersection*

$$F = P_r^{\mathsf{Kostka}} \cap \left(\left(\bigcap_{i=1}^{z_1 - 1} H_i \right) \cap \left(\bigcap_{j = z_1 + z_2}^{r} H_j \right) \cap \left(\bigcap_{k = z_1}^{r - z_3} \widehat{H}_k \right) \right)$$

is a face of P_r^{Kostka} of dimension at most d with $z_1 z_2 z_3$ vertices.

Proof. We begin by determining the set of vertices contained in F. Let v be a vertex of $\mathrm{P}_r^{\mathsf{Kostka}}$ labeled by (a, b, ℓ). We have $v \in \left(\bigcap_{i=1}^{z_1-1} H_i\right) \cap \left(\bigcap_{j=z_1+z_2}^{r} H_j\right)$ if and only if $z_1 \leqslant b \leqslant z_1 + z_2 - 1$. Similarly, we have $v \in \bigcap_{k=z_1}^{r-z_3} \widehat{H}_k$ if and only if $a, \ell \notin \{z_1, \ldots, r - z_3\}$. By assumption, we have $r - z_3 \geqslant z_1 + z_2 - 1$ and $\ell \leqslant b$, hence $v \in F$ if and only if

$$0 \leqslant \ell < z_1 \leqslant b \leqslant z_1 + z_2 - 1 \leqslant r - z_3 < a \leqslant r \,.$$

The ranges for ℓ, b, and a are disjoint and of sizes z_1, z_2, and z_3, respectively. Therefore, there are $z_1 \cdot z_2 \cdot z_3$ vertices of $\mathrm{P}_r^{\mathsf{Kostka}}$ contained in F, each associated to a triple (a, b, ℓ) satisfying the inequalities above.

It remains to show that dimension of F is at most d. This follows because any element (λ, μ) in F lies in the affine subspace of $\mathbb{R}^{2r}$ where

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{z_1},\ \lambda_{z_1+z_2} = \cdots = \lambda_r,\ \mu_{z_1} = \cdots = \mu_{r-z_3},\ \text{and}\ \sum_{i=1}^{r} \lambda_i = \sum_{j=1}^{r} \mu_j = \frac{1}{2}\,,$$

which has dimension $2r - (z_1 - 1) - (r - z_1 - z_2 + 1) - (r - z_3 - z_1 + 1) - 2 = d$. ☐

Using the language of $m(d)$ and $m(r, d)$, we restate the result stated in Theorem 1.

Corollary 21. *For $r > d + 1$, we have*

$$m(r, d) = m(d) = \prod_{i=1}^{3} \left\lfloor \frac{d + 2 + i}{3} \right\rfloor .$$

Proof. The upper bound follows directly from Theorem 19. For the lower bound, consider the face constructed in Theorem 20 with $z_i = \left\lfloor \frac{d+1+i}{3} \right\rfloor$. Since this face achieves the upper bound on the number of vertices in a d-face from Theorem 19 and $m(r, d)$ is strictly increasing in d, we can conclude that this face has dimension exactly d. ☐

Remark 22. The values of $m(d)$ appear as the sequence A006501 in the OEIS [10], with generating function $\dfrac{1 + x^2}{(1 - x)^2(1 - x^3)^2}$. The quantity $m(d)$ can be alternatively characterized as the maximum product of three positive integers summing to $d + 3$.

4 Characterization of Edges

Here we present a procedure for characterizing the faces of a fixed dimension in the Kostka polytope $\mathrm{P}_r^{\mathsf{Kostka}}$, where r can vary. We carry out this characterization explicitly for dimension 1. This characterization yields an enumeration of the faces of these dimensions, which is handled in the following section. It seems very feasible that these methods could be extended to higher dimensions, though the conditions seem to get increasingly complicated.

Proposition 23. *The minimal face of* $\mathrm{P}_r^{\mathsf{Kostka}}$ *containing a set of vertices with labels* $\{(a_i, b_i, \ell_i)\}_{1\leqslant i\leqslant m}$ *is formed by the set of all vertices whose label* (a, b, ℓ) *satisfies that*

1. b *is an element of* $\bigcup_{i=1}^m \{b_i\}$,
2. ℓ *and* a *are both elements of* $\bigcup_{i=1}^m \{\ell_i, a_i\}$,
3. *the open interval* (ℓ, a) *is contained in* $\bigcup_{i=1}^m (\ell_i, a_i)$*, and*
4. $0 \leqslant \ell < b < a \leqslant r$ *or* $1 \leqslant a = b = \ell \leqslant r$.

Proof. Comparing these conditions to those in Theorem 18, we can exactly determine the hyperplanes H_i, $\hat{H}_k$, and J_j that contain our chosen set of vertices. The conditions of the lemma precisely encode the condition that the vertex labeled by (a, b, ℓ) is contained in all these bounding hyperplanes that contain the vertices with labels $\{(a_i, b_i, \ell_i)\}_{1\leqslant i\leqslant m}$. By taking the intersection of these hyperplanes with $\mathrm{P}_r^{\mathsf{Kostka}}$, we obtain the minimal face containing the chosen set of vertices. □

Remark 24. For convenience, when considering the labels of a list of vertices, we follow the convention that the labels are ordered lexicographically.

Let $\mathcal{B}_n$ denote the *braid arrangement* in $\mathbb{R}^n$, consisting of all hyperplanes of the form $x_i = x_j$ where $i, j \in \{1, \ldots, n\}$ and $i < j$. We will now show that whether a collection of vertices is the vertex set of some face of the Kostka cone depends only on the cell of the braid arrangement that the vertex label list lies in, i.e., the relative order of the vertex label entries. We say that two tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{Z}^n$ are *order-isomorphic* provided that $x_i > x_j$ (resp. $x_i > x_j$) if and only if $y_i > y_j$ (resp. $y_i < y_j$) for any $i, j \in \{1, \ldots, n\}$.

Lemma 25. *Suppose we have a pair of order-isomorphic tuples* $(a_1, b_1, \ell_1, \ldots, a_m, b_m, \ell_m)$ *and* $(a'_1, b'_1, \ell'_1, \ldots, a'_m, b'_m, \ell'_m)$ *in* $\{0, \ldots, r\}^3$ *such that the triples* (a_i, b_i, ℓ_i) *and* (a'_i, b'_i, ℓ'_i) *are labels of vertices of* $\mathrm{P}_r^{\mathsf{Kostka}}$*. Then the vertices labeled by* $\{(a_i, b_i, \ell_i)\}_{1\leqslant i\leqslant m}$ *form the vertex set of a* d*-dimensional face of* $\mathrm{P}_r^{\mathsf{Kostka}}$ *if and only if the vertices that are labeled by* $\{(a'_i, b'_i, \ell'_i)\}_{1\leqslant i\leqslant m}$ *do.*

Proof. In order to determine if a set V of vertices in $\mathrm{P}_r^{\mathsf{Kostka}}$ labeled by $\{(a_i, b_i, \ell_i)\}_{1\leqslant i\leqslant m}$ is the vertex set of a d-face of $\mathrm{P}_r^{\mathsf{Kostka}}$, we test whether any other vertex of $\mathrm{P}_r^{\mathsf{Kostka}}$ lies in the intersection of the hyperplanes containing V. In order to lie in this intersection, the new vertex labeled by (a, b, ℓ) must satisfy the conditions of Theorem 23.

These conditions, and hence the existence of such a tuple, only depend on the order-isomorphism class of the tuple $(a_1, b_1, \ell_1, \ldots, a_m, b_m, \ell_m)$. Moreover, all vertex sets corresponding to a given ordering have convex hulls of the same dimension, since the set of bounding hyperplanes of $\mathrm{P}_r^{\mathsf{Kostka}}$ containing a vertex is determined entirely by this ordering. □

Thus, in order to determine if a set of vertices is the vertex set of some face of $\mathrm{P}_r^{\mathsf{Kostka}}$, it is sufficient to test this for any set of vertices with an order-isomorphic list of labels.

That is, a list in $\{0, \ldots, r\}^{3m}$ being the set of labels of a face of $\mathrm{P}_r^{\mathsf{Kostka}}$ is constant across open cells of the braid arrangement $\mathcal{B}_{3m}$. We can combine this fact with the well-known Upper Bound Theorem for polytopes, proved by McMullen [8] in 1970 (see [14, Chapter 2, Section 3] for more details). This yields an upper bound on the dimension of open cells in $\mathcal{B}_{3m} \cap \{0, \ldots, r\}^{3m}$ that correspond to vertex labels of d-faces of $\mathrm{P}_r^{\mathsf{Kostka}}$. We state the Upper Bound Theorem under the additional assumption that the face dimension is less than half the polytope dimension, which is sufficient for our purposes.

Theorem 26. [Upper Bound Theorem, [8]] *For $0 \leqslant i \leqslant \left\lfloor \frac{m}{2} \right\rfloor$, the number of i-faces of an m-polytope with n vertices is at most $\binom{n}{i+1}$. Moreover, this bound is realized by $\Delta(n, m)$, the m-dimensional cyclic polytope with n vertices.*

We now prove an upper bound on the number of distinct values of the triples labeling the vertices of a face of fixed dimension in $\mathrm{P}_r^{\mathsf{Kostka}}$. Of course, we already have an upper bound of $3 \prod_{i=1}^{3} \left\lfloor \frac{d+2+i}{3} \right\rfloor$ from Theorem 19, which bounds the number of vertices. However, we can obtain a tight bound using the Upper Bound Theorem.

Lemma 27. *If the vertices of a d-face of P_r^{Kostka} are labeled by $\{(a_i, b_i, \ell_i)\}_{1 \leqslant i \leqslant n}$, then there are at most $3d + 3$ distinct values among the parameters $a_1, b_1, \ell_1, \ldots, a_n, b_n, \ell_n$.*

Proof. Let $t = |\{a_1, b_1, \ell_1, \ldots, a_n, b_n, \ell_n\}|$. By Theorem 25, the number of d-faces of $\mathrm{P}_r^{\mathsf{Kostka}}$ generated by tuples order-isomorphic to $(a_1, b_1, \ell_1, \ldots, a_n, b_n, \ell_n)$ is $\binom{r+1}{t}$, which has degree t as a polynomial in r.

We now apply the Upper Bound Theorem for polytopes (see Theorem 26). Since the number of vertices is $\binom{r}{3} + \binom{r}{2} + \binom{r}{1}$ by Theorem 10, then the number of faces of dimension d is bounded by the corresponding number of d-faces of the cyclic $2r$-polytope with $\binom{r}{3} + \binom{r}{2} + \binom{r}{1}$ vertices. By Theorem 26, this quantity is a polynomial in r of degree $3d + 3$. Therefore, we must have $t \leqslant 3d + 3$, as desired. □

We can now see that in order to characterize the d-dimensional faces of $\mathrm{P}_r^{\mathsf{Kostka}}$ for arbitrary r, one must merely determine the d-faces of $\mathrm{P}_{3d+3}^{\mathsf{Kostka}}$.

Lemma 28. *A set of vertices form a d-face of P_r^{Kostka} if and only if their label set is order-isomorphic to the vertex label set of a d-face of $P_{3d+3}^{\mathsf{Kostka}}$.*

Proof. First, note that $\mathrm{P}_{3d+3}^{\mathsf{Kostka}}$ is a face of $\mathrm{P}_r^{\mathsf{Kostka}}$ for $r > 3d + 3$ by intersecting with the hyperplanes H_i for $i \in \{3d + 4, 3d + 5, \ldots, r\}$ and J_{3d+3} from Theorem 7. Theorem 27 implies that the label set of any d-face in $\mathrm{P}_r^{\mathsf{Kostka}}$ is order-isomorphic to a set of vertex labels in $\mathrm{P}_{3d+3}^{\mathsf{Kostka}}$. Combining this with Theorem 25 and the inclusion of $\mathrm{P}_{3d+3}^{\mathsf{Kostka}}$ in $\mathrm{P}_r^{\mathsf{Kostka}}$ yields the desired statement. □

Theorem 29. *Let u and v be vertices of P_r^{Kostka} labeled (a, b, ℓ) and (a', b', ℓ'), where $a - b \leqslant a' - b'$. Then $\{u, v\}$ is a face of P_r^{Kostka} if and only if*

(1) $a = b$ and at least one of the following holds:

(i) $a' = b'$,

(ii) $a = b'$,

(iii) $a \geqslant a'$, *or*

(iv) $\ell' \geqslant a$.

(2) $a \neq b$ *and at least one of the following holds:*

(i) two of the three equalities $a = a'$, $b = b'$, *and* $\ell = \ell'$ *hold,*

(ii) $\ell \geqslant a'$, *or*

(iii) $\ell' \geqslant a$.

Proof. By Theorem 28, it is enough to check that this is the case for the vertices of $\mathrm{P}_6^{\mathsf{Kostka}}$. This can be readily completed with the aid of a computer. □

By similarly examining the 2-faces of $\mathrm{P}_9^{\mathsf{Kostka}}$, one could determine a characterization of the 2-faces of all Kostka polytopes. While the conditions seem rather complex, we shall see in the next section that these methods yield nice enumerative results.

5 Enumeration of Faces of a Fixed Dimension

In this section, we derive formulas for the number of faces of a fixed dimension d in $\mathrm{P}_r^{\mathsf{Kostka}}$ for $d = 1, 2, 3$. We then asymptotically determine the number of d-faces of $\mathrm{P}_r^{\mathsf{Kostka}}$ for arbitrary d. As mentioned in Theorem 26, it is well known that the number of d-faces of a k-polytope with n vertices is maximized by the cyclic polytope $\Delta(n, k)$ for sufficiently large k. We show that, as r increases, the number of d-faces of $\mathrm{P}_r^{\mathsf{Kostka}}$ grows asymptotically at the same rate as the number of d-faces of $\Delta\left(\binom{r}{3} + \binom{r}{2} + \binom{r}{1}, 2r - 2\right)$ up to a constant factor depending on d, and we furthermore determine this constant for all d (see Theorem 36).

Definition 30. Let $f_d(r)$ denote the number of d-dimensional faces of $\mathrm{P}_r^{\mathsf{Kostka}}$.

In the previous section, we showed that whether a set of m vertices of $\mathrm{P}_r^{\mathsf{Kostka}}$ forms the vertex set of a face depends only on the order-isomorphism class of the vertex labels (see Theorem 25). In other words, it depends only on the cell of the braid arrangement $\mathcal{B}_{3m}$ that the list of m vertex label triples lies in. By examining the integer points in each cell, we obtain the following lemma.

Lemma 31. *The function* $f_d(r)$ *is a polynomial in* r *of degree at most* $3d + 3$ *and has a positive integer expansion in terms of the of basis* $\left\{\binom{r}{k}\right\}_{0 \leqslant k \leqslant 3d+3}$.

Proof. The number of integer points in any collection of cells of $\mathcal{B}_{3m} \cap \{0, \ldots, r\}^{3m}$ has a positive integer expansion in the basis $\left\{\binom{r}{k}\right\}_{1 \leqslant k \leqslant 3m}$. Thus, it follows directly from Theorem 25 that $f_d(r)$ is a polynomial with a positive integer expansion in terms of the of basis $\left\{\binom{r}{k}\right\}_{0 \leqslant k}$. The claim about the degree then follows from Theorem 27. □

Theorem 32. *Fix $d \geqslant 0$. Setting $d_{\min} = \left\lfloor \frac{d+3}{2} \right\rfloor$, we have*

$$f_d(r) = \sum_{k=d_{\min}}^{3d+3} \alpha_k \binom{r}{k}$$

where $\alpha_k = f_d(k) - \left(\sum_{j=d_{\min}}^{k-1} \binom{k}{j} \alpha_j \right)$ for $k > d_{\min}$ and

$$\alpha_{d_{\min}} = f_d(d_{\min}) = \begin{cases} 3d-2 & \text{if } d \text{ odd and } d > 1\,, \\ 1 & \text{if } d \text{ even,} \\ 3 & \text{if } d = 1\,. \end{cases}$$

Proof. If d is odd, then the value of $\alpha_{d_{\min}}$ is the number of facets of $\mathrm{P}^{\mathsf{Kostka}}_{d_{\min}}$, which we calculate in Theorem 7. If d is even, then $\alpha_{d_{\min}}$ is the number of top-dimensional faces, which is 1 since $\mathrm{P}^{\mathsf{Kostka}}_r$ is a polytope. The recursive formula for the other values of α_k follows from Theorem 25, Theorem 6, and Theorem 31 by evaluating $f_d(k)$ as a sum of terms of the form $\alpha_k \binom{r}{k}$. □

Thus, if one can compute the values $f_d(0), \ldots, f_d(3d+3)$, then Theorem 32 implies that we can determine the entire function $f_d(r)$. Using SageMath, we were able to compute some initial terms of $f_d(r)$ (see Table 2). The number of vertices, $f_0(r)$, is also shown in Table 2 for $r \leqslant 13$, with the general formula given in [3]

	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	3	7	14	25	41	63	92	129	175	231	298	377
1	0	3	16	52	132	288	567	1036	1788	2949	4686	7216	10816
2	0	1	16	89	328	961	2427	5517	11584	22846	42812	76868	133068
3	0	0	7	81	466	1898	6253	17803	45502	106946	234964	488229	967863

Table 2: The values of $f_d(r)$, the number of d-faces in $\mathrm{P}^{\mathsf{Kostka}}_r$, are shown for the cases where $0 \leqslant d \leqslant 3$ and $1 \leqslant r \leqslant 13$. Here, d is given by the row label and r is given by the column label.

These computations allow us to derive formulas for the number of d-faces of $\mathrm{P}^{\mathsf{Kostka}}_r$ for $d = 1, 2, 3$, given in Theorem 2.

Proof of Theorem 2. Each formula can be obtained by applying Theorem 32 to the values in a fixed row of Table 2. The final claim follows directly from Theorem 31. □

We now shift our focus to determining the asymptotic behavior of the function $f_d(r)$. To achieve this, we determine the degree and leading coefficient of the polynomial $f_d(r)$.

Lemma 33. *Fix positive integers d, r such that $r \geqslant 3d+3$. If a set of $d+1$ vertices in P^{Kostka}_r with labels $\{(a_i, b_i, \ell_i)\}_{1 \leqslant i \leqslant d+1}$ satisfies that the intervals $[\ell_i, a_i]$ are all disjoint, then it is the vertex set of a d-face of Kostka_r.*

Proof. We prove this by induction on d, with the base case $d = 1$ following from Theorem 29. We first show that such a set of vertices is the vertex set of a face of $\mathrm{P}_r^{\mathsf{Kostka}}$, and then determine its dimension. We prove the former by showing there is no other vertex in the minimal face F containing the vertices labeled by $\{(a_i, b_i, \ell_i)\}_{1\leqslant i\leqslant d+1}$ via the conditions of Theorem 23. Let (a, b, ℓ) be the label of a vertex in F. By Condition (2), the parameters a, ℓ must be chosen from within intervals $[\ell_i, a_i]$. If a and ℓ are chosen from different intervals, then in order to satisfy Condition (3), we must have $a = \ell + 1$. However, then b cannot be chosen to satisfy Condition (4). On the other hand, if a and ℓ are chosen within the same interval $[\ell_j, a_j]$, then Condition (2) implies that $a = a_j$ and $\ell = \ell_j$. But then Condition (1) and the required ordering of ℓ, b, and a imply that we also have $b = b_j$, so (a, b, ℓ) was in the original list of vertex labels. Hence, the minimal face containing the vertices labeled by $\{(a_i, b_i, \ell_i)\}_{1\leqslant i\leqslant d+1}$ contains no other vertices, so these form the vertex set of F.

The fact that the dimension of F is d follows from the induction. In particular, we know that the vertices with labels $\{(a_i, b_i, \ell_i)\}_{1\leqslant i\leqslant d}$ form the vertex set of a face of dimension $d - 1$. Since we have added one additional vertex and formed another face of $\mathrm{P}_r^{\mathsf{Kostka}}$, the dimension of F must be d. □

Lemma 34. *Fix positive integers d, r such that $r \geqslant 3d + 3$, and suppose L is the set of vertex labels of a d-face F of P_r^{Kostka}. Then either*

(i) F is a simplex whose vertex labels satisfy the conditions of Theorem 33, or

(ii) there are at most $3d + 2$ distinct values among the vertex label entries.

Proof. We proceed by induction on d, with the base case $d = 1$ following from Theorem 29.

Let F be a d-face of $\mathrm{P}_r^{\mathsf{Kostka}}$, and let $L = \{(a_i, b_i, \ell_i)\}_{1\leqslant i\leqslant n}$ be the set of labels of the vertices of F. Let t denote the number of distinct values among the label entries a_i, b_i, and ℓ_i. Fix a bounding hyperplane of F of type H_i or $\widehat{H}_i$ (see Theorem 9 for hyperplane descriptions), i.e.,

$$H \in \left\{H_i : 1 \leqslant i \leqslant r,\ \dim(H_i \cap F) = d - 1\right\} \cup \left\{\widehat{H}_i : 1 \leqslant i \leqslant r,\ \dim(\widehat{H}_i \cap F) = d - 1\right\},$$

such that the number of vertices of F contained in H is minimal.

Suppose $H = H_j$ (the case for $\widehat{H}_j$ proceeds analogously). By Theorem 18, the vertices of F that are contained in H are precisely those whose label (a_i, b_i, ℓ_i) does not have $b_i = j$. We now consider the number of distinct values among the label entries of the vertices in $F \cap H$. If a label entry m appears among the vertices of F but not $F \cap H$, then all vertices whose label contains the entry m must also contain the entry j. Moreover, by the minimality condition, $F \cap \widehat{H}_m$ has at least as many vertices as $F \cap H$, so $F \cap \widehat{H}_m = F \cap H_j$. Since each label has three entries, it is either the case that

(a) there are at most two label entries that appear among the vertices of F but not $F \cap H$, or

(b) there are exactly three label entries that appear among the vertices F but not $F \cap H$, and these entries appear in the label of a unique vertex of F.

In Case (a), the face $F \cap H$ is then a $(d-1)$-dimensional face of $\mathrm{P}_r^{\mathsf{Kostka}}$ with at least $t-2$ distinct entries among the labels of its vertices. By the inductive hypothesis, this implies $t \leqslant 3d+2$.

In Case (b), the face $F \cap H$ is a $(d-1)$-dimensional face of $\mathrm{P}_r^{\mathsf{Kostka}}$ with $t-3$ distinct entries among the labels of its vertices and one fewer vertex than F. Thus, by the inductive hypothesis, we have $t \leqslant 3d+3$.

It remains to show that, if $t = 3d+3$ in Case (b), then F is a simplex satisfying the conditions of Theorem 33 (up to reordering of the vertices). In this case, the inductive hypothesis implies that $F \cap H$ is a simplex whose labels satisfy the conditions of Theorem 33. So it is enough to show that the label (a, b, ℓ) of the unique vertex of F that is not in $F \cap H$ satisfies $a < \ell'$ or $a' < \ell$ for any label (a', b', ℓ') of a vertex of $F \cap H$. This must hold because otherwise (ℓ, b', a') or (ℓ', b, a) is the label of an additional vertex in F, contradicting that there is only one vertex of F not contained in H. Therefore, F is indeed a simplex whose labels satisfy the conditions of Theorem 33. □

Theorem 35. *The function $f_d(r)$ is a polynomial of degree $3d+3$ with leading coefficient $\frac{1}{(3d+3)!}$.*

Proof. By Theorem 31, $f_d(r) = \sum_{k=1}^{3d+3} \alpha_k \binom{r}{k}$ for nonnegative integers α_k. By Theorem 33, the coefficient α_{3d+3} is at least 1. By Theorem 34, the coefficient α_{3d+3} is at most 1, and hence we can conclude $\alpha_{3d+3} = 1$. Expanding this out as a polynomial in r, we see that the top degree coefficient is $\alpha_{3d+3}/(3d+3)! = 1/(3d+3)!$. □

Corollary 36. *For $r \geqslant 1$, let $n_r = \binom{r}{3} + \binom{r}{2} + \binom{r}{1}$. We have*

$$\lim_{r \to \infty} \frac{f_d(r)}{f_d\left(\Delta\left(n_r, 2r-2\right)\right)} = \frac{6^{d+1}(d+1)!}{(3d+3)!},$$

where $f_d\left(\Delta\left(n_r, 2r-2\right)\right)$ is the number of d-faces of the cyclic polytope $\Delta\left(n_r, 2r-2\right)$.

Proof. By Theorem 26, the leading coefficient of the polynomial $f_d\left(\Delta\left(n_r, 2r-2\right)\right)$ is $\frac{1}{6^{d+1}(d+1)!}$. By Theorem 35, the leading coefficient of the polynomial $f_d(r)$ is $\frac{1}{(3d+3)!}$. Since both polynomials have degree $3d+3$, we can directly compute the limit of their quotient. □

6 Initial Partition Entries of Hilbert Basis Elements

Lastly, we study some families of Hilbert basis elements of $\mathsf{Kostka}_r^{\mathbb{Z}}$ in the context of their relation to the face structure. We begin by recalling the statement of Theorem 3, our last main result which we prove in this section. It consists of three cases, the first of which is due to Gao, Kiers, Orelowitz, and Yong and the others proven below in Theorem 42 and Theorem 44.

Theorem 1.3. If any of the following conditions hold:

- λ_1 and μ_1 are coprime [3, Theorem 1.4], or

- $\lambda_1 + 1$ and μ_1 are coprime, or
- $\lambda_1 + 1$ and $\mu_1 + 1$ are coprime with $2\mu_1 \geqslant \lambda_1$,

then the pair (λ_1, μ_1) is $(\lambda_1 + 1)$-initial. Moreover, this holds even if we consider only those Hilbert basis elements contained in the 2-skeleton of Kostka_r.

This result builds upon the "Width Bound" proved by Gao, Kiers, Orelowitz, and Yong. See, for example, [3, Table 1] for the Hilbert basis elements of $\mathsf{Kostka}_4^{\mathbb{Z}}$.

Theorem 37 ([3, Theorem 1.4], Width Bound)**.** *Suppose (λ, μ) is a Hilbert basis element of $\mathsf{Kostka}_r^{\mathbb{Z}}$. Then $\lambda_1 \leqslant r$. Moreover, if $\lambda_1 = r$ then λ and μ are both rectangles.*

We now further study the initial entries of Hilbert basis elements of $\mathsf{Kostka}_r^{\mathbb{Z}}$, recalling the following definition.

Definition 38. We say that an integer pair (λ_1, μ_1) is *r-initial* if there is an element (λ, μ) in the Hilbert basis of Kostka_r such that λ has first element λ_1 and μ has first element μ_1.

By the dominating condition for λ and μ, an r-initial pair must satisfy $\lambda_1 \geqslant \mu_1$. Moreover, note that if (λ_1, μ_1) is r-initial, then it is also r'-initial for any $r' > r$. This is because any $(\lambda, \mu) \in \mathsf{Kostka}_r$ can be embedded in $\mathsf{Kostka}_{r'}$ by appending zeroes to λ and μ (see Theorem 6), and this map preserves the Hilbert basis elements.

Remark 39. It follows from Theorem 37 and Theorem 10 that

- if (λ_1, μ_1) is r-initial then $r \geqslant \lambda_1$, and
- a pair (r, μ_1) is r-initial if and only if r and μ_1 are coprime.

While the first point is immediate from Theorem 37, the second uses the fact that when λ and μ are both rectangles, then (λ, μ) lies on some extremal ray.

Thus, it remains to determine when (λ_1, μ_1) is r-initial for $r > \lambda_1$. Theorem 10 implies that the pair (λ_1, λ_1) is r-initial for any $\lambda_1 < r$, as realized by the extremal rays. It may seem tempting to expect that any pair (λ_1, μ_1) is $(\lambda_1 + 1)$-initial, but there is a counterexample when $\lambda_1 = 14$. This is currently the only counterexample known to the author.

Example 40. We have checked computationally that $(14, 6)$ is not 15-initial. Moreover, $r = 15$ is the smallest value such that there is a pair $(r - 1, \mu_1)$ with $\mu_1 < r - 1$ that is not r-initial.

The main result of this section is Theorem 3, which states that a pair (λ_1, μ_1) is $(\lambda_1 + 1)$-initial if $\lambda_1 \geqslant \mu_1$ and any of the following conditions holds

- λ_1 and μ_1 are coprime, or
- $\lambda_1 + 1$ and μ_1 are coprime, or

- $\lambda_1 + 1$ and $\mu_1 + 1$ are coprime with $2\mu_1 \geqslant \lambda_1$.

Example 41. The pairs (λ_1, μ_1) with $\mu_1 < \lambda_1 \leqslant 30$ for which the conditions of Theorem 3 do not hold are $(14, 6)$, $(15, 6)$, $(20, 6)$, $(20, 14)$, $(21, 6)$, $(24, 10)$, $(25, 10)$, $(26, 6)$, $(26, 12)$, $(27, 6)$, $(27, 12)$, and $(27, 21)$.

Theorem 42. *Fix $\lambda_1 > \mu_1$. Let*

$$r(\lambda_1, \mu_1) = \min\{z \in \mathbb{N} : z \geqslant \lambda_1,\ \gcd(z, \mu_1) = 1\}\,.$$

Then (λ_1, μ_1) is $r(\lambda_1, \mu_1)$-initial. In particular, (λ_1, μ_1) is $(\lambda_1 + \mu_1 - 1)$-initial.

Proof. Let $r = r(\lambda_1, \mu_1)$. Since some entry among the μ_1 integers $\lambda_1, \ldots, \lambda_1 + \mu_1 - 1$ must be equivalent to 1 modulo μ_1, we have $r \leqslant \lambda_1 + \mu_1 - 1$.

Let λ and μ be the partitions

$$\lambda = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\mu_1}) \quad \text{and} \quad \mu = (\underbrace{\mu_1, \ldots, \mu_1}_{r-\mu_1}, \underbrace{\mu_1 - (r - \lambda_1), \ldots, \mu_1 - (r - \lambda_1)}_{\mu_1})\,.$$

If (λ, μ) were reducible, then, by Theorem 15, we could choose a proper subset of the columns of λ with the same size as a proper subset of the columns of μ. The μ_1 columns of μ are all equivalent to r modulo μ_1, and $\gcd(r, \mu_1) = 1$, and hence there is no way to choose a proper subset of the columns of μ such that their size is divisible by μ_1. However, any subset of the columns of λ is divisible by μ_1. Therefore, (λ, μ) is irreducible in Kostka_r and hence is in the Hilbert basis. □

Example 43. Let $\lambda_1 = 20$ and $\mu_1 = 15$. Since $\gcd(20, 15) = 5$, $\gcd(21, 15) = 3$, and $\gcd(22, 15) = 1$, we have $r(\lambda_1, \mu_1) = 22$.

The construction in the proof of Theorem 42 yields the Hilbert basis element (λ, μ) in Kostka_{22}, where

$$\lambda = (\underbrace{20, \ldots, 20}_{15}) \text{ and } \mu = (\underbrace{15, \ldots, 15}_{7}, \underbrace{13, \ldots, 13}_{15})\,.$$

Thus the pair $(20, 15)$ is 22-initial.

The second sufficient condition of Theorem 3 follows immediately from Theorem 42, as in this case we have $r(\lambda_1, \mu_1) \leqslant \lambda_1 + 1$. We can now construct another family of examples to account for the last case of Theorem 3.

Theorem 44. *Suppose* $\gcd(\lambda_1 + 1, \mu_1 + 1) = 1$ *and* $2\mu_1 \geqslant \lambda_1$. *Then the pair (λ_1, μ_1) is $(\lambda_1 + 1)$-initial.*

Proof. Let

$$\lambda = (\underbrace{\lambda_1, \ldots, \lambda_1}_{2\mu_1 - \lambda_1 + 1}, \underbrace{\lambda_1 - 1, \ldots, \lambda_1 - 1}_{\lambda_1 - \mu_1})$$

and

$$\mu = (\underbrace{\mu_1, \ldots, \mu_1}_{\lambda_1+1}).$$

See Figure 1 for a visualization of this construction. It is straightforward to check that λ dominates μ, so (λ, μ) is in $\mathsf{Kostka}_{\lambda_1+1}$. Observe that all but one of the columns of λ have size $\mu_1 + 1$, while the last column has size $2\mu_1 - \lambda_1 + 1$. The columns of μ all have size $\lambda_1 + 1$.

By Theorem 15, if (λ, μ) is reducible, then we can choose a proper subset of the columns of λ with the same size as a proper subset of the columns of μ. If such a choice exists, note that the complement of the chosen columns also satisfies this property. Thus, we can choose a subset of the columns of λ excluding the smallest column of size equal to some subset of columns of μ. Note that the total size of any collection of columns of μ is divisible by $\lambda_1 + 1$. Since we assume $\mu_1 + 1$ is coprime to $\lambda_1 + 1$, then a collection of at most $\lambda_1 - 1$ columns of size $\mu_1 + 1$ will not be divisible by $\lambda_1 + 1$. Therefore, no such set of columns exist. We can conclude (λ, μ) is irreducible and hence is in the Hilbert basis of $\mathsf{Kostka}_{\lambda_1+1}$. □

Example 45. As in Theorem 43, we consider $\lambda_1 = 20$ and $\mu_1 = 15$. Since 21 and 16 are coprime, Theorem 44 applies to the pair (λ_1, μ_1). The construction in the proof yields the Hilbert basis element $(\lambda, \mu) \in \mathsf{Kostka}_{21}$, where

$$\lambda = (\underbrace{20, \ldots, 20}_{11}, \underbrace{19, \ldots, 19}_{5}) \text{ and } \mu = (\underbrace{15, \ldots, 15}_{21}).$$

Thus the pair $(20, 15)$ is 21-initial, which is stronger than the statement yielded in Theorem 43.

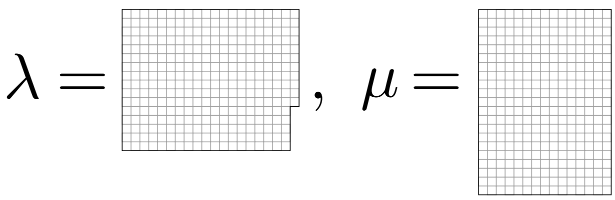


Figure 1: The pair of partitions (λ, μ) obtained from the construction in the proof of Theorem 44.

Lastly, we show that the Hilbert basis elements we constructed lie on the 2-skeleton of the Kostka cone by examining elements consisting of few distinct entries in Kostka_r. The proof relies on the characterization of the 2-faces of Kostka_r given by Theorem 29.

Lemma 46. *If $\lambda, \mu \in \mathsf{Par}_r(n)$ are partitions such that $K_{\lambda,\mu} > 0$ and one is rectangular while the other has exactly two part sizes, then the point (λ, μ) lies on a 2-dimensional face of Kostka_r.*

Proof. By Theorem 6, we can assume that the length of μ is r.

We first handle the case when λ is rectangular. Fix $x \geqslant y > z > 0$ such that

$$\lambda = (\underbrace{x, \ldots, x}_{s}, \underbrace{0, \ldots, 0}_{r-s}) \text{ and } \mu = (\underbrace{y, \ldots, y}_{t}, \underbrace{z, \ldots, z}_{u}).$$

Fix $h \in \mathbb{R}$ such that $0 < h < y$ and $ht + (h - y + z)u - hs = 0$, noting that such an h must exist since the latter expression has opposite signs at the endpoints $h = 0$ and $h = y$. Let R_1 be the point with first entry h along the extremal ray labeled by $(t + u, s, t)$, and let R_2 be the point with first entry $x - h$ along the extremal ray labeled by $(t + u, s, 0)$. It is then straightforward to check that $(\lambda, \mu) = R_1 + R_2$. The extremal rays labeled by $(t + u, s, t)$ and $(t + u, s, 0)$ satisfy Condition (2)(i) of Theorem 29 and their span contains (λ, μ), so (λ, μ) lies on a 2-face of Kostka_r.

We now handle the remaining case in which μ is rectangular. Fix $x \geqslant y > z > 0$ such that

$$\lambda = (\underbrace{x, \ldots, x}_{s}, \underbrace{y, \ldots, y}_{t}, \underbrace{0, \ldots, 0}_{r-s-t}) \text{ and } \mu = (\underbrace{z, \ldots, z}_{r}).$$

Let R_3 be the point with first entry $x - y$ along the extremal ray labeled by $(r, s, 0)$, and let R_4 be the point with first entry $\frac{s(x-y)}{r}$ along the extremal ray labeled by $(r, s + t, 0)$. It is then straightforward to check that $(\lambda, \mu) = R_3 + R_4$. The extremal rays labeled by $(r, s, 0)$ and $(r, s + t, 0)$ satisfy Condition (2)(i) of Theorem 29 and their span contains (λ, μ), so we can conclude that (λ, μ) lies on a 2-face of Kostka_r. ☐

The Hilbert basis elements we constructed satisfy the hypotheses of Theorem 46. In fact, the third family of Hilbert basis elements lies strictly on the 2-faces of $\mathrm{P}_r^{\mathsf{Kostka}}$, as opposed to the first two families. We can conclude the following.

Corollary 47. *The (λ, μ) constructed in the proofs of Theorem 42 and Theorem 44 lie on a two-dimensional face of their respective Kostka cones.*

We can now combine these results to prove the main result.

Proof of Theorem 3. The first sufficient condition follows from the Width Bound of Gao-Kiers-Orelowitz-Yong (Theorem 37) and the fact that if a pair is r-initial, then it is r'-initial for any $r' > r$. The second and third sufficient conditions follow from Theorem 42 and Theorem 44, respectively. The final claim is a result of Theorem 47 and the fact that the Hilbert basis elements in Theorem 37 are primitive vectors of extremal rays. ☐

Remark 48. One could also consider the initial elements of Hilbert basis elements lying in the d-skeleton of Kostka_r, for a fixed dimension d. The final claim of Theorem 3 shows that the earlier claims hold even if we restrict to looking at the d-skeleton of Kostka_r for any $d \geqslant 2$. For $d = 1$, the characterization of the extremal rays given by Gao-Kiers-Orelowitz-Yong [3, Proposition 4.1] implies that the portion of integer pairs $\lambda_1 \geqslant \mu_1$ arising as the initial elements of a Hilbert basis elements in the 1-skeleton of Kostka_r is asymptotically $\frac{6}{\pi^2} \approx 0.607$ (see OEIS sequence A059956 [10]). It would be interesting to know if the portion of integer pairs $\lambda_1 \geqslant \mu_1$ arising as initial entries of Hilbert basis elements of $\mathsf{Kostka}_{\lambda_1+1}$ depends on d for $d \geqslant 2$.

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