Graphs of bounded chordality

Aristotelis Chaniotis[a] Babak Miraftab[b] Sophie Spirkl[a,c]

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Abstract

A *hole* in a graph is an induced subgraph which is a cycle of length at least four. A graph is *chordal* if it contains no holes. Following McKee and Scheinerman (1993), we define the *chordality* of a graph G to be the minimum number of chordal graphs on $V(G)$ such that the intersection of their edge sets is equal to $E(G)$. In this paper we study classes of graphs of bounded chordality.

In the 1970s, Buneman, Gavril, and Walter, proved independently that chordal graphs are exactly the intersection graphs of subtrees in trees. We generalize this result by proving that the graphs of chordality at most k are exactly the intersection graphs of convex subgraphs of median graphs of tree-dimension k.

A hereditary class of graphs $\mathcal{A}$ is χ-*bounded* if there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that for every graph $G \in \mathcal{A}$, we have $\chi(G) \leqslant f(\omega(G))$. In 1960, Asplund and Grünbaum proved that the class of all graphs of boxicity at most two is χ-bounded. In his seminal paper "Problems from the world surrounding perfect graphs," Gyárfás (1987), motivated by the above result, asked whether the class of all graphs of chordality at most two, which we denote by $\mathcal{C} \mathbin{\dot\cap} \mathcal{C}$, is χ-bounded. We discuss a result of Felsner, Joret, Micek, Trotter and Wiechert (2017), concerning tree-decompositions of Burling graphs, which implies an answer to Gyárfás' question in the negative. We prove that two natural families of subclasses of $\mathcal{C} \mathbin{\dot\cap} \mathcal{C}$ are polynomially χ-bounded.

Finally, we prove that for every $k \geqslant 3$ the k-Chordality Problem, which asks to decide whether a graph has chordality at most k, is NP-complete.

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[a] Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada (aristotelis.chaniotis@uwaterloo.ca, sophie.spirkl@uwaterloo.ca).

[b] School of Computer Science, Carleton University, Ottawa, Ontario, K1S 5B6, Canada (bobby.miraftab@gmail.com).

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1 Introduction

For basic notions and notation not defined here we refer readers to [35]. In this paper we consider finite, undirected graphs with no loops or parallel edges. For a set S we denote the power set of S by 2^S, and the set of all size-two elements of 2^S by $\binom{S}{2}$. Let G be a graph. We denote the complement of G by G^c. We call a subset of $V(G)$ a *clique* (respectively a *stable* set) of G if it is a set of pairwise adjacent (respectively non-adjacent) vertices. A clique of size three is called a *triangle*. The *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique in G. Let $A, B \subseteq V(G)$. An (A,B)*-path* in G is a path in G which has one of its ends in A and its other end in B; if $A = \{u\}$ we write (u, B)-path instead of $(\{u\}, B)$-path. A *non-edge* of G is an element of the set $\binom{V(G)}{2} \setminus E(G)$. Given a graph H we say that G is H*-free* (respectively *contains* H) if it contains no (respectively contains an) induced subgraph isomorphic to H. For a set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G which is induced by X. A class of graphs is *hereditary* if it is closed under isomorphism and under taking induced subgraphs.

Let $G_1, \ldots, G_k$ be graphs. Then, their *intersection* (respectively *union*), which we denote by $\cap_{i\in[k]}G_i$ (respectively $\cup_{i\in[k]}G_i$), is the graph $(\cap_{i\in[k]}V(G_i), \cap_{i\in[k]}E(G_i))$ (respectively $(\cup_{i\in[k]}V(G_i), \cup_{i\in[k]}E(G_i))$). Given graph classes $\mathcal{G}_1, \ldots, \mathcal{G}_k$, we denote by $\mathcal{G}_1 \Cap \ldots \Cap \mathcal{G}_k$ the class $\{G : \exists G_i \in \mathcal{G}_i \text{ such that } G = G_1 \cap \ldots \cap G_k\}$, which we call the *graph-intersection* of $\mathcal{G}_1, \ldots, \mathcal{G}_k$. The *graph-union* of $\mathcal{G}_1, \ldots, \mathcal{G}_k$, which we denote by $\mathcal{G}_1 \Cup \ldots \Cup \mathcal{G}_k$, is the class $\{G : \exists G_i \in \mathcal{G}_i \text{ such that } G = G_1 \cup \ldots \cup G_k\}$.

Given a class of graphs $\mathcal{A}$ and a graph G, we follow Kratochvíl and Tuza [21], and define the *intersection dimension of* G *with respect to* $\mathcal{A}$ to be the minimum integer k such that $G \in \Cap_{i\in[k]}\mathcal{A}$ if such a k exists, and $+\infty$ otherwise. We remark that the intersection dimension of graphs with respect to graph classes has been also studied by by Cozzens and Roberts [9] under a different name: they called a graph property P *dimensional* if for every graph G, the intersection dimension of G with respect to the class $\mathcal{A}(P) := \{G : G \text{ has the property } P\}$ is finite. For a positive integer n, we denote by K_n the complete graph on n vertices, and by K_n^- the graph we obtain from K_n by deleting an edge. It is easy to observe that a graph property P is dimensional if and only if for every positive integer n, both the graphs K_n and K_n^- have the property P.

A *hole* in a graph G is an induced cycle of length at least four. A graph is *chordal* if it contains no holes, and we denote the class of chordal graphs by $\mathcal{C}$. Following McKee and Scheinerman [25] we call the intersection dimension of a graph G with respect to $\mathcal{C}$ the *chordality* of G and we denote it by $\mathsf{chor}(G)$. Since, for every positive integer n, both the graphs K_n and K_n^- are chordal, it follows that the chordality of every graph is finite (and upper bounded by the number of its non-edges). To the best of our knowledge, chordality was first studied by Cozzens and Roberts [9] under the name rigid circuit dimension.

Given a finite family of nonempty sets $\mathcal{S}$, the *intersection graph* of $\mathcal{S}$ is the graph which has as vertices the elements of $\mathcal{S}$ and two vertices are adjacent if and only if they have a non-empty intersection. Given a graph G and a family $\mathcal{S}$ of subgraphs of G, the intersection graph of $\mathcal{S}$ is the intersection graph of the family $\{V(H) : H \in \mathcal{S}\}$.

In the 1970s, Buneman [5], Gavril [15], and Walter [34, 33], proved independently that chordal graphs are exactly the intersection graphs of subtrees in trees. Let H be a chordal graph. A tree T is a *representation tree* of H if there exists a function $\beta : V(T) \to 2^{V(H)}$ such that for every $v \in V(H)$, the subgraph $T[\{t \in V(T) : v \in \beta(t)\}]$ of T is connected, and H is isomorphic to the intersection graph of the family $\{\{t \in V(T) : v \in \beta(t)\} : v \in V(H)\}$. In this case we call the pair (T, β) a *representation* of H. By the aforementioned characterization of chordal graphs, it follows that every chordal graph has a representation. In section 2, we prove a characterization of graphs of chordality at most k which generalizes the above characterization of chordal graphs. We continue with some definitions before we state the main result of section 2.

An *interval graph* is any graph which is isomorphic to the intersection graph of a family of intervals on the real line. We denote the hereditary class of interval graphs by $\mathcal{I}$. It is easy to see that the intersection graphs of subpaths in paths are exactly the interval graphs, and thus every interval graph is also chordal.

Let G be a graph. A *chordal completion* (respectively *interval completion*) of G is a supergraph of G on the same vertex which is chordal (respectively interval). Since every complete graph is an interval graph, it follows that every graph has an interval and thus a chordal completion.

A *tree-decomposition* of G is a representation (T, β) of a chordal completion H of G. Fix a chordal completion H of G and a representation (T, β) of H. For every $t \in V(T)$, we call the set $\beta(t)$ the *bag* of t. It is easy to see that every bag is a clique of H and that every clique of H is contained in a bag of T. We say that (T, β) is a *complete tree-decomposition* of G if for every $t \in V(T)$, the set $\beta(t)$ is a clique of G. If T is a path, then H is an interval completion of G and we call tree-decomposition (T, β) a *path-decomposition* of G. It is easy to see that a graph has a complete tree-decomposition (respectively complete path-decomposition) if and only if it is chordal (respectively interval). The *width* of a tree-decomposition is the the clique number of the corresponding chordal completion minus one[1]. The *tree-width* (respectively *path-width*) of G, denoted by $\mathsf{tw}(G)$ (respectively $\mathsf{pw}(G)$) is the minimum width of a tree-decomposition (respectively path-decomposition) of G. That is, $\mathsf{tw}(G) := \min\{\omega(H) - 1 : H \text{ is a chordal completion of } G\}$, and $\mathsf{pw}(G) := \min\{\omega(I) - 1 : I \text{ is an interval completion of } G\}$. A tree-decomposition *separates a non-edge* e if e is a non-edge of the chordal completion which corresponds to this tree-decomposition. Let $\mathcal{T}$ be a family of tree-decompositions of G. We say that $\mathcal{T}$ is a *non-edge-separating* family of tree-decompositions if for every non-edge e of G, there exists a tree-decomposition in $\mathcal{T}$ which separates e.

Below is the main result of section 2, we postpone some definitions for section 2.

Theorem 1. *Let G be a graph and k be a positive integer. Then the following are equivalent:*

1. *The graph G has chordality k.*
2. *The minimum size of a non-edge-separating family of tree-decompositions of G is k.*
3. *k is the minimum integer such that the graph G is the intersection graph of a family*

[1]The "minus one" in the definition of the width serves so that trees have tree-width one.

of convex subgraphs of the Cartesian product of k trees.

4. *k is the minimum integer such that the graph G is the intersection graph of a family of convex subgraphs of a median graph of tree-dimension k.*
5. *The graph G has tree-median-dimension k.*

In section 3 we focus on the chromatic number of graphs of bounded chordality.

For a positive integer k we denote by $[k]$ the set of integers $\{1, \ldots, k\}$. A *k-coloring* of G is a function $f\colon V(G) \to [k]$ such that for every $i \in [k]$ we have that $f^{-1}(i)$ is a stable set. A graph is *k-colorable* if it admits a k-coloring, and the *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum integer k, for which G is k-colorable.

It is immediate that for every graph G we have $\chi(G) \geqslant \omega(G)$, and it is easy to see that there are graphs G for which we have $\chi(G) > \omega(G)$ (for example odd cycles). Moreover, the gap between the chromatic number and the clique number can be arbitrarily large. Indeed, Tutte [11, 12] first proved in the 1940s that there exist triangle-free graphs of arbitrarily large chromatic number (for other such constructions see also [6, 26, 37]). Thus, in general, the chromatic number is not upper-bounded by a function of the clique number.

A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. Berge [4] proved in 1960 that chordal graphs are perfect. What can we say for the connection between χ and ω for graphs of bounded chordality?

In his seminal paper “Problems from the world surrounding perfect graphs”, Gyárfás [19] introduced the χ-bounded graph classes as “natural extensions of the world of perfect graphs”. We say that a hereditary class $\mathcal{A}$ is *χ-bounded* if there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that for every graph $G \in \mathcal{A}$, we have $\chi(G) \leqslant f(\omega(G))$. Such a function f is called a *χ-bounding function* for $\mathcal{A}$. For more on χ-boundedness we refer the readers to the surveys of Scott and Seymour [31], and Scott [30]. The examples of triangle-free graphs of arbitrarily large chromatic number that we mention above imply that the class of all graphs is not χ-bounded.

A natural direction of research on χ-boundedness is to consider operations that we can apply among graphs of different classes in order to obtain new classes of graphs, and study (from the perspective of χ-boundedness) graph classes which are obtained via this way from χ-bounded classes.

Gyárfás [19, Section 5] considered graph-intersections and graph-unions of χ-bounded graph classes from the perspective of χ-boundedness. Graph-unions of χ-bounded graph classes are χ-bounded[2]. The situation with intersections of graphs is different. We refer the interested reader to [8] where Chaniotis, Koerts, and Spirkl, study further the interplay between graph-intersection and χ-boundedness, and to [2] where Adenwalla, Braunfeld, Sylvester, and Zamaraev considered this topic in the context of their broader study on Boolean combinations of graphs.

[2] It is easy to observe that for any two graphs G_1 and G_2, we have $\chi(G_1 \cup G_2) \leqslant \chi(G_1)\chi(G_2)$ and that $\omega(G_1 \cup G_2) \geqslant \max\{\omega(G_1), \omega(G_2)\}$. Thus, if for each $i \in [k]$ we have that f_i is a χ-bounding function for a class $\mathcal{G}_i$, then $f := \prod_{i\in[k]} f_i$ is a χ-bounding function for the class $\biguplus_{i\in[k]}\mathcal{G}_i$.

Since interval graphs are chordal, it follows that they are perfect as well. Following [28], we define the *boxicity* of G to be the minimum integer k such that G is isomorphic to the intersection graph of a family of axis-aligned boxes in $\mathbb{R}^k$. We denote the boxicity of a graph G by $\mathsf{box}(G)$. It easy to see that the boxicity k of a graph is equal to its intersection dimension with respect to the class of interval graphs.

In 1965, in his Ph.D. thesis [6] Burling introduced a sequence $\{\mathcal{B}_k\}_{k \geqslant 1}$ of families of axis-aligned boxes in $\mathbb{R}^3$ such that for each k the intersection graph of $\mathcal{B}_k$ is triangle-free and has chromatic number at least k. Thus, for every $k \geqslant 3$ the class of all graphs of boxicity at most k, that is, the class $\dot\cap_{i \in [k]} \mathcal{I}$, is not χ-bounded. Hence, for every $k \geqslant 3$ the class of graphs of chordality at most k is not χ-bounded.

What about the class $\mathcal{C} \mathbin{\dot\cap} \mathcal{C}$? Asplund and Grünbaum [3], in one of the first results which provides an upper bound of the chromatic number in terms of the clique number for a class of graphs, proved in 1960 that every intersection graph of axis-aligned rectangles in the plane with clique number ω is $\mathcal{O}(\omega^2)$-colorable. Hence the class $\mathcal{I} \mathbin{\dot\cap} \mathcal{I}$ is χ-bounded (see also [7] for a better χ-bounding function).

Since the class $\mathcal{I} \mathbin{\dot\cap} \mathcal{I}$ is χ-bounded it is natural to ask whether any proper superclasses of this class are χ-bounded as well. Gyárfás, asked the following question:

Problem 2 (Gyárfás, [19, Problem 5.7])**.** Is the class $\mathcal{C} \mathbin{\dot\cap} \mathcal{C}$ χ-bounded? In particular, is $\mathcal{C} \mathbin{\dot\cap} \mathcal{I}$ χ-bounded?

In subsection 3.1 we discuss a result of Felsner, Joret, Micek, Trotter and Wiechert [14] which implies that Burling graphs are contained in $\mathcal{C} \mathbin{\dot\cap} \mathcal{I}$, and thus that the answer to Gyárfás' question is negative.

In the rest of section 3 we consider two families of subclasses of the class $\mathcal{C} \mathbin{\dot\cap} \mathcal{C}$, which we prove are χ-bounded. In subsection 3.2 we prove the following:

Theorem 3. *Let k_1 and k_2 be positive integers, and let G_1 and G_2 be chordal graphs such that for each $i \in [2]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \leqslant k_i$. If G is a graph such that $G = G_1 \cap G_2$, then G is $\mathcal{O}(\omega(G) \log(\omega(G)))(k_1 + 1)(k_2 + 2)$-colorable.*

We remark that each of the classes which satisfies the assumptions of Theorem 3 is a proper superclass of $\mathcal{I} \mathbin{\dot\cap} \mathcal{I}$.

Let u and v be two vertices of a graph G. Then their *distance*, which we denote by $d_G(u, v)$, is the length of a shortest (u, v)-path in G; we will often omit the subscript G from $d_G(u, v)$ unless there is ambiguity. A *rooted tree* is a tree T with one fixed vertex $r \in V(T)$ which we call the *root* of T. The *height* of a rooted tree T with root r is $h(T, r) := \max\{d(r, t) : t \in V(T)\}$. The *radius* of a tree T, which we denote by $\mathsf{rad}(T)$, is the nonnegative integer $\min\{h(T, r) : r \in V(T)\}$. In subsection 3.3 we prove the following:

Theorem 4. *Let k be a positive integer, and let G_1 and G_2 be chordal graphs such that the graph G_1 has a representation (T_1, β_1) where $\mathsf{rad}(T_1) \leqslant k$. If G is a graph such that $G = G_1 \cap G_2$, then $\chi(G) \leqslant k \cdot \omega(G)$.*

In section 4, we consider the recognition problem for the class of graphs of chordality at most k. The k-Chordality Problem is the following: Given a graph G as an input, decide whether or not $\mathsf{chor}(G) \leqslant k$. We prove the following:

Theorem 5. *For every $k \geqslant 3$, the k-CHORDALITY PROBLEM is NP-complete.*

Since chordal graphs can be recognized efficiently (see, for example, [16, 24]), the only open case in order to fully classify the complexity of the k-CHORDALITY PROBLEM, is the case $k = 2$. Recently Abueida, Busch and Sritharan [1], independently of our work, examined the computational complexity of the k-CHORDALITY PROBLEM, and of the recognition problems for several subclasses of the class of graphs of chordality at most two. In [1] Abueida, Busch and Sritharan discuss how Theorem 5 follows from a result of Yannakakis [36]. However they did not resolve whether the 2-CHORDALITY PROBLEM is NP-complete, and thus this remains an open question.

2 A characterization of the graphs of chordality k

In this section we prove Theorem 1 which provides different characterizations of graphs of bounded chordality. We also point out how our proof of Theorem 1 can be adapted (in a straight-forward way) to provided analogous characterizations for graphs of bounded boxicity.

The key notion that we use for the proof of Theorem 1 is that of the *tree-median-dimension* of a graph, introduced by Stavropoulos [32], which we prove is equivalent to chordality. We introduce the tree-median-dimension of a graph in subsection 2.1. We first need some definitions.

Let G be a graph. For $u, v \in V(G)$, a (u, v)-*geodesic* is a shortest (u, v)-path. We denote by $I_G(u, v)$ the set of all vertices of G which lie in a (u, v)-geodesic, that is, $I_G(u, v) := \{x \in V(G) \mid d(u, v) = d(u, x) + d(x, v)\}$. We will often omit the subscript G from $I_G(u, v)$ unless there is ambiguity. Given three distinct vertices $u, v, w \in V(G)$ we denote by $I(u, v, w)$ the set $I(u, v) \cap I(u, w) \cap I(v, w)$.

A graph M is a *median graph* if it is connected and for every choice of three distinct vertices $u, v, w \in V(M)$, there exists a vertex x with the property that $I(u, v, w) = \{x\}$. In this case, the vertex x is called the *median* of u, v, w. For three distinct vertices $u, v, w \in V(G)$, we denote their median vertex by $\mathsf{median}(u, v, w)$. It is immediate that trees are median graphs.

Given two graphs G and H, their *Cartesian product* is the graph $G \Box H := (V, E)$ where $V := V(G) \times V(H)$ and $\{(v_1, h_1), (v_2, h_2)\} \in E$ if and only if $v_1 = v_2$ and $h_1 h_2 \in E(H)$, or $h_1 = h_2$ and $v_1 v_2 \in E(G)$. A graph is G *isometrically embeddable* into a graph H if there exists a map $\phi : V(G) \to V(H)$ such that for every $u, v \in V(G)$ we have $d_G(u, v) = d_H(\phi(u), \phi(v))$. In this case we write $G \hookrightarrow H$ and we call the map ϕ an *isometric embedding*. The *tree-dimension (respectively* path-dimension*)* of a graph G, denoted by $\mathsf{tdim}(G)$ (respectively $\mathsf{pdim}(G)$) is the minimum k such that G has an isometric embedding into the Cartesian product of k trees (respectively paths) if such an embedding exists, and infinite otherwise.

For every positive integer n the *hypercube* Q_n is a graph isomorphic to the Cartesian product of n copies of K_2. A *partial cube* is a graph which is isometrically embeddable into a hypercube. Median graphs form a proper subclass of partial cubes (see, for example, [27, Theorem 5.75]). Hence, both the tree-dimension and the path-dimension of every median graph are finite. We observe that for any graph G, we have $\mathsf{tdim}(G) \leqslant \mathsf{pdim}(G)$. We note that the class of median graphs of tree-dimension one is exactly the class of trees.

We say that a set $S \subseteq V(G)$ is *geodesically convex* or simply *convex* if for every $u, v \in S$, we have $I(u, v) \subseteq S$. We remark that if G is a connected graph and $S \subseteq V(G)$ is a convex set, then $G[S]$ is connected. A subgraph H of G is a *convex subgraph* if the set $V(H)$ is convex. We note that the convex subgraphs of a tree are exactly its subtrees.

2.1 The tree-median-dimension of a graph

In his Ph.D. thesis, Stavropoulos [32] introduced median-decompositions of graphs, and a variant of those, k-median-decompositions. We find it more convenient for the context of this paper to use the term k-tree-median-decomposition for the notion of k-median-decomposition.

Let k be a positive integer. We say that a graph H has *the property* $\mathcal{M}_k$ if H is the intersection graph of a family of convex subgraphs of a median graph of tree-dimension at most k. By the characterization of chordal graphs that we discussed in section 1 as intersection graphs of subtrees of a tree, we have that property $\mathcal{M}_1$ is the property of being a chordal graph. A *representation* of a graph G with the property $\mathcal{M}_k$ is a pair (M, γ), where M is a median graph, and $\gamma : V(M) \to 2^{V(G)}$ is a function such that for every $v \in V(G)$, we have that $M[\{x \in V(M) : v \in \gamma(x)\}]$ is a convex subgraph of M, and G is isomorphic to the intersection graph of the family $\{\{x \in V(M) : v \in \gamma(x)\} : v \in V(G)\}$.

Let G be a graph. A *k-tree-median-completion* of G is a supergraph H of G such that $V(H) = V(G)$, and H has the property $\mathcal{M}_k$. As we discussed in section 1, every graph has a chordal completion (the complete supergraph on the same vertex set), thus every graph has a 1-tree-median-completion, and hence every graph has a k-tree-median-completion. A *k-tree-median-decomposition* of G is a representation (M, γ) of a k-tree-median-completion H of G. For every $x \in V(M)$, we call the set $\gamma(x)$ the *bag* of x. We say that (M, γ) is a *complete k-tree-median-decomposition* of G if every bag of M is a clique of G. Following Stavropoulos [32], a *median-decomposition* of a graph G is a k-tree-median-decomposition for some k. Stavropoulos proved the following:

Theorem 6 (Stavropoulos, [32, Theorem 5.12])**.** *Every graph G has a complete median-decomposition.*

We define the *tree-median-dimension* of a graph G, denoted by $\mathsf{tmd}(G)$, as the minimum integer k such that G has a complete k-tree-median-decomposition. By Theorem 6 it follows that the tree-median-dimension is well defined.

By the following theorem we have that every 1-tree-median-decomposition is a tree-decomposition, and vice versa.

Theorem 7 (Buneman [5], Gavril [15], and Walter [34, 33])**.** *A graph G is the intersection graph of subtrees of a tree if and only if G is chordal.*

We omit the proof of the following proposition as it follows immediately from the corresponding definitions.

Proposition 8. *Let G be a graph and k be a positive integer. Then G has a complete k-tree-median-decomposition if and only if G has the property $\mathcal{M}_k$.*

Corollary 9 and Corollary 10 follow immediately by Theorem 7 and Proposition 8.

Corollary 9. *A graph G has a complete tree-decomposition if and only if G is chordal.*

Corollary 10. *Let G be a graph. Then $\mathsf{tmd}(G) = k$ if and only if k is the minimum integer for which G has the property $\mathcal{M}_k$.*

2.2 A characterization of the graphs of chordality k

The main ingredient that we need for the proof of Theorem 1 is the following:

Theorem 11. *If G is a graph, then the tree-median-dimension of G is equal to its chordality.*

We begin with the following easy observation about chordality.

Lemma 12. *Let G be a graph. Then the chordality of G is equal to the minimum size of a non-edge-separating family of tree-decompositions of G.*

Proof of Lemma 12. Let m be the minimum size of a non-edge-separating family of tree-decompositions of G.

Let $k := \mathsf{chor}(G)$, and let $G_1, \dots, G_k$ be k chordal graphs such that $G = \bigcap_{i \in [k]} G_i$. For each $i \in [k]$ let (T_i, β_i) be the tree-decomposition of G which is obtained from the chordal completion G_i, and let $\mathcal{T} := \{(T_i, \beta_i) : i \in [k]\}$. Let $\{u, v\}$ be a non-edge of G. Then there exists i such that $\{u, v\}$ is a non-edge of G_i. Since the bags of (T_i, β_i) are cliques in G_i, it follows that (T_i, β_i) separates $\{u, v\}$. Thus $m \leqslant \mathsf{chor}(G)$.

Let $\mathcal{T} = \{(T_i, \beta_i)\}_{i \in [m]}$ be a non-edge-separating family of tree-decompositions of G such that $|\mathcal{T}|$ is minimized. For each $i \in [k]$, let G_i be the chordal completion of G which corresponds to the tree-decomposition (T_i, β_i). Let $\{u, v\}$ be a non-edge of G and let (T_i, β_i) be the tree-decomposition of G which separates $\{u, v\}$. Then $\{u, v\} \notin E(G_i)$. Thus, $G = \bigcap_{i \in [k]} G_i$, and $\mathsf{chor}(G) \leqslant m$. $\square$

In light of Lemma 12, in order to prove Theorem 11, it suffices to prove that the tree-median-dimension of a graph G is equal to the minimum size of a non-edge-separating family of tree-decompositions of G. We begin with the following lemma:

Lemma 13. *Let M be a median graph, and let $T_1, \dots, T_k$ be trees such that there exists an isometric embedding $\phi \colon V(M) \to T_1 \square \cdots \square T_k$. Let $a, b \in V(M)$, let $\pi_i : V(T_1 \square \cdots \square T_k) \to V(T_i)$ be the projection to the i-th coordinate, and let $Q = x_1, \dots, x_l$ be a shortest (a, b)-path in M. Then the following hold:*

1. *Let $i \in [k]$. If $\pi_i(\phi(a)) = \pi_i(\phi(b)) =: t_i$, then for every $j \in [l]$ we have $\pi_i(\phi(x_j)) = t_i$.*

2. *For every $i \in [k]$, we have that $W_i := \pi_i(\phi(x_1)), \dots, \pi_i(\phi(x_l))$ is a sequence of vertices of T_i which contains exactly the vertices of the $(\pi_i(\phi(x_1)), \pi_i(\phi(x_l)))$-path=: P_i in T_i, and these vertices appear in W_i, possibly with repetitions, in the same order as in P_i.*

Proof Sketch of Lemma 13. Let $\phi(a) = (t_1, \dots, t_k)$, $\phi(b) = (t'_1, \dots, t'_k)$, and let P be a shortest $(\phi(a), \phi(b))$-path in $T_1 \square \cdots \square T_k$. The main observation is the following:

$$d_{T_1 \square \cdots \square T_k}(\phi(a), \phi(b)) = \sum_{i}^{k} d_{T_i}(t_i, t'_i).$$

Intuitively, by the definition of the Cartesian product any two adjacent vertices in a path correspond to exactly one "move" in exactly one of the factors of the product. Now in a shortest $(\phi(a), \phi(b))$-path the goal is to "transform" each coordinate of $\phi(a) = (t_1, \dots, t_k)$ to the corresponding coordinate of $\phi(b) = (t'_1, \dots, t'_k)$ in a way that minimizes the number of "moves" in each factor, and thus minimizes the sum of the all the "moves", that is, the length of the path.

The first statement of our lemma says that if for some $i \in [k]$ we have $\pi_i(\phi(a)) = \pi_i(\phi(b))$, then a shortest path does not waste "moves" on unnecessary changes to the i-th coordinate of its vertices. The second statement follows from the facts that each factor in our Cartesian product is a tree, and that in each tree there is a unique path between two vertices. Thus the changes in the i-th coordinate of the vertices of P follow exactly the unique $(\pi_i(\phi(x_1)), \pi_i(\phi(x_l)))$-path in T_i, maybe with some pauses (when an edge of P corresponds to a change in a coordinate $j \neq i$). □

Lemma 14. *Let G be a graph. If $\mathcal{T}$ is a non-edge-separating family of tree-decompositions of G, then the tree-median-dimension of G is at most $|\mathcal{T}|$.*

Proof of Lemma 14. Let $\mathcal{T} := \{(T_i, \beta_i)\}_{i \in [k]}$ be a family of k tree-decompositions of G as in the statement of the lemma. We construct a complete i-tree-median-decomposition of G, with $i \leqslant k$. Let $M := T_1 \square \dots \square T_k$. Then M is a median graph of tree-dimension at most k. For every $i \in [k]$, let $\pi_i : V(T_1 \square \cdots \square T_k) \to V(T_i)$ be the projection to the i-th coordinate. Let $\gamma : V(M) \to 2^{V(G)}$ defined as follows: for every $x \in V(M)$, we have $\gamma(x) := \cap_{i \in [k]} \beta_i(\pi_i(x))$.

We claim that for every $v \in V(G)$, the subgraph $M[\{x \in V(M) : v \in \gamma(x)\}]$ of M is convex. Let $v \in V(G)$. Since (T_i, β_i) is a tree-decomposition it follows that $T_i[\{t \in V(T_i) : v \in \beta_i(t)]$ is a connected (and thus a convex) subgraph of T. Note that

$$M[\{x \in V(M) : v \in \gamma(x)\}] = \square_{i \in [k]} T_i[\{t \in V(T_i) : v \in \beta_i(t)\}].$$

Our claim now follows by the fact that the Cartesian product of convex graphs is a convex graph and from Lemma 13.

We claim that for every $x \in V(M)$, the bag $\gamma(x)$ is a clique. Indeed, this follows from the definition of γ and the fact that for every non-edge of G there exists $i \in [k]$ such that no bag of the tree-decomposition (T_i, β_i) contains both u and v.

By the above it follows that (M, γ) is a complete i-tree-median-decomposition of G which witnesses that $\mathsf{tmd}(G) \leqslant k$. □

In order to complete the proof of Theorem 11, it remains to prove that the minimum size of a non-edge-separating family of tree-decompositions of a graph G is upper-bounded by the tree-median-dimension of G. To this end we need some preliminary results. We begin with the statement of a theorem of Stavropoulos [32] which states that given a k-tree-median-decomposition of a graph G, one can obtain a family of k tree-decompositions of G which satisfy certain nice properties. We then show that if we apply this theorem to a complete k-tree-median-decomposition, then the family of k tree-decompositions that we get is non-edge-separating.

Theorem 15 (Stavropoulos, [32, Lemma 6.1, Theorem 6.7])**.** *Let G be a graph, and let (M, γ) be a k-tree-median-decomposition of G. Then there exists a family $\mathcal{T} = \{(T_i, \beta_i)\}_{i\in[k]}$ of k tree-decompositions of G such that:*

1. *There exists an isometric embedding ϕ of M to the graph $T_1 \square \cdots \square T_k$.*
2. *For every $i \in [k]$ and for every $t \in V(T_i)$, we have $\phi(V(M)) \cap \pi_i^{-1}(t) \neq \emptyset$, where $\pi_i : V(T_1 \square \cdots \square T_k) \to V(T_j)$ is the projection to the i-th coordinate.*
3. *For every $x \in V(M)$, we have $\gamma(x) = \bigcap_{\pi_i(\phi(x)),i\in[k]} \beta_i(\pi_i(\phi(x)))$.*
4. *For every $i \in [k]$, and for every $t \in V(T_i)$, we have $\beta_i(t) = \bigcup_{\{x\in V(M):\pi_i(\phi(x))=t\}} \gamma(x)$.*

Given a set X, we say that a family $\mathcal{X} := \{X_i\}_{i\in I}$ of subsets of X satisfies the *Helly property* if for every $I' \subseteq I$ the following holds: if $X_i \cap X_j \neq \emptyset$ for all $i, j \in I'$, then we have that $\bigcap_{i\in I'} X_i \neq \emptyset$. The following is a folklore (see, for example, [18, Proposition 4.7]):

Proposition 16. *Every family of subtrees of a tree satisfies the Helly property.*

Lemma 17. *Let G be a graph, let (T, β) be a tree-decomposition of G, and let $\{u, v\}$ be a non-edge of G such that (T, β) does not separate $\{u, v\}$. Let $t_1, t_2 \in V(T)$ be such that $u \in \beta(t_1)$ and $v \in \beta(t_2)$, and let P be the (t_1, t_2)-path in T. Then, there exists $p \in V(P)$ such that $\{u, v\} \subseteq \beta(p)$.*

Proof of Theorem 17. Since (T, β) does not separate $\{u, v\}$, we have that $\{t \in V(T) : u \in \beta(t)\} \cap \{t \in V(T) : v \in \beta(t)\} \neq \emptyset$. Moreover, since $u \in \beta(t_1)$ and $v \in \beta(t_2)$, we have that $\{t \in V(T) : u \in \beta(t)\} \cap V(P) \neq \emptyset$ and $\{t \in V(T) : v \in \beta(t)\} \cap V(P) \neq \emptyset$. Hence, by Theorem 16, it follows that $\{t \in V(T) : u \in \beta(t)\} \cap \{t \in V(T) : v \in \beta(t)\} \cap V(P) \neq \emptyset$. □

We are now ready to prove that the minimum size of a non-edge-separating family of tree-decompositions of a graph G is upper bounded by the tree-median-dimension of G.

Lemma 18. *Let G be a graph, and let (M, γ) be a complete k-tree-median-decomposition of G. Let $\mathcal{T} = \{(T_i, \beta_i)\}_{i\in[k]}$ be a family of k tree-decompositions of G which satisfies the conditions of Theorem 15. Then for every non-edge e of G, there exists $i \in [k]$ such that (T_i, β_i) separates e.*

Proof of Lemma 18. Let us suppose towards a contradiction that the lemma does not hold. Let $\{u, v\} \in \binom{V(G)}{2} \setminus E(G)$ be such that no tree-decomposition in $\mathcal{T}$ separates

$\{u, v\}$, and let ϕ be an isometric embedding of M to the graph $T_1 \square \cdots \square T_k$, as in the statement of Theorem 15. In what follows we show that there exists a vertex of M whose bag, in (M, γ), contains both the vertices u and v, contradicting the fact that every bag of M is a clique of G and that $\{u, v\} \notin E(G)$.

Claim 19. *For every $j \in [k]$ there exist, not necessarily distinct $a, b \in V(M)$ such that the following hold:*

- *$u \in \gamma(a)$ and $v \in \gamma(b)$; and*
- *for all $i \in [j]$, we have $\pi_i(\phi(a)) = \pi_i(\phi(b))$.*

Proof of Theorem 19. We prove the claim by induction on j. For the basis of the induction: Since (T_1, β_1) does not separate the non-edge $\{u, v\}$, it follows that there exists $t_1 \in V(T_1)$ such that $\{u, v\} \subseteq \beta_1(t_1)$. Thus, by Theorem 15 (4), there exist not necessarily distinct $a, b \in V(M)$ such that $\pi_1(\phi(a)) = \pi_1(\phi(b)) = t_1$, $u \in \gamma(a)$, and $v \in \gamma(b)$.

Let $j \geqslant 1$, and let us suppose that there exist $a, b \in V(M)$ as in the statement of the claim. For each $i \in [j]$, let $t_i := \pi_i(\phi(a)) = \pi_i(\phi(b)) \in V(T_i)$. Since $u \in \gamma(a)$ we have that $u \in \beta_{j+1}(\pi_{j+1}(\phi(a)))$. Since $v \in \gamma(b)$ we have that $v \in \beta_{j+1}(\pi_{j+1}(\phi(b)))$. Let P be the $\Big(\pi_{j+1}(\phi(a)), \pi_{j+1}(\phi(b))\Big)$-path in T_{j+1}. Since $u \in \beta_{j+1}(\pi_{j+1}(\phi(a)))$, $v \in \beta_{j+1}(\pi_{j+1}(\phi(b)))$, and the tree-decomposition (T_{j+1}, β_{j+1}) does not separate the non-edge $\{u, v\}$, by Theorem 17, it follows that there exists $t \in V(P) \subseteq V(T_{j+1})$ such that $\{u, v\} \subseteq \beta_{j+1}(t)$. Let t_{j+1} be such a vertex.

Let Q be a shortest (a, b)-path in M. We claim that there exists $z \in V(Q)$ such that for each $i \in [j+1]$ we have $\pi_i(\phi(z)) = t_i$. Indeed, by (1) of Lemma 13, we know that for every vertex $q \in V(Q)$ and for every $i \in [j]$ we have $\pi_i(\phi(q)) = t_i$. Since t_{j+1} lies in P, by (2) of Lemma 13, there exists $z \in V(Q)$ such that $\pi_{j+1}(\phi(z)) = t_{j+1}$. Let z be such a vertex. Then z satisfies our claim.

Since $\{u, v\} \subseteq \beta_{j+1}(t_{j+1})$, by Theorem 15 (4), it follows that there exist not necessarily distinct vertices $x, y \in V(M)$ such that $u \in \gamma(x), v \in \gamma(y)$ and $\pi_{j+1}(\phi(x)) = \pi_{j+1}(\phi(y)) = t_{j+1}$.

Let $a' := \mathsf{median}(a, x, z)$. Then, since a' lies in a shortest (a, x)-path of M and in a shortest (x, z)-path of M, by Lemma 13, we have that $\pi_i(\phi(a')) = t_i$ for all $i \in [j+1]$. Let $U := \{w \in V(M) : u \in \gamma(w)\}$, and recall that U is convex. Since $a, z \in U$ and a' lies in a shortest (a, z)-path of M, it follows that $a' \in U$. In particular, we have that $u \in \gamma(a')$.

Let $b' := \mathsf{median}(b, y, z)$. Similarly, with a', we have that $\pi_i(\phi(b')) = t_i$ for all $i \in [j+1]$, and that $v \in \gamma(a')$. Thus, the vertices a' and b' of M witness that the statement of the claim holds for $j+1$. This concludes the induction, and thus the proof of Theorem 19. ■

Let $a, b \in V(M)$ be such that:

- $u \in \gamma(a)$ and $v \in \gamma(b)$; and
- for all $i \in [k]$, we have $\pi_i(\phi(a)) = \pi_i(\phi(b))$.

Then, by Theorem 15, we have that $\gamma(a) = \gamma(b)$. In particular, $\{u, v\} \subseteq \gamma(a)$, and, since $\gamma(a)$ is a clique of G, we have that $\{u, v\} \in E(G)$, which is a contradiction. This concludes the proof of Lemma 18. □

Corollary 20 follows immediately from Lemma 18 and Lemma 14.

Corollary 20. *Let G be a graph. Then the tree-median-dimension of G is equal to the minimum size of a non-edge-separating family of tree-decompositions of G.*

Now Theorem 11, which states that the tree-median-dimension of a graph is equal to its chordality, follows immediately by Lemma 12 and Corollary 20.

Theorem 1 is an immediate corollary of Theorem 11, Proposition 8, Corollary 10 and Lemma 12. We remark that Theorem 1 generalizes Theorem 7 and Corollary 9.

The notion of k-path-median-decomposition can be defined similarly with that of k-tree-median-decomposition, by considering completions which are intersection graphs of convex subgraphs of median graphs of path-median-dimension k. By modifying the proofs of this section in a trivial way we can derive the following characterizations of boxicity.

Theorem 21. *Let G be a graph and k be a positive integer. Then the following are equivalent:*

1. *The graph G has boxicity k.*
2. *The minimum size of a non-edge-separating family of path-decompositions of G is k.*
3. *k is the minimum integer such that the graph G is the intersection graph of a family of convex subgraphs of the Cartesian product of k paths.*
4. *k is the minimum integer such that the graph G is the intersection graph of a family of convex subgraphs of a median graph of path-dimension k.*
5. *The graph G has path-median-dimension k.*

3 Chordality and χ-boundedness

We study classes of graphs of bounded chordality from the perspective of χ-boundedness.

3.1 The class $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{I}$ is not χ-bounded

In [13], Dujmovic, Joret, Morin, Norin, and Wood studied graphs which have two tree-decompositions such that "each bag of the first decomposition has a bounded intersection with each bag of the second decomposition". Following [13], we say that two tree-decompositions (T_1, β_1) and (T_2, β_2) of a graph G are *k-orthogonal* if for every $t_1 \in T_1$ and $t_2 \in T_2$, we have $|\beta_1(t_1) \cap \beta_2(t)| \leqslant k$. Dujmovic, Joret, Morin, Norin, and Wood [13] proved that this is the case for graphs which belong to a proper minor-closed class, for string graphs with a linear number of crossings in a fixed surface, and for graphs with linear crossing number in a fixed surface. In a more recent work Liu, Norin, and Wood [23] proved that for graphs which exclude a fixed graph as odd-minor there exists an integer k such that these graphs have a tree-decomposition and a path-decomposition which are k-orthogonal. Here we are interested in connections of this concept with the concept of χ-boundedness.

Observation 22 (Dujmović, Joret, Morin, Norin, and Wood [13, Observation 27])**.** *Let G be a graph and k be a positive integer. Then G has two k-orthogonal path-decompositions if and only if G is a subgraph of a graph H such that H has boxicity at most two, and $\omega(H) \leqslant k$.*

Lemma 23. *Let G be a graph and k be a positive integer. Then the following hold:*

1. *The graph G has two k-orthogonal tree-decompositions if and only if G is a subgraph of a graph H such that H has chordality at most two, and $\omega(H) \leqslant k$.*
2. *The graph G has a tree-decomposition and a path-decomposition which are k-orthogonal if and only if G is a subgraph of a graph H such that $H \in \mathcal{C} \Cap \mathcal{I}$, and $\omega(H) \leqslant k$.*

Proof of Lemma 23. Follows immediately by the corresponding definitions and the facts that every bag of a tree-decomposition is a clique of the corresponding chordal completion, and that every clique of a chordal completion is contained in a bag of the corresponding tree-decomposition. □

The following is an immediate corollary of Theorem 22 and Lemma 23.

Proposition 24. *Let $\mathcal{C}$ be the class of chordal graphs and $\mathcal{I}$ be the class of interval graphs. The following hold:*

1. *The class $\mathcal{I} \Cap \mathcal{I}$ is χ-bounded if and only if there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that for every graph G which has two k-orthogonal path-decompositions, we have $\chi(G) \leqslant f(k)$.*
2. *The class $\mathcal{C} \Cap \mathcal{C}$ is χ-bounded if and only if there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that for every graph G which has two k-orthogonal tree-decompositions, we have $\chi(G) \leqslant f(k)$.*
3. *The class $\mathcal{C} \Cap \mathcal{I}$ is χ-bounded if and only if there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that for every graph G which has a tree-decomposition and a path-decomposition which are k-orthogonal, we have $\chi(G) \leqslant f(k)$.*

The authors of [13] posed the following question, for which they conjectured a positive answer.

Problem 25 (Dujmović, Joret, Morin, Norin, and Wood, [13, Open Problem 3])**.** Is there a function f such that every graph G that has two k-orthogonal tree-decompositions is $f(k)$-colorable?

By Proposition 24, it follows that the above question is equivalent to the first part of the question of Gyárfás that we metioned in the Introduction (Problem 2), which asks whether the class of all graphs of chordality at most two is χ-bounded.

In [14] Felsner, Joret, Micek, Trotter and Wiechert, answered Problem 25 in the negative, and in particular they answered Gyárfás's question (Problem 2), in the negative.

Felsner, Joret, Micek, Trotter and Wiechert [14] proved the following, which answers in the negative both the questions in Problem 2 and Problem 25.

Theorem 26 (Felsner, Joret, Micek, Trotter and Wiechert, [14, Theorem 2])**.** *For every positive integer k, there is a graph with chromatic number at least k which has a tree-decomposition (T, β) and a path-decomposition (P, γ), which are 2-orthogonal. That is, for every $t \in V(T)$ and for every $p \in V(P)$, we have $|\beta(t) \cap \gamma(p)| \leqslant 2$.*

The following is an immediate corollary of Lemma 23 and Theorem 26.

Corollary 27. *For every positive integer k, there exist a graph $H_k \in \mathcal{C} \mathbin{\dot{\cap}} \mathcal{I}$ such that H_k is triangle-free and has chromatic number at least k.*

Corollary 28. *The class $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{I}$ is not χ-bounded. In particular, since $\mathcal{I} \subseteq \mathcal{C}$, it follows that the class of all the graphs of chordality at most two is not χ-bounded.*

3.2 Subclasses of $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{C}$: When each chordal graph has a representation tree of bounded path-width

In subsection 3.1, we saw that the class $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{I}$ is not χ-bounded. From the characterization of chordal (respectively interval) graphs as intersection graphs of subtrees (respectively subpaths) of trees (respectively paths) that we presented in the Introduction, it follows that $\mathcal{I} \mathbin{\dot{\cap}} \mathcal{I}$ is the subclass of $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{C}$ in which each of the two chordal graphs in the intersection has a representation tree which is a path.

In this subsection we consider the family of subclasses of $\mathcal{C} \mathbin{\dot{\cap}} \mathcal{C}$ (and superclasses of $\mathcal{I} \mathbin{\dot{\cap}} \mathcal{I}$) in which each of the two chordal graphs in the intersection has a representation tree of bounded path-width. We prove that these classes are χ-bounded.

Theorem 29. *Let k_1 and k_2 be positive integers, and let G_1 and G_2 be chordal graphs such that for each $i \in [2]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \leqslant k_i$. If G is a graph such that $G = G_1 \cap G_2$, then G is $\mathcal{O}(\omega(G) \log(\omega(G)))(k_1 + 1)(k_2 + 1)$-colorable.*

The main step towards our proof of Theorem 29 is to prove that the vertex set of a graph G as in the statement of Theorem 29 can be partitioned into a constant number of sets so that each of these sets induces a graph of boxicity at most two. Then we use the fact that the class $\mathcal{I} \mathbin{\dot{\cap}} \mathcal{I}$ is χ-bounded and we color each of these induced subgraphs with a different palette of colors.

Lemma 30. *Let k_1 and k_2 be positive integers, and let G_1 and G_2 be chordal graphs such that for each $i \in [2]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \leqslant k_i$. If G is a graph such that $G = G_1 \cap G_2$, then there exists a partition $\mathcal{P}$ of $V(G)$ such that $|\mathcal{P}| \leqslant (k_1 + 1)(k_2 + 1)$ and for every $V \in \mathcal{P}$, the graph $G[V]$ has boxicity at most two.*

In 2021, Chalermsook and Walczak [7] provided an improvement on the upper bound of Asplund and Grünbaum [3] for the chromatic number of graphs of boxicity at most two.

Theorem 31 (Chalermsook and Walczak, [7]). *Every family of axis-parallel rectangles in the plane with clique number ω is $\mathcal{O}(\omega \log(\omega))$-colorable, and an $\mathcal{O}(\omega \log(\omega))$-coloring of it can be computed in polynomial time.*

Since Theorem 29 follows immediately by Lemma 30 and Theorem 31, in order to prove Theorem 29 it remains to prove Lemma 30. The main observation that we need is that if a graph has path-width at most k, then it can decomposed into a family of $k+1$ disjoint subgraphs, each of which is a disjoint union of induced paths.

We first need a result about tree-decompositions. Let G be a graph, and let $A, B, X \subseteq V(G)$. We say that X *separates A from B in G* if for every (A, B)-path P in G we have $V(P) \cap X \neq \emptyset$.

Lemma 32 (Robertson and Seymour [29, (2.4)]). *Let G be a graph, (T, β) be a tree-decomposition of G, let $\{t_1, t_2\}$ be an edge of T. If T_1 and T_2 are the components of $T \setminus \{t_1, t_2\}$, where $t_1 \in V(T_1)$ and $t_2 \in V(T_2)$, then $\beta(t_1) \cap \beta(t_2)$ separates $V_1 := \bigcup_{t \in V(T_1)} \beta(t)$ from $V_2 := \bigcup_{t \in V(T_2)} \beta(t)$ in G.*

Lemma 33. *Let G be a connected graph and let k be a positive integer. If G has path-width at most k, then there exists an induced path Q which is a subgraph of G such that $G \setminus V(Q)$ has path-width at most $k-1$.*

Proof of Lemma 33. Consider a path-decomposition (P, β) of G which realizes its path-width. Let $p_1, \ldots, p_l$ be the elements of $V(P)$ enumerated in the order that they appear in P. Let $v_1 \in V(G) \cap \beta(p_1)$ and $v_l \in V(G) \cap \beta(p_l)$, and let Q be an induced (v_1, v_l)-path in G. We define the function $\beta' : V(P) \to 2^{V(G)}$ as follows: for every $p \in V(P)$ we have $\beta'(p) := \beta(p) \setminus V(Q)$. Then (P, β') is a path-decomposition of the graph $G \setminus V(Q)$. Moreover, by Lemma 32, it follows that for each $i \in [l]$ we have $V(Q) \cap \beta(p_i) \neq \emptyset$. Thus, the width of (P, β') is at most $k-1$. □

Corollary 34. *Let k be a positive integer. If G is a graph of path-width at most k, then there exist (possibly null) induced subgraphs $P_1, \ldots, P_{k+1}$ of G such that the following hold:*

1. *For each $i \in [k+1]$, every component of the graph P_i is a path.*
2. *For each $i \in [2, k+1]$, we have that P_i is an induced subgraph of $G \setminus (V(P_1) \cup \ldots \cup V(P_{i-1}))$, and every component of $G \setminus \cup_{j<i} V(P_j)$ contains exactly one component of P_i.*
3. *$V(G) = \bigcup_{i \in [k+1]} V(P_i)$.*

Proof of Corollary 34. We prove the statement by induction on k. If the graph G has path-width equal to one, then G is the disjoint union of paths, and letting $P_1 := G$ we see that the statement of Corollary 34 holds.

Let $k > 1$ and suppose that the statement of Corollary 34 holds for every positive integer $k' < k$. Let $C_1, \ldots, C_l$ be the connected components of G. For each $j \in [l]$, we have that C_j is a connected graph of path-width at most k. Let P_1^j be a subgraph of C_j which is a path as in the statement of Lemma 33. Let $P_1 := \cup_{j \in [l]} P_1^j$. Consider the graph $G' := G \setminus V(P_1)$ which, by Lemma 33, has path-width at most $k' := k-1 < k$. Then, by applying the induction hypothesis to the graph G', we obtain subgraphs $P_2, \ldots, P_{k+1}$ of G' such that the subgraphs $P_1, \ldots, P_{k+1}$ of G satisfy the statement of Corollary 34. $\square$

We are now ready to prove Lemma 30.

Proof of Lemma 30. Let $P_1, \ldots, P_{k_1+1}$ and $Q_1, \ldots, Q_{k_2+1}$ be subgraphs of T_1 and T_2 respectively, chosen as in Corollary 34.

Let X be a subtree of T_1. We define the level of X, denoted by $L_1(X)$, as follows:

$$L_1(X) := \min\{i \in [k_1 + 1] : V(X) \cap V(P_i) \neq \emptyset\}.$$

Similarly, we define the level of a subtree X of T_2 as follows:

$$L_2(X) := \min\{i \in [k_2 + 1] : V(X) \cap V(Q_i) \neq \emptyset\}.$$

Claim 35. *Let X and Y be subtrees of T_1 such that $L_1(X) = L_1(Y) = l$. Then the following hold:*

1. *Both $X \cap P_l$ and $Y \cap P_l$ are paths; and*
2. *$V(X) \cap V(Y) \neq \emptyset$ if and only if $V(X) \cap V(Y) \cap V(P_l) \neq \emptyset$.*

Similarly for subtrees of T_2.

Proof of Claim 35. We prove the claim for T_1; the proof for T_2 is identical. Since $L_1(X) = l$, we have that $V(X) \cap (V(P_1) \cup \ldots \cup V(P_{l-1})) = \emptyset$. Thus X is contained in a connected component of the forest $T \setminus (V(P_1) \cup \ldots \cup V(P_{l-1}))$. Let C be this component. By Corollary 34, we have that $Z := P_l \cap C$ is a path, and thus $X \cap P_l$ is a path as well. With identical arguments we get that $Y \cap P_l$ is a path.

For the second statement of our claim: The reverse implication is immediate. For the forward implication: Since $V(X) \cap V(Y) \neq \emptyset$, both the subtrees X and Y are contained in the same connected component of the forest $T \setminus (V(P_1) \cup \ldots \cup V(P_{l-1}))$. Let C be this component. By Corollary 34, we have that $Z := P_l \cap C$ is a path. Consider the tree C and its family of subtrees $\{X, Y, Z\}$. Since $V(X) \cap V(Z) \neq \emptyset$, $V(Y) \cap V(Z) \neq \emptyset$ and $V(X) \cap V(Y) \neq \emptyset$, by Theorem 16, it follows that $V(X) \cap V(Y) \cap V(Z) \neq \emptyset$. In particular $V(X) \cap V(Y) \cap V(P_l) \neq \emptyset$. This concludes the proof of Claim 35. $\blacksquare$

In what follows in this proof, for every $v \in V(G)$ and $i \in [2]$, we denote by T_i^v the subtree $T_i[\{t \in V(T_i) : v \in \beta_i(t)\}]$ of T_i. For each $i \in [k_1 + 1]$ and for each $j \in [k_2 + 1]$, we define a subset of $V(G)$ as follows:

$$V_{i,j} := \{v \in V(G) : L_1(T_1^v) = i \text{ and } L_2(T_2^v) = j\}.$$

Let $\mathcal{P} := \{V_{i,j}\}_{i \in [k_1+1], j \in [k_2+1]}$ and observe that $\mathcal{P}$ is a partition of $V(G)$.

Claim 36. *For each $i \in [k_1+1]$ the graph $G_1[\bigcup_{j \in [k_2+1]} V_{i,j}]$ is an interval graph. Similarly for each $j \in [k_2 + 1]$, and $G_2[\bigcup_{i \in [k_1+1]} V_{i,j}]$.*

Proof of Claim 36. We prove the claim for G_1; the proof for G_2 is identical. Let $i \in [k_1+1]$. For each vertex $v \in G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$, let $P_i^v := P_i \cap T_1^v$. Then, by Claim 35, we have that P_i^v is a path. Let u and v be distinct vertices of the graph $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$. Then u is adjacent to v in $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ if and only if $V(T_1^v) \cap V(T_1^u) \neq \emptyset$. By Claim 35, we have that $V(T_1^v) \cap V(T_1^u) \neq \emptyset$ if and only if $V(T_1^v) \cap V(T_1^u) \cap V(P_i) \neq \emptyset$. Since $V(T_1^v) \cap V(T_1^u) \cap V(P_i) = V(P_i^u) \cap V(P_i^v)$, it follows that u is adjacent to v in $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ if and only if $V(P_i^u) \cap V(P_i^v) \neq \emptyset$.

Hence, the graph $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ is the intersection graph of the family $\{P_i^v : v \in G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]\}$ of subpaths P_i. Since P_i is the disjoint union of paths it follows that every component of $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ is an interval graph, and so $G_1[\bigcup_{j\in[k_2+1]} V_{i,j}]$ is an interval graph as well. ■

Let $i \in [k_1+1]$ and $j \in [k_2+1]$. Then, by Claim 36, we have $G[V_{i,j}] \in \mathcal{I} \mathbin{\dot\cap} \mathcal{I}$. Hence $\mathcal{P}$ is the desired partition. □

We remark that, using the arguments of the above proof and induction, we can get the following:

Lemma 37. *Let $k_1, \ldots, k_l$ be positive integers, and let $G_1, \ldots, G_l$ be chordal graphs such that for each $i \in [l]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \leqslant k_i$. If G is a graph such that $G = G_1 \cap \ldots \cap G_l$, then there exists a partition $\mathcal{P}$ of $V(G)$ such that $|\mathcal{P}| \leqslant \prod_{i\in l}(k_i+1)$ and for every $V \in \mathcal{P}$, the graph $G[V]$ has boxicity at most l.*

Davies and Yuditsky [10] proved that the Gyárfas-Sumner conjecture holds with polynomial bounds for graphs of bounded boxicity:

Theorem 38 (Davies and Yuditsky [10])**.** *For every positive integer d and forest F, the class of all F-free graphs of boxicity at most d is polynomially χ-bounded.*

Now the following strengthening of Theorem 38 is an immediate corollary of Lemma 37 and Theorem 38.

Theorem 39. *Let $k_1, \ldots, k_l$ be positive integers, and let $\mathcal{C}$ be the class of all graphs G for which there exist chordal graphs $G_1, \ldots, G_l$ such that for each $i \in [l]$ the graph G_i has a representation (T_i, β_i), where $\mathsf{pw}(T_i) \leqslant k_i$, and $G = G_1 \cap \ldots \cap G_l$. Then, the for every forest F the class of F-free graphs in $\mathcal{C}$ is polynomially χ-bounded.*

3.3 Subclasses of $\mathcal{C} \mathbin{\dot\cap} \mathcal{C}$: When at least one chordal graph has a representation tree of bounded radius

For each positive integer k we consider the subclass of $\mathcal{C} \mathbin{\dot\cap} \mathcal{C}$ in which one of the two chordal graphs in the intersection has a representation tree of radius at most k, and we prove that this class is χ-bounded.

Theorem 40. *Let k be a positive integer, and let G_1 and G_2 be chordal graphs such that the graph G_1 has a representation (T_1, β_1) where $\mathsf{rad}(T_1) \leqslant k$. If G is a graph such that $G = G_1 \cap G_2$, then $\chi(G) \leqslant k \cdot \omega(G)$.*

The main observation that we need for the proof of Theorem 40 is the following:

Lemma 41. *Let G be a chordal graph and k be a positive integer. If G has a representation (T, β) such that $\mathsf{rad}(T) \leqslant k$, then there exists a partition $\mathcal{P}$ of $V(G)$ such that $|\mathcal{P}| \leqslant k$ and for each $V \in \mathcal{P}$ we have that $G[V]$ is a disjoint union of complete graphs.*

We show how Theorem 40 follows from Lemma 41.

Proof of Theorem 40 assuming Lemma 41. Let G be a graph as in the statement of Theorem 40, and let $\mathcal{P}$ be a partition of $V(G_1)$ as in the statement of Lemma 41.

We claim that for each $V \in \mathcal{P}$, we have $\chi(G[V]) \leqslant \omega(G)$. Indeed, let $V \in \mathcal{P}$. Then the graph $G_1[V]$ is a disjoint union of complete graphs. Hence, the graph $G[V] = G_1[V] \cap G_2[V]$ is the intersection of a chordal graph with a disjoint union of cliques, and thus a chordal graph. Hence, $\chi(G[V]) \leqslant \omega(G[V]) \leqslant \omega(G)$.

For each $V \in \mathcal{P}$, we can color the graph $G[V]$ with a different palette of $\omega(G)$ colors, and obtain a $(k \cdot \omega(G))$-coloring of G. Hence $\chi(G) \leqslant k \cdot \omega(G)$. ☐

It remains to prove Lemma 41.

Proof of Lemma 41. Let r be a vertex of T which, chosen as a root, realizes the radius of T. For each vertex $v \in V(G)$, we denote by T^v the subtree $T[\{t \in V(T) : v \in \beta(t)\}]$ of T. Furthermore, for each subtree X of T, we denote by $L(X)$ the value $\min\{d(r, x) : x \in V(X)\}$, and by $r(X)$ the root of X, which is the unique element of the set $\arg\min_{x \in V(X)} d(r, x)$. We refer to the value $L(X)$ as the level of X.

Let $S := \{T^v : v \in V(G)\}$, and for each $i \in [k]$, let $L_i := \{X \in S : L(X) = i\}$. Observe that, since T has radius at most k, we have that $\{L_i\}_{i \in [k]}$ is a partition of S.

The main observation that we need is that two subtrees X and Y of the same level intersect if and only if they have the same root (and no other common vertex).

Thus, for each level $i \in [k]$, the relation of intersection of subtrees is an equivalence relation in L_i, and the corresponding induced subgraph of G is a disjoint union of complete graphs.

For each $i \in [k]$, let $V_i := \{v \in V(G) : T^v \in L_i\}$. Then $\mathcal{P} := \{V_i : i \in [k] \text{ and } V_i \neq \emptyset\}$ is the desired partition of $V(G)$. ☐

4 The k-Chordality Problem is NP-complete for $k \geqslant 3$

We recall from the section 1 that for a fixed positive integer k, the k-Chordality Problem is the following: Given a graph G as an input, decide whether $\mathsf{chor}(G) \leqslant k$. In this section we study the computational complexity of this problem. For $k = 1$, the 1-Chordality Problem is to decide whether a given graph is chordal, and there exists a polynomial-time algorithm for this problem (see, for example, [16, 22]). In this section we prove Theorem 5 which we restate here.

Theorem 42. *For every $k \geqslant 3$, the k-Chordality Problem is NP-complete.*

For a fixed positive integer k the k-Coloring Problem is the following: Given a graph G as an input, decide whether G has a k-coloring.

Theorem 43 (Karp, [20, Main Theorem]). *For every $k \geqslant 3$, the k-Coloring Problem is* NP*-complete.*

We immediately see that for every positive integer k, the k-Chordality Problem is in NP. We prove Theorem 42 by proving a polynomial-time reduction of the k-Coloring Problem to the k-Chordality Problem. We first state some preliminary definitions and results.

Theorem 44 (McKee and Scheinerman, [25, Corollary 4]). *Let G be a graph. Then $\mathsf{chor}(G) \leqslant \chi(G)$.*

Given two graphs G and H, the *lexicographic product* of G with H, denoted by $G \cdot H$, is the graph which has as vertices the elements of the set $V(G) \times V(H)$, and where two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $\{x_1, x_2\} \in E(G)$, or $x_1 = x_2$ and $\{y_1, y_2\} \in E(H)$. The graph $G \cdot H$ can be though as the graph that we obtain if in G we "substitute" a copy of H for each vertex of G.

Theorem 45 (Geller and Stahl, [17, Theorem 3]). *Let G and H be two graphs. If $\chi(H) = n$, then $\chi(G \cdot H) = \chi(G \cdot K_n)$.*

Proposition 46. *Let G be a graph. Then $\chi(G) \leqslant k$ if and only if $\mathsf{chor}(G \cdot K_2^c) \leqslant k$.*

Proof of Proposition 46. For the forward direction: By Theorem 44 and Theorem 45 we have that $\mathsf{chor}(G \cdot K_2^c) \leqslant \chi(G \cdot K_2^c) = \chi(G \cdot K_1) = \chi(G) \leqslant k$.

For the reverse direction: Suppose that $\mathsf{chor}(G \cdot K_2^c) \leqslant k$ and let $H_1, \ldots, H_k$ be chordal graphs such that $G \cdot K_2^c = H_1 \cap \ldots \cap H_k$. Let $V(K_2^c) = \{1, 2\}$. Let $f : V(G) \to [k]$ be defined as follows: $f(v) = i \in [k]$, where i is chosen so that it satisfies $\{(v, 1), (v, 2)\} \notin E(H_i)$. We claim that f is a proper k-coloring of G. Suppose not. Let $\{u, v\} \in E(G)$ be such that $f(u) = f(v) =: i$. Then we have that $\{(v, 1), (v, 2)\} \notin E(H_i)$ and $\{(u, 1), (u, 2)\} \notin E(H_i)$. Thus, $H_i[\{(v, 1), (v, 2), (u, 1), (u, 2)\}]$ is a hole in H_i which is a contradiction. □

Corollary 47. *Let G be a graph. Then, in polynomial-time in the size of G we can construct a graph G' such that the following hold: $\chi(G) = k$ if and only if $\mathsf{chor}(G') = k$.*

Now Theorem 42 follows immediately by Theorem 43 and Corollary 47.

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