Linear Bounds on Treewidth in Terms of Excluded Planar Minors

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Abstract

One of the fundamental results in graph minor theory is that for every planar graph H, there is a minimum integer $f(H)$ such that graphs with no minor isomorphic to H have treewidth at most $f(H)$. A lower bound for $f(H)$ can be obtained by considering the maximum integer k such that H contains k vertex-disjoint cycles. There exists a graph of treewidth $\Omega(k \log k)$ which does not contain k vertex-disjoint cycles, from which it follows that $f(H) = \Omega(k \log k)$. In particular, if $f(H)$ is linear in $|V(H)|$ for graphs H from a subclass of planar graphs, it is necessary that n-vertex graphs from the class contain at most $O(n/\log n)$ vertex-disjoint cycles. We ask whether this is also a sufficient condition, and demonstrate that this is true for classes of planar graphs with bounded component size. For an n-vertex graph H which is a disjoint union of r cycles, we show that $f(H) \leqslant 3n/2 + O(r^2 \log r)$, and improve this to $f(H) \leqslant n + O(\sqrt{n})$ when $r = 2$. In particular this bound is linear when $r = O(\sqrt{n}/\log n)$. We present a linear bound for $f(H)$ when H is a subdivision of an r-edge planar graph for any constant r. We also improve the best known bounds for $f(H)$ when H is the wheel graph or the 4×4 grid, obtaining a bound of 160 for the latter.

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1 Introduction

A *tree decomposition* of a graph G consists of a tree T and a subtree S_v of T for each vertex v of G such that for every edge uv of G, S_u and S_v have a common node. For

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each node t of the tree T, we let $X_t = \{v \mid t \in V(S_v)\}$ and define the *width* of the tree decomposition as the maximum of $|X_t| - 1$ over the nodes t of the tree. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum of the widths of its tree decompositions.

One of the fundamental results in graph minor theory, proved by Robertson and Seymour [20], is that for every planar graph H, there is a minimum integer $f(H)$ such that graphs which do not contain H as a minor have treewidth at most $f(H)$. When H is a $k \times k$ grid, the best known upper bound on $f(H)$, obtained by Chuzhoy and Tan [6], is $O(k^9 \operatorname{poly}\log k)$. This implies that $f(H) = O(|V(H)|^9 \operatorname{poly}\log|V(H)|)$ for arbitrary planar graphs H, as Robertson, Seymour, and Thomas [22] proved that every planar H is a minor of a $k \times k$ grid for $k = O(|V(H)|)$.

It is natural to ask for a better bound on $f(H)$ for H in various classes of planar graphs. Any bound must be $\Omega(|V(H)|)$ as the complete graph on $|V(H)| - 1$ vertices has treewidth $|V(H)| - 2$ and does not contain H as a minor. This paper focuses on H for which $f(H)$ is $O(|V(H)|)$.

Several authors have presented results showing that $f(H) = O(|V(H)|)$ for various special H. Bienstock, Robertson, Seymour, and Thomas [1] showed that when H is a forest, $f(H)$ is $|V(H)| - 2$. Fellows and Langston [13] showed that if H is a cycle, then $f(H)$ is again $|V(H)| - 2$, which was also shown later by Birmelé [2] independently. Bodlaender, van Leeuwen, Tan, and Thilikos [4] showed that $f(K_{2,t}) \leqslant 2t - 2$ for every integer $t \geqslant 2$. Raymond and Thilikos [18, Theorem 5.1] proved that

$$f(H) \leqslant 36|V(H)| - 39 \tag{1}$$

for every wheel graph H. Leaf and Seymour [15, 4.4] proved that for an apex forest H with at least two vertices, which is a graph that becomes a forest by deleting some vertex, $f(H) \leqslant \frac{3}{2}|V(H)| - 3$. Liu and Yoo [16] informed us that, in a manuscript under preparation, they proved $f(H) \leqslant |V(H)| - 2$ for an apex forest H, improving the bound of Leaf and Seymour.

Not every planar graph H has the property that $f(H) = O(|V(H)|)$. Robertson, Seymour, and Thomas [22] pointed out that $f(H) = \Omega(g^2 \log g)$ for the $g \times g$ grid H. For this, they use a probabilistic argument of Erdős [10] to show the following. For the completeness of this paper, we include a proof of Proposition 1 in Section 2. In our paper, log denotes the natural logarithm.

Proposition 1. *There is a positive ε such that for every sufficiently large integer n, there are n-vertex graphs with treewidth at least εn and girth at least $\varepsilon \log n$.*

Here is how they showed that $f(H) = \Omega(g^2 \log g)$ for the $g \times g$ grid H. For all sufficiently large integers g, if we choose $n = \lceil \frac{1}{9}\varepsilon g^2 \log g \rceil$ and obtain an n-vertex graph G from Proposition 1, then G has no $g \times g$ grid H as a minor because H contains $\lfloor g^2/9 \rfloor$ vertex-disjoint cycles and each cycle has length at least $\varepsilon \log n$. This implies that $f(H) \geqslant \mathrm{tw}(G) \geqslant \varepsilon n \geqslant \frac{1}{9}\varepsilon^2 g^2 \log g$. As was implicitly pointed out by Cames van Batenburg, Huynh, Joret, and Raymond [5], by the same method, we deduce the following generalization.

Proposition 2. *For every $c > 0$, there is $d > 0$ such that for every graph H with at least two vertices, if $f(H) \leqslant c|V(H)|$, then H contains at most $\frac{d|V(H)|}{\log|V(H)|}$ vertex-disjoint cycles.*

Proof. By Proposition 1, there is $\varepsilon \in (0,1)$ and an integer $n_0 > 1$ such that for all integers $n \geqslant n_0$, there is an n-vertex graph G with $\operatorname{tw}(G) > \varepsilon n$ and girth at least $\varepsilon \log n$. We may assume that $c > 1$. We set $d := \max(\frac{3}{2}c\varepsilon^{-2}, \lceil \log n_0 \rceil)$. We may assume that $|V(H)| \geqslant n_0$, since otherwise $\frac{d|V(H)|}{\log|V(H)|}$ is a trivial upper bound on the number of disjoint cycles in H.

Let $n := \lceil c\varepsilon^{-1}|V(H)| \rceil \geqslant n_0$. Let G be an n-vertex graph which has treewidth more than $\varepsilon n \geqslant c|V(H)|$ and girth at least $\varepsilon \log n$. Then G has at most $n/(\varepsilon \log n)$ vertex-disjoint cycles. Since H is a minor of G, the maximum number of vertex-disjoint cycles in H is at most

$$\frac{n}{\varepsilon \log n} \leqslant \frac{\frac{3}{2}c\varepsilon^{-1}|V(H)|}{\varepsilon \log n} \leqslant \frac{d|V(H)|}{\log|V(H)|}. \qquad \square$$

We conjecture that Proposition 2 in fact gives a precise characterisation of families of planar graphs H for which $f(H)$ is linear.

Conjecture 3. For every constant d, $f(H) = O(|V(H)|)$ for the family of planar graphs H containing at most $\frac{d|V(H)|}{\log|V(H)|}$ vertex-disjoint cycles.

Our first theorem verifies this conjecture for graphs whose non-tree components have bounded size.

Theorem 4. *Let r be a fixed positive integer and H be a planar graph with at least two vertices. If every component of H is a tree or has at most r vertices, then $f(H) = O(|V(H)|)$ precisely if H has at most $O(\frac{|V(H)|}{\log|V(H)|})$ components having cycles.*

Our second result is a step towards verifying Conjecture 3 for 2-regular graphs. It shows that $f(H)$ is $O(|V(H)|)$ whenever H is the disjoint union of $O(\frac{\sqrt{|V(H)|}}{\log|V(H)|})$ cycles:

Theorem 5. *There is an absolute constant c such that for every $r \geqslant 3$, if H is the disjoint union of r cycles, then*

$$f(H) \leqslant \frac{3|V(H)|}{2} + cr^2 \log r.$$

If H is the disjoint union of two cycles, then

$$f(H) \leqslant |V(H)| + \frac{9}{2}\left\lceil \sqrt{1 + |V(H)|} \right\rceil - 3.$$

Previously, Mousset, Noever, Škorić, and Weissenberger [17, Corollary 1.3] showed that if H is the disjoint union of r cycles each of length exactly ℓ, then $f(H) \leqslant (6 + \frac{1}{r})|V(H)| + 10r\log_2 r + O(r)$. For $r \in o(\frac{\sqrt{|V(H)|}}{\log(V(H))})$, our result gives a constant factor improvement on their result. In addition, our result does not require the cycles of H to be the same length. This is an important distinction when considering 2-regular graphs H for which the maximum cycle length is much larger than the average cycle length.

Our third result shows that $f(H)$ is $O(|V(H)|)$ whenever H is the subdivision of a planar graph with $O(1)$ edges.

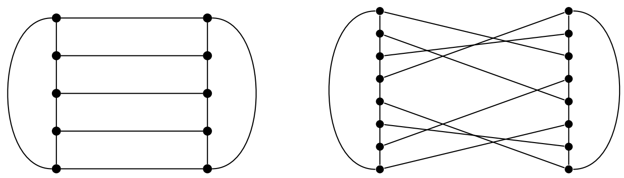

Figure 1: The 5-prism and an instance of a twisted 8-prism.

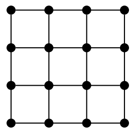

Figure 2: The 4×4 grid.

Theorem 6. *For every integer $r \geqslant 2$, there is a constant b_r such that if H is a subdivision of a planar graph with at most r edges, then $f(H) \leqslant \frac{r+1}{2}|V(H)| + b_r$.*

It would be interesting to know whether the coefficient of $|V(H)|$ in the above result can be replaced by an absolute constant, as this would be implied by Conjecture 3[1].

The *ℓ-prism* is a graph that is the Cartesian product of K_2 and the cycle of length ℓ. A twisted ℓ-prism is a graph that consists of two vertex-disjoint cycles of length ℓ joined by a matching of size ℓ. See Figure 1 for an illustration of an ℓ-prism and a twisted ℓ-prism. Birmelé, Bondy, and Reed [3] showed that $f(H) \leqslant 60\ell^2 - 120\ell + 62$ for the ℓ-prism H. They used this to show that the treewidth of any graph without a 4×4 grid minor was at most 7262. See Figure 2 for an illustration of the 4×4 grid.

Their approach was to show that if a graph G does not contain a minor isomorphic to a twisted ℓ-prism H, then its treewidth is at most $60\ell - 58$. They then combine this with a well-known theorem of Erdős and Szekeres [12], which immediately implies that a graph with a twisted $((\ell-1)^2+1)$-prism minor contains an ℓ-prism minor.

We prove the following theorem, which is tight up to a $o(1)$ factor.

Theorem 7. *Every graph without a twisted ℓ-prism as a minor has treewidth at most $2\ell + 18\lceil \frac{1+\sqrt{2\ell+1}}{4} \rceil - 8$.*

Since the wheel graph on $\ell + 1$ vertices is a minor of a twisted ℓ-prism, we deduce the following corollary, which improves the bound (1) for the wheel by Raymond and Thilikos [18, Theorem 5.1].

Corollary 8. *Every graph without the k-vertex wheel graph as a minor has treewidth at most $2k + 18\lceil \frac{1+\sqrt{2k-1}}{4} \rceil - 10$.* □

We also obtain the following.

Theorem 9. *Every graph without a twisted ℓ-prism or a 4×4 grid as a minor has treewidth at most $2\ell + 10$.*

[1] Set $b_r := \max\{f(H') : |V(H')| \leqslant r \log r\}$ and note that H has at most r disjoint cycles.

We use the latter result to show a new upper bound on the treewidth of graphs without a 4×4 grid minor, improving the previous bound 7262 by Birmelé, Bondy, and Reed [3].

Theorem 10. *Every graph without a 4×4 grid minor has treewidth at most 160.*

This paper is organized as follows. Section 2 recalls brambles. In Section 3, we consider graphs H having a few non-tree components where each non-tree component has at most r vertices, and prove Theorem 4. In Section 4, we consider twisted ℓ-prisms and prove Theorems 7, 9 and 10. In Section 5, we consider graphs H that are the disjoint union of a few cycles and prove Theorem 5. In Section 6, we consider graphs H that are subdivisions of planar graphs with at most r edges, and prove Theorem 6. We conclude this paper by presenting a few open problems in Section 7.

2 Preliminaries

In proving these results, we often focus on the dual of treewidth, the *bramble number*. A *bramble* $\mathcal{B}$ in a graph G is a set of sets $B \subseteq V(G)$ that induce a connected subgraph $G[B]$ such that for every two $B, B' \in \mathcal{B}$, we have that $G[B \cup B']$ is connected. A *hitting set* for a bramble is a set of vertices intersecting all of its elements. The *order* of a bramble $\mathcal{B}$, denoted by $\operatorname{ord}(\mathcal{B})$, is the minimum size of a hitting set for $\mathcal{B}$. The *bramble number of* G is the largest order of a bramble in G. Any subset of a bramble $\mathcal{B}$ yields a new bramble, which is called a *subbramble* of $\mathcal{B}$. Seymour and Thomas [25] showed the following duality theorem, see [19].

Theorem 11 (Seymour and Thomas [25])**.** *The treewidth of a graph is exactly one less than its bramble number.*

We will use the following result from Birmelé, Bondy, and Reed [3].

Lemma 12 (Birmelé, Bondy, and Reed [3, Theorem 2.4])**.** *Let G be a graph having a bramble $\mathcal{B}$ of order at least three. Then, there is a cycle C meeting every element of $\mathcal{B}$.*

Here is a restatement of Proposition 1 with its proof.

Proposition 13. *There is a positive ε such that for every sufficiently large integer n, there are n-vertex graphs with treewidth at least εn and girth at least $\varepsilon \log n$.*

Proof. Let N be a minimum integer such that for all $n \geqslant N$,

$$\frac{1}{5}n^{0.9}\log n < \frac{1}{2}\frac{n}{6} \quad\text{and}\quad 3^n e^{-4(n-5)^2/n} < \frac{1}{2}.$$

Let $n \geqslant N$ be an integer and let $m := \lceil \frac{6n}{5} \rceil \geqslant N$. It is easy to see that $\lceil \frac{m}{6} \rceil = \lceil \frac{n}{5} \rceil$. We claim that there is such an n-vertex graph with $\varepsilon = \frac{1}{5}$.

Let $p := \frac{81}{m}$ and $G = G(m,p)$ be the Erdős-Rényi random graph with m vertices and edge probability p. Let $\ell := \lfloor \varepsilon \log m \rfloor$. Let S be the set of vertices of G in a cycle of length at most ℓ. The expectation of $|S|$ is at most

$$\sum_{i=3}^{\ell} \binom{m}{i} \frac{i!}{2i} p^i i < \sum_{i=3}^{\ell} \frac{(mp)^i}{2} < \ell (mp)^\ell \leqslant \ell (81)^{\varepsilon \log m} \leqslant \ell m^{0.9}.$$

By Markov's inequality, the probability that $|S| \geqslant \frac{m}{6}$ is at most $\frac{\frac{1}{5} m^{0.9} \log m}{(m/6)} < \frac{1}{2}$.

If G has treewidth at most $2\lceil \frac{m}{6} \rceil - 1$, then it contains a set W of at most $2\lceil \frac{m}{6} \rceil$ vertices such that every component of $G - W$ has at most $\frac{2}{3}|V(G) - W|$ vertices, see Reed [19, Fact 2.6]. This means we can partition $V(G) - W$ into two sets X and Y, each of size at least $\frac{1}{3}|V(G) - W| \geqslant \frac{2(m-5)}{9}$ so that there are no edges between X and Y. The total number of choices for a partition of $V(G)$ into (X, Y, W) is less than 3^m and the probability for each such a partition that there are no edges between X and Y is $(1-p)^{|X||Y|} \leqslant e^{-p|X||Y|} \leqslant e^{-4(m-5)^2/m}$. Therefore, the probability that G has treewidth at most $2\lceil \frac{m}{6} \rceil - 1$ is less than $3^m e^{-4(m-5)^2/m} < \frac{1}{2}$.

Hence, with positive probability, G has treewidth at least $2\lceil \frac{m}{6} \rceil$ and $|S| < \lceil \frac{m}{6} \rceil = \lceil \frac{n}{5} \rceil = m - n$. Let T be a subset of $V(G)$ of size $\lceil \frac{m}{6} \rceil$ containing S. Trivially, we observe that $|V(G-T)| = n$ and that the treewidth of $G - T$ is at least $2\lceil \frac{m}{6} \rceil - \lceil \frac{m}{6} \rceil = \lceil \frac{m}{6} \rceil = \lceil \frac{n}{5} \rceil$. Furthermore, $G - T$ has no cycle of length at most $\ell \geqslant \lfloor \varepsilon \log m \rfloor \geqslant \lfloor \varepsilon \log n \rfloor$ and hence the girth of $G - T$ is at least $\lceil \varepsilon \log n \rceil$. □

3 Excluding H with Bounded Size Components

In this short section, we prove Theorem 4. As we discussed in Proposition 2, in order to have a linear bound on $f(H)$, H cannot have too many vertex-disjoint cycles. So if we limit our attention to graphs with at most r vertices in each non-tree component, then the number of non-tree components should be small to have a linear bound on $f(H)$.

To state the next lemma, we first define the pathwidth of graphs. A *path decomposition* is a tree decomposition in which the underlying tree is a path. The *pathwidth* of a graph G is the minimum width of its path decompositions. Clearly, the pathwidth of a graph is always greater than or equal to its treewidth.

To deal with the tree components, we use the following lemma of Diestel [8], which was used in his short proof of the theorem of Bienstock, Robertson, Seymour, and Thomas [1] that every graph of pathwidth at least $t - 1$ contains every tree on t vertices as a minor. A better presentation of its proof can be found in the proof of Theorem 12.4.5 in the first edition of the book by Diestel [9]. Seymour [24] wrote this more explicitly.

Lemma 14 (Diestel [8, 9])**.** *Let T be a tree with t vertices and let G be a graph of pathwidth at least $t - 1$. Then there is a separation (A, B) of G such that*

(P1) $|A \cap B| = t$,

(P2) $G[A]$ contains T as a minor where each vertex of $A \cap B$ appears as a distinct vertex of T, and

(P3) $G[A]$ has a path decomposition of width $t-1$ with $A \cap B$ as the last bag.

By using Lemma 14, we deduce the following.

Lemma 15. *For any tree component T of H, $f(H) \leqslant f(H - V(T)) + |V(T)|$.*

Proof. Let G be a graph that does not contain H as a minor and let T be a tree that is a component of H. If the pathwidth of G is less than $|V(T)| - 1$, then so is its treewidth, and we are done. Otherwise, we apply Lemma 14 to obtain a separation (A, B) of G satisfying (P1), (P2), and (P3). We know $G - A$ does not contain $H - V(T)$ as a minor and therefore the treewidth of $G - A$ is at most $f(H - V(T))$. We take a tree decomposition of $G - A$ having width at most $f(H - V(T))$, add $A \cap B$ to every bag, and combine it with the path decomposition of $G[A]$ by adding an edge from the endpoint x of the path decomposition to a node of the tree. Thus we obtain a tree decomposition of G of width at most $f(H - V(T)) + |V(T)|$. □

So, we need only consider H that are the disjoint union of bounded size components, all of which contain a cycle. We will apply the following lemma.

Lemma 16 (Cames van Batenburg, Huynh, Joret, and Raymond [5, Corollary 2.2])**.** *For every integer r, there is an integer s such that for every integer k, every graph of treewidth at least $sk\log(k+1)$ contains k vertex-disjoint subgraphs of treewidth at least r.*

We now present the proof of Theorem 4.

Theorem 4. *Let r be a fixed positive integer and H be a planar graph with at least two vertices. If every component of H is a tree or has at most r vertices, then $f(H) = O(|V(H)|)$ precisely if H has at most $O(\frac{|V(H)|}{\log|V(H)|})$ components having cycles.*

Proof. By Proposition 2, it is enough to prove that if H has at most $O(\frac{|V(H)|}{\log|V(H)|})$ components having cycles, then $f(H) = O(|V(H)|)$.

Let $R := \max\{f(G) \mid G \text{ is a planar graph with } r \text{ vertices}\}$. Let H' be the induced subgraph of H consisting of all components of H which are not trees. Let k be the number of components of H'. By Lemma 16, there is an integer s depending only on R such that every graph of treewidth at least $sk\log(k+1)$ contains k vertex-disjoint subgraphs of treewidth at least $R+1$. By the choice of R, we deduce that $f(H') < sk\log(k+1)$. Since $k = O(|V(H)|/\log|V(H)|)$, we have $f(H') = O(|V(H)|)$. By applying Lemma 15 to each tree component of H, we deduce $f(H) \leqslant f(H') + |V(H) - V(H')| = O(|V(H)|)$. □

4 Excluding a Twisted ℓ-Prism

In this section, we prove Theorems 7, 9 and 10. Our approach is to try to find two vertex-disjoint cycles to which we can build a twisted ℓ-prism.

Here is a lemma implicitly used by Birmelé, Bondy, and Reed [3].

Lemma 17. *Let G be a graph with a bramble $\mathcal{B}$. Let S and T be two vertex-disjoint subgraphs such that for each of $V(S)$ and $V(T)$ there is a subbramble of order at least ℓ of $\mathcal{B}$ each member of which intersects $V(S)$ or $V(T)$, respectively. Then there are ℓ vertex-disjoint paths from $V(S)$ to $V(T)$. In particular, if both S and T are cycles, then G has a twisted ℓ-prism as a minor.*

Proof. Suppose for a contradiction that there are no ℓ vertex-disjoint paths between $V(S)$ and $V(T)$. By Menger's Theorem, there is a cutset X of size less than ℓ separating $V(S)$ from $V(T)$. Then there exists an element B of $\mathcal{B}$ disjoint from X because the order of $\mathcal{B}$ is at least ℓ. Since $G[B]$ is connected, $V(S)$ or $V(T)$ does not intersect the component of $G - X$ containing B. By symmetry, we may assume that S does not intersect the component of $G - X$ containing B. Let $\mathcal{B}'$ be the subbramble of $\mathcal{B}$ consisting of all elements intersecting S. Since all elements of $\mathcal{B}'$ either intersect B or are joined by an edge to B, they all intersect X. But then X is a hitting set for $\mathcal{B}'$ and therefore $\mathcal{B}'$ has order at most $|X| < \ell$, which is a contradiction. □

The following theorem due to Seymour [23] will allow us to specify the planar minor we should consider.

Theorem 18 (Seymour [23, 4.1])**.** *Let G be a graph and let $T = \{s_1, t_1, s_2, t_2\}$ be a subset of $V(G)$. If G does not contain two vertex-disjoint paths, one linking s_1 to t_1 and the other linking s_2 to t_2, then there is a graph J satisfying all of the following.*

(S1) $T \subseteq V(J) \subseteq V(G)$.

(S2) For every component U of $G - V(J)$, the set $S_U := N_G(V(U))$ has at most three vertices.

(S3) J is obtained from $G[V(J)]$ by adding edges minimally so that each S_U is a clique for every component U of $G - V(J)$.

(S4) J has an embedding in the closed disk with the vertices of T appearing around the boundary of the disk in the order s_1, s_2, t_1, t_2.

The following easy corollary of this result shows that we can insist that J is a minor of G.

Corollary 19. *Let G be a graph and let $T = \{s_1, t_1, s_2, t_2\}$ be a subset of $V(G)$. If G does not contain two vertex-disjoint paths one linking s_1 to t_1 and the other linking s_2 to t_2, then there is a graph J satisfying (S1), (S2), (S3), (S4) and furthermore J is a minor of G.*

Proof. We proceed by the induction on $|V(G)|$. Suppose that G has a minimal cutset S of size at most two separating some component U of $G - S$ from T. Then let G' be the graph obtained from $G - V(U)$ by adding an edge if necessary to turn S into a clique. By the induction hypothesis, G' has a minor J satisfying (S1), (S2), (S3), (S4) for G'. Then it follows that J satisfies (S1), (S2), (S3), (S4) for G, regardless of the size of $S \cap V(J)$.

Now observe that J is a minor of G. So, we can assume that no cutset of size at most 2 separates a component of $G - S$ from T.

We choose a graph J guaranteed to exist by Theorem 18 with maximal $V(J)$. If $V(J) = V(G)$, then $J = G$ and we are done. Otherwise, consider any component U of $G - V(J)$. Observe that $G[V(U) \cup S_U]$ cannot be a forest because then we could draw it in the plane with S_U on the infinite face, so $G[V(U) \cup S_U]$ could be added to J, contradicting the maximality of $V(J)$. So, U contains a cycle C_U. By the previous paragraph and Menger's theorem, there are three vertex-disjoint paths from C_U to S_U. It follows that J is a minor of G. □

The following theorem will allow us to find the desired structure in that planar minor. Initially, this theorem was proved by Robertson, Seymour, and Thomas [22] with a slightly worse bound.

Theorem 20 (Gu and Tamaki [14])**.** *Let g be a positive integer. Every planar graph of treewidth at least $\frac{9}{2}g - 5$ has a $g \times g$ grid as a minor.*

Lemma 21. *Let c_1, c_2 be positive integers. Let G be a graph and let P be a path from x to y. Let $\mathcal{B}$ be a bramble of order at least $c_1 + c_2$ in G. If $V(P)$ intersects every element of $\mathcal{B}$, then P can be partitioned into two edge-disjoint subpaths P_1, P_2 such that*

- *(i) $x \in V(P_1)$, $y \in V(P_2)$,*
- *(ii) the subbramble $\mathcal{B}_1$ of $\mathcal{B}$ consisting of all elements of $\mathcal{B}$ intersecting $V(P_1)$ has order exactly c_1,*
- *(iii) the subbramble $\mathcal{B}_1'$ of $\mathcal{B}$ consisting of all elements of $\mathcal{B}$ intersecting $V(P_1) - V(P_2)$ has order at most $c_1 - 1$,*
- *(iv) the subbramble $\mathcal{B} - \mathcal{B}_1$ of $\mathcal{B}$ has order at least c_2 and $V(P_2) - V(P_1)$ intersects every element of $\mathcal{B} - \mathcal{B}_1$, and*
- *(v) the subbramble $\mathcal{B} - \mathcal{B}_1'$ of $\mathcal{B}$ has order at least $c_2 + 1$ and $V(P_2)$ intersects every element of $\mathcal{B} - \mathcal{B}_1'$.*

Proof. We choose a minimal subpath P_1 of P starting at x such that $\mathcal{B}_1$ has order at least c_1. Then $\mathcal{B}_1$ has order exactly c_1 because otherwise, we could shorten P_1 by removing its last vertex, contradicting its minimality. By the minimality, $\mathcal{B}_1'$ has order at most $c_1 - 1$. Note that there are c_1 vertices intersecting all members of $\mathcal{B}_1$ and therefore the order of $\mathcal{B} - \mathcal{B}_1$ is at least c_2 because the order of $\mathcal{B}$ is at least $c_1 + c_2$. Similarly, the order of $\mathcal{B} - \mathcal{B}_1'$ is at least $c_2 + 1$. □

Lemma 22. *Let ℓ_1, ℓ_2 be positive integers. If a graph G has a bramble $\mathcal{B}$ of order $m \geqslant \ell_1 + \ell_2 + 5$, then at least one of the following holds.*

- *(1) G has two vertex-disjoint cycles C_1, C_2 such that for each $i \in \{1, 2\}$, the subbramble of $\mathcal{B}$ consisting of elements intersecting C_i has order at least ℓ_i.*

(2) G has a planar minor J having a bramble of order at least $m - \ell_1 - \ell_2 + 2$.

Proof. Let G be a graph. Let $\mathcal{B}$ be a bramble of G of order $m \geqslant \ell_1 + \ell_2 + 5$, and let $t := m - \ell_1 - \ell_2$. We apply Lemma 12 to obtain a cycle C which is a hitting set for $\mathcal{B}$.

Let $f = vw$ be an edge of C. We obtain an edge-partition of C into four subpaths P_1, P_3, P_2, P_4, in this cyclic order with $V(P_1) \cap V(P_4) = \{v\}$ and $w \in V(P_4)$, by applying Lemma 21 several times as follows.

(a) We first apply it to $C - f$ and $\mathcal{B}$ to obtain paths P_1 and Q_1 such that the subbramble $\mathcal{B}_1$ consisting of all elements of $\mathcal{B}$ intersecting $V(P_1)$ has order exactly ℓ_1, the subbramble $\mathcal{B}'_1$ consisting of all elements of $\mathcal{B}$ intersecting $V(P_1) - V(Q_1)$ has order at most $\ell_1 - 1$, and $\mathcal{B}_{3,2,4} := \mathcal{B} - \mathcal{B}'_1$ has order at least $m - \ell_1 + 1$.

(b) Secondly, we apply it to $C[V(Q_1) \cup \{v\}]$ and $\mathcal{B}_{3,2,4}$ to obtain $Q_{3,2}$ and P_4 such that the subbramble $\mathcal{B}_{3,2}$ consisting of all elements of $\mathcal{B}_{3,2,4}$ intersecting $V(Q_{3,2})$ has order exactly $\ell_2 + 3$, the subbramble $\mathcal{B}'_{3,2}$ consisting of all elements of $\mathcal{B}_{3,2,4}$ intersecting $V(Q_{3,2}) - V(P_4)$ has order at most $\ell_2 + 2$, and $\mathcal{B}_4 := \mathcal{B}_{3,2,4} - \mathcal{B}'_{3,2}$ has order at least $m - \ell_1 - \ell_2 - 1 \geqslant 4$.

(c) Lastly, we apply it to $Q_{3,2}$ and $\mathcal{B}_{3,2}$ to obtain P_2 and P_3 such that the subbramble $\mathcal{B}_2$ consisting of all elements of $\mathcal{B}_{3,2}$ intersecting $V(P_2)$ has order exactly ℓ_2, the subbramble $\mathcal{B}'_2$ consisting of all elements of $\mathcal{B}_{3,2}$ intersecting $V(P_2) - V(P_3)$ has order at most $\ell_2 - 1$, and $\mathcal{B}_3 := \mathcal{B}_{3,2} - \mathcal{B}'_2$ has order at least 4.

For each $i \in \{1, 2\}$ let s_i and t_i be the endvertices of P_i labelled such that t_i is an endvertex of P_{i+2}. Let H be the induced subgraph of G obtained by deleting all internal vertices of both P_1 and P_2. If H has two vertex-disjoint paths R_1, R_2 linking s_1 to t_1 and s_2 to t_2 respectively, then property (1) is witnessed by the two vertex-disjoint cycles $P_1 \cup R_1$ and $P_2 \cup R_2$ with $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively.

Thus we may assume that H does not have two vertex-disjoint paths linking s_1 to t_1 and s_2 to t_2, respectively. By Corollary 19 applied to H, there is a planar minor J of H satisfying (S1), (S2), (S3), (S4) of Theorem 18.

Let $\mathcal{B}_{3,4} := \mathcal{B}_{3,2,4} - \mathcal{B}'_2$ and $\mathcal{B}^* := \{B \cap V(J) \mid B \in \mathcal{B}_{3,4}\}$. Observe that since $\mathcal{B}_{3,2,4}$ has order at least $m - \ell_1 + 1$ and $\mathcal{B}'_2$ has a hitting set of size at most $\ell_2 - 1$, the order of $\mathcal{B}_{3,4}$ is at least $(m - \ell_1 + 1) - (\ell_2 - 1) = t + 2$.

Claim 23. *$B \cap V(J) \neq \emptyset$ for all $B \in \mathcal{B}_{3,4}$.*

Proof. Suppose not. Since $G[B] = H[B]$ is connected, some component U of $H - V(J)$ contains B. By (S2), $S_U := N_H(V(U))$ is a clique in J with at most three vertices. There is some $i \in \{3, 4\}$ such that $|V(P_i) \cap S_U| \leqslant 1$. By (S1), both ends of P_i are in J and therefore $V(P_i) \cap V(U) = \emptyset$. We claim that S_U is a hitting set for $\mathcal{B}_i$. Suppose not. Then there is some $B' \in \mathcal{B}_i$ such that $B' \cap S_U = \emptyset$. Since $B' \cap V(P_i) \neq \emptyset$, we have $B' \cap V(U) = \emptyset$. Since both B and B' are in $\mathcal{B}_{3,4}$ and $B \subseteq V(U)$, H has an edge joining a vertex of B to a vertex of B', contradicting the fact that B' is disjoint from S_U. Thus, we deduce that S_U is a hitting set for $\mathcal{B}_i$. This is a contradiction because $\mathcal{B}_i$ has order at least 4. $\square$

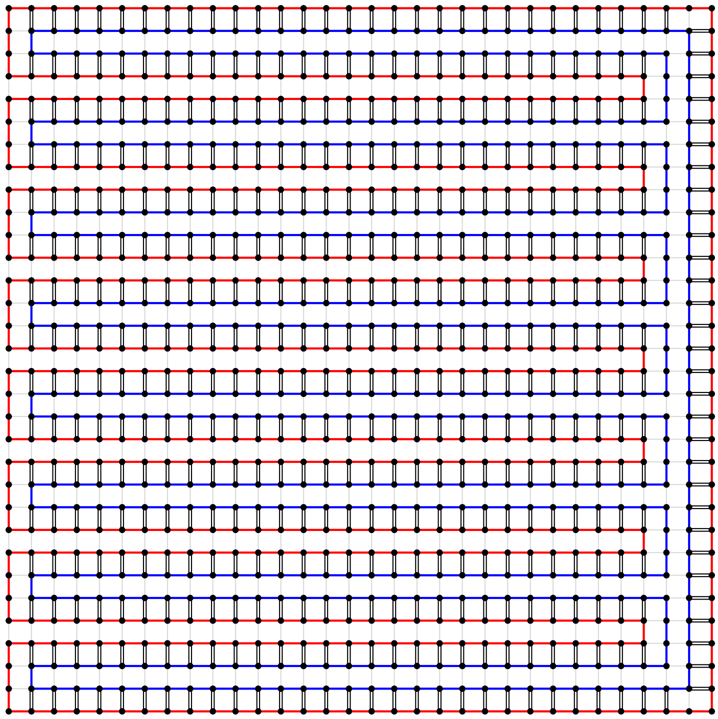

Figure 3: A $4r \times 4r$ grid contains a $(8r^2 - 4r)$-prism as a minor. Here, $r = 8$.

Claim 24. *$\mathcal{B}^*$ is a bramble in J.*

Proof. We aim to show thatboth $J[B_1 \cap V(J)]$ and $J[(B_1 \cup B_2) \cap V(J)]$ are connected for all $B_1, B_2 \in \mathcal{B}_{3,4}$. Let $B_1, B_2 \in \mathcal{B}_{3,4}$, not necessarily distinct, and let $B := B_1 \cup B_2$. Let

$$\mathcal{U} := \{U \mid U \text{ is a component of } H - V(J) \text{ with } B \cap V(U) \neq \emptyset\}.$$

For each component $U \in \mathcal{U}$, let $X_U := E(H[B \cap V(U)]) \cup \{e_U\}$ where e_U is an edge of $H[B]$ joining S_U with a vertex in $B \cap V(U)$. Since S_U is a clique in J, we have that $H[B]/\bigcup_{U \in \mathcal{U}} X_U$ is a connected spanning subgraph of $J[B \cap V(J)]$. Thus, $J[B \cap V(J)]$ is connected. □

Since any hitting set for $\mathcal{B}^*$ is also a hitting set of $\mathcal{B}_{3,4}$, the order of $\mathcal{B}^*$ is at least $t + 2$. □

Now it is straightforward to prove Theorems 7 and 9.

Theorem 7. *Every graph without a twisted ℓ-prism as a minor has treewidth at most $2\ell + 18\lceil \frac{1+\sqrt{2\ell+1}}{4} \rceil - 8$.*

Proof. Let $r := \lceil \frac{1+\sqrt{2\ell+1}}{4} \rceil$. Suppose that a graph G has treewidth at least $2\ell + 18r - 7$ and has no twisted ℓ-prism as a minor. Let $\mathcal{B}$ be a maximum order bramble of G. By Theorem 11, the order of $\mathcal{B}$ is at least $2\ell + 18r - 6$. By Lemma 17, we may assume that G has no two vertex-disjoint cycles such that each of them intersects every member of some subbramble of order at least ℓ of $\mathcal{B}$. By Lemma 22, G has a planar minor J having a bramble of order at least $18r - 4$. By Theorem 11, the treewidth of J is at least $18r - 5 = \frac{9}{2} \cdot 4r - 5$. By Theorem 20, J has a $4r \times 4r$ grid as a minor. Observe from Figure 3 that a $4r \times 4r$ grid contains a $(8r^2 - 4r)$-prism and $8r^2 - 4r = \frac{1}{2}(4r-1)^2 - \frac{1}{2} \geqslant \ell$. It follows that J has a ℓ-prism as a minor, contradicting our assumption. □

Theorem 9. *Every graph without a twisted ℓ-prism or a 4×4 grid as a minor has treewidth at most $2\ell + 10$.*

Proof. Suppose that a graph G has treewidth at least $2\ell + 11$ and has no twisted ℓ-prism as a minor. Let $\mathcal{B}$ be a maximum order bramble of G. By Theorem 11, the order of $\mathcal{B}$ is at least $2\ell + 12$. By Lemma 17, we may assume that G has no two vertex-disjoint cycles such that each of them intersects every member of some subbramble of order at least ℓ of $\mathcal{B}$. By Lemma 22, G has a planar minor J having a bramble of order at least 14. By Theorem 11, the treewidth of J is at least 13. By Theorem 20, J has a 4×4 grid as a minor. □

Now let us prove Theorem 10. To do so, we prove the following lemma, which, combined with Theorem 9, implies the theorem immediately.

Lemma 25. *Every twisted 75-prism contains a 4×4 grid minor.*

Proof. Let G be a twisted 75-prism, and let C_1, C_2 be two cycles of G with a matching of size 75 between them. Let $v_1, \ldots, v_{75}$ be the vertices of C_1 in cyclic order, let $w_1, \ldots, w_{75}$ be the vertices of C_2 in cyclic order, and let $\pi\colon \{1, \ldots, 75\} \to \{1, \ldots, 75\}$ be the permutation for which $e_i := v_i w_{\pi(i)}$ is an edge of G for all $i \in \{1, \ldots, 75\}$.

Suppose that G has a cycle C of length 4. Without loss of generality, we may assume that C contains e_1 and e_{75}. Since $|\{e_2, \ldots, e_{74}\}| = 8 \cdot 9 + 1$, we can apply the Erdős-Szekeres theorem [12] to find

(i) an increasing sequence of integers $2 \leqslant i_1 < i_2 < \cdots < i_9 \leqslant 74$ such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_9)$, or

(ii) an increasing sequence of integers $2 \leqslant i_1 < i_2 < \cdots < i_{10} \leqslant 74$ such that $\pi(i_1) > \pi(i_2) > \cdots > \pi(i_{10})$.

Using both e_1 and e_{75} in case (i) and using e_1 in case (ii), G contains a planar minor consisting of two vertex disjoint cycles of length 12 and a matching of size 11 between them. By allowing one of these cycles to form the outer face and contracting some edges of the other cycle, we observe that G contains the 4×4 grid as a minor.

Thus we may assume that G has no cycle of length 4. Without loss of generality, we may assume that $\pi(75) = 75$. Since $74 > 8^2$, by applying the Erdős-Szekeres theorem [12] we find

(i) an increasing sequence of integers $1 \leqslant i_1 < i_2 < \cdots < i_9 \leqslant 74$ such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_9)$, or

(ii) an increasing sequence of integers $1 \leqslant i_1 < i_2 < \cdots < i_9 \leqslant 74$ such that $\pi(i_1) > \pi(i_2) > \cdots > \pi(i_9)$.

Using e_{75} as well as $e_{i_1}, \ldots, e_{i_9}$ we obtain that G contains a subdivision of the 10-prism H' in each case. Let C_i' denote the cycle of H' corresponding to C_i in G for each $i \in \{1,2\}$. For an edge e of H', let $\ell(e)$ be the length of the path in G corresponding to e.

We claim that if $xyzw$ is a path of length three in C_i' for some $i \in \{1,2\}$, and $\ell(xy), \ell(zw) > 1$, then G has a 4×4 grid as a minor. By symmetry, we may assume that $i = 1$. Let x', y', z', w' be the vertices of C_2' that are matched to x, y, z, w by the edges of H'. Then it is easy to observe that H' contains a minor isomorphic to the 4×4 grid, which can be seen by contracting edges of the subpaths of C_2' from x' to y' and from z' to w'.

Thus, if we assume for contradiction that G contains no 4×4 grid minor, it is straightforward to observe that at most 4 edges of C_i' are subdivided in G for each $i \in \{1,2\}$. Thus G has a cycle of length 4, contradicting our assumption. □

Theorem 10. *Every graph without a 4×4 grid minor has treewidth at most 160.*

Proof. Let G be a graph with treewidth at least $2 \cdot 75 + 11$. By Theorem 9, G contains a twisted 75-prism or a 4×4 grid as a minor. By Lemma 25, if G contains a twisted 75-prism as a minor, then G contains a 4×4 grid as a minor. □

5 Excluding Disjoint Unions of Cycles

In this section, we prove Theorem 5. Our approach to the $r = 2$ case is similar to that used in the proof of Theorem 7. Let us state it as a separate proposition and then prove it.

Proposition 26. *Let H be the disjoint union of two cycles. Every graph without an H minor has treewidth at most $|V(H)| + \frac{9}{2}\lceil\sqrt{1 + |V(H)|}\rceil - 3$.*

Proof. We let C_1 and C_2 be the two cycles whose disjoint union is H. We set $\ell_i := |E(C_i)|$ for each $i \in \{1,2\}$ and let $\ell := \ell_1 + \ell_2 \geqslant 6$. Let G be a graph whose treewidth is at least $\ell + \frac{9}{2}\lceil\sqrt{1+\ell}\rceil - \frac{5}{2} \geqslant \ell + 5$. Let $\mathcal{B}$ be a maximum order bramble of G. By Theorem 11, the order of $\mathcal{B}$ is at least $\ell + \frac{9}{2}\lceil\sqrt{1+\ell}\rceil - \frac{3}{2}$. Note that if a cycle intersects every element of some bramble of order at least m, its length is at least m. Since G has no H minor, by Lemma 22, G has a planar minor J having a bramble of order at least $\frac{9}{2}\lceil\sqrt{1+\ell}\rceil + \frac{1}{2}$. By Theorem 11, the treewidth of J is at least $\frac{9}{2}\lceil\sqrt{1+\ell}\rceil - \frac{1}{2}$. Let $g := 1 + \lceil\sqrt{1+\ell}\rceil$. By Theorem 20, J has a $g \times g$ grid as a minor. Since $g(g-2) \geqslant \ell_1 + \ell_2$, we deduce that

$$g \geqslant \frac{\ell_1 + g}{g} + \frac{\ell_2 + g}{g} \geqslant \left\lceil \frac{\ell_1 + 1}{g} \right\rceil + \left\lceil \frac{\ell_2 + 1}{g} \right\rceil,$$

and since $\ell \geqslant 6$, we deduce that $g \geqslant 4$.

Note that if $a, b > 1$ are integers, then the $a \times b$ grid has a Hamiltonian cycle or its subgraph obtained by deleting one corner vertex has a Hamiltonian cycle. Thus, the $a \times b$ grid has a cycle of length at least $ab - 1$.

The first $\max(2, \lceil \frac{\ell_1+1}{g} \rceil)$ rows of the $g \times g$ grid has a cycle of length at least ℓ_1. The next $\max(2, \lceil \frac{\ell_2+1}{g} \rceil)$ rows of this grid contains a cycle of length at least ℓ_2. So G has H as a minor, as required. □

To handle the $r \geqslant 3$ case of Theorem 5, we need the following famous theorem of Erdős and Pósa.

Theorem 27 (Erdős and Pósa [11])**.** *There is a constant c^* such that, for every positive integer r, every graph contains either a set of vertices of size at most $c^* r \log r$ which hits every cycle or a packing of r vertex-disjoint cycles.*

Proposition 28. *There is an absolute constant c such that for every positive integer r, if H is the disjoint union of r cycles, then every graph without an H minor has treewidth less than*

$$\frac{3|V(H)|}{2} + cr^2 \log r.$$

Proof. It is enough to show that there is a constant c such that if H is the disjoint union of r cycles, every graph G without an H minor has treewidth less than

$$\frac{3|V(H)|}{2} + c \sum_{k=1}^{r} k \log k.$$

Let c^* be a positive integer that is the constant in Theorem 27 and let $c := 4c^*$. We proceed by induction on r.

Let H be a disjoint union of r cycles of lengths $c_1, c_2, \ldots, c_r$, with $c_i \geqslant c_{i+1}$ for all $i \in \{1, 2, \ldots, r-1\}$. If $r = 1$, the result follows from Lemma 12. Thus we may assume that $r > 1$. Let G be a graph whose treewidth is at least $\frac{3|V(H)|}{2} + c \sum_{k=1}^{r} k \log k$. Let $\mathcal{B}$ be a maximum order bramble of G. By Theorem 11, the order of $\mathcal{B}$ is $\operatorname{tw}(G) + 1$.

Let C by a cycle in G that is a hitting set for $\mathcal{B}$, as guaranteed by Lemma 12.

Let $f \in E(C)$. We apply Lemma 21 repeatedly, we obtain vertex-disjoint consecutive subpaths P_0, P_1, P_2 of C in the cyclic order in the following way.

(a) We apply it to $C - f$ and $\mathcal{B}$ to obtain a path P_1 such that the subbramble $\mathcal{B}_1$ of elements of $\mathcal{B}$ intersecting P_1 has order exactly $c_1 - 2$.

(b) We apply it to $C - V(P_1)$ and $\mathcal{B} - \mathcal{B}_1$ to obtain a path P_2 adjacent to P_1 such that the subbramble $\mathcal{B}_2$ of elements of $\mathcal{B} - \mathcal{B}_1$ intersecting P_2 has order exactly $2\lfloor c^* r \log r \rfloor + 2$.

(c) We apply it to $C - V(P_1 \cup P_2)$ and $(\mathcal{B} - \mathcal{B}_1) - \mathcal{B}_2$ to obtain a path P_0 adjacent to P_1 such that the subbramble $\mathcal{B}_0$ consisting of elements of $(\mathcal{B} - \mathcal{B}_1) - \mathcal{B}_2$ intersecting P_0 has order exactly $2\lfloor c^* r \log r \rfloor + 2$.

(d) Moreover, the subbramble $\mathcal{B}_3 := \mathcal{B} - \mathcal{B}_0 - \mathcal{B}_1 - \mathcal{B}_2$ has order at least $\operatorname{tw}(G) + 1 - c_1 - 4\lfloor c^* r \log r \rfloor - 2$ and no element of $\mathcal{B}_3$ insersects $P_0 \cup P_1 \cup P_2$.

By Lemma 17, there exists a set S of at least $2\lfloor c^* r \log r \rfloor + 2$ vertex-disjoint paths from $V(P_0)$ to $V(P_2)$ in $G - V(P_1)$. If any path Q in S has at most $\frac{1}{2}c_1 - 2$ internal vertices, then consider a cycle O in $G[V(Q \cup P_0 \cup P_1 \cup P_2)]$ containing P_1 and Q. Since O has length at least $|V(P_1)| + 2 \geqslant c_1$, it suffices to embed the graph H' consisting of $r-1$ vertex-disjoint cycles of lengths $c_2, c_3, \ldots, c_r$ as a minor in $G - V(O)$. By (d) and Theorem 11, the treewidth of $G - V(O)$ is at least $\operatorname{tw}(G) - c_1 - 4\lfloor c^* r \log r \rfloor - 2 - (\frac{1}{2}c_1 - 2)$, which is at least

$$\left(\frac{3}{2}\sum_{i=2}^{r} c_i\right) + c\sum_{k=1}^{r-1} k \log k.$$

Thus, $G - V(O)$ has H' as a minor by the induction hypothesis. This implies that G has H as a minor.

Hence, we may assume that every path in S has at least $c_1/2$ vertices. It follows that every cycle in $G' := P_0 \cup P_2 \cup \bigcup S$ has length at least c_1. Therefore it suffices to find a packing of r vertex-disjoint cycles in G'. It is easy to see that a spanning tree of G' can be obtained by deleting a single edge of all but one path in S, so $|E(G')| - |V(G')| = |S| - 2 = -1 + (|S| - 1)$, and by construction G' has maximum degree 3. Deleting a vertex decreases the difference between the number of edges and the number of vertices by at most 2. Therefore every hitting set for the cycles in G' has size at least $(|S| - 1)/2 > \lfloor c^* r \log r \rfloor$. By Theorem 27, G' has a packing of r vertex-disjoint cycles, and hence G has H as a minor. □

Theorem 5 follows from Propositions 26 and 28.

6 Excluding Subdivisions of Small Planar Graphs

In this section, we prove Theorem 6 using an approach similar to that applied to prove Theorem 7.

In place of Theorem 18, we apply a corollary of the following result due to Robertson and Seymour [21, (5.3)]. For a graph H, an H-*model* in a graph G is a collection $(T_v)_{v \in V(H)}$ of vertex-disjoint trees in G such that for every edge xy of H, there is an edge joining T_x and T_y in G. We note that a graph contains an H-model if and only if it has H as a minor.

Theorem 29 (Robertson and Seymour [21, (5.3)]). *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = z$. Let $k \geqslant \lfloor 3z/2 \rfloor$ be an integer and let $G_1, G_2, \ldots, G_k$ be pairwise vertex-disjoint subgraphs of G satisfying the following.*

(i) *For each $i \in \{1, 2, \ldots, k\}$, either G_i is connected or every component of G_i has a vertex in Z.*

(ii) *For each* $1 \leqslant i < j \leqslant k$, *either* $V(G_i)$ *and* $V(G_j)$ *both meet* Z *or there is an edge between* G_i *and* G_j.

(iii) *For each* $i \in \{1, 2, \ldots, k\}$, *there is no separation* (A, B) *of* G *of order less than* z *with* $Z \subseteq A$ *and* $A \cap G_i = \emptyset$.

Then for some μ *with* $0 \leqslant \mu \leqslant z$, *there are* $\ell = k - \lfloor \frac{1}{2}(z - \mu) \rfloor$ *non-null pairwise vertex-disjoint connected subgraphs* $H_1, \ldots, H_\ell$ *of* G *such that*

(a) $|V(H_i) \cap Z| = 1$ *for all* $1 \leqslant i \leqslant z$ *and* $V(H_j) \cap Z = \emptyset$ *for all* $z + 1 \leqslant j \leqslant \ell$, *and*

(b) *for all* $1 \leqslant i < j \leqslant \ell$, *if there is no edge of* G *between* H_i *and* H_j, *then* $i, j \leqslant \mu$.

Corollary 30. *Let* c *be a positive integer and* Z *be a set of at most* c *vertices of a graph* G. *Let* $(T_v)_{v \in V(K_{2c})}$ *be a* K_{2c}*-model in* G. *If* $G - Y$ *has a component containing at least one* T_v *and at least one vertex of* Z *for all* $Y \subseteq V(G)$ *with* $|Y| < |Z|$, *then* G *has a* K_c*-model such that each vertex of* Z *is contained in a distinct tree of the model.*

Proof. Let $z = |Z| \leqslant c$. Let $t_1, t_2, \ldots, t_z$ be the vertices in Z. By applying Theorem 29 with $k = 2c$ and $(T_v)_{v \in V(K_{2c})}$, we obtain pairwise vertex-disjoint connected subgraphs $H_1, \ldots, H_\ell$ for some $\ell = 2c - \lfloor \frac{1}{2}(z - \mu) \rfloor$ where $0 \leqslant \mu \leqslant z$, satisfying (a) and (b) of Theorem 29. Now, observe that $\lfloor \frac{1}{2}(z - \mu) \rfloor + \mu \leqslant z$, so $\ell \geqslant c + \mu$. We may assume that $V(H_i) \cap Z = \{t_i\}$ for all $1 \leqslant i \leqslant z$ by relabeling vertices in Z. For each i with $1 \leqslant i \leqslant \mu$, let J_i be a spanning tree of the connected subgraph formed from the union of H_i and H_{c+i} by adding some edge of G joining H_i and H_{c+i}. For each i with $\mu < i \leqslant c$, let J_i be a spanning tree of H_i. Then $(J_i)_{1 \leqslant i \leqslant c}$ is a K_c-model in G such that every vertex of Z is contained in a distinct tree of the model. □

Corollary 31. *Let* c *be a positive integer and let* Z *be a set of at most* c *vertices of a graph* G. *Let* $\mathcal{B}$ *be a bramble of* G *of order at least* $3c$. *If there does not exist a cutset* Y *of size less than* c *for which the unique component of* $G - Y$ *containing an element of* $\mathcal{B}$ *is disjoint from* Z, *then either*

(a) there is a K_c*-model such that every vertex of* Z *is contained in a distinct element of the model, or*

(b) there is a minor of G *which contains no* K_{3c}*-minor and which contains a bramble whose order is at least the order of* $\mathcal{B}$.

Proof. Suppose for contradiction that G is a graph containing a set Z of vertices and a bramble $\mathcal{B}$ violating the statement of the corollary, and subject to this suppose that $|V(G)|$ is as small as possible. Then G has K_{3c} as a minor because otherwise (b) holds. Furthermore, given a K_{3c}-model in G, there must be a cutset Y of size less than c such that $G - Y$ has a component containing a tree of the model and no vertex of Z, because otherwise we are done by applying Corollary 30.

We consider a cutset Y of minimum size such that $G - Y$ has a component U containing a tree of the model and no vertex of Z, we note $|Y| < c$. So, since $G[U \cup Y]$

contains at least $3c - |Y| \geqslant 2c$ trees from the model, we can apply Corollary 30 to Y and $G[Y \cup U]$, and so the graph obtained from $G - U$ by adding edges so Y is a clique is a minor G' of G.

Let $\mathcal{B}' = \{X \setminus V(U) \mid X \in \mathcal{B}\}$. Since no element of $\mathcal{B}$ is contained in U and Y is a clique, we deduce that $\mathcal{B}'$ is a bramble of G'. Since any hitting set of $\mathcal{B}'$ is also a hitting set of $\mathcal{B}$, the order of $\mathcal{B}'$ in G' is at least the order of $\mathcal{B}$. Since $N_G(U) = Y$ which is a clique in G', it holds for any set $Y' \subseteq V(G')$, that vertices in separate components of $G' - Y$ are in separate components of $G - Y'$. Hence, there is no cutset Y' in G' of size less than c for which the unique component of $G' - Y'$ containing an element of $\mathcal{B}'$ is disjoint from Z. Since $|V(G')| < |V(G)|$, the statement of the corollary holds for G' with Z and $\mathcal{B}'$. Both (a) and (b) for G' contradict the assumption that G is a counterexample. □

In place of Theorem 20, we apply the following result of Demaine and Hajiaghayi [7].

Theorem 32 (Demaine and Hajiaghayi [7])**.** *For every positive integer a, there is an integer $c_a > 1$ such that for every positive integer g, every graph of treewidth at least $c_a g$ has K_a or the $g \times g$ grid as a minor.*

Lemma 33. *Let H be a minor of a $g \times g$ grid. Let ℓ be a positive integer. If G is a subdivision of H obtained by subdividing each edge less than ℓ times, then G is a minor of a $\lceil 2\sqrt{\ell}\rceil g \times \lceil 2\sqrt{\ell}\rceil g$ grid.*

Proof. Let G be a subdivision of H such that each edge of H is subdivided less than ℓ times. If $\ell = 1$, then $G = H$ and the result is trivial. Thus we may assume that $\ell > 1$. Now, the $g\ell \times g\ell$ grid has a subdivision of the $g \times g$ grid in which each edge is subdivided exactly $\ell - 1$ times. So, we can assume $2\sqrt{\ell} \leqslant \ell - 1$, and hence $\ell \geqslant 6$ and $\lceil 2\sqrt{\ell}\rceil \geqslant 5$. Let $r := \lceil \frac{1}{2}\sqrt{2\ell - 1} - \frac{1}{2}\rceil \geqslant 2$.

We now present a mapping φ from the $g \times g$ grid to the $(2r+1)g \times (2r+1)g$ grid such that if a graph J is a subgraph of the $g \times g$ grid, then its image $\varphi(J)$ is a subdivision of J in which each edge is subdivided at least ℓ times. We regard

$$\{0, 1, 2, \ldots, g-1\} \times \{0, 1, 2, \ldots, g-1\}$$

as the vertex set of the $g \times g$ grid. For vertices, we define $\varphi(i, j) := ((2r+1)i, (2r+1)j)$. Let $P_{i,j}$ be a path from $\varphi(i, j)$ to $\varphi(i, j+1)$ whose set of edges is

$$\begin{aligned}
&\{(x, y)(x+1, y) \mid (2r+1)i \leqslant x < (2r+1)i + r,\ (2r+1)j < y < (2r+1)(j+1)\} \\
&\cup \{(x, y)(x, y+1) \mid x = (2r+1)i,\ (2r+1)j \leqslant y < (2r+1)(j+1), \\
&\qquad y - (2r+1)j \equiv 0 \pmod 2\} \\
&\cup \{(x, y)(x, y+1) \mid x = (2r+1)i + r,\ (2r+1)j \leqslant y < (2r+1)(j+1), \\
&\qquad y - (2r+1)j \equiv 1 \pmod 2\}.
\end{aligned}$$

Let $Q_{i,j}$ be a path from $\varphi(i,j)$ to $\varphi(i+1,j)$ whose set of edges is

$$\begin{aligned}
&\{(x,y)(x+1,y) \mid (2r+1)i \leqslant x \leqslant (2r+1)i+r,\ y=(2r+1)j\}\\
&\cup \{(x,y)(x,y+1) \mid x=(2r+1)i+r+1,\\
&\qquad\qquad (2r+1)j \leqslant y < (2r+1)(j+1)-1\}\\
&\cup \{(x,y)(x+1,y) \mid (2r+1)i+r+1 < x < (2r+1)(i+1)-2,\\
&\qquad\qquad (2r+1)j < y < (2r+1)(j+1),\ \}\\
&\cup \{(x,y)(x+1,y) \mid x=(2r+1)i+r+2,\ (2r+1)j \leqslant y < (2r+1)(j+1)-1,\\
&\qquad\qquad y-(2r+1)j \equiv 0 \pmod 2\}\\
&\cup \{(x,y)(x+1,y) \mid x=(2r+1)(i+1)-1,\ (2r+1)j \leqslant y < (2r+1)(j+1)-1,\\
&\qquad\qquad y-(2r+1)j \equiv 1 \pmod 2\}\\
&\cup \{(x,y)(x+1,y) \mid (x,y)=((2r+1)i+r+1,(2r+1)(j+1)-1)\}\\
&\cup \{(x,y)(x+1,y) \mid (x,y)=((2r+1)(i+1)-1,(2r+1)j)\}.
\end{aligned}$$

For a vertical edge $(i,j)(i,j+1)$ of the $g \times g$ grid, we map it to a path $P_{i,j}$ of length $2r^2+2r+1$ from $\varphi(i,j)$ to $\varphi(i,j+1)$. For a horizontal edge $(i,j)(i+1,j)$ of the $g \times g$ grid, we map it to a path $Q_{i,j}$ of length $2r^2+2r+1$ from $\varphi(i,j)$ to $\varphi(i+1,j)$. See Figure 4. Note that $2r^2+2r \geqslant \frac{2\ell-1}{2} - \frac{1}{2} = \ell - 1$.

Now it follows that the $(2r+1)g \times (2r+1)g$ grid has a G-model because we can map the H-model in the $g \times g$ grid by φ. Note that

$$2r+1 \leqslant \lceil \sqrt{2\ell-1}+1 \rceil \leqslant \lceil 2\sqrt{\ell} \rceil. \qquad \square$$

With these preliminaries out of the way, we turn to the proof of the theorem.

Theorem 6. *For every integer $r \geqslant 2$, there is a constant b_r such that if H is a subdivision of a planar graph with at most r edges, then $f(H) \leqslant \frac{r+1}{2}|V(H)| + b_r$.*

Proof. Let c_{6r} be the constant guaranteed by Theorem 32 for $a = 6r$. Robertson, Seymour, and Thomas [22] showed that there exists an integer g_{2r} such that every planar graph with at most $2r$ vertices is a minor of the $g_{2r} \times g_{2r}$ grid. Let $b_r := \max\{2r^2, 6r, 12c_{6r}^2 g_{2r}^2\}$.

We claim that for every positive integer n, we have $\frac{r-1}{2}n + b_r \geqslant c_{6r}\lceil 2\sqrt{n}\rceil g_{2r}$. To prove this, we may assume that $n < 2c_{6r}\lceil 2\sqrt{n}\rceil g_{2r}$. Then $n < 2c_{6r}(3\sqrt{n})g_{2r}$ and therefore $2\sqrt{n} \leqslant 12c_{6r}g_{2r}$. This implies that $c_{6r}\lceil 2\sqrt{n}\rceil g_{2r} \leqslant 12c_{6r}^2 g_{2r}^2 \leqslant b_r$.

Assume for contradiction that the theorem is false for this choice of b_r, and consider a minimal counterexample H which is a subdivision of a graph H' with $d := |E(H')| \leqslant r$. We can assume that H' does not contain an isolated vertex x as then so does H and

$$f(H-x) \geqslant f(H) - 1 \geqslant \frac{r+1}{2}|V(H-x)| + b_r$$

contradicting the minimality of H. So $|V(H')| \leqslant 2d$.

We let $Q_1, \ldots, Q_d$ be the paths of H corresponding to the edges of H' and let ℓ_i be the length of Q_i. We pick this labelling so that $\ell_i \geqslant \ell_{i+1}$ for all $i \in \{1, 2, \ldots, d-1\}$.

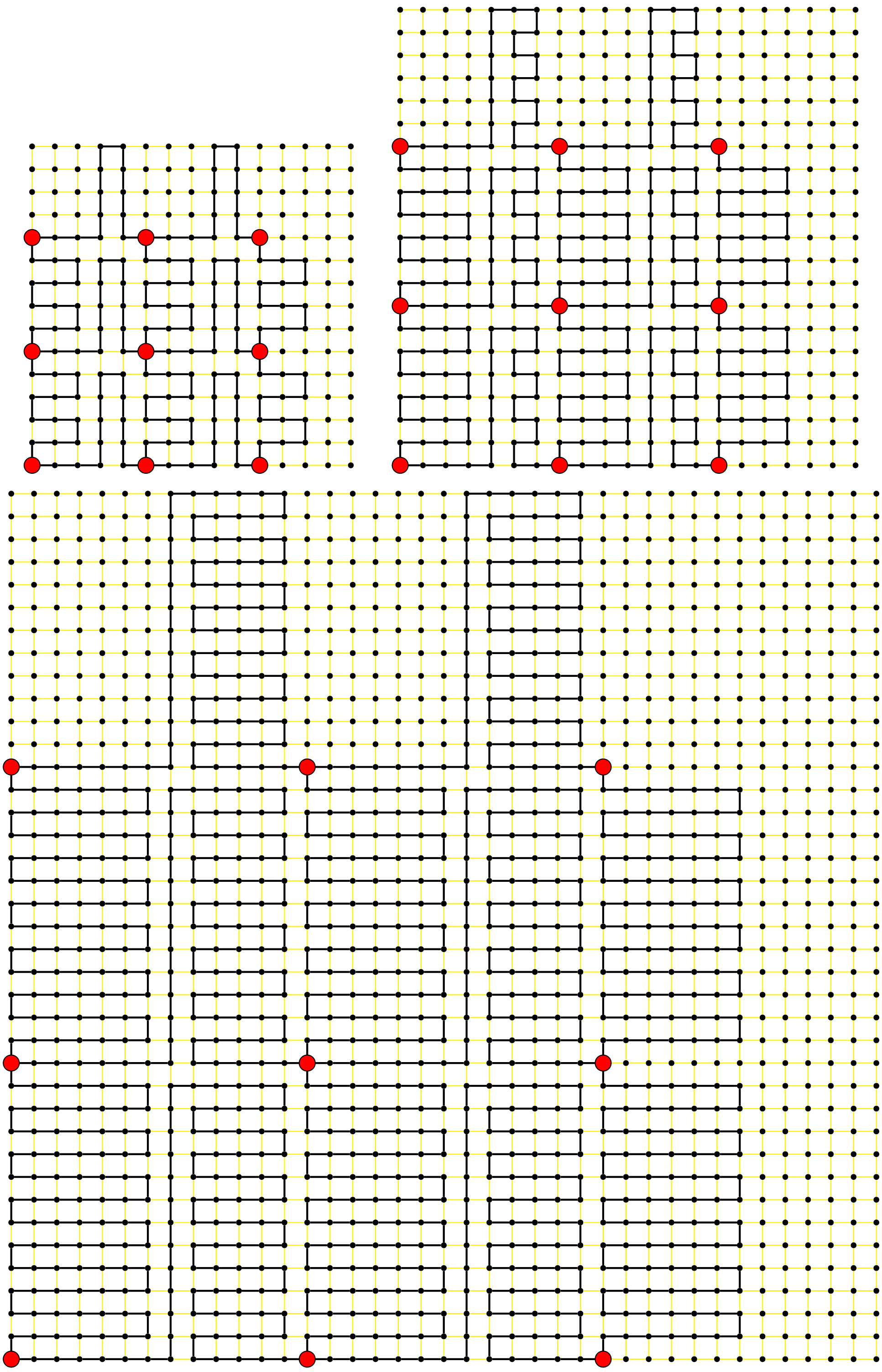

Figure 4: Embedding a subdivision of the grid in the proof of Lemma 33.

Let G be a graph that does not contain H as a minor such that the treewidth of G is at least $\frac{r+1}{2}|V(H)| + b_r$. Let $\mathcal{B}$ be a maximum order bramble in G. By Theorem 11, $\mathcal{B}$ has order at least $\frac{r+1}{2}|V(H)| + b_r + 1$. By Lemma 12, there is a cycle C intersecting all elements of $\mathcal{B}$.

Note that

$$\begin{aligned}\sum_{i=1}^{d}\left(\sum_{j=i}^{d}(\ell_j - 1) + \left\lfloor \frac{b_r}{d} \right\rfloor\right) &\leqslant \sum_{i=1}^{d}(\ell_i - 1)i + b_r \\ &\leqslant \frac{1}{d}\left(\sum_{i=1}^{d}(\ell_i - 1)\right)\left(\sum_{i=1}^{d} i\right) + b_r \\ &\leqslant \frac{|V(H)|}{d}\frac{d(d+1)}{2} + b_r \\ &\leqslant \frac{r+1}{2}|V(H)| + b_r,\end{aligned}$$

where the second inequality follows from Chebyshev's sum inequality. Therefore, by Lemma 21, there exists a partition of C into vertex-disjoint subpaths $R_1, \ldots, R_d$ such that the order of the subbramble $\mathcal{B}_i$ of $\mathcal{B}$ consisting of elements of $\mathcal{B}$ intersecting R_i and not intersecting $\bigcup_{j=1}^{i-1} R_j$ is at least $\sum_{j=i}^{d}(\ell_j - 1) + \lfloor \frac{b_r}{d} \rfloor$ for each $i \in \{1, 2, \ldots, d\}$.

Let us fix an orientation of C. For each $i \in \{1, 2, \ldots, d\}$, let P_i be the path formed by the last $\ell_i - 1$ vertices of R_i. We let $P'_i := R_i - V(P_i)$. Let Z be the set of vertices that are endpoints of P'_i for some $i \in \{1, 2, \ldots, d\}$. Let $\mathcal{B}'$ be the subbramble of $\mathcal{B}$ consisting of the elements of $\mathcal{B}$ not intersecting $\bigcup_{i=1}^{d} V(P_i)$. Since $\sum_{i=1}^{d}(\ell_i - 1) < |V(H)|$, the order of $\mathcal{B}'$ is at least $\frac{r-1}{2}|V(H)| + b_r + 1$.

We note that for every $i \in \{1, 2, \ldots, d\}$, the order of the subbramble $\mathcal{B}'_i$ of $\mathcal{B}'$ consisting of all elements of $\mathcal{B}'$ intersecting $V(P'_i)$ is at least $\lfloor \frac{b_r}{d} \rfloor \geqslant 2r \geqslant 2d$. This is because the order of $\mathcal{B}_i$ is at least $\sum_{j=i}^{d}(\ell_j - 1) + \lfloor \frac{b_r}{d} \rfloor$ and so the subbramble of $\mathcal{B}_i$ consisting of all elements of $\mathcal{B}_i$ not intersecting $\bigcup_{j=i}^{d} V(P_j)$ has order at least $\lfloor \frac{b_r}{d} \rfloor$ and this subbramble is a subbramble of $\mathcal{B}'_i$.

Let $G' := G - \bigcup_{i=1}^{d} V(P_i)$. Now, any set $Y \subseteq V(G)$ of size at most $2d - 1$ intersects some P'_i in at most one vertex. Since the order of $\mathcal{B}'_i$ is at least $2d$, there is some $B \in \mathcal{B}'_i$ disjoint from Y. Some endpoint of P'_i is in the same component K of $G' - Y$ as B, because B intersects $V(P'_i)$. Since $\mathcal{B}'$ is a bramble, K is the unique component of $G' - Y$ containing an element of $\mathcal{B}'$. Thus G' has no cutset Y of size less than $2d$ that such that the unique component of $G' - Y$ containing an element of $\mathcal{B}'$ is disjoint from Z.

Now, if there is a K_{2d}-model in G' such that each element of Z is contained in a distinct tree of the model, we can find H as a minor of G as follows. We first contract each tree of the model so that Z becomes a clique. We then embed the internal vertices of each path Q_i using the path P_i, and contract appropriate subcliques of Z to obtain the vertices in $V(H')$.

Note that $\mathcal{B}'$ is a bramble of G' and the order of $\mathcal{B}'$ is at least $b_r \geqslant 6r \geqslant 6d$. Hence, by Corollary 31 applied to $\mathcal{B}'$ with $c = 2d$, we may assume that G' has a minor G''

such that G'' has treewidth at least $\frac{r-1}{2}|V(H)| + b_r$ and G'' does not contain K_{6r} as a minor. Now, since $\frac{r-1}{2}|V(H)| + b_r \geqslant c_{6r}\lceil 2\sqrt{|V(H)|}\rceil g_{2r}$, Theorem 32 implies that G'' has a $(\lceil 2\sqrt{|V(H)|}\rceil g_{2r}) \times (\lceil 2\sqrt{|V(H)|}\rceil g_{2r})$ grid as a minor. By Lemma 33 and the definition of g_{2r}, it follows that G'' contains H as a minor, a contradiction. □

7 Open problems

By Lemma 15, if H is the disjoint union of H_1 and H_2 where H_2 is a forest, then $f(H) \leqslant f(H_1) + f(H_2)$. Just by considering disjoint unions of cycles, we see that there exist graphs H_1 and H_2 such that for the disjoint union H of H_1 and H_2, we have $f(H) > f(H_1) + f(H_2)$. It is natural to ask if there is a constant c such that $f(H) \leqslant c(f(H_1) + f(H_2))$. This would follow immediately if the following conjecture were to be proven true:

Conjecture 34. There is a constant $\varepsilon > 0$ such that the vertex set of every graph of positive treewidth w can be partitioned into two sets each inducing a subgraph of treewidth at least $\lfloor \varepsilon w \rfloor$.

We ask two further questions about how $f(H)$ grows with small changes to H.

Question 35. Given an n-vertex planar graph H and a vertex $v \in V(H)$, what is the maximum possible difference between $f(H)$ and $f(H - v)$?

Question 36. Given an n-vertex planar graph H and an edge $e \in E(H)$, what is the maximum possible difference between $f(H)$ and $f(H - e)$?

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