Brun's Inequality for a Geometric Lattice

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Abstract

V. Brun introduced Brun's sieve in his seminal paper, which is based on Brun's inequality for the Möbius function and is a very powerful tool in modern number theory. The importance of the Möbius function in enumeration problems led G.-C. Rota to introduce the concept of the Möbius function for partially ordered sets. In this article, we prove Brun's inequality for geometric lattices and develop a combinatorial sieve in this context. One of the main ingredients is a recent work of K. Adiprasito, J. Huh, and E. Katz on the log-concavity of absolute values of the Whitney numbers associated with matroids. Further, we study shifted convolutions of the Whitney numbers associated with Dowling lattices and derive an asymptotic formula for generalized Dowling numbers.

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1 Introduction

One of the most powerful tools in analytic number theory is the sieve method. Classically, the sieve methods of Eratosthenes, Brun and Selberg are combinatorial in nature, whereas the large sieve inequality due to Linnik is Fourier analytic. The former class of sieve methods could, in principle, be formulated in a general combinatorial setting. The first step in this direction was due to Wilson [25], [24] who formulated a "Selberg sieve inequality" for partially ordered sets satisfying some standard properties. This was subsequently developed by Chow [6]. Later, Liu and Murty [12] isolated an idea of Turán and formulated a very general combinatorial sieve that they called "the Turán sieve" and applied it to an assortment of questions in graph theory and combinatorics. Though interesting, these results have had limited impact. It would seem that the classical Brun's sieve would afford a similar generalization. However, this was not the case since the simple seed idea which generates the classical Brun's sieve required a unimodal property of certain Whitney numbers before it could be generalized to a geometric lattice. The

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required unimodal property is now a theorem due to Adiprasito-Huh-Katz [1]. Using this theory, we generalize Brun's inequalities and then develop a general combinatorial sieve. We apply this to the study of Dowling lattices.

In 1919, Brun developed his sieve method by making the following elementary observation concerning the classical Möbius function $\mu(n)$. Let $\omega(n) = r$ denote the number of distinct prime factors of n. Then,

$$\sum_{\substack{d|n \\ \omega(d)\leqslant k}} \mu(d) \;=\; (-1)^k \binom{r-1}{k}$$

so the sum on the left is positive when k is even and negative when k is odd. That is,

$$\sum_{\substack{d|n \\ \omega(d)\leqslant k}} \mu(d) = \begin{cases} \geqslant 0 & \text{if } k \text{ is even} \\ \leqslant 0 & \text{if } k \text{ is odd.} \end{cases} \tag{1}$$

Brun's sieve method is then developed from this fundamental observation.

To generalize this to a combinatorial setting, one would require some form of "rank" function that would serve as an analog of $\omega(n)$. The other difficulty is to probe an oscillatory theorem similar to the one stated for the classical case. This is the content of our first theorem described in the next section.

2 Statements of Results

The concept of a Möbius function for a partially ordered set originates in the 1964 work of Rota [18], though the idea is nascent in earlier works of Hall [9] and Weisner [21]. We refer the reader to Rota's paper for the basic background, though we give a brief review in later sections. The following generalizes Brun's inequality (1) in the context of a geometric lattice.

Theorem 1 (Brun's inequality for a geometric lattice)**.** *Let L be a geometric lattice with minimal element $\widehat{0}$. Then for any positive integer k, we have*

$$\sum_{\substack{x\in L \\ r(x)\leqslant 2k}} \mu(\widehat{0}, x) \;\geqslant\; 0 \qquad \text{and} \qquad \sum_{\substack{x\in L \\ r(x)\leqslant 2k-1}} \mu(\widehat{0}, x) \;\leqslant\; 0.$$

Here μ denotes the Möbius function on L and r denotes a rank function on L.

The following theorem on unimodal sequences is an important tool for incorporating the work of [1] in our context.

Theorem 2. *Let $\{a_i\}_{i=0}^{n}$ be a unimodal sequence of non-negative real numbers such that*

$$\sum_{i=0}^{n} (-1)^i a_i \;=\; 0.$$

Then for any positive integer k, we have

$$\sum_{i=0}^{2k}(-1)^i a_i \ \geqslant \ 0 \qquad \textit{and} \qquad \sum_{i=0}^{2k-1}(-1)^i a_i \ \leqslant \ 0.$$

We use the properties of the Móbius function of partially ordered sets to develop the following combinatorial sieve for geometric lattices.

Theorem 3. *Let L be a geometric lattice of rank n and $\mathcal{A} \subseteq L$. Let $cr(y)$ denote the co-rank of y given by $cr(y) = n - r(y)$. Also let $\mathcal{T}$ be a set of atoms in L whose join is τ. Suppose there exists a function*

$$f \ : \ \mathbb{N} \cup \{0\} \ \longrightarrow \ [0, \infty)$$

and a positive real number X such that

$$\#\{a \in \mathcal{A} \ : \ a \geqslant y\} \ = \ f(cr(y))X \ + \ \mathcal{E}(y)$$

and

$$\mathcal{E}(y) \ \ll \ cr(y)f(cr(y)).$$

Let

$$\mathcal{S}(\mathcal{A}, \mathcal{T}) \ = \ \#\{a \in \mathcal{A} \ : \ a \wedge \tau \ = \ \widehat{0}\}.$$

Then we have

$$\mathcal{S}(\mathcal{A}, \mathcal{T}) \ = \ X \sum_{k=0}^{r(\tau)} f(n-k) w_k \left([\widehat{0}, \tau]\right) \ + \ E(\mathcal{A}, \mathcal{T}),$$

where

$$E(\mathcal{A}, \mathcal{T}) \ \ll \ \sum_{k=0}^{r(\tau)} (n-k) f(n-k) |w_k([\widehat{0}, \tau])|.$$

Remark 4. One can apply Theorem 1 to derive lower and upper bounds for $\mathcal{S}(\mathcal{A}, \mathcal{T})$.

Now we apply our results to study Dowling lattices (defined below in section 3.4). This leads us to the investigation of shifted convolution of Whitney numbers associated with these lattices.

Let n be a positive integer and G be a multiplicative group of order m. Let $Q_n(G)$ be the Dowling lattice of rank n (see subsection 3.4). It is well-known that $Q_n(G)$ is a geometric lattice. We study relations between certain sifted sets and shifted convolution of Whitney numbers of these lattices. Let

$$w_m(n,k) \ = \ \sum_{\substack{x \in Q_n(G) \\ r(x) = n-k}} \mu(\widehat{0}, x) \quad \text{and} \quad W_m(n,k) \ = \ \sum_{\substack{x \in Q_n(G) \\ r(x) = n-k}} 1,$$

where $w_m(n,k)$ and $W_m(n,k)$ denote the Whitney numbers of $Q_n(G)$ of the first kind and the second kind respectively. Here μ denotes the Möbius function of $Q_n(G)$ and r denotes a rank function on $Q_n(G)$. In this context, we prove the following result.

Proposition 5. *Let n be a positive integer. For non-negative integers s, t, let*

$$c_{n,t}(s) \;=\; \sum_{k\geqslant 0} w_m(n,k)W_m(k+s,t),$$

denote shifted convolution of Whitney numbers of $Q_n(G)$. Then we have

$$c_{n,t}(s) \;=\; 0 \quad \text{if} \quad t<n.$$

If $t \geqslant n$, then we have

$$\sum_{s\geqslant 0} c_{n,t}(s)x^s \;=\; \frac{1}{x}\prod_{j=n}^{t}\frac{x}{1-(1+jm)x}.$$

In particular, we have

$$\sum_{k\geqslant 0} w_m(n,k)W_m(k,t) \;=\; \delta(n,t).$$

Here $\delta(n,t)=1$ if $n=t$ and $\delta(n,t)=0$ otherwise.

Further, the study of estimation of sifted sets and shifted convolution of Whitney numbers of $Q_n(G)$ leads us to the study of r-Dowling numbers $D_{m,r}(n)$ (see section 7). In this setup, we prove the following asymptotic formula for r-Dowling numbers (see [13, Corollary 3.1] for an asymptotic expression of $D_{m,r}(n)$ without any error term, and we want to point out to the reader that there is a typo in the main term).

Theorem 6. *Let m, n and r be positive integers. For $n > e^m$, we have*

$$D_{m,r}(n) \;=\; \frac{e^{g_0}}{\sqrt{4\pi g_2}}\cdot\frac{n!}{\delta^n}\left(1 \;+\; O\left(\frac{(\log n)^{12}}{\sqrt{n}}\right)\right),$$

where δ is a positive real number such that $\delta(r+e^{m\delta}) \;=\; n$ and the implied constant is absolute. Here

$$g_0 \;=\; r\delta \;+\; \frac{e^{m\delta}-1}{m} \qquad \text{and} \qquad g_2 \;=\; \frac{n \;+\; m\delta^2 e^{m\delta}}{2}.$$

3 Preliminaries

3.1 Unimodality and log-concavity

A sequence $\{a_i\}_{i=0}^{n}$ of real numbers is said to be **unimodal** if there exists an index $0 \leqslant j \leqslant n$ such that

$$a_0 \;\leqslant\; a_1 \;\leqslant\; \cdots \;\leqslant\; a_{j-1} \;\leqslant\; a_j \;\geqslant\; a_{j+1} \;\geqslant\; \cdots \;\geqslant\; a_n.$$

There are numerous naturally occurring sequences which are unimodal. For example, the classical sequence $\{\binom{n}{k}\}_{k=0}^{n}$ of binomial coefficients is unimodal.

A sequence $\{a_i\}_{i=0}^{n}$ is said to be **log-concave** if

$$a_k^2 \;\geqslant\; a_{k-1}a_{k+1} \qquad \text{for all} \;\; 0 < k < n.$$

The sequence of binomial coefficients is also log-concave (as is easily checked). One can also show that any log-concave sequence is unimodal. Thus, log-concavity is a stronger property than unimodality. Other important examples of log-concave sequences that occur in combinatorics are **Stirling numbers of the first kind** $s(n,k)$ defined by the polynomial identity

$$x(x-1)\cdots(x-(n-1)) \;=\; \sum_{k=1}^{n} s(n,k)x^k.$$

Recall that $|s(n,k)|$ is the number of permutations of the symmetric group S_n which can be written as a product of k-disjoint cycles. One can see the log-concavity of these numbers from an important result of Isaac Newton (see [10, p. 52] [16]) which states that if all the roots of a polynomial

$$f(x) \;=\; \sum_{k=0}^{n} a_k x^k$$

are real, then the sequence $\{a_k\}_{k=0}^{n}$ of coefficients of $f(x)$ is log-concave. This result produces a rich number of log-concave sequences, since the characteristic polynomial of a real symmetric matrix has all its roots real.

3.2 Prerequisites from theory of partially ordered sets (posets)

Let $(P, \leqslant)$ be a **partially ordered set** (**poset**). For any two elements $x, z \in P$, the **interval** $[x,z]$ is given by

$$[x,z] \;=\; \{y \in P \;:\; x \leqslant y \leqslant z\}.$$

A poset P is called **locally finite** if every interval of P is a finite set. In 1964, Rota [18] introduced the concept of Möbius function on locally finite posets and proved a fundamental property of the Möbius function, namely Möbius inversion formula and thereby initiating a combinatorial theory of locally finite posets. The **Möbius function** $\mu : P \times P \to \mathbb{Z}$ is defined recursively as follows: $\mu(x,z) = 0$ if $x \not\leqslant z$, $\mu(x,x) = 1$ for all $x \in P$ and if $x < z$,

$$\mu(x,z) \;=\; -\sum_{x \leqslant y < z} \mu(x,y).$$

It is clear from the definition that

$$\sum_{x \leqslant y \leqslant z} \mu(x,y) \;=\; \delta(x,z),$$

where δ denotes the **Kronecker delta function** given by

$$\delta(x,z) \;=\; \begin{cases} 1 & \text{if } \; x = z, \\ 0 & \text{otherwise.} \end{cases}$$

Given two elements $x, y \in P$, we say that $z \in P$ is an **upper bound** for x and y if $x \leqslant z$ and $y \leqslant z$. We say that z is a **least upper bound** for x and y if z is an upper bound for x and y and if w is an upper bound for x and y, then $z \leqslant w$. Clearly, if a least upper bound for x and y exists, then it is unique and is denoted by $x \vee y$ (called x **join** y). In a similar way, one defines the **greatest lower bound** for x and y (also called **meet** of x and y) and it is denoted by $x \wedge y$, if it exists. A poset P is said to have a **least element** (usually denoted by $\hat{0}$) if $\hat{0} \leqslant x$ for all $x \in P$. In a similar way, a poset P is said to have a **greatest element** (usually denoted by $\hat{1}$) if $x \leqslant \hat{1}$ for all $x \in P$.

A **lattice** L is a poset in which the least upper bound and the greatest lower bound for any two elements of L exist. Henceforth, we will focus on finite lattices. Notice that a finite lattice has least and greatest elements. An element $\widehat{0} \neq x \in L$ is called an **atom** if there exists no element $y \in L$ with $\widehat{0} < y < x$, in other words $[\widehat{0}, x] = \{\widehat{0}, x\}$. A lattice is said to be **atomistic** if every non-zero element is a join of atoms. Given $x < y$ in L, we say that y **covers** x if $[x, y] = \{x, y\}$ and we denote it by $x <: y$. A lattice L is said to be **graded** if there exists a function

$$r \;:\; L \;\longrightarrow\; \mathbb{R}_{\geqslant 0}$$

such that $r(\hat{0}) = 0$ and $r(y) = r(x) + 1$ if $x <: y$. A graded lattice is said to be **semimodular** if

$$r(x \wedge y) + r(x \vee y) \;\leqslant\; r(x) + r(y)$$

for any $x, y \in L$. A **geometric lattice** is a finite atomistic, semimodular lattice. For example, the set Π_n of all partitions of $\{1, 2, \ldots, n\}$ ordered by refinement and the set of all subspaces of an n-dimensional vector space over a finite field ordered by inclusion are two examples of familiar geometric lattices (see [7, Ch. 6] [15] [20, Ch. 3] for more details).

3.3 Prerequisites from theory of matroids

A **matroid** M is pair $(E, \mathcal{F})$, where E is a finite set and $\mathcal{F}$ is a collection of subsets of E (called **independent sets**) satisfying the following axioms:

i) $\emptyset \in \mathcal{F}$;

ii) If $A \in \mathcal{F}$ and $B \subseteq A$, then $B \in \mathcal{F}$;

iii) If $A, B \in \mathcal{F}$ and $|A| < |B|$, then there exists $b \in B \backslash A$ such that $A \cup \{b\} \in \mathcal{F}$ (**exchange property**).

A maximal independent set of E is called a **basis** of M. As a consequence of the exchange property, all bases of M have the same cardinality which is called the **rank** of M and is denoted by $r(M)$. For any $A \subseteq E$, the rank of A is defined to be the cardinality of a maximal independent subset of A and is denoted by $r(A)$. A subset of E which is not independent is called a **dependent set** and a minimal dependent set is called a **circuit**.

One defines the **characteristic polynomial** of a matroid M as follows:

$$\chi_M(\lambda) \;=\; \sum_{A \subseteq E} (-1)^{|A|} \lambda^{r(M)-r(A)}.$$

As the name suggests, $\chi_M(\lambda)$ is a polynomial in λ of degree equal to the rank of M and all of its coefficients are integers. Hence one can write it as

$$\chi_M(\lambda) \;=\; \sum_{i=0}^{r(M)} w_i(M) \lambda^{r(M)-i}.$$

The above integers $w_i(M)$ are called **Whitney numbers of the first kind** associated with the matroid M. In 1971, Rota [19] conjectured that the sequence of absolute values of these numbers is **unimodal**. A few years later. a stronger conjecture was proposed (independently) by Heron [11] and Welsh [22].

Conjecture 7 (Heron-Rota-Welsh)**.** The sequence $\{|w_i(M)|\}_{i=0}^{r(M)}$ of absolute values of the coefficients of $\chi_M(\lambda)$ is log-concave.

This conjecture was resolved in 2018 by Adiprasito-Huh-Katz [1].

We have seen the definition of a matroid using independent axioms that abstract the notion of linear independence. Now we will see another way to define a matroid in terms of **closure operators** which abstract the notion of linear span.

Let $\mathcal{P}(E)$ be the power set of E. A matroid M is a finite set E together with a function (closure operator)

$$\begin{aligned} \mathrm{cl} \;:\; \mathcal{P}(E) &\longrightarrow \mathcal{P}(E) \\ X &\longmapsto \overline{X} \end{aligned}$$

satisfying the following axioms:

i) $X \subseteq \overline{X}$;

ii) If $Y \subseteq X$, then $\overline{Y} \subseteq \overline{X}$;

iii) $\overline{\overline{X}} = \overline{X}$;

iv) If $y \in \overline{X \cup \{x\}}$ and $y \notin \overline{X}$, then $x \in \overline{X \cup \{y\}}$.

The above two definitions of a matroid are equivalent. To see this, given a matroid with independent axioms, one defines the closure of a set $X \subseteq E$ by

$$\overline{X} \; := \; \{x \in E : r(X \cup \{x\}) = r(X)\}.$$

It is easy to check that the closure operator satisfies the span axioms. Conversely, given a matroid with span axioms, one declares a subset $A \subseteq E$ to be independent if $x \in A$ implies $x \notin \overline{A\backslash\{x\}}$.

A subset $X \subseteq E$ is called **flat** if $\overline{X} = X$. The elements of $\overline{\emptyset}$ are called **loops**. Two elements $x, y \in E$ are said to be **parallel** if $\{x, y\}$ is a circuit. A matroid is said to be **simple** if it has no loops and parallel points. Every matroid has a canonical **simplification** $\widehat{M}$ obtained by removing loops and identifying parallel points. More precisely, if E' denotes the set of all elements of E which are not loops, then one defines an equivalence relation on E' by:

$$x \; \sim \; y \quad \Longleftrightarrow \quad x \text{ and } y \text{ are parallel.}$$

Let $\widehat{E}$ denotes the set of all equivalence classes of E', then $\widehat{E} = \{\overline{x} : x \in E'\}$. We say a subset $\{\overline{x_1}, \overline{x_2}, \cdots, \overline{x_k}\} \subseteq \widehat{E}$ is independent if $\{x_1, x_2, \ldots, x_k\}$ is independent. Let $\widehat{\mathcal{F}}$ be the set of all independent subsets of $\widehat{E}$ along with empty set, then one can check that $\widehat{M} = (\widehat{E}, \widehat{\mathcal{F}})$ is a simple matroid. The reader can refer to [2, 15, 17] for more details.

3.4 Prerequisites from Dowling lattices

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of n elements. Let P_n be the collection of partitions of X. Then the set P_n is partially ordered by **refinement** : we say $\alpha \leqslant \beta$, if every block of β is a union of blocks of α. For example,

$$\{x_1, x_2\}\{x_3\}\{x_4\}\{x_5, x_6, \ldots, x_n\} \; \leqslant \; \{x_1, x_2, x_3\}\{x_4\}\{x_5, x_6, \ldots, x_n\}.$$

By a **partial partition** of X, we mean a collection $\mathcal{F} = \{A_1, A_2, \ldots, A_r\}$ of mutually disjoint non-empty subsets of X. For example, if we take $A_1 = \{x_1, x_2\}$ and $A_2 = \{x_n\}$, then $\{A_1, A_2\}$ is a partial partition of X. The subsets A_i's are called **blocks** of $\mathcal{F}$. Let Q_n be the set of all partial partitions of X. As before, the set Q_n is partially ordered by refinement. Let x_0 be an element which does not belong to X. We have an isomorphism

$$\begin{aligned} Q_n \; &\longrightarrow \; P_{n+1} \\ \{A_1, A_2, \ldots, A_r\} \; &\longmapsto \; \{A_0 \cup \{x_0\}, A_1, A_2, \ldots, A_r\}, \end{aligned}$$

where $A_0 = X \backslash \left(\cup_{i=1}^r A_i\right)$. The block $A_0 \cup \{x_0\}$ is called the **zero block** of the partition $\{A_0 \cup \{x_0\}, A_1, A_2, \ldots, A_r\}$ of $X \cup \{x_0\}$. If $\mathcal{F}$ is a partial partition of X, then the rank of $\mathcal{F}$ is given by

$$r(\mathcal{F}) \; = \; n - |\mathcal{F}|,$$

where $|\mathcal{F}|$ denotes the number of blocks of $\mathcal{F}$.

Let G be a finite multiplicative group. By a **partial G-partition**, we mean a collection $\mathcal{F} = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of functions given by

$$\alpha_i \;:\; A_i \longrightarrow G$$

for $i \in \{1, 2, \ldots, r\}$, where $\{A_1, A_2, \ldots, A_r\}$ is a partial partition of X. Let $\tilde{Q}_n(G)$ be the set of all partial G-partitions of X. We have a map

$$\begin{aligned} \psi : \tilde{Q}_n(G) \;&\longrightarrow\; Q_n \\ \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \;&\longmapsto\; \{A_1, A_2, \ldots, A_r\}. \end{aligned}$$

Let $\mathcal{F} = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and $\mathcal{G} = \{\beta_1, \beta_2, \ldots, \beta_s\}$ be two partial G-partitions of X, where $\alpha_i : A_i \to G$ and $\beta_j : B_j \to G$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$. Suppose that $\psi(\mathcal{F}) \leqslant \psi(\mathcal{G})$. Then for each $j \in \{1, 2, \ldots, s\}$, there exists a subset $M_j \subseteq \{1, 2, \ldots, r\}$ such that

$$B_j \;=\; \bigcup_{i \in M_j} A_i$$

We say that β_j is a G-linear combination of α_i's if

$$\beta_j|_{A_i} \;=\; \lambda_i \alpha_i$$

for some scalars $\lambda_i \in G$, $i \in M_j$ and we write β_j as

$$\beta_j \;=\; \sum_{i \in M_j} \lambda_i \alpha_i.$$

One defines a **preorder** on $\tilde{Q}_n(G)$ as follows: we write

$$\mathcal{F} \;\leqslant\; \mathcal{G},$$

if $\psi(\mathcal{F}) \leqslant \psi(\mathcal{G})$ and β_j is a G-linear combinations of α_i's for every $j \in \{1, 2, \ldots, s\}$. Clearly, the relation $\leqslant$ is reflexive and transitive, but not antisymmetric unless $G = \{1\}$. Hence we define an equivalence relation on $\tilde{Q}_n(G)$ by

$$\mathcal{F} \;\sim \mathcal{G} \quad \text{if } \mathcal{F} \;\leqslant\; \mathcal{G} \text{ and } \mathcal{G} \;\leqslant\; \mathcal{F}.$$

The preorder $\leqslant$ on $\tilde{Q}_n(G)$ induces a partial order on the quotient set

$$Q_n(G) \;=\; \tilde{Q}_n(G)/\sim .$$

In [8], Dowling proved that $Q_n(G)$ is a geometric lattice. Note that if G is the trivial group, then $Q_n(G) \cong Q_n \cong P_{n+1}$. The equivalence class of $\mathcal{F}$ is usually denoted by $(\mathcal{F})$.

Let $E_i = \{x_i\}$ for $i \in \{1, 2, \ldots, n\}$ and $e_i : E_i \to G$ be the function defined by $e_i(x_i) = 1$. Also let $\varepsilon = \{e_1, e_2, \ldots, e_n\}$, a partial G-partition of X. The zero element $\hat{0}$ of $Q_n(G)$ is (ε) and the unit element $\hat{1}$ is the equivalence class of empty partial G-partitions. The rank function on $Q_n(G)$ is given by

$$r(\mathcal{F}) \;=\; n - |\mathcal{F}|.$$

Dowling proved the following result regarding the structure of intervals in $Q_n(G)$ (see [8, Theorem 2]).

Theorem 8. *Let $(\mathcal{F}) \in Q_n(G)$ be of co-rank r and let $(\mathcal{G}) \in Q_n(G)$, where $\mathcal{G} = \{\beta_j : 1 \leqslant j \leqslant s\}$ with $\beta_j : B_j \to G$ for $1 \leqslant j \leqslant s$.*

i) We have

$$[(\mathcal{F}), \hat{1}] \;\cong\; Q_r(G).$$

ii) Let $B_0 = X \setminus \left(\bigcup_{j=1}^{s} B_j\right)$ and $n_j = |B_j|$ for $0 \leqslant j \leqslant s$. Then we have

$$[\hat{0}, (\mathcal{G})] \;\cong\; Q_{n_0}(G) \times P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}.$$

We have the following theorem regarding convolution of Whitney numbers (see [8, Theorem 6]).

Theorem 9. *For non-negative integers n, s, we have*

$$\sum_{r \geqslant 0} W_m(n,r) w_m(r,s) \;=\; \delta(n,s) \;=\; \sum_{r \geqslant 0} w_m(n,r) W_m(r,s).$$

4 Proofs of main theorems

4.1 Proof of Theorem 2

Let $\{a_i\}_{i=0}^{n}$ be a unimodal sequence of non-negative real numbers such that

$$\sum_{i=0}^{n} (-1)^i a_i \;=\; 0. \tag{2}$$

We will prove Theorem 2 by contradiction. Suppose there exists an integer $k \geqslant 0$ such that

$$\sum_{i=0}^{2k} (-1)^i a_i \;<\; 0. \tag{3}$$

Since the sequence $\{a_i\}_{i=0}^{n}$ is unimodal, there exists an index $0 \leqslant j \leqslant n$ such that

$$a_0 \;\leqslant\; a_1 \;\leqslant\; \cdots \;\leqslant\; a_{j-1} \;\leqslant\; a_j \;\geqslant\; a_{j+1} \;\geqslant\; \cdots \;\geqslant\; a_n.$$

If possible $2k \leqslant j$, then we have

$$\sum_{i=0}^{2k} (-1)^i a_i \;=\; a_0 \underbrace{-a_1 + a_2}_{\geqslant 0} - \cdots \underbrace{-a_{2k-1} + a_{2k}}_{\geqslant 0} \;\geqslant\; 0.$$

Hence we get $2k > j$. We split the following sum into two parts:

$$\sum_{i=0}^{n} (-1)^i a_i \;=\; \sum_{i=0}^{2k} (-1)^i a_i \;+\; \sum_{i=2k+1}^{n} (-1)^i a_i. \tag{4}$$

From the fact that $2k > j$, we get

$$\sum_{i=2k+1}^{n} (-1)^i a_i \;=\; \underbrace{-a_{2k+1} + a_{2k+2}}_{\leqslant 0} - \cdots + (-1)^n a_n \;\leqslant\; 0. \tag{5}$$

From (3), (4) and (5), we get

$$\sum_{i=0}^{n} (-1)^i a_i \;<\; 0,$$

a contradiction to (2). This completes the proof of first part of Theorem 2. The second part follows in a similar manner. Suppose that for some positive integer k,

$$\sum_{i=0}^{2k-1} (-1)^i a_i \;>\; 0.$$

Then as before, we must have $2k - 1 > j$. Now, we split the sum as

$$\sum_{i=0}^{n} (-1)^i a_i \;=\; \sum_{i=0}^{2k-1} (-1)^i a_i \;+\; \sum_{i=2k}^{n} (-1)^i a_i$$

and note that

$$\sum_{i=2k}^{n} (-1)^i a_i \;=\; \underbrace{a_{2k} - a_{2k+1}}_{\geqslant 0} + \cdots + (-1)^n a_n \;\geqslant\; 0.$$

This leads to

$$\sum_{i=0}^{n} (-1)^i a_i \;>\; 0,$$

a contradiction to (2). This completes the proof of Theorem 2. □

Now as an application of Heron-Rota-Welsh conjecture (now a theorem due to Adiprasito-Huh-Katz) and Theorem 2, we give a proof Brun's inequality for a geometric lattice.

4.2 Proof of Theorem 1

The outline of proof is as follows : First we notice that it is equivalent to prove Brun's inequality for lattices associated to simple matroids. Then, we see that Whitney numbers of the first kind can be expressed in terms of values of the Möbius function on such lattices. Finally, we complete the proof of Theorem 1 using the result of [1] and Theorem 2.

Let $M = (E, \mathcal{F})$ be a matroid. The set $L(M)$ of flats of M is a poset ordered by inclusion. One can show that $L(M)$ is a geometric lattice. In fact, every geometric lattice arises in this way up to isomorphism (see [17, Theorem 1.7.5]). More precisely, given any geometric lattice L, there exists a matroid M such that

$$L \;\cong\; L(M).$$

Further, one can suppose that M is simple, since $L(M) \cong L(\widehat{M})$.

Henceforth, let M denote a simple matroid and $L(M)$ denote the lattice of flats of M. Let $\mu = \mu_{L(M)}$ denote the Möbius function on $L(M)$. Unlike in the classical case where the Möbius function takes the values 0 or ± 1, computing values of Möbius functions on posets is a difficult problem in general. In this context, we have the following important formula (see [9, 21, 23], [26, Prop. 7.1.4]):

$$\mu(\emptyset, F) \;=\; \sum_{\substack{A \subseteq F \\ \overline{A} = F}} (-1)^{|A|}. \tag{6}$$

If we consider the polynomial

$$\chi_{L(M)}(\lambda) \;=\; \sum_{F \in L(M)} \mu(\emptyset, F) \lambda^{r(M)-r(F)}$$

called the characteristic polynomial of the lattice $L(M)$, then one has

$$\chi_{L(M)}(\lambda) \;=\; \chi_M(\lambda).$$

This can be shown by using the formula (6) as follows:

$$\sum_{F \in L(M)} \mu(\emptyset, F) \lambda^{r(M)-r(F)} \;=\; \sum_{F \in L(M)} \sum_{\substack{A \subseteq F \\ \overline{A} = F}} (-1)^{|A|} \lambda^{r(M)-r(F)}. \tag{7}$$

Note that $r(F) = r(A)$, since $\overline{A} = F$. Hence the right hand side of (7) is equal to

$$\sum_{F \in L(M)} \sum_{\substack{A \subseteq F \\ \overline{A} = F}} (-1)^{|A|} \lambda^{r(M)-r(A)} \;=\; \sum_{A \subseteq E} (-1)^{|A|} \lambda^{r(M)-r(A)} \;=\; \chi_M(\lambda).$$

Thus we have

$$\sum_{F \in L(M)} \mu(\emptyset, F) \lambda^{r(M)-r(F)} \;=\; \sum_{i=0}^{r(M)} w_i(M) \lambda^{r(M)-i}. \tag{8}$$

Now by comparing the coefficients on both sides, we get

$$w_i(M) \;=\; \sum_{\substack{F \in L(M) \\ r(F) = i}} \mu(\emptyset, F). \tag{9}$$

This in turn gives

$$\sum_{\substack{F \in L(M) \\ r(F) \leqslant k}} \mu(\emptyset, F) \;=\; \sum_{i=0}^{k} w_i(M). \tag{10}$$

To complete the proof of Theorem 1, we need to show that $\sum_i w_i(M)$ is 'alternating in sign'. That is, we need to show

$$(-1)^k \sum_{i=0}^{k} w_i(M) \;\geqslant\; 0.$$

Thus, it is important to know the 'sign' of Whitney numbers of the first kind. In [18, Theorem 4], Rota proved the following fundamental result on the sign of Möbius function of a geometric lattice : If L is a geometric lattice, then the Möbius function is non-zero and alternates in sign, i.e., for any $x \leqslant y$ in L,

$$(-1)^{r(y)-r(x)}\mu_L(x,y) \;>\; 0. \tag{11}$$

From this result and (9), it follows that the numbers $w_i(M)$ alternate in sign. That is,

$$(-1)^i w_i(M) \;\geqslant\; 0. \tag{12}$$

But this alone will not be sufficient for the completion of the proof, we need to establish that the sums

$$\sum_i w_i(M)$$

alternate in sign. By substituting $\lambda = 1$ in (8), we get

$$\sum_{i=0}^{r(M)} w_i(M) \;=\; \sum_{F\in L(M)} \mu(\emptyset, F) \;=\; 0. \tag{13}$$

By using the fact that $|w_i(M)| = (-1)^i w_i(M)$, we rewrite the sum in (10) as

$$\sum_{\substack{F\in L(M)\\ r(F)\leqslant k}} \mu(\emptyset, F) \;=\; \sum_{i=0}^{k} w_i(M) \;=\; \sum_{i=0}^{k} (-1)^i |w_i(M)|$$

Now using the result of [1] along with (12) and (13), we apply Theorem 2 to conclude that

$$\sum_{\substack{F\in L(M)\\ r(F)\leqslant 2k}} \mu(\emptyset, F) \;\geqslant\; 0 \qquad \text{and} \qquad \sum_{\substack{F\in L(M)\\ r(F)\leqslant 2k-1}} \mu(\emptyset, F) \;\leqslant\; 0.$$

This completes the proof of Theorem 1. □

5 A sieve on a geometric lattice

In this section, we will establish a sieve for a geometric lattice.

5.1 Proof of Theorem 3

Let L be a geometric lattice of rank n and $\mathcal{A} \subseteq L$. Also let $\mathcal{T}$ be a set of atoms in L and τ be the join of atoms in $\mathcal{T}$. Set

$$\mathcal{S}(\mathcal{A}, \mathcal{T}) = \#\{a \in \mathcal{A} : a \wedge \tau = \hat{0}\}.$$

Then we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}, \mathcal{T}) &= \sum_{\substack{a \in \mathcal{A} \\ a \wedge \tau = \widehat{0}}} 1 = \sum_{a \in \mathcal{A}} \sum_{y \leqslant a \wedge \tau} \mu(\widehat{0}, y) \\ &= \sum_{y \leqslant \tau} \mu(\widehat{0}, y) \sum_{\substack{a \in \mathcal{A} \\ a \geqslant y}} 1 = \sum_{y \leqslant \tau} \mu(\widehat{0}, y) \# \mathcal{A}_y, \end{aligned} \tag{14}$$

where $\mathcal{A}_y = \{a \in \mathcal{A} : a \geqslant y\}$. Suppose there exists a function $f : \mathbb{N} \cup \{0\} \to [0, \infty)$ such that

$$\#\mathcal{A}_y = f(\mathrm{cr}(y))X + \mathcal{E}(y) \qquad \text{and} \qquad \mathcal{E}(y) \ll \mathrm{cr}(y) f(\mathrm{cr}(y)).$$

Then, we get

$$\sum_{y \leqslant \tau} \mu(\widehat{0}, y) \# \mathcal{A}_y = X \sum_{y \leqslant \tau} \mu(\widehat{0}, y) f(\mathrm{cr}(y)) + \sum_{y \leqslant \tau} \mu(\widehat{0}, y) \mathcal{E}(y).$$

Note that $y \leqslant \tau$ implies that $r(y) \leqslant r(\tau)$. By collecting the terms $y \leqslant \tau$ with same rank, we get

$$\begin{aligned} \sum_{y \leqslant \tau} \mu(\widehat{0}, y) f(\mathrm{cr}(y)) &= \sum_{k=0}^{r(\tau)} \sum_{\substack{y \leqslant \tau \\ r(y) = k}} \mu(\widehat{0}, y) f(n-k) \\ &= \sum_{k=0}^{r(\tau)} f(n-k) \sum_{\substack{y \in [\widehat{0}, \tau] \\ r(y) = k}} \mu(\widehat{0}, y) \\ &= \sum_{k=0}^{r(\tau)} f(n-k) w_k\left([\widehat{0}, \tau]\right), \end{aligned}$$

where $w_k\left([\widehat{0}, \tau]\right)$ denotes the k-th Whitney number of the first kind of the geometric lattice $[\widehat{0}, \tau]$ and note that any interval in a geometric lattice is geometric. By arguing in a similar way and using the fact that $(-1)^{r(y)} \mu(\widehat{0}, y) \geqslant 0$, we deduce that

$$\sum_{y \leqslant \tau} \mu(\widehat{0}, y) \mathcal{E}(y) \ll \sum_{k=0}^{r(\tau)} (n-k) f(n-k) |w_k([\widehat{0}, \tau])|.$$

This completes the proof of Theorem 3. □

6 Sifted sets and Shifted convolution of Whitney numbers

In this section, we study relations between the cardinality of sifted sets and shifted convolution of Whitney numbers in the special case of Dowling lattices. Consider $L = \mathcal{A} = Q_n(G)$ and $\mathcal{T}$ be a set of atoms whose join is τ. Also let $k = r(\tau)$. Then from Theorem 8, we have

$$\#\mathcal{A}_y \;=\; \#[y, \hat{1}] \;=\; \#Q_{n-r(y)}(G). \tag{15}$$

We set

$$X \;=\; \#Q_n(G) \quad \text{and} \quad f(m) \;=\; \frac{\#Q_m(G)}{\#Q_n(G)} \quad \text{for} \quad m \in \mathbb{N} \cup \{0\}.$$

From (14), we have

$$\begin{aligned} \mathcal{S}(\mathcal{A}, \mathcal{T}) \;&=\; \#Q_n(G) \sum_{s=0}^{k} f(n-s) \sum_{\substack{y \leqslant \tau \\ r(y)=s}} \mu(\hat{0}, y) \;=\; \sum_{s=0}^{k} \#Q_{n-s}(G) \sum_{\substack{y \leqslant \tau \\ r(y)=s}} \mu(\hat{0}, y) \\ &=\; \sum_{s=0}^{k} \#Q_{n-k+s}(G) \sum_{\substack{y \leqslant \tau \\ r(y)=k-s}} \mu(\hat{0}, y). \end{aligned}$$

For simplicity, we suppose that all blocks of τ are trivial except the zero-block, so that by Theorem 8, we have

$$[\hat{0}, \tau] \;\cong\; Q_k(G) \qquad \text{and} \qquad \sum_{\substack{y \leqslant \tau \\ r(y)=k-s}} \mu(\hat{0}, y) \;=\; w_m(k, s).$$

Thus we get

$$\mathcal{S}(\mathcal{A}, \mathcal{T}) \;=\; \sum_{s=0}^{k} \#Q_{n-k+s}(G) \cdot w_m(k, s) \;=\; \sum_{s=0}^{k} \sum_{r=0}^{n-k+s} w_m(k, s) W_m(n-k+s, r). \tag{16}$$

Hence the estimation of cardinality of sifted sets leads us to the study of **shifted convolution** of Whitney numbers of Dowling lattices:

$$\sum_{k \geqslant 0} w_m(n, k) W_m(k+s, t).$$

6.1 Proof of Proposition 5

Let n, s, t be non-negative integers and set

$$c_{n,t}(s) \;=\; \sum_{k \geqslant 0} w_m(n, k) W_m(k+s, t). \tag{17}$$

Notice that the sum in (17) is a finite sum, since $w_m(n,k) = 0$ whenever $k > n$. We know that (see [8, Theorem 7]) the numbers $w(n,k)$ satisfy the recurrence relation

$$w_m(n,k) \;=\; w_m(n-1,k-1) \;-\; (1+m(n-1))w_m(n-1,k)$$

and hence we get

$$c_{n,t}(s) \;=\; c_{n+1,t}(s-1) \;+\; (1+mn)c_{n,t}(s-1), \qquad s \in \mathbb{N}. \tag{18}$$

Now consider the rational generating function

$$F_{n,t}(x) \;=\; \sum_{s=0}^{\infty} c_{n,t}(s)x^s.$$

From (18) and Theorem 9, we get

$$F_{n,t}(x) \;=\; \frac{1-(1+m(n-1))x}{x} \cdot F_{n-1,t}(x) \;-\; \frac{\delta(n-1,t)}{x}. \tag{19}$$

By induction on n, we deduce that

$$F_{n,t}(x) \;=\; \prod_{i=0}^{n-1} \frac{1-(1+im)x}{x} \cdot F_{0,t}(x) \;-\; \sum_{j=0}^{n-1} \prod_{i=j+1}^{n-1} \frac{1-(1+im)x}{x} \cdot \frac{\delta(j,t)}{x}. \tag{20}$$

From [3, Theorem 5], we get

$$F_{0,t}(x) \;=\; \sum_{s=0}^{\infty} c_{0,t}(s)x^s \;=\; \sum_{s=t}^{\infty} W_m(s,t)x^s \;=\; \frac{1}{x}\prod_{i=0}^{t} \frac{x}{1-(1+im)x}. \tag{21}$$

From (20) and (21), we have

$$F_{n,t}(x) \;=\; \frac{1}{x}\prod_{i=0}^{n-1} \frac{1-(1+im)x}{x} \cdot \prod_{j=0}^{t} \frac{x}{1-(1+jm)x} \;-\; \sum_{j=0}^{n-1} \prod_{i=j+1}^{n-1} \frac{1-(1+im)x}{x} \cdot \frac{\delta(j,t)}{x}. \tag{22}$$

If $t < n$, then from (22), we get

$$F_{n,t}(x) \;=\; \frac{1}{x}\prod_{i=t+1}^{n-1} \frac{1-(1+im)x}{x} \;-\; \prod_{i=t+1}^{n-1} \frac{1-(1+im)x}{x} \cdot \frac{1}{x} \;=\; 0. \tag{23}$$

Thus we conclude that if $t < n$, then we have

$$c_{n,t}(s) \;=\; 0 \quad \forall \;\; s \in \mathbb{N} \cup \{0\}. \tag{24}$$

Suppose that $t \geqslant n$. Then from (22), we get

$$F_{n,t}(x) \;=\; \frac{1}{x}\prod_{j=n}^{t} \frac{x}{1-(1+jm)x}. \tag{25}$$

This completes the proof of Proposition 5. ☐

7 Sifted sets and Dowling numbers

In this section, we study relations between certain sifted sets and generalized Dowling numbers. Further, we derive an asymptotic expression for the generalized Dowling numbers.

For any non-negative integer r, one defines (see [5, 14]) r-Whitney numbers of the second kind of $Q_n(G)$ by

$$(mx+r)^n \;=\; \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k,$$

where $(x)_k = x(x-1)\cdots(x-k+1)$ denotes the k-th falling factorial. Then the numbers $W_{m,r}(n,k)$ satisfy the recurrence relation

$$W_{m,r}(n,k) \;=\; W_{m,r}(n-1,k-1) + (km+r)W_{m,r}(n-1,k)$$

and one can show that its rational generating function is given by

$$\sum_{s=0}^{\infty} W_{m,r}(s,k)x^s \;=\; \frac{1}{x}\prod_{i=0}^{k} \frac{x}{1-(r+im)x}. \tag{26}$$

From (24), (25) and (26), we conclude that

$$c_{n,t}(s) \;=\; W_{m,1+mn}(s,t-n) \quad \forall \;\; s \in \mathbb{N}\cup\{0\}. \tag{27}$$

From (16) and (27), we get

$$\begin{aligned} \mathcal{S}(\mathcal{A},\mathcal{T}) \;&=\; \sum_r \sum_s w_m(k,s) W_m(n-k+s,r) \;=\; \sum_r c_{k,r}(n-k) \\ &=\; \sum_r W_{m,1+mk}(n-k,r-k) \;=\; \sum_{r=0}^{n-k} W_{m,1+mk}(n-k,r). \end{aligned} \tag{28}$$

The Dowling numbers associated with $Q_n(G)$ (see [3, p. 22] [5, Sec. 5]) are defined by

$$D_m(n) \;=\; \sum_{k=0}^{n} W_m(n,k).$$

More generally, one defines r-Dowling numbers associated with $Q_n(G)$ by

$$D_{m,r}(n) \;=\; \sum_{k=0}^{n} W_{m,r}(n,k).$$

From (28), we conclude that

$$\mathcal{S}(\mathcal{A},\mathcal{T}) \;=\; D_{m,1+mk}(n-k). \tag{29}$$

In the next subsection, we will give a proof of Theorem 6 which gives an explicit expression for r-Dowling numbers $D_{m,r}(n)$ using the Cauchy's residue theorem and the method of steepest descent/saddle-point method (see for example [3, Sec. 4.2] and [4]) and the expression obtained is uniform in m and r.

7.1 Proof of Theorem 6

The exponential generating function of $D_{m,r}(n)$ (see [5, Eq. 26]) is given by

$$\sum_{n=0}^{\infty} D_{m,r}(n)\frac{z^n}{n!} \;=\; \exp\left(rz + \frac{e^{mz}-1}{m}\right) \quad \text{for} \quad z \in \mathbb{C}.$$

By Cauchy's residue theorem, we get

$$\begin{aligned} D_{m,r}(n) \;&=\; \frac{n!}{2\pi i}\int_{|z|=\delta} \frac{\exp\left(rz + \frac{e^{mz}-1}{m}\right)}{z^{n+1}}\, dz \\ &=\; \frac{n!}{2\pi\delta^n}\int_{-\pi}^{\pi} \exp\left(r\delta e^{i\theta} - in\theta + \frac{e^{m\delta e^{i\theta}}-1}{m}\right)\, d\theta, \end{aligned} \tag{30}$$

where $\delta > 0$ is a constant which will be chosen later. Set

$$G(z) \;=\; r\delta e^{iz} - inz + \frac{e^{m\delta e^{iz}}-1}{m}.$$

Then G is an entire function and the power series expansion of G around $z = 0$ is given by

$$G(z) \;=\; G(0) + G'(0)z + G''(0)\frac{z^2}{2} + H(z)z^3$$

for some entire function H. We choose δ such that $G'(0) = 0$ i.e., δ is a positive constant such that

$$r\delta \;+\; \delta e^{m\delta} \;=\; n. \tag{31}$$

Then we have

$$g_0 \;:=\; G(0) \;=\; r\delta + \frac{e^{m\delta}-1}{m} \quad \text{and} \quad g_2 := -\frac{G''(0)}{2} \;=\; \frac{n + m\delta^2 e^{m\delta}}{2}. \tag{32}$$

Let

$$\epsilon \;=\; \frac{(\log n)^5}{n^{1/2}}. \tag{33}$$

We split the integral in (30) into three parts as follows:

$$D_{m,r}(n) \;=\; \frac{n!}{2\pi\delta^n}\left(\int_{-\pi}^{-\epsilon} + \int_{-\epsilon}^{\epsilon} + \int_{\epsilon}^{\pi}\right)\exp\left(g_0 - g_2\theta^2 + H(\theta)\theta^3\right)\, d\theta. \tag{34}$$

Now we will show that

$$\left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi}\right)\exp\left(-g_2\theta^2 + \Re\{H(\theta)\}\theta^3\right)\, d\theta \;\ll\; \exp\left(-\frac{(\log n)^9}{20}\right). \tag{35}$$

Note that

$$\begin{aligned} -g_2\theta^2 + \Re\{H(\theta)\}\theta^3 \;&=\; \Re\{G(\theta)\} - \Re\{G(0)\} \\ &=\; r\delta(\cos\theta - 1) \;+\; \frac{e^{m\delta\cos\theta}\cos(m\delta\sin\theta) - e^{m\delta}}{m}. \end{aligned} \tag{36}$$

Note that

$$\cos\theta - 1 \;\leqslant\; -\frac{\theta^2}{5} \quad \text{for} \quad \theta \in [-\pi, \pi] \tag{37}$$

and

$$\begin{aligned} e^{m\delta\cos\theta}\cos(m\delta\sin\theta) - e^{m\delta} \;&\leqslant\; e^{m\delta\cos\theta} - e^{m\delta} \;=\; \sum_{k=1}^{\infty} \frac{(m\delta\cos\theta)^k - (m\delta)^k}{k!} \\ &=\; \sum_{k=1}^{\infty} \frac{(m\delta)^k}{k!}\left(\cos^k\theta - 1\right). \end{aligned} \tag{38}$$

Thus for $\theta \in [0, \pi/2]$, we have

$$e^{m\delta\cos\theta} - e^{m\delta} \;\leqslant\; (\cos\theta - 1)\sum_{k=1}^{\infty} \frac{(m\delta)^k}{k!} \;\leqslant\; -\frac{\theta^2}{5}\left(e^{m\delta} - 1\right). \tag{39}$$

If $\theta \in [\pi/2, \pi]$, then we have

$$e^{m\delta\cos\theta} - e^{m\delta} \;\leqslant\; 1 - e^{m\delta}. \tag{40}$$

From (38), (39) and (40), we deduce that

$$e^{m\delta\cos\theta}\cos(m\delta\sin\theta) - e^{m\delta} \;\leqslant\; 1 - \frac{\theta^2}{10}\left(e^{m\delta} - 1\right) \quad \text{for} \quad \theta \in [-\pi, \pi]. \tag{41}$$

From (36), (37) and (41), we get

$$-g_2\theta^2 + \Re\{H(\theta)\}\theta^3 \;\leqslant\; -\frac{r\delta\theta^2}{5} + \frac{1}{m} - \frac{\theta^2}{10m}\left(e^{m\delta} - 1\right). \tag{42}$$

Thus we have

$$\left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi}\right)\exp\left(-g_2\theta^2 + \Re\{H(\theta)\}\theta^3\right)\,d\theta \;\leqslant\; 2\int_{\epsilon}^{\pi}\exp\left(1 + \frac{\theta^2}{10} - \left(2r\delta + \frac{e^{m\delta}}{m}\right)\frac{\theta^2}{10}\right)\,d\theta. \tag{43}$$

Suppose that $n > e^m$. From (31), we get

$$\delta \;\leqslant\; \frac{\log n}{m}.$$

Further, we get

$$2r\delta + \frac{e^{m\delta}}{m} \;\geqslant\; \frac{n}{2\log n}. \tag{44}$$

Therefore, we have

$$\begin{aligned} \int_{\epsilon}^{\pi}\exp\left(-\left(2r\delta + \frac{e^{m\delta}}{m}\right)\frac{\theta^2}{10}\right)\,d\theta \;&\leqslant\; \int_{\epsilon}^{\pi}\exp\left(-\frac{n}{20\log n}\theta^2\right)\,d\theta \\ &\leqslant\; \pi\exp\left(-\frac{n}{20\log n}\epsilon^2\right) \;\leqslant\; \pi\exp\left(-\frac{(\log n)^9}{20}\right). \end{aligned} \tag{45}$$

Now the estimate in (35) follows from (43) and (45). Thus, we get

$$D_{m,r}(n) \;=\; \frac{e^{g_0} n!}{2\pi\delta^n}\left(\int_{-\epsilon}^{\epsilon} \exp\left(-g_2\theta^2 + H(\theta)\theta^3\right)\, d\theta \;+\; O\left(\exp\left(-\frac{(\log n)^9}{20}\right)\right)\right). \tag{46}$$

Consider the entire function

$$h(z) \;=\; \exp\left(H(z)z^3\right) \quad \text{for} \quad z \in \mathbb{C}.$$

and write

$$h(z) \;=\; 1 + \tilde{h}(z)z$$

for some entire function $\tilde{h}$. Thus we have

$$\begin{aligned}\int_{-\epsilon}^{\epsilon} \exp\left(-g_2\theta^2 + H(\theta)\theta^3\right)\, d\theta \;&=\; \int_{-\epsilon}^{\epsilon} \exp\left(-g_2\theta^2\right)\left(1 + \tilde{h}(\theta)\theta\right)\, d\theta \\ &=\; \int_{-\epsilon}^{\epsilon} \exp\left(-g_2\theta^2\right)\, d\theta \;+\; \int_{-\epsilon}^{\epsilon} \exp\left(-g_2\theta^2\right)\tilde{h}(\theta)\theta\, d\theta.\end{aligned} \tag{47}$$

By noting that $|e^w - 1| \leqslant \max\{|w|, |w|e^{\Re\{w\}}\} \leqslant |w|e^{|w|}$ for any $w \in \mathbb{C}$, we get

$$|\tilde{h}(z)| \;=\; \left|\frac{h(z)-1}{z}\right| \;=\; \left|\frac{e^{H(z)z^3}-1}{z}\right| \;\leqslant\; |H(z)||z|^2 e^{|H(z)||z|^3} \quad \text{for} \quad z \in \mathbb{C}\backslash\{0\} \tag{48}$$

and by continuity, the inequality in (48) holds for all $z \in \mathbb{C}$. For $z \in \mathbb{C}\backslash\{0\}$, we have

$$\begin{aligned}H(z) \;&=\; \frac{G(z) - g_0 + g_2 z^2}{z^3} \\ &=\; \frac{n}{z^3}\sum_{k=3}^{\infty}\frac{(iz)^k}{k!} \;+\; \frac{im\delta^2 e^{m\delta}}{z^2}\sum_{k=2}^{\infty}\frac{(iz)^k}{k!} \;+\; \frac{m\delta^2 e^{m\delta}}{2z^3}\left(\sum_{k=2}^{\infty}\frac{(iz)^k}{k!}\right)^2 \;+\; \frac{e^{m\delta}}{mz^3}\sum_{k=3}^{\infty}\frac{(m\delta(e^{iz}-1))^k}{k!}\end{aligned}$$

Thus for $\theta \in [-\epsilon, \epsilon]\backslash\{0\}$, we get

$$\begin{aligned}|H(\theta)| \;&\ll\; n \;+\; m\delta^2 e^{m\delta} \;+\; \frac{e^{m\delta}}{m|\theta|^3}\sum_{k=3}^{\infty}\frac{(m\delta|\theta|)^k}{k!} \\ &\ll\; n \;+\; m\delta n \;+\; m^2\delta^3 e^{(1+\epsilon)m\delta}.\end{aligned} \tag{49}$$

From (31) and (33), we get

$$|H(\theta)| \;\ll\; (1 + m\delta)n \;+\; m^2\delta^2 n \;\ll\; n(\log n)^2, \tag{50}$$

where the implied constant is absolute. From (48) and (50), we get

$$|\tilde{h}(\theta)| \;\ll\; (\log n)^{12} \quad \text{for} \quad \theta \in [-\epsilon, \epsilon]. \tag{51}$$

Hence we get

$$\int_{-\epsilon}^{\epsilon} \exp\left(-g_2\theta^2\right)|\tilde{h}(\theta)\theta|\, d\theta \;\ll\; \frac{(\log n)^{12}}{g_2}. \tag{52}$$

Also, we have

$$\int_{-\epsilon}^{\epsilon} \exp\left(-g_2\theta^2\right)\, d\theta \;=\; \frac{1}{\sqrt{g_2}}\int_{-\epsilon\sqrt{g_2}}^{\epsilon\sqrt{g_2}} e^{-u^2}\, du \;=\; \frac{1}{\sqrt{g_2}}\int_{-\infty}^{\infty} e^{-u^2}\, du \;-\; \frac{2}{\sqrt{g_2}}\int_{\epsilon\sqrt{g_2}}^{\infty} e^{-u^2}\, du$$
$$= \frac{\sqrt{\pi}}{\sqrt{g_2}} \;+\; O\left(\frac{e^{-\epsilon^2 g_2}}{\epsilon g_2}\right). \tag{53}$$

From (46), (47), (52) and (53), we get

$$D_{m,r}(n) \;=\; \frac{e^{g_0}n!}{\sqrt{4\pi g_2}\delta^n}\left(1 \;+\; O\left(\frac{e^{-\epsilon^2 g_2}}{\epsilon\sqrt{g_2}} \;+\; \frac{(\log n)^{12}}{\sqrt{g_2}} \;+\; \sqrt{g_2}\exp\left(-\frac{(\log n)^9}{20}\right)\right)\right).$$

From (32), we note that $n/2 \leqslant g_2 \leqslant n\log n$. Hence for $n > e^m$, we have

$$D_{m,r}(n) \;=\; \frac{e^{g_0}n!}{\sqrt{4\pi g_2}\delta^n}\left(1 \;+\; O\left(\frac{(\log n)^{12}}{\sqrt{n}}\right)\right),$$

where the implied constant is absolute. This completes the proof of Theorem 6. □

8 Concluding remarks

Above, we have shown that the classical Brun's sieve generalizes to the combinatorial context and this generalization is dependent on the deep work of June Huh and his school. Unlike the combinatorial Selberg sieve due to Wilson [24], the generalization of Brun's sieve is deeper. Our application to the theory of Dowling lattices is a humble beginning. We envisage a larger theory to emerge from these modest initiatives. There are many questions in combinatorics that can be formulated in some sort of sieve theoretic terms. Coloring problems come immediately to mind where these ideas could have potential applications. This is the hope for the future.

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