Ratliff Property of Edge Ideals of Weighted Oriented Graphs

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Abstract

Let D be a weighted oriented graph and $I(D)$ be its edge ideal. In this paper, we prove that $I(D)$ satisfies the Ratliff (strong persistence) property in the following three cases: (i) D has an outward leaf; (ii) D has an inward leaf $(u, v) \in E(D)$, where v is a sink vertex; (iii) D has an inward leaf $(u, v) \in E(D)$ with $w(v) = 1$. We further show that $(I(D)^2 : I(D)) = I(D)$ if D contains a vertex with in-degree less than or equal to 1, and $(I(D)^3 : I(D)) = I(D)^2$ when D is either a weighted oriented cycle, or a tree. Finally, if D contains no source vertex, then any associated prime of $I(D)^k$, other than the irrelevant maximal ideal, is also an associated prime of $I(D)^{k+1}$. In addition, if D contains a vertex of in-degree one and all the vertices of D have non-trivial weights, we show that the persistence property holds.

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1 Introduction

Monomial ideals are fundamental in commutative algebra, serving as a bridge between algebraic and combinatorial objects. They play a central role in ideal theory, Gröbner bases, and algebraic geometry, particularly in the study of toric varieties. In combinatorics, monomial ideals encode properties of graphs and other combinatorial structures. Additionally, they are used in coding theory and computational complexity, facilitating the construction of error-correcting codes, providing insights into the complexity of polynomial systems, and solving integer programming problems. Various classes of monomial ideals are associated with the combinatorics of a graph, including edge ideals and cover ideals. An edge ideal is a monomial ideal generated by quadratic square-free monomials associated to the edges of a simple graph. Recently, a generalization of simple graphs

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known as weighted oriented graphs has attracted considerable attention due to its connections with algebraic coding theory. A weighted oriented graph is a directed graph with weights (natural numbers) associated with each vertex. The edge ideal of a weighted oriented graph is generated by monomials of the form $x_i {x_j}^{w_j}$, where there is a directed edge from x_i to x_j and the weight of the vertex x_j is w_j. Algebraic properties of weighted oriented graphs such as Cohen-Macaulayness, unmixedness, Castelnuovo-Mumford regularity, etc. have been studied in [4], [5], [6], [10], [13], [15], [17], [24]; whereas symbolic power has been studied in [1], [8], [16], and [18]. One of the motivations for studying edge ideals of weighted oriented graphs is their connection to coding theory, particularly in the study of Reed-Muller-type codes (see [20]).

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, and $I \subseteq R$ be a monomial ideal. The set of all associated primes of I will be denoted by $\operatorname{Ass}(R/I)$. Although the associated primes of powers of ideals behave erratically, they stabilize for larger powers. In 1979, Brodmann (see [3]) showed that if I is an ideal in a Noetherian ring, then the associated primes of I^n stabilize for large values of n. But in general, for a given ideal, it is often difficult to determine this stable set or to identify when the sets of associated primes become stable. Thus, Brodmann's result suggests the following weaker property, known as the *persistence property* of ideals: for which ideals does the containment $\operatorname{Ass}(R/I^k) \subseteq \operatorname{Ass}(R/I^{k+1})$ hold for all $k \geqslant 1$? This question has attracted much attention in the literature, and has been investigated for various special classes of monomial ideals (see [2], [12], [14], [19], [21], [23], [27] etc.). For large $k \geqslant 1$, Ratliff [26] proved that $(I^{k+1} : I) = I^k$ holds, which in turn implies that $\operatorname{Ass}(R/I^k) \subseteq \operatorname{Ass}(R/I^{k+1})$. An ideal $I \subseteq R$ is said to satisfy the *Ratliff condition* (or, the *strong persistence property*) if $(I^{k+1} : I) = I^k$ for all $k \geqslant 1$. Hence, the Ratliff condition implies the persistence property. Many interesting classes of ideals satisfy the Ratliff condition, for instance, edge ideals of simple graphs [19], normally torsion-free monomial ideals [25], polymatroidal ideals [12], and the cover ideals of some imperfect graphs, such as, cycle graphs of odd order, wheel graphs of even order, and helm graphs of odd order greater than or equal to five [22]. On the other hand, the Ratliff condition and the persistence property are not equivalent, even for the class of square-free monomial ideals (see Example 10). Continuing this line of investigation, for the class of edge ideals of weighted oriented graphs, one can ask the following questions.

Question 1. Let D be a weighted oriented graph, and let $I(D)$ denote its edge ideal.

1. Does $I(D)$ satisfy the Ratliff condition?
2. If the answer to the above question is negative, does $I(D)$ still satisfy the persistence property?

In this article, we primarily study the above two questions for the class of edge ideals of weighted oriented graphs. We show that the Ratliff condition is satisfied by the following classes of ideals:

Theorem 2 (Theorem 11, Theorem 12, and Theorem 13)**.** *Let D be a weighted oriented graph, and let $I(D)$ denote the edge ideal of D. Then $I(D)$ satisfies the Ratliff condition in the following cases:*

1. *D contains an edge $(u,v) \in E(D)$ such that $\deg_D(v) = 1$.*

2. *D contains an edge $(u,v) \in E(D)$ with $\deg_D(u) = 1$ and v is a sink vertex.*

3. *D contains an edge $(u,v) \in E(D)$ with $\deg_D(u) = 1$ and $w(v) = 1$.*

Next, we establish partial results related to the Ratliff condition and the persistence property for smaller powers of edge ideals of weighted oriented graphs. In particular, we show that for any weighted oriented graph D containing a vertex v with $\text{indeg}_D(v) \leqslant 1$, the equality $(I(D)^2 : I(D)) = I(D)$ holds (see Theorem 16). Moreover, if D is a weighted oriented tree with an inward leaf edge, or a naturally oriented cycle, then $(I(D)^3 : I(D)) = I(D)^2$ holds (see Theorem 18 and Theorem 19).

While studying the depth function of ideals, Herzog and Hibi [11] constructed monomial ideals whose depth function can be any non-decreasing convergent numerical function. In particular, they came up with the following family of ideals:

Example 3. ([11, Theorem 4.1] and [9, Example 3.1]) Let $S = \mathbb{K}[x,y,z]$. For any integer $d \geqslant 2$, consider

$$I_d = (x^{d+2}, x^{d+1}y, xy^{d+1}, y^{d+2}, x^d y^2 z).$$

Then $\mathfrak{m} \in \text{Ass}(S/I_d^n)$ if and only if $n \leqslant d-1$.

Let $\mathfrak{m}$ denote the irrelevant maximal ideal of the polynomial ring R. In general, the following interesting result was proved by Há, Nguyen, Trung, and Trung [9], which answers a question raised by Ratliff.

Theorem 4. [9, Corollary 4.2] *Let Γ be a set of positive integers which is either finite or contains all sufficiently large integers. Then there exists a monomial ideal Q in the polynomial ring R such that $\mathfrak{m} \in \text{Ass}(R/Q^n)$ if and only if $n \in \Gamma$, where $\mathfrak{m}$ is the maximal homogeneous ideal of R.*

Moreover, for the above example, computational evidence suggests that $\text{Ass}(S/I_d^k) \setminus \{\mathfrak{m}\} \subseteq \text{Ass}(S/I_d^{k+1}) \setminus \{\mathfrak{m}\}$ for all $k \geqslant 1$. This observation motivates the following definition.

Definition 5. Let $I \subseteq R$ be an ideal. We say that I has the *punctured persistence property* if $\text{Ass}(R/I^k) \setminus \{\mathfrak{m}\} \subseteq \text{Ass}(R/I^{k+1}) \setminus \{\mathfrak{m}\}$ holds for all $k \geqslant 1$.

For a sufficiently general class of weighted oriented graphs, we show that if we do not take the irrelevant maximal ideal $\mathfrak{m}$ into our consideration, the punctured persistence property holds. More precisely, we prove the following.

Theorem 6 (Theorem 21)**.** *Let D be a weighted oriented graph without any source vertex. Then $I(D)$ satisfies the punctured persistence property.*

This article is organized in the following way. In Section 2, we recall the properties related to weighted oriented graphs and the persistence property of ideals. Section 3 contains all of our main results. In this section, we show that the Ratliff property holds for certain classes of edge ideals of weighted oriented graphs (Theorems 11,12, and 13). We also discuss persistence property for some smaller powers of edge ideals. Finally, we prove that if a weighted oriented graph D has no source vertices, then the punctured persistence property holds (Theorem 21).

2 Preliminaries

In this section, we recall all the known results and properties related to weighted oriented graphs, as well as the persistence and Ratliff properties of monomial ideals.

Let D be a weighted oriented graph with vertex set $V(D) = \{x_1, \dots, x_n\}$, edge set $E(D) \subseteq V(D) \times V(D)$, and the weight function $w : V(D) \to \mathbb{N}$. The *edge ideal* of D is denoted by $I(D)$, and is defined by

$$I(D) = (\{x_i x_j^{w(x_j)} \mid (x_i, x_j) \in E(D)\}).$$

Let D be an oriented graph, and let x be a vertex of D. The *out-neighborhood* of x, denoted by $N_D^+(x)$, consists of all y in $V(D)$ such that $(x, y) \in E(D)$. The *in-neighborhood* of x, denoted by $N_D^-(x)$, is the set of all y in $V(D)$ such that $(y, x) \in E(D)$. We define the *out-degree* of the vertex x to be $\operatorname{outdeg}_D(x) = |N_D^+(x)|$, and similarly, the *in-degree* of x as $\operatorname{indeg}_D(x) = |N_D^-(x)|$. The *neighborhood* of x is the set $N_D(x) = N_D^+(x) \cup N_D^-(x)$. In a similar manner, if $S \subseteq V(D)$, we define $N_D^+(S) = \bigcup_{x \in S} N_D^+(x)$ and $N_D^-(S) = \bigcup_{x \in S} N_D^-(x)$. The *degree* of the vertex x is $\deg_D(x) = |N_D(x)|$. A non-isolated vertex $x \in V(D)$ is called a *source* (respectively, a *sink*) if it has no in-neighbors (respectively, no out-neighbors). We denote by $V^*(D)$ the set of vertices of D with non-trivial weight; that is, $V^*(D) = \{x \in V(D) \mid w(x) > 1\}$. To simplify the notation, we shall write w_j to denote $w(x_j)$ for a vertex $x_j \in V(D)$. To avoid trivial cases, we assume that D contains no isolated vertices.

Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, and $I \subseteq R$ be an ideal.

Definition 7. An ideal $I \subseteq R$ is said to satisfy the *persistence property* if

$$\operatorname{Ass}(R/I^k) \subseteq \operatorname{Ass}(R/I^{k+1}) \text{ for all } k \geqslant 1.$$

Definition 8. An ideal I is said to satisfy the *Ratliff condition* if $(I^{k+1} : I) = I^k$ for all $k \geqslant 1$.

The strong persistence property was introduced in [12, Definition 1.1] for arbitrary ideals in Noetherian rings, and it was later shown to be equivalent to the Ratliff property [12, Theorem 1.3]. Satisfying the Ratliff condition is a stronger property, which implies that the ideal has the persistence property.

Lemma 9. [12] Let $I \subseteq R$ be an ideal. If $(I^{k+1} : I) = I^k$ for some $k \geqslant 1$, then $\operatorname{Ass}(R/I^k) \subseteq \operatorname{Ass}(R/I^{k+1})$. In particular, if I satisfies the Ratliff condition, then I satisfies the persistence property.

As the following example suggests, the converse is not true even for the class of square-free monomial ideals.

Example 10. [19, Example 2.18] Let $R = \mathbb{K}[x_1, \dots, x_6]$ and let I be the square-free monomial ideal

$$I = (x_1x_2x_5, x_1x_3x_4, x_1x_2x_6, x_1x_3x_6, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5, x_2x_4x_6, x_3x_5x_6, x_4x_5x_6).$$

Then one can verify that $(I^2 : I) = I$ and $(I^3 : I) \neq I^2$. Although, the sets of associated primes of the powers of I form an ascending chain.

3 Main Results

This section contains all of our main results and has been subdivided into three subsections. In the first subsection, we study the Ratliff property of edge ideals of weighted oriented graphs.

3.1 The Ratliff Condition

The Ratliff condition holds for the edge ideal of a simple graph. On the other hand, there are many examples of square-free monomial ideals for which even the persistence property fails. In this subsection, we show that for various classes of edge ideals of weighted oriented graphs, the Ratliff condition holds. In the following theorem, we show that if there is a outward leaf edge in a weighted oriented graph D, then the edge ideal $I(D)$ is Ratliff.

Theorem 11. *Let D be a weighted oriented graph such that there is an edge $(u, v) \in E(D)$ with $\deg_D(v) = 1$. Then $(I(D)^{k+1} : I(D)) = I(D)^k$ for all $k \geqslant 1$.*

Proof. Let $(x_i, x_j) \in E(D)$ be such that $\deg_D(x_j) = 1$. Let $f \in (I(D)^{k+1} : I(D))$. Then

$$f \cdot x_i {x_j}^{w_j} = e_1 e_2 \cdots e_{k+1} \mathbf{x}^\alpha$$

for some $e_i \in \mathcal{G}(I(D)), 1 \leqslant i \leqslant k+1$, and $\mathbf{x}^\alpha$ is a monomial. We then consider the following two cases:
Case 1: If ${x_j}^{w_j} \mid \mathbf{x}^\alpha$, then $f \cdot x_i = e_1 e_2 \cdots e_{k+1} \mathbf{x}^\beta$ where $\mathbf{x}^\beta = \frac{\mathbf{x}^\alpha}{{x_j}^{w_j}}$. Now if $x_i \mid e_t$ for some $1 \leqslant t \leqslant k+1$, then $f \in I(D)^k$, and if $x_i \nmid e_t$ for all $1 \leqslant t \leqslant k+1$, then $x_i \mid \mathbf{x}^\beta$ which in turn implies that $f \in I(D)^{k+1}$, and hence $f \in I(D)^k$.
Case 2: If ${x_j}^{w_j} \nmid \mathbf{x}^\alpha$, then $x_j \mid e_t$ for some $1 \leqslant t \leqslant k+1$. Without any loss of generality, assume $x_j \mid e_1$. Then, $e_1 = x_i {x_j}^{w_j}$ as x_i is the only neighbor of x_j, and hence we have $f = e_2 \cdots e_{k+1} \mathbf{x}^\alpha$. This again implies that $f \in I(D)^k$. □

Next, we consider the case where D contains a leaf that is not directed outward. Under some additional assumptions, we show that the Ratliff property holds.

Theorem 12. *Let D be a weighted oriented graph such that there is an edge $(u, v) \in E(D)$ with $\deg_D(u) = 1$. If v is a sink vertex, then $(I(D)^{k+1} : I(D)) = I(D)^k$ for all $k \geqslant 1$.*

Proof. Let $(x_i, x_j) \in E(D)$, where $\deg_D(x_i) = 1$, and x_j be a sink vertex. Then for any $f \in (I(D)^{k+1} : I(D))$, we have

$$f \cdot x_i {x_j}^{w_j} = e_1 e_2 \cdots e_{k+1} \mathbf{x}^\alpha,$$

where $e_i \in \mathcal{G}(I(D)), 1 \leqslant i \leqslant k+1$, and $\mathbf{x}^\alpha$ is a monomial. Now, we have the following two cases:
Case 1: If ${x_j}^{w_j} \mid \mathbf{x}^\alpha$, then $f x_i = e_1 \cdots e_{k+1} \mathbf{x}^\alpha$. Note that either $x_i \mid \mathbf{x}^\alpha$, or $x_i \mid e_i$ for some $1 \leqslant i \leqslant k+1$. So x_i divides at most one of $e_1, \ldots, e_{k+1}$, yielding the fact that $f \in I(D)^k$.
Case 2: Assume that ${x_j}^{w_j} \nmid \mathbf{x}^\alpha$. If $e_t = x_i {x_j}^{w_j}$ for some $t \in \{1, \ldots, k+1\}$, then $f = e_1 \cdots e_{t-1} e_{t+1} \cdots e_{k+1} \mathbf{x}^\alpha$, and hence $f \in I(D)^k$. Now if ${x_j}^{w_j} \nmid \mathbf{x}^\alpha$ and $x_i {x_j}^{w_j} \neq e_t$ for

all $t \in \{1, \ldots, k+1\}$, then $x_i \nmid e_t$ for all $t \in \{1, \ldots, k+1\}$. This implies $x_i \mid \mathbf{x}^\alpha$, and $x_j \mid e_s$ for some $s \in \{1, \ldots, k+1\}$. Since x_j is a sink vertex, $e_s = x_\ell {x_j}^{w_j}$ for some $x_\ell \in V(D)$ with $x_\ell \neq x_i$, and thus, $f = e_1 \cdots e_{s-1} e_{s+1} \cdots e_{k+1} \mathbf{x}^\beta$ for some suitable monomial $\mathbf{x}^\beta$. Hence $f \in I(D)^k$, which concludes the proof. □

Theorem 13. *Let D be a weighted graph such that there is an edge $(u, v) \in E(D)$ such that $\deg_D(u) = 1$ and $w(v) = 1$, then $(I(D)^{k+1} : I(D)) = I(D)^k$ for all $k \geqslant 1$.*

Proof. Let $(x_i, x_j) \in E(D)$ where $\deg_D(x_i) = 1$ and $w(x_j) = 1$. Suppose $f \in (I(D)^{k+1} : I(D))$. Then

$$f \cdot x_i x_j = e_1 \cdots e_{k+1} \mathbf{x}^\alpha,$$

where $e_i \in \mathcal{G}(I(D))$ for all $1 \leqslant i \leqslant k+1$ and $\mathbf{x}^\alpha \in R$ is a monomial. If $x_i \mid e_t$ for some $1 \leqslant t \leqslant k+1$, then $e_t = x_i x_j$, and hence $f = e_1 \cdots e_{t-1} e_{t+1} \cdots e_{k+1} \mathbf{x}^\alpha \in I(D)^k$. On the other hand, if $x_i \mid \mathbf{x}^\alpha$, then note that x_j can divide at most one edge among $e_1, \ldots, e_{k+1}$, say e_t. Then again, $f = e_1 \cdots e_{t-1} e_{t+1} \cdots e_{k+1} \mathbf{x}^\beta \in I(D)^k$ for some monomial $\mathbf{x}^\beta$, which implies that $f \in I(D)^k$. □

The following remark is a consequence of [27, Theorem 5], which we include here for the reader's convenience.

Remark 14. If D is a weighted oriented graph such that all vertices except sink are of weight 1. Then we have $(I(D)^{k+1} : I(D)) = I(D)^k$ for all $k \geqslant 1$.

Remark 15. If D is a weighted oriented graph as given in Theorem 11, Theorem 12, or Theorem 13, then we have $\operatorname{Ass}(R/I(D)^n) \subseteq \operatorname{Ass}(R/I(D)^{n+1})$ for all $n \in \mathbb{N}$.

3.2 Ratliff Condition for Small Powers

In this subsection, we study the persistence property for small powers of edge ideals of weighted oriented graphs. As observed in the previous section, the existence of a leaf in a weighted graph plays an important role in ensuring that the Ratliff condition holds. We now show that, under a weaker structural condition, the Ratliff condition still holds for the second power of the ideal.

Theorem 16. *Let D be an weighted oriented graph such that there is a vertex $v \in V(D)$ with $\operatorname{indeg}_D(v) \leqslant 1$. Then $(I(D)^2 : I(D)) = I(D)$. Consequently, $\operatorname{Ass}(R/I(D)) \subseteq \operatorname{Ass}(R/I(D)^2)$.*

Proof. First, let us assume that $x_i \in V(D)$ be such that $\operatorname{indeg}_D(x_i) = 0$, that is, x_i is a source vertex. Then there is $x_j \in V(D)$ such that $(x_i, x_j) \in E(D)$. Let $f \in (I(D)^2 : I(D))$. Now we have

$$f \cdot x_i {x_j}^{w_j} = e_1 e_2 \mathbf{x}^\alpha,$$

where $e_1, e_2 \in \mathcal{G}(I(D))$, and $\mathbf{x}^\alpha$ is a monomial. We assume that $e_1, e_2 \neq x_i {x_j}^{w_j}$ and ${x_j}^{w_j} \nmid \mathbf{x}^\alpha$; otherwise, we would have $f \in I(D)$, as desired. Without any loss of generality, assume that $x_j \mid e_1$. Then there is $x_k \in V(D), k \neq i, j$ such that $e_1 = x_j {x_k}^{w_k}$, or $e_1 = x_k {x_j}^{w_j}$. We now consider the following two cases.

CASE 1: Assume that $e_1 = x_j{x_k}^{w_k}$. Then we have $f{\cdot}x_i{x_j}^{w_j} = x_j{x_k}^{w_k}e_2\mathbf{x}^\alpha$. This implies that ${x_k}^{w_k} \mid f$. Set $f = {x_k}^{w_k}h$ for some monomial h. Since $x_j{x_k}^{w_k} \in I(D)$, we have

$$f{\cdot}x_j{x_k}^{w_k} = e'_1e'_2\mathbf{x}^\beta$$

for some $e'_1, e'_2 \in \mathcal{G}(I(D))$ and $\mathbf{x}^\beta$ a monomial. Again if $x_j{x_k}^{w_k} = e'_1, e'_2$, or ${x_k}^{w_k} \mid \mathbf{x}^\beta$, then $f \in I(D)$. So assume that none of these happens. Without any loss of generality, we assume that $x_k \mid e'_1$. Then there is $x_\ell \in V(D), \ell \neq j, k$ such that $e'_1 = x_k{x_\ell}^{w_\ell}$ or $e'_1 = x_\ell{x_k}^{w_k}$. We further consider the following subcases.
SUBCASE 1: Assume that $e'_1 = x_k{x_\ell}^{w_\ell}$. Then we have $f{\cdot}x_j{x_k}^{w_k} = x_k{x_l}^{w_l}e'_2\mathbf{x}^\beta$. It is evident that ${x_\ell}^{w_\ell} \mid f$. Therefore, $f = h{x_k}^{w_k} = h'{x_k}^{w_k}{x_\ell}^{w_\ell} = h'{x_k}^{w_k-1}e'_1$ for some monomial h'. This shows that $f \in I(D)$.
SUBCASE 2: Assume that $e'_1 = x_\ell{x_k}^{w_k}$. Then we have $f{\cdot}x_j{x_k}^{w_k} = x_\ell{x_k}^{w_k}e'_2\mathbf{x}^\beta$. It is evident that $x_\ell \mid f$. Therefore, $f = h{x_k}^{w_k} = h'{x_k}^{w_k}x_\ell = h'e'_1$ for some monomial h'. This proves that $f \in I(D)$.
CASE 2: Assume that $e_1 = x_k{x_j}^{w_j}$. Then we have $f{\cdot}x_i{x_j}^{w_j} = x_k{x_j}^{w_j}e_2\mathbf{x}^\alpha$. If $x_i \nmid e_2$, then again $f \in I(D)$ and we are through. Now assume that $x_i \mid e_2$. Since x_i is a source vertex, there is $x_m \in V(D), m \neq j$ such that $e_2 = x_i{x_m}^{w_m}$. Then we have

$$f{\cdot}x_i{x_j}^{w_j} = (x_k{x_j}^{w_j})(x_i{x_m}^{w_m})\mathbf{x}^\alpha.$$

Again, it is evident that ${x_m}^{w_m} \mid f$. So, we can write $f = h{x_m}^{w_m}$ for some monomial h. Now $x_i{x_m}^{w_m} \in I(D)$ and hence, $f{\cdot}x_i{x_m}^{w_m} \in I(D)^2$. Thus, $f{\cdot}x_i{x_m}^{w_m} = e'_1e'_2\mathbf{x}^\beta$ for some $e'_1, e'_2 \in \mathcal{G}(I(D))$ and $\mathbf{x}^\beta$ a monomial. Then proceeding similarly as in CASE 1, one can show that $f \in I(D)$.

Finally, we assume that $\operatorname{indeg}_D(x_i) = 1$ for some $x_i \in V(D)$. Let $(x_j, x_i) \in E(D)$ for some $x_j \in V(D)$. Let $f \in (I(D)^2 : I(D))$. We now have

$$f{\cdot}x_j{x_i}^{w_i} = e_1e_2\mathbf{x}^\alpha$$

where $e_1, e_2 \in \mathcal{G}(I(D))$ and $\mathbf{x}^\alpha$ a monomial. We assume that $e_1, e_2 \neq x_j{x_i}^{w_i}$ and ${x_i}^{w_i} \nmid \mathbf{x}^\alpha$; otherwise, we would have $f \in I(D)$, as desired. Without any loss of generality, assume that $x_i \mid e_1$. Since $\operatorname{indeg}_D(x_i) = 1$ and $e_1 \neq x_j{x_i}^{w_i}$, we have $e_1 = x_i{x_k}^{w_k}$ for some $x_k \in V(D), k \neq j$. Now $f{\cdot}x_j{x_i}^{w_i} = (x_i{x_k}^{w_k})e_2\mathbf{x}^\alpha$ and this implies that ${x_k}^{w_k} \mid f$. Thus we can write $f = h{x_k}^{w_k}$ for some monomial h. Again since $x_i{x_k}^{w_k} \in I(D)$, we have

$$f{\cdot}x_i{x_k}^{w_k} = e'_1e'_2\mathbf{x}^\beta$$

for some $e'_1, e'_2 \in \mathcal{G}(I(D))$ and $\mathbf{x}^\beta$ a monomial. To avoid triviality, we assume that $e'_1, e'_2 \neq x_i{x_k}^{w_k}$ and ${x_k}^{w_k} \nmid \mathbf{x}^\beta$. Without any loss of generality, we may assume that $x_k \mid e'_1$. Then either $e'_1 = x_k{x_\ell}^{w_\ell}$, or $e'_1 = x_\ell{x_k}^{w_k}$ for some $x_\ell \in V(D), \ell \neq i$. We again consider the relation $f{\cdot}e'_1 \in I(D)^2$. Then proceeding similarly as in Subcases 1 and 2, we can conclude that $e'_1 \mid f$, and hence, $f \in I(D)$. □

Corollary 17. *Let D be a weighted oriented graph such that there is a vertex $v \in V(D)$ with $\deg_D(v) = 1$. Then $(I(D)^2 : I(D)) = I(D)$. Consequently,* $\operatorname{Ass}(R/I(D)) \subseteq \operatorname{Ass}(R/I(D)^2)$.

The assumption that there exists a vertex $v \in V(D)$ with $\operatorname{indeg}_D(v) \leqslant 1$ in essential in the above theorem. To see this, consider the weighted oriented graph D whose edge ideal is given by

$$I(D) = ({x_2 x_1}^5, {x_3 x_2}^5, {x_4 x_3}^5, {x_4 x_1}^5, {x_1 x_3}^5, {x_4 x_2}^5, {x_5 x_4}^5, {x_6 x_4}^5, {x_5 x_6}^5, {x_1 x_5}^5, {x_3 x_6}^5, {x_3 x_5}^5).$$

A direct computation shows that $I(D)^{k+1} : I(D) \neq I(D)^k$ for $k = 1, 2, 3$. Therefore, in general, the Ratliff condition does not hold for edge ideals of weighted oriented graphs. Nevertheless, in case of weighted oriented forests, the Ratliff condition is satisfied for the third power as well.

Theorem 18. *Let D be a weighted oriented forest. Then $(I(D)^3 : I(D)) = I(D)^2$. Consequently,* $\operatorname{Ass}(R/I(D)^2) \subseteq \operatorname{Ass}(R/I(D)^3)$.

Proof. Since D is a weighted oriented forest, there is a vertex $x_i \in V(D)$ such that $\deg_D(x_i) = 1$. Let $N_D(x_i) = \{x_j\}$. If $(x_j, x_i) \in E(D)$, then it follows from Theorem 11 that $(I(D)^3 : I(D)) = I(D)^2$. Now we consider the case when $(x_i, x_j) \in E(D)$. Let $f \in (I(D)^3 : I(D))$. Then

$$f \cdot x_i {x_j}^{w_j} = e_1 e_2 e_3 \mathbf{x}^\alpha$$

where $e_t \in \mathcal{G}(I(D)), t = 1, 2, 3$ and $\mathbf{x}^\alpha \in R$ a monomial. Note that if $x_i \mid e_t$ for some $1 \leqslant t \leqslant 3$, then $e_t = x_i {x_j}^{w_j}$. Then we have $f \in I(D)^2$, as desired. So we assume that $x_i \nmid e_t$, or equivalently, $e_t \neq x_i {x_j}^{w_j}$ for $t = 1, 2, 3$. Then note that $x_i \mid \mathbf{x}^\alpha$, and we can rewrite the above equation as $f \cdot {x_j}^{w_j} = e_1 e_2 e_3 \mathbf{x}^\beta$, where $\mathbf{x}^\beta = \frac{\mathbf{x}^\alpha}{x_i}$. Now again, if $x_j \nmid e_t$ for all $t = 1, 2, 3$, or if $x_j^{w_j} \mid e_t$ for some $t = 1, 2, 3$, then we get $f \in I(D)^2$. So, without any loss of generality, assume that $x_j \mid e_1, e_2$, and $e_1 = x_j x_k^{w_k}$, $e_2 = x_j x_\ell^{w_\ell}$ for some $x_k, x_\ell \in V(D)$. We now consider the following cases.

Case 1: Assume that $e_1 \neq e_2$. Then we have $k \neq \ell$, and from the equality $f \cdot {x_j}^{w_j} = (x_j {x_k}^{w_k})(x_j {x_\ell}^{w_\ell}) e_3 \mathbf{x}^\beta$ we get, ${x_k}^{w_k} {x_\ell}^{w_\ell} \mid f$. Therefore, $f = {x_k}^{w_k} {x_\ell}^{w_\ell} h$ for some monomial $h \in R$. Since $e_1 = x_j {x_k}^{w_k} \in I(D)$, we get

$$f \cdot x_j {x_k}^{w_k} = e_1' e_2' e_3' \mathbf{x}^{\alpha_1},$$

where $e_t' \in \mathcal{G}(I(D))$ for $1 \leqslant t \leqslant 3$ and $\mathbf{x}^{\alpha_1} \in R$ a monomial. As before, we assume ${x_k}^{w_k} \nmid \mathbf{x}^{\alpha_1}$ or $e_t' \neq x_j {x_k}^{w_k}$ for any $t = 1, 2, 3$. Without any loss of generality, assume that $x_k \mid e_1'$. Then $e_1' = x_k {x_p}^{w_p}$, or $e_1' = x_p {x_k}^{w_k}$ for some $x_p \in V(D), p \neq j$. We now consider the following subcases:

Subcase 1.1: Assume that $e_1' = x_k {x_p}^{w_p}$. Then we have the equality $f \cdot x_j {x_k}^{w_k} = x_k {x_p}^{w_p} e_2' e_3' \mathbf{x}^{\alpha_1}$, and it follows that ${x_p}^{w_p} \mid f$. Hence $f = {x_k}^{w_k} {x_\ell}^{w_\ell} h = {x_k}^{w_k} {x_\ell}^{w_\ell} {x_p}^{w_p} h'$ where $h' = \frac{h}{x_p^{w_p}}$. Thus we conclude that $f = e_1' {x_k}^{w_k - 1} {x_\ell}^{w_\ell} h'$.

Subcase 1.2: Assume that $e_1' = x_p {x_k}^{w_k}$. Then we have the equality $f \cdot x_j {x_k}^{w_k} = x_p {x_k}^{w_k} e_2' e_3' \mathbf{x}^{\alpha_1}$, and it follows that, $x_p \mid f$. Then $f = {x_k}^{w_k} {x_\ell}^{w_\ell} h = {x_k}^{w_k} {x_\ell}^{w_\ell} x_p h'$, where $h' = \frac{h}{x_p}$. Hence, $f = e_1' {x_\ell}^{w_\ell} h'$.

So, in either of the above subcases, we have $f = e_1' g$, for some monomial g. Thus, either $x_k {x_p}^{w_p} \mid f$, or $x_p {x_k}^{w_k} \mid f$ Now, proceeding similarly as in the above subcases with

$e_2 = x_j x_\ell^{w_\ell}$ and $f{\cdot}x_j x_\ell^{w_\ell} \in I(D)^3$, we would obtain the following: either $x_\ell {x_q}^{w_q} \mid f$, or $x_q {x_\ell}^{w_\ell} \mid f$, where $x_q \in V(D)$ and $q \neq j$. Note that $p \neq q$, as otherwise the vertices $x_j, x_k, x_p{=}x_q, x_l$ forms a cycle in D, which is a contradiction. Therefore f is divisible by two disjoint edges of D, which implies that $f \in I(D)^2$.
Case 2: Assume that $e_1 = e_2$. Then $k = \ell$ and $f{\cdot}x_i {x_j}^{w_j} = (x_j {x_k}^{w_k})^2 e_3 \mathbf{x}^\alpha$. This implies that ${x_k}^{2w_k} \mid f$ and hence, $f = {x_k}^{2w_k} h$ for some monomial $h \in R$. Since $x_j {x_k}^{w_k} \in I(D)$, we get

$$f{\cdot}x_j {x_k}^{w_k} = e_1' e_2' e_3' \mathbf{x}^\beta$$

where $e_t' \in \mathcal{G}(I(D))$ for $t = 1, 2, 3$ and $\mathbf{x}^\beta$ a monomial. As before, we assume that $x_k^{w_k} \nmid \mathbf{x}^\beta$ and $e_t{}' \neq x_j {x_k}^{w_k}$ for any $t = 1, 2, 3$. Without any loss of generality, assume that $x_k \mid e_1'$. Then $e_1' = x_k {x_m}^{w_m}$ or $e_1' = x_m {x_k}^{w_k}$, for some $x_m \in V(D), m \neq j$. We consider the following subcases:
subcase 2.1: Assume that $e_1' = x_k {x_m}^{w_m}$. Then we have the equality $f{\cdot}x_j {x_k}^{w_k} = x_k {x_m}^{w_m} e_2' e_3' \mathbf{x}^\beta$. This implies that ${x_m}^{w_m} \mid f$ and hence $f = x_k^{2w_k} h = x_k^{2w_k} x_m^{w_m} h'$, where $h' = \frac{h}{x_m^{w_m}}$. We consider the following subcases:
subcase 2.1.1: Assume that $x_k \mid e_2' e_3'$. Without any loss of generality, assume that $x_k \mid e_2'$. Then either $e_2' = x_k {x_p}^{w_p}$, or $e_2' = x_p {x_k}^{w_k}$ for some $x_p \in V(D), p \neq j$. In either case we get $f = (x_k {x_m}^{w_m}) e_2' g$ for some monomial g, and consequently $f \in I(D)^2$.
subcase 2.1.2: Assume that $x_k \nmid e_2' e_3'$. Now from the equality

$$f{\cdot}x_j {x_k}^{w_k} = x_k {x_m}^{w_m} e_2' e_3' \mathbf{x}^\beta,$$

observe that x_j can divide at most one of the e_2' and e_3'. If $x_j \nmid e_2' e_3'$, then $e_2' e_3' \mid f$ and we are through. Without any loss of generality, assume that $x_j \mid e_3'$. Since $x_k \nmid e_2'$, we have ${x_m}^{w_m} e_2' \mid f$. Therefore, $f = x_k^{2w_k} h = x_k^{2w_k} {x_m}^{w_m} e_2' h'$ since $x_k \nmid {x_m}^{w_m} e_2'$, where h' is a monomial. Thus, we can write $f = (x_k {x_m}^{w_m}) e_2' {x_k}^{2w_k - 1}$, proving that $f \in I(D)^2$.
Subcase 2.2: Assume that $e_1' = x_m {x_k}^{w_k}$. Then we have $f{\cdot}x_j {x_k}^{w_k} = x_m {x_k}^{w_k} e_2' e_3' \mathbf{x}^\beta$, and this implies that

$$f{\cdot}x_j = x_m e_2' e_3' \mathbf{x}^\beta.$$

If $x_j \nmid e_2' e_3'$, then $e_2' e_3' \mid f$ and therefore $f \in I(D)^2$. On the other hand, if $x_j \mid e_2' e_3'$, then we may assume that $x_j \mid e_2'$. Then $x_m e_3' \mid f$, and hence $x_m \mid \frac{f}{e_3'}$. Moreover, since $x_k^{2w_k} \mid f$, observe that for any choice of e_3', $x_m x_k^{w_k} \mid \frac{f}{e_3'}$. Therefore $f \in I(D)^2$, and this competes the proof. □

Let D be a weighted oriented cycle with natural orientation, that is, the underlying simple graph of D is a cycle, and the orientation of the edges of D are either clockwise, or anticlockwise. More precisely, if $V(D) = \{x_1, \ldots, x_n\}$, we set $E(D) = \{(x_i, x_{i+1}) \mid 1 \leqslant i \leqslant n-1\} \cup \{(x_n, x_1)\}$. Note that for every vertex $x_i \in V(D)$, we have $\operatorname{indeg}_D(x_i) = \operatorname{outdeg}_D(x_i) = 1$. Thus, it follows from Theorem 16 that $(I(D)^2 : I(D)) = I(D)$ holds. In the next result, we show that the Ratliff Condition also holds for the third power.

Theorem 19. *Let D be a weighted oriented cycle with natural orientation. Then $(I(D)^3 : I(D)) = I(D)^2$. Consequently,* $\operatorname{Ass}(R/I(D)^2) \subseteq \operatorname{Ass}(R/I(D)^3)$*.*

Proof. Let $f \in (I(D)^3 : I(D))$. If $w_i = 1$ for all $1 \leqslant i \leqslant n$, then $I(D)$ is the edge ideal of a simple graph, and hence the assertion follows. Now we take $x_1{x_2}^{w_2} \in \mathcal{G}(I(D))$ with $w_2 \geqslant 2$ and consider

$$f \cdot x_1{x_2}^{w_2} = e_1e_2e_3\mathbf{x}^\alpha,$$

where $e_1, e_2, e_3 \in \mathcal{G}(I(D))$ and $\mathbf{x}^\alpha$ a monomial. We assume that $e_i \neq x_1x_2^{w_2}$ for $i = 1, 2, 3$ and ${x_2}^{w_2} \nmid \mathbf{x}^\alpha$, as otherwise we would have $f \in I(D)^2$. Then we have $x_2 \mid e_1e_2e_3$. Without any loss of generality, assume that $x_2 \mid e_1$. Then, $e_1 = x_2{x_3}^{w_3}$. Consider the following cases:

Case 1: Assume that $x_2 \mid e_2e_3$. Without any loss of generality, let $x_2 \mid e_2$. Then $e_2 = x_2{x_3}^{w_3}$. Hence, from the equality $f \cdot x_1{x_2}^{w_2} = (x_2{x_3}^{w_3})^2e_3\mathbf{x}^\alpha$ we get $f = {x_3}^{2w_3}h$ for some monomial h. Now consider

$$f \cdot x_2{x_3}^{w_3} = e'_1e'_2e'_3\mathbf{x}^\beta,$$

where $e'_i \in \mathcal{G}(I(D))$ for $1 \leqslant i \leqslant 3$ and $\mathbf{x}^\beta$ a monomial. As before, we assume that $e'_i \neq x_2{x_3}^{w_3}$ for $i = 1, 2, 3$ and ${x_3}^{w_3} \nmid \mathbf{x}^\beta$. Then $x_3 \mid e'_1e'_2e'_3$. Without any loss of generality, assume that $x_3 \mid e'_1$. Then we have, $e'_1 = x_3{x_4}^{w_4}$ and the equality $f \cdot x_2{x_3}^{w_3} = x_3{x_4}^{w_4}e'_2e'_3\mathbf{x}^\beta$. Now it follows that ${x_4}^{w_4} \mid f$, and hence we can write $f = {x_3}^{2w_3}h = {x_3}^{2w_3}{x_4}^{w_4}h'$ for some monomial h'. We now consider the following subcases:

Subcase 1.1: Assume that $x_3 \mid e'_2e'_3$. Without any loss of generality, let $x_3 \mid e'_2$. Then $e'_2 = x_3{x_4}^{w_4}$. Thus we have $f \cdot x_2{x_3}^{w_3} = (x_3{x_4}^{w_4})^2e'_3\mathbf{x}^\beta$. Then it follows that ${x_4}^{2w_4} \mid f$. Hence $f = {x_3}^{2w_3}h = {x_3}^{2w_3}{x_4}^{2w_4}g$ for some monomial g, and consequently $f \in I(D)^2$.

Subcase 1.2: Assume that $x_3 \nmid e'_2e'_3$. If $x_2 \nmid e'_2e'_3$, then $e'_2e'_3 \mid f$ and we are through. Now assume that $x_2 \mid e'_2$. Then $e'_2 = x_1{x_2}^{w_2}$ and consequently,

$$f \cdot x_2{x_3}^{w_3} = (x_3{x_4}^{w_4})(x_1{x_2}^{w_2})e'_3\mathbf{x}^\beta.$$

Since $w_2 \geqslant 2$, it follows that $x_2 \mid f$, and hence $f = {x_3}^{2w_3}{x_4}^{w_4}h' = x_2{x_3}^{2w_3}{x_4}^{w_4}h''$, where $h'' = \frac{h'}{x_2}$. Thus $(x_2{x_3}^{w_3})(x_3{x_4}^{w_4}) \mid f$, and hence $f \in I(D)^2$.

Case 2: Assume that $x_2 \nmid e_2e_3$. Now if $x_1 \nmid e_2e_3$, then $e_2e_3 \mid f$ and we are through. Now, assume that $x_1 \mid e_2e_3$. Without any loss of generality, $x_1 \mid e_2$. Then $e_2 = x_n{x_1}^{w_1}$ and we have

$$f \cdot x_1{x_2}^{w_2} = (x_2{x_3}^{w_3})(x_n{x_1}^{w_1})e_3\mathbf{x}^\alpha.$$

We now consider the following subcases:

Subcase 2.1: Let $x_1 \mid e_3$. Then $e_3 = x_n{x_1}^{w_1}$, and $f \cdot x_1{x_2}^{w_2} = (x_2{x_3}^{w_3})(x_n{x_1}^{w_1})^2\mathbf{x}^\alpha$. This gives ${x_1}^{w_1}{x_3}^{w_3}{x_n}^2 \mid f$. So, we can write $f = {x_1}^{w_1}{x_3}^{w_3}{x_n}^2h$ for some monomial h. Now, we consider

$$f \cdot x_n{x_1}^{w_1} = e'_1e'_2e'_3\mathbf{x}^\beta$$

where $e'_i \in \mathcal{G}(I(D))$ for $1 \leqslant i \leqslant 3$ and $\mathbf{x}^\beta \in R$ a monomial. Likewise, we assume $e'_i \neq x_n{x_1}^{w_1}$ for $i = 1, 2, 3$ and ${x_1}^{w_1} \nmid \mathbf{x}^\beta$, as otherwise, $f \in I(D)^2$. Without any loss of generality assume that, $x_1 \mid e'_1$. Then $e'_1 = x_1{x_2}^{w_2}$ and therefore, $f \cdot x_n{x_1}^{w_1} = x_1{x_2}^{w_2}e'_2e'_3\mathbf{x}^\beta$. Consequently, ${x_2}^{w_2} \mid f$ and hence $f = {x_1}^{w_1}{x_3}^{w_3}{x_n}^2h = {x_1}^{w_1}x_2^{w_2}{x_3}^{w_3}{x_n}^2h'$, where $h' = \frac{h}{x_2^{w_2}}$. Thus $(x_n{x_1}^{w_1})(x_2{x_3}^{w_3}) \mid f$ and hence $f \in I(D)^2$.

Subcase 2.2: Assume that $x_1 \nmid e_3$. Then since $x_2 \nmid e_3$, from the equality $f \cdot x_1 {x_2}^{w_2} = (x_2 {x_3}^{w_3})(x_n {x_1}^{w_1}) e_3 \mathbf{x}^{\alpha}$, it follows that $e_3 {x_3}^{w_3} \mid f$. Thus, $f = {x_3}^{w_3} x_n e_3 h$ for some monomial h. Now, consider $f \cdot x_n {x_1}^{w_1} \in I(D)^3$. Then by the same arguments as in Subcase 2.1, we conclude that ${x_2}^{w_2} \mid f$, and consequently, $f = {x_3}^{w_3} x_n e_3 h = x_2^{w_2} {x_3}^{w_3} x_n e_3 h'$, where $h' = \frac{h}{x_2^{w_2}}$. Thus $(x_2 {x_3}^{w_3}) e_3 \mid f$ and hence $f \in I(D)^2$. $\square$

3.3 Punctured Persistence Property

Let I be an ideal in the polynomial ring R, and let $\mathfrak{m}$ denote the irrelevant maximal ideal generated by all the variables of R. We say that I satisfies the *punctured persistence property* if

$$\operatorname{Ass}(R/I^k) \setminus \{\mathfrak{m}\} \subseteq \operatorname{Ass}(R/I^{k+1}) \setminus \{\mathfrak{m}\} \text{ for all } k \geqslant 1.$$

In this subsection, we show that for a large class of weighted oriented graphs, the punctured persistence property holds. Before going into our main result, we need the following lemma.

Lemma 20. *Let D be a weighted oriented graph, and let $P \in \operatorname{Ass}(R/I(D)^k), k \geqslant 1$ be such that $x_i \notin P$ for some $x_i \in V(D)$. Let D' be a weighted oriented graph such that*

$$\begin{aligned} V(D') &= V(D) \cup \{y\}, \\ E(D') &= E(D) \cup \{(x_i, y)\} \text{ with } w(y) = 1. \end{aligned}$$

If we set $R' = \mathbb{K}[x_1, \ldots, x_n, y]$, then $P + (y) \in \operatorname{Ass}(R'/I(D')^k)$.

Proof. It is clear that $I(D') = I(D) + (x_i y)$. Since $P \in \operatorname{Ass}(R/I(D)^k)$, there is a monomial $f \in R$ such that $P = (I(D)^k : f)$. Now, it suffices to prove the following claim: $P + (y) = (I(D')^k : x_i f)$. We first show that $x_i f \notin I(D')^k$. On the contrary, if $x_i f \in I(D')^k$, then since $y \nmid x_i f$, we have $x_i f \in I(D)^k$, which in turn implies that $x_i \in (I(D)^k : f) = P$, a contradiction. Hence, $x_i f \notin I(D')^k$. Now for any $x_j \in P$, $x_j f \in I(D)^k$, which implies $x_i x_j f \in I(D)^k \subseteq I(D')^k$, and thus $x_j \in (I(D')^k : x_i f)$. This proves that $P \subseteq (I(D')^k : x_i f)$. Also, $y x_i f = f \cdot x_i y \in I(D')^k$, as $f \in I(D)^{k-1}$ and $x_i y \in I(D')$. Hence, we conclude that $P + (y) \subseteq (I(D')^k : x_i f)$. On the other hand, if possible, let us assume that $g \in (I(D')^k : x_i f)$ be a monomial such that $g \notin P + (y)$. Then $x_i f g \in I(D')^k$. Note that $y \nmid g$ and $y \nmid f$, and therefore $x_i f g \in I(D)^k$. This implies that $x_i g \in (I(D)^k : f) = P$, and hence, $g \in P$ as $x_i \notin P$. This is a contradiction. Therefore, $P + (y) = (I(D')^k : x_i f)$, and this completes the proof of the claim. $\square$

The next theorem is one of the main results of this article. We show that if a weighted oriented graph has no source vertex and if P is an associated prime of $I(D)^k$ such that $P \neq \mathfrak{m}$, then P is also an associated prime of $I(D)^{k+1}$. In other words, the punctured persistence property holds for any weighted oriented graph with no source vertex.

Theorem 21. *Let D be a weighted oriented graph without any source vertex. Then $I(D)$ satisfies the punctured persistence property.*

Proof. Let $P \in \operatorname{Ass}(R/I(D)^k)$ be such that $P \neq \mathfrak{m}$. Then there is some $x_i \in V(D)$ such that $x_i \notin P$. Let D' be the weighted oriented graph where

$$V(D') = V(D) \cup \{y\},$$
$$E(D') = E(D) \cup \{(x_i, y)\} \text{ with } w(y) = 1.$$

Set $R' = \mathbb{K}[x_1, \ldots, x_n, y]$. Then, by Lemma 20, $P + (y) \in \operatorname{Ass}(R'/I(D')^k)$. Since (x_i, y) is an outward leaf in D', by Theorem 11, $I(D')$ satisfies the Ratliff condition. Therefore, $P + (y) \in \operatorname{Ass}(R'/I(D')^{k+1})$. So, there exists a monomial $g \in R'$ and $g \notin I(D')^{k+1}$ such that $P + (y) = (I(D')^{k+1} : g)$. We now make the following claim.
Claim: $P + (y) = (I(D')^{k+1} : g{x_i}^m)$ for all $m \geqslant 0$.
Proof of the claim. We proceed by induction on m. The case $m = 0$ is true. We assume $m \geqslant 1$ and $P + (y) = (I(D')^{k+1} : g{x_i}^{m-1})$. Since $(I(D')^{k+1} : g{x_i}^{m-1}) \subseteq (I(D')^{k+1} : g{x_i}^m)$, we already have $P + (y) \subseteq (I(D')^{k+1} : g{x_i}^m)$. Conversely, let $f \in (I(D')^{k+1} : g{x_i}^m)$. Then $f' \cdot g{x_i}^n \in I(D')^{k+1}$, and hence $f'x_i \in (I(D')^{k+1} : g{x_i}^{n-1}) = P + (y)$ by the induction hypothesis. But $x_i \notin P + (y)$ implies that $f \in P + (y)$, and this completes the proof of the claim.

We now consider the following cases:
Case 1: Assume that $y \nmid g$. Then we claim that $P = (I(D)^{k+1} : g)$. Indeed, if $x_m \in P$ then $x_m g \in I(D')^{k+1}$. But $y \nmid x_m g$ implies that $x_m g \in I(D)^{k+1}$. Therefore $x_m \in (I(D)^{k+1} : g)$, and hence $P \subseteq (I(D)^{k+1} : g)$. On the other hand, if possible, let us take a monomial $f \in R$ such that $f \in (I(D)^{k+1} : g)$ and $f \notin P$. Then $fg \in I(D)^{k+1} \subseteq I(D')^{k+1}$. Thus $f \in (I(D')^{k+1} : g)$, and hence $f \in P + (y)$. But this implies that $f \in P$, which is a contradiction. Therefore in this case, $P = (I(D)^{k+1} : g)$ and hence, $P \in \operatorname{Ass}(R/I(D)^{k+1})$.

Case 2: Assume that $y \mid g$. Since $y \in (I(D')^{k+1} : g)$, we must have $x_i \mid g$, as otherwise, if $x_i \nmid g$ then $yg \in I(D')^{k+1}$ implies that $g \in I(D')^{k+1}$, which is a contradiction. Thus, $x_i y \mid g$ and assume that $g = x_i y g'$ for some monomial $g' \in R'$. Now, D does not have any source vertex, so there is $x_m \in V(D)$ such that $(x_m, x_i) \in E(D)$. Set

$$h = x_m {x_i}^{w_i} g' = x_m {x_i}^{w_i} \frac{g}{x_i y}.$$

Claim: $h \notin I(D')^{k+1}$ and $P + (y) = (I(D')^{k+1} : h)$.
Proof of the claim. If possible, let us assume that $h = e_1 \cdots e_{k+1} h'$ for some $e_i \in \mathcal{G}(I(D')), 1 \leqslant i \leqslant k+1$ and $h' \in R'$. If $e_t = x_m {x_i}^{w_i}$ for some $1 \leqslant t \leqslant k+1$, then we have

$$g' = \frac{h}{x_m x_i^{w_i}} = e_1 \cdots e_{t-1} e_{t+1} \cdots e_{k+1} h'.$$

Now, multiply both sides of the this equation by $x_i y$ to get back

$$g = e_1 \cdots e_{t-1} (x_i y) e_{t+1} \cdots e_{k+1} h' \in I(D')^{k+1},$$

which is a contradiction. So, we may assume that $e_t \neq x_k {x_i}^{w_i}$ for all $1 \leqslant t \leqslant k+1$. Moreover, we may assume that $x_m \nmid h'$, as otherwise, in virtue of Lemma 20, we can assume that $x_i^{w_i} \mid h'$. Then we have $x_m x_i^{w_i} \mid h'$ and hence $h \in I(D')^{k+2}$. Which in turn implies that $g = x_i y \frac{h}{x_m x_i^{w_i}} \in I(D')^{k+2}$, and this is a contradiction. Therefore, $x_m \mid e_1 \cdots e_{k+1}$. Without any loss of generality, assume that $x_m \mid e_1$. Then we can write

$$h = e_2 e_3 \cdots e_{k+1} x_m h'' = e_2 e_3 \cdots e_{k+1} (x_m x_i^{w_i}) \frac{h''}{x_i^{w_i}},$$

where h'' is a monomial, and the second equality makes sense because of Lemma 20. We multiply both sides of the above equation by $\frac{x_i y}{x_m x_i^{w_i}}$ to get back g, and then it would imply that $g \in I(D')^{k+1}$, again a contradiction.

It now remains to show that $P + (y) = (I(D')^{k+1} : h)$. Let $x_\ell \in P$. Now, $x_\ell h = x_\ell x_m x_i^{w_i} \frac{g}{x_i y} = \frac{x_\ell g}{x_i y} x_m x_i^{w_i}$. As $x_\ell g \in I(D')^{k+1}$, note that $\frac{x_\ell g}{x_i y} \in I(D')^k$. Indeed, in view of Lemma 20, we can assume that sufficiently large power of x_i divides the monomial g, and moreover, the vertex y can appear in at most one edge in the expression of $x_\ell g \in I(D')^{k+1}$. Thus $\frac{x_\ell g}{x_i y} \in I(D')^k$, and hence $\frac{x_\ell \cdot g}{x_i y} x_m x_i^{w_i} \in I(D')^{k+1}$. This proves that $P \subseteq (I(D')^{k+1} : h)$. By similar arguments, one can show that $y \in (I(D')^{k+1} : h)$. Thus $P + (y) \subseteq (I(D')^{k+1} : h)$. On the other hand, if possible, let us suppose that $f \in (I(D')^{k+1} : h)$, but $f \notin P + (y)$. Then $fh \in I(D')^{k+1}$, and so there are $e_i \in I(D'), 1 \leqslant i \leqslant k+1$ and $h' \in R'$ a monomial such that

$$\begin{aligned} fh &= e_1 \cdots e_{k+1} h' \\ \Rightarrow f(x_m x_i^{w_i}) g' &= e_1 \cdots e_{k+1} h'. \end{aligned}$$

We now consider the following cases:

Subcase 2.1: Assume that $x_m \nmid e_i$ for all $1 \leqslant i \leqslant k+1$. Then we have $x_m \mid h'$, and by Lemma 20, we can assume that $x_i^{w_i} \mid h'$. Then multiply both the sides of the above equation by $\frac{x_i y}{x_m x_i^{w_i}}$ to get $f(x_i y) g' = (x_i y) e_1 \cdots e_{k+1} h''$ where $h'' = \frac{h}{x_m x_i^{w_i}}$. Thus, we get $fg \in I(D')^{k+2} \subseteq I(D')^{k+1}$, and hence, $f \in P + (y)$, a contradiction.

Subcase 2.2: Without any loss of generality, assume that $x_m \mid e_1$. We further consider the following subcases:

Subcase 2.2.1: Assume that $e_1 = x_m x_i^{w_i}$. Again, proceeding as in Subcase 2.1, we get $f(x_i y) g' = (x_i y) e_2 e_3 \cdots e_{k+1} h'$, which implies that $fg \in I(D')^{k+1}$. Then again, $f \in P + (y)$, a contradiction.

Subcase 2.2.2: Assume that $e_1 = x_m x_\ell^{w_\ell}$ for some $x_\ell \in V(D')$ and $\ell \neq i$. So, we have

$$f(x_m x_i^{w_i}) g' = (x_m x_\ell^{w_\ell}) e_2 e_3 \cdots e_{k+1} h'.$$

Then multiply both sides of the above equation by $\frac{x_i y}{x_m x_i^{w_i}}$ to get

$$f(x_i y) g' = (x_i y) e_2 e_3 \cdots e_{k+1} h'',$$

where $h'' = \frac{x_\ell^{w_\ell} h'}{x_i^{w_i}}$. This proves that $fg \in I(D')^{k+1}$, and hence, $f \in P + (y)$, a contradiction.

Subcase 2.2.3: Assume that $e_1 = x_\ell x_m^{w_m}$ for some $x_\ell \in V(D')$ and $\ell \neq i$. Then

$$f(x_m x_i^{w_i}) g' = (x_\ell x_m^{w_m}) e_2 e_3 \cdots e_{k+1} h'.$$

Then multiply both sides of the above equation by $\frac{x_i y}{x_m x_i^{w_i}}$ to get

$$f(x_i y)g' = (x_i y)e_2 e_3 \cdots e_{k+1} h'',$$

where $h'' = \frac{x_\ell x_m^{w_m - 1} h'}{x_i^{w_i}}$. This proves that $fg \in I(D')^{k+1}$, and hence, $f \in P + (y)$, a contradiction. This completes the proof of the claim.

Note that the exponent of y in h is one less than the exponent of y in g. If $y \nmid h$, then $P + (y) = (I(D')^{k+1} : h)$ implies that $P = (I(D)^{k+1} : h)$, and hence $P \in \mathrm{Ass}(R/I(D)^{k+1})$, as desired. If $y \mid h$, then we continue the above process, and after a finite number of steps, we arrive at a situation where $P + (y) = (I(D')^{k+1} : \tilde{h})$, where $\tilde{h}$ is a monomial with $y \nmid \tilde{h}$. Then $P = (I(D)^{k+1} : \tilde{h})$, and this completes the proof of the theorem. $\square$

Finally, we prove that if a weighted oriented graph has no source vertex and the weight of each vertex is at least 2, then the irrelevant maximal ideal $\mathfrak{m}$ is an associated prime of $I(D)^k$ for all $k \geqslant 1$.

Theorem 22. *Let D be a weighted oriented graph such that $w(v) \geqslant 2$ for all $v \in V(D)$. Assume that D does not contain any source vertex, and there is a vertex $u \in V(D)$ such that* $\mathrm{indeg}_D(u) = 1$. *Then $\mathfrak{m} \in \mathrm{Ass}(R/I(D)^k)$ for all $k \geqslant 1$.*

Proof. If D does not contain any source vertex and $w(v) \geqslant 2$ for all $v \in V(D)$, then the assertion $\mathfrak{m} \in \mathrm{Ass}(R/I(D))$ follows from [1, Lemma 3.1]. For our purpose, we shall prove a stronger statement. Let $V(D) = \{x_1, \ldots, x_n\}$ and set $f = \prod_{i=1}^{n} {x_i}^{w_i - 1}$. We shall show that $\mathfrak{m} = (I(D) : f)$. First, note that $f \notin I(D)$ and thus $(I(D) : f)$ is a proper ideal contained in $\mathfrak{m}$. Now for any $1 \leqslant t \leqslant n$,

$$x_t f = x_t^{w_t} \prod_{\substack{i \in [n] \\ i \neq t}} {x_i}^{w_i - 1}.$$

Since x_t is not a source vertex, there is $x_s \in V(D)$ such that $x_s {x_t}^{w_t} \in I(D)$, and moreover, as $w_s \geqslant 2$, it follows that $x_s {x_t}^{w_t} \mid x_t f$. Thus, $x_t f \in I(D)$ for all $1 \leqslant t \leqslant n$, and therefore $\mathfrak{m} = (I(D) : f)$.

Since D contains a vertex of in-degree 1, without any loss of generality, assume that $(x_1, x_2) \in E(D)$ and $\mathrm{indeg}_D(x_2) = 1$. It now suffices to prove the following claim.
Claim: $\mathfrak{m} = (I(D)^{k+1} : (x_1 {x_2}^{w_2})^k f)$.
Proof of the claim. We first show that $(x_1 {x_2}^{w_2})^k f \notin I(D)^{k+1}$. On the contrary, assume that $(x_1 {x_2}^{w_2})^k f \in I(D)^{k+1}$. Since $w_i \geqslant 2$, observe that if $e \in \mathcal{G}(I(D))$ and $e \mid (x_1 {x_2}^{w_2})^k f$, then either $e = x_1 {x_2}^{w_2}$ or $e = x_i {x_1}^{w_1}$ for some $1 \leqslant i \leqslant n, i \neq 2$. Thus we have

$$(x_1 {x_2}^{w_2})^k f = (x_1 {x_2}^{w_2})^a (x_{i_1} {x_1}^{w_1})^{b_1} \cdots (x_{i_l} {x_1}^{w_1})^{b_l} g,$$

where g is a monomial, $3 \leqslant i_1, \ldots, i_l \leqslant n$, and $a, b_1, \ldots, b_l$ are positive integers such that $a + b_1 + \cdots + b_l = k + 1$. Note that $a \neq k + 1$, as otherwise, $x_2^{(k+1)w_2} \mid (x_1 {x_2}^{w_2})^k f$, which is a contradiction. Set $b = b_1 + \cdots + b_\ell$. Then $b \geqslant 1$ and it follows from the above expression that ${x_1}^{a + b w_1}$ divides $(x_1 {x_2}^{w_2})^k f$. But $a + b w_1 = (k + 1) - b + b w_1 =$

$(k + w_1) + (w_1 - 1)(b - 1) \geqslant k + w_1$ as $w_1 \geqslant 2$ and $b \geqslant 1$. This is a contradiction since the exponent of x_2 in $(x_1 {x_2}^{w_2})^k f$ is $k + w_1 - 1$. Therefore, $(x_1 {x_2}^{w_2})^k f \notin I(D)^{k+1}$. It now remains to show that $x_i (x_1 {x_2}^{w_2})^k f \in I(D)^{k+1}$ for all $1 \leqslant i \leqslant n$. Note that $x_i f \in I(D)$ for all $1 \leqslant i \leqslant n$, and therefore, $x_i (x_1 {x_2}^{w_2})^k f \in I(D)^{k+1}$ for all $1 \leqslant i \leqslant n$. Thus $\mathfrak{m} = (I(D)^{k+1} : (x_1 {x_2}^{w_2})^k f)$, and this completes the proof of the claim. □

As an immediate consequence of Theorem 21 and the above theorem, we have the following.

Corollary 23. *Let D be a weighted oriented graph as in Theorem 22. Then $I(D)$ has the persistence property.*

Finally, based on the results of this article and the computational evidence, we conclude this section with the following conjecture.

Conjecture 24. Let D be a weighted oriented graph and $I(D)$ be the edge ideal. Then $I(D)$ satisfies the persistence property.

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