On cospectral graphons

Jan Hladký[a] Daniel Il'kovič[b] Jared León[c] Xichao Shu[b]

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Abstract

In this short note, we study the notion of cospectral graphons, paralleling the notion of cospectral graphs. As in the graph case, we give three equivalent definitions: by equality of spectra, by equality of cycle densities, and by a unitary transformation. We also give an example of two cospectral graphons that cannot be approximated by two sequences of cospectral graphs in the cut-distance.

Mathematics Subject Classifications: 05C50, 05C60, 05C80

1 Introduction

In this paper, we deal with finite, undirected, simple graphs, and graphons as their limit counterparts. Cospectrality, which is the focus of this paper, is a certain way of grouping graphs (or graphons, as we will see later) into equivalence classes. This grouping actually follows a broad scheme, and we will first introduce it abstractly from two perspectives. Let G and H be two graphs on vertex set $[n]$.

(F1) Suppose that $\mathcal{M}_n \subset \mathbb{C}^{n\times n}$ is a monoid with respect to matrix multiplication. We say that G and H are *$\mathcal{M}_n$-similar* for the adjacency matrix A_G of G and for the adjacency matrix A_H of H if there exists a suitable $n\times n$ matrix $T\in\mathcal{M}_n$ such that $TA_H = A_GT$.

(F2) Suppose that $\mathcal{F}$ is a (finite or infinite) family of graphs. We say that G and H are *$\mathcal{F}$-indistinguishable* if $\hom(F,G)=\hom(F,H)$ for all $F\in\mathcal{F}$.[1]

[a] Institute of Computer Science of the Czech Academy of Sciences, Pod Vodárenskou věží 2, 182 07 Prague, Czechia. With institutional support RVO:67985807. (hladky@cs.cas.cz).

[b] Institute of Mathematics, Leipzig University, Augustusplatz 10, 04109 Leipzig, Germany. *Previous affiliation:* Faculty of Informatics, Masaryk University, Botanická 68A, 602 00 Brno, Czech Republic. (493343@mail.muni.cz, 540987@mail.muni.cz).

[c] Mathematics Institute and DIMAP, University of Warwick, Coventry CV4 7AL, UK (jared.leon@warwick.ac.uk)

[1] Recall that a *homomorphism of a graph F to a graph G* is any map $h: V(F)\to V(G)$ such that $h(u)h(v)\in E(G)$ for every $uv\in E(F)$. The symbol $\hom(F,G)$ denotes the size of the set of all homomorphisms from F to G.

Note that $\mathcal{M}_n$-similarity and $\mathcal{F}$-indistinguishability are equivalence relations. Even though (F1) and (F2) look very different, several prominent equivalence relations can be expressed in both ways. These equivalence relations are graph isomorphism, fractional isomorphism, cospectrality, and quantum isomorphism. Let us give details and specify the families of matrices $\mathcal{M}_n$ and of graphs $\mathcal{F}$.

- The most important of such equivalence relations is the usual *graph isomorphism*. In this case, $\mathcal{M}_n$ is the set of all $n \times n$ permutation matrices. Remarkably, characterization of graph isomorphism using indistinguishability is also possible, with $\mathcal{F} = \{\text{all graphs}\}$. Indeed, a famous result of Lovász [8] states that if $\hom(F, G) = \hom(F, H)$ for all graphs F, then G and H are isomorphic.

- The concept of *fractional isomorphism* was introduced in [13] using (F1) for $\mathcal{M}_n$ equal to the set of all doubly stochastic matrices. Later, several further equivalent definitions were given in [12]. Finally, Dvořák [4] gave a characterization as in (F2), with $\mathcal{F}$ being all trees.

- *Cospectrality* (sometimes called *isospectrality*) is derived from linear algebra, where this notion denotes the equality of spectra (including multiplicities) of matrices. For graphs, this concept is applied to adjacency matrices.[2] Basic linear algebra then asserts that for a characterization using (F1) we need to take $\mathcal{M}_n$ to be all orthonormal $n \times n$ matrices. For a characterization using (F2) we take $\mathcal{F}$ to be all cycles. The key insight to infer the equivalence relations of these two definitions is that when we write C_k for the cycle of length k, $\hom(C_k, G)$ is equal to the trace of A_G^k, which is in turn equal to the sum of the k-th powers of the eigenvalues.

- *Quantum isomorphism* is the newest addition to the list. It was introduced in [1]. Here, the matrices $\mathcal{M}_n$ are actually not over the field of complex numbers. Rather, their entries are operators (and projections, more specifically) on a fixed Hilbert space. The conditions imposed on $\mathcal{M}_n$ are known in the field of quantum algebra as 'magic unitaries'. In a remarkable paper [11] it was shown that this definition is equivalent to (F2) with $\mathcal{F}$ being all planar graphs.

In this note, we translate the concept of cospectrality to graphons. Graphons are analytic objects introduced by Borgs, Chayes, Lovász, Sós, Szegedy, and Vesztergombi [10, 2] to represent limits of sequences of graphs. We recall the exact definition and properties later in Section 2. At this moment, it suffices that each graphon is a symmetric bounded Lebesgue measurable function on $[0, 1]^2$. In particular, it can be viewed as an integral kernel operator on $L^2([0, 1])$ (see (2)). There are several directions in which graphons can be studied. Below are the two most important ones.

- Graphons can be studied *per se*, often motivated by concepts from finite graphs. The seminal paper [10] introduced the notion of *homomorphism density* $t(F, W)$ (here, F

[2] Less often, some other matrices, such as the normalized Laplacian, are used. These alternative definitions lead to different notions of cospectrality.

is a graph and W is a graphon), which is motivated by the notion of homomorphism counts $\hom(F,G)$ (or by the notion of homomorphism density $\hom(F,G)v(G)^{-v(F)}$). Other concepts that have been extended to graphons include, for example, notions of the max-cut [2] and the chromatic number [7].

Our attention is on (F1) and (F2). Note that there is a very easy way to introduce the graphon counterparts of graph isomorphism, fractional isomorphism, cospectrality, and quantum isomorphism from their definitions using the concept of indistinguishability in (F2). Namely, we will say that two graphons U and W are (i) *weakly isomorphic*, (ii) *fractionally isomorphic*, (iii) *cospectral*, and (iv) *quantum isomorphic* if $t(F,U) = t(F,W)$ for (i) every graph F, (ii) every tree F, (iii) every cycle F, (iv) every planar graph F, respectively. Let us now turn to graphon counterparts to (F1). Here, we should view each graphon W as an integral kernel operator $T_W : L^2([0,1]) \to L^2([0,1])$ (defined by (2)). Similarity is not facilited by a family $\mathcal{M}_n$ of $n \times n$ matrices, but rather by a set of bounded operators $\mathcal{M} \subset \mathcal{B}(L^2([0,1]))$ on $L^2([0,1])$, and the counterpart of the matrix equality $TA_H = A_GT$ for some $T \in \mathcal{M}_n$ is the equality of operators $TT_U = T_WT$ for some $T \in \mathcal{M}$. (As we shall see in Footnote 3, an additional twist is sometimes needed.) Let us see counterparts of (F1) relating to the above equivalence relations for graphons. The main result of [3] asserts that two graphons U and W are weakly isomorphic if and only if for every $\varepsilon > 0$ there exists a unitary Koopman operator T so that the operator norm of $TU - WT$ is less than ε. Recall that being a unitary Koopman operator amounts to the existence of a measure-preserving bijection $\pi : [0,1] \to [0,1]$ such that $(Tf)(x) = f(\pi(x))$ for every $f \in L^2([0,1])$ and $x \in [0,1]$. This can be viewed as an analogue of (F1).[3] The theory of fractional isomorphism for graphons was introduced by Grebík and Rocha [5]. Specifically, [5] gives graphon counterparts to all known characterizations of graph fractional isomorphism and proves their equivalence relations to the above definition which requires the equality of all tree densities. The characterization of (F1) is $TU = WT$ for a suitable 'Markov operator' T. One of the main contributions in this note is that in Theorem 3 we give several definitions of 'cospectral graphons' and prove their equivalence relations. As far as we know, no attempt to extend the notion of quantum isomorphism to graphons has been made.

- A particularly important line of research is in investigating continuity properties of various graph(on) parameters. The most prominent example is again the homomorphism density. In particular, the so-called Counting lemma asserts that if $(G_n)_n$ is a sequence of graphs converging to a graphon W, then for every graph F we have $\hom(F,G_n)v(G_n)^{-v(F)} \to t(F,W)$.

  This yields the following result. Namely, suppose that $(G_n)_n$ and $(H_n)_n$ are sequences of graphs so that for each n, the graphs G_n and H_n are of the same order and are isomorphic, or fractionally isomorphic, or cospectral, or quantum isomorphic. Suppose further that $(G_n)_n$ converges to a graphon U and $(H_n)_n$ converges

[3] Note that we cannot achieve precisely $TU = WT$ in some situations, see Figure 7.1 in [9].

to a graphon W. Then by the above, G_n and H_n have the same counts (and thus also densities) of homomorphisms from each graph, or from each tree, or from each cycle, or from each planar graph, respectively. By the continuity of homomorphism densities, these equalities are inherited to U and W. We conclude that U and W are weakly isomorphic, or fractionally isomorphic, or cospectral, or quantum isomorphic.

The main result of [6] goes in the opposite direction in the case of fractional isomorphism: If U and W are fractionally isomorphic then we can find sequences of graphs $G_n \to U$ and $H_n \to W$ such that for each n, the graphs G_n and H_n are fractionally isomorphic. The second result of this note, Theorem 5, is that the counterpart of this result for cospectrality does not hold.

2 Preliminaries

For a graph G we denote by $v(G)$ the number of vertices of G and $e(G)$ the number of edges. By C_k we denote the cycle of length k.

2.1 Graphons

We now review the basic concepts of the theory of graphons, using common notation from the excellent monograph [9]. A *graphon* is a symmetric measurable function $W : [0,1]^2 \to [0,1]$. When we view the sets $[0,1]$ or $[0,1]^2$ as measure spaces, we take the underlying measure to be the Lebesgue measure. The *cut-norm distance* is defined as

$$d_\square(U,W) = \sup_{S,T\subset[0,1]} \left| \int_{S\times T} U(x,y) - W(x,y)\,dx\,dy \right|, \tag{1}$$

where the supremum ranges over all measurable subsets S and T of $[0,1]$. The *cut-distance* is defined as

$$\delta_\square(U,W) = \inf_{\varphi} d_\square(U^\varphi, W),$$

where the infimum ranges over all measure preserving bijections $\varphi : [0,1] \to [0,1]$ and $U^\varphi(x,y) = U(\varphi(x),\varphi(y))$. The cut-distance is symmetric, satisfies the triangle inequality, and $\delta_\square(U,U) = 0$ for all U. In addition, note that the pairs of graphons of the form U and U^φ are at cut-distance 0, yet there are many instances when they are not equal. That means that $\delta_\square$ is pseudometric but not metric. More details can be found in Section 8.2.2 of [9].

Recall that each graph can be represented as a graphon. For a graph G on a vertex set V, we define a graphon W_G in the following way. Partition $[0,1]$ into sets $\{\Omega_v\}_{v\in V}$ of measure $\frac{1}{|V|}$ each. On each rectangle $\Omega_u \times \Omega_v$, the function W_G is constant-1 or constant-0, depending on whether uv forms an edge of G or not. The representation W_G is not unique as it depends on the choice of the partition $\{\Omega_v\}_{v\in V}$. However, any two representations are at cut-distance 0. The *sequence of graphs* $(G_n)_n$ *converges to* W if and only if $\delta_\square(W_{G_n}, W) \to 0$ as $v(G_n) \to \infty$.

The density of a graph G in a graphon W is defined as

$$t(G,W) = \int_{(x_v)_{v\in v(G)}\in[0,1]^{V(G)}} \prod_{uv\in E(G)} W(x_u,x_v) \prod_{v\in V(G)} dx_v \ .$$

So far, the analogies between the adjacency matrix of a graph and a graphon were made in the original domain $[0,1]^2$. In the language of Fourier transform, this corresponds to the spatial perspective. We now move to the dual perspective. Namely, we recall a formalism that allows us to study spectral properties of graphons. The details can be found in Section 7.5 of [9]. Each graphon W can be associated with an operator $T_W : L^2([0,1]) \to L^2([0,1])$ defined by

$$(T_W f)(x) = \int_0^1 W(x,y)f(y)\,dy \quad \text{for every } x \in [0,1]. \tag{2}$$

It is known that T_W is a symmetric real Hilbert-Schmidt operator. In particular, we say that $f \in L^2([0,1])$ is an *eigenvector* of W with *eigenvalue* $\lambda \in \mathbb{R}$ if $T_W f = \lambda f$. Since T_W is a compact operator it has a discrete real spectrum (that is, the multiset of eigenvalues), denoted by $Spec(W)$. It is well known that for each $\varepsilon > 0$ the number of eigenvalues of modulus at least ε (including multiplicities) is finite. Further, Parseval's Theorem (see e.g. (7.23) in [9]) asserts that

$$\|W\|_2^2 = \sum_{\lambda\in Spec(W)} \lambda^2 \ . \tag{3}$$

We mentioned in Section 1 that for $k \geqslant 3$ and a graph G, the quantity $\hom(C_k, G)$ is the sum of the k-th powers of the eigenvalues of the adjacency matrix of G. The graphon counterpart to this is (see (7.22) in [9])

$$t(C_k, W) = \sum_{\lambda\in Spec(W)} \lambda^k \quad \text{for each } k \geqslant 3. \tag{4}$$

2.2 Spectral theorem

Here we recall the notion of unitary operators. This is a general concept which applies to every Hilbert space. We need a definition only in the complex Hilbert space $L^2([0,1])$. A bounded operator $T : L^2([0,1]) \to L^2([0,1])$ is *unitary* if T is surjective and for every $f, g \in L^2([0,1])$, we have $\langle Tf, Tg\rangle = \langle f, g\rangle$ and $\|Tf\|_2 = \|f\|_2$. In fact, the last two properties in the previous definition are known to be equivalent. That is, the property of the preservation of the inner product implies the property of the preservation of the norm and vice versa. Unitary operators are functional-analytic counterparts to orthonormal matrices.

Next, we state the Spectral theorem for self-adjoint compact operators. Recall that a bounded operator $T : L^2([0,1]) \to L^2([0,1])$ is *self-adjoint* if for every $f, g \in L^2([0,1])$, we have $\langle Tf, g\rangle = \langle f, Tg\rangle$. For every graphon W, the associated operator T_W in (2) is self-adjoint. This can be easily verified, using the fact that W is a symmetric function. Since T_W is a Hilbert-Schmidt operator, it is also a compact operator.

Theorem 1 (Spectral theorem)**.** *Let $T : L^2([0,1]) \to L^2([0,1])$ be a compact self-adjoint linear operator.*

For each non-zero eigenvalue λ, let K_λ be the eigenspace corresponding to λ, i.e., $K_\lambda = \{x \in L^2([0,1]) \mid Tx = \lambda x\}$. Let P_{K_λ} be the orthogonal projection onto the eigenspace K_λ.

Then the following spectral decomposition of T *holds,*

$$T = \sum_{\substack{\lambda \in Spec(T)\\ \lambda \neq 0}} \lambda P_{K_\lambda} ,$$

where the sum converges in the operator norm if $Spec(T)$ is infinite.

3 Equivalent definitions of cospectral graphons

We defined cospectral graphons in Section 1. Since not all the concepts were defined back then, we repeat the definition here.

Definition 2. Graphons U and W are *cospectral* if and only if $t(C_k, U) = t(C_k, W)$ for all integers $k \geqslant 3$.

In our first main theorem, we prove that graphon cospectrality has several equivalent definitions.

Theorem 3. *For any two graphons U and W, the following statements are equivalent:*

(i) For all integers $k \geqslant 3$, we have $t(C_k, U) = t(C_k, W)$.

(ii) There are infinitely many odd numbers k and infinitely many even numbers k, such that $t(C_k, U) = t(C_k, W)$.

(iii) Graphons U and W have the same spectra, that is, $Spec(U) = Spec(W)$.

(iv) There exists a unitary operator $T : L^2([0,1]) \to L^2([0,1])$, such that $T \circ T_W = T_U \circ T$.

To prove Theorem 3, we establish the implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$ and $(iv) \Rightarrow (iii) \Rightarrow (iv)$. The implication $(i) \Rightarrow (ii)$ is immediate. The implication $(iii) \Rightarrow (i)$ follows directly from (4).

Proof of Theorem 3, $(ii) \Rightarrow (iii)$. Suppose for contradiction that (ii) holds but $Spec(U) \neq Spec(W)$. Let $\nu > 0$ be the largest number such that there exists an eigenvalue of modulus ν with different multiplicities in $Spec(U)$ and $Spec(W)$. Let $\alpha := \sup\{|\lambda| : \lambda \in Spec(U) \cup Spec(W), |\lambda| < \nu\}$. If the set $\{|\lambda| : \lambda \in Spec(U) \cup Spec(W), |\lambda| < \nu\}$ is empty, then we let $\alpha := 0$. We have that $\alpha < \nu$. Further, let $\beta > 0$ be small enough so that

$$\sum_{\lambda \in Spec(U), |\lambda| \leqslant \beta} \lambda^2 \leqslant \alpha^2 \quad \text{and} \quad \sum_{\lambda \in Spec(W), |\lambda| \leqslant \beta} \lambda^2 \leqslant \alpha^2 . \tag{5}$$

Such a number β exists since the eigenvalues are square-summable by (3).

Let $h_U = |\{\lambda \in Spec(U) : |\lambda| \in (\beta, \nu)\}|$, $h_W = |\{\lambda \in Spec(W) : |\lambda| \in (\beta, \nu)\}|$, and $h = h_U + h_W$. Recall that the spectra of U and W are real and thus the only eigenvalues of modulus ν may be ν and $-\nu$. Let m_U^+ and m_U^- be the multiplicities of ν and $-\nu$ in $Spec(U)$ and let m_W^+ and m_W^- be the multiplicities of ν and $-\nu$ in $Spec(W)$. The key is the following observation, which follows for every $k \geqslant 3$ by substituting into (4),

$$t(C_k, U) = \underbrace{\sum_{\lambda \in Spec(U):|\lambda|>\nu} \lambda^k}_{(\mathsf{T1})_U} + m_U^+ \nu^k + (-1)^k m_U^- \nu^k + \underbrace{\sum_{\lambda \in Spec(U):|\lambda| \in (\beta,\nu)} \lambda^k}_{(\mathsf{T2})_U} + \underbrace{\sum_{\lambda \in Spec(U):|\lambda| \in (0,\beta)} \lambda^k}_{(\mathsf{T3})_U} .$$

We can write an analogous formula for $t(C_k, W)$, and get the terms $(\mathsf{T1})_W$, $(\mathsf{T2})_W$, and $(\mathsf{T3})_W$. We can express $|t(C_k, U) - t(C_k, W)|$, by grouping $(\mathsf{T1})_U$ with $(\mathsf{T1})_W$, then $(\mathsf{T2})_U$ with $(\mathsf{T2})_W$, and lastly $(\mathsf{T3})_U$ with $(\mathsf{T3})_W$. The terms $(\mathsf{T1})_U$ and $(\mathsf{T1})_W$ cancel perfectly due to the way we chose ν. Next, we deal with $(\mathsf{T2})$. Each summand contributes between $-\alpha^k$ and α^k. There are h_U summands in $(\mathsf{T2})_U$. Similarly, there are h_W summands in $(\mathsf{T2})_W$. Hence, $|(\mathsf{T2})_U - (\mathsf{T2})_W| \leqslant h\alpha^k$. Finally, we deal with the terms $(\mathsf{T3})$. We use the well-known inequality between the ℓ^p-norm and the ℓ^q-norm, $\|\cdot\|_p \geqslant \|\cdot\|_q$ for $p \leqslant q$. For $p = 2$ and $q = k$ this gives

$$|(\mathsf{T3})_U| \leqslant \left| \sum_{\lambda \in Spec(U):|\lambda| \in (0,\beta)} \lambda^k \right| \leqslant \sum_{\lambda \in Spec(U):|\lambda| \in (0,\beta)} |\lambda|^k \leqslant \left(\sum_{\lambda \in Spec(U):|\lambda| \in (0,\beta)} |\lambda|^2 \right)^{k/2} \overset{(5)}{\leqslant} \alpha^k ,$$

and similarly $|(\mathsf{T3})_W| \leqslant \alpha^k$. Putting all these bounds together, we conclude that

$$|t(C_k, U) - t(C_k, W)| = \left(m_U^+ - m_W^+ + (-1)^k (m_U^- - m_W^-) \right) \nu^k \pm (h+2)\alpha^k . \tag{6}$$

We emphasize that for k large, $(h+2)\alpha^k$ is negligible with respect to ν^k. Thus, in order to arrive at the contradiction that for arbitrary large even k or for arbitrary large odd k we have $t(C_k, U) - t(C_k, W) \neq 0$, we only need to prove that for a suitable parity $P = 0$ (corresponding to k even) or $P = 1$ (corresponding to k odd) we have

$$m_U^+ - m_W^+ + (-1)^P (m_U^- - m_W^-) \neq 0. \tag{7}$$

We distinguish two cases. First, if $m_U^+ + m_U^- \neq m_W^+ + m_W^-$ then (7) holds for $P = 0$. Second, assume that $m_U^+ + m_U^- = m_W^+ + m_W^-$. In that case, $m_U^+ - m_U^- \neq m_W^+ - m_W^-$ (indeed, otherwise, we would have $m_U^+ = m_W^+$ and $m_U^- = m_W^-$, a contradiction to the way we chose ν). In this case (7) holds for $P = 1$. □

Proof of Theorem 3, (iv) ⇒ (iii). Assume (iv), that is, $T_U = T T_W T^{-1}$ for a unitary operator T.

For $\lambda \neq 0$, let K_λ and L_λ be the corresponding eigenspaces for λ with respect to T_W and to T_U defined as:

$$K_\lambda = \{x \in L^2([0,1]) \mid T_W x = \lambda x\} \quad \text{and} \quad L_\lambda = \{x \in L^2([0,1]) \mid T_U x = \lambda x\} .$$

(Obviously, $K_\lambda = \{0\}$ if λ is not an eigenvalue of T_W, and $L_\lambda = \{0\}$ if λ is not an eigenvalue of T_U.)

We need to prove that for every $\lambda \neq 0$ we have $\dim(K_\lambda) = \dim(L_\lambda)$. To this end, it suffices to show that $L_\lambda = TK_\lambda$, since a linear bijection preserves the dimension of subspaces. Equivalently, we will prove that

$$L_\lambda \supseteq TK_\lambda \quad \text{and} \quad K_\lambda \supseteq T^{-1}L_\lambda \ . \tag{8}$$

For the first part of (8), consider $z \in K_\lambda$ arbitrary. We need to prove that $Tz \in L_\lambda$. We have

$$T_U(Tz) = TT_WT^{-1}(Tz) = TT_W(z) \overset{z \in K_\lambda}{=} T(\lambda z) = \lambda T(z) \ ,$$

as was needed. The second part of (8) is analogous. This time, we use that $T_W = T^{-1}T_UT$. Consider $z \in L_\lambda$ arbitrary. We need to prove that $T^{-1}z \in K_\lambda$. We have

$$T_W(T^{-1}z) = T^{-1}T_UT(T^{-1}z) = T^{-1}T_U(z) \overset{z \in L_\lambda}{=} T^{-1}(\lambda z) = \lambda T^{-1}(z) \ ,$$

as was needed. □

Proof of Theorem 3, (iii) ⇒ (iv). By Theorem 1, we can decompose $L^2([0,1])$ according to the orthogonal eigenspaces of T_W,

$$L^2([0,1]) = \bigoplus_{\lambda \in Spec(W)} K_\lambda \ .$$

Here, $\{K_\lambda\}_{\lambda \in Spec(W)}$ are mutually orthogonal spaces, each of dimension equal to the multiplicity of the eigenvector λ, and writing $P_K : L^2([0,1]) \to L^2([0,1])$ for the orthogonal projection on a closed subspace $K \subset L^2([0,1])$, we have $T_W = \sum_{\lambda \in Spec(W)} \lambda P_{K_\lambda}$. Likewise, we have $L^2([0,1]) = \bigoplus_{\lambda \in Spec(U)} L_\lambda$ and $T_U = \sum_{\lambda \in Spec(U)} \lambda P_{L_\lambda}$ for the eigenspaces of T_U. Since the spectra of U and W are the same including multiplicities, we can fix linear isometries $b_\lambda : K_\lambda \to L_\lambda$ for each $\lambda \in Spec(U) = Spec(W)$. It is clear that the operator $Tf := \sum_\lambda b_\lambda \circ P_{L_\lambda}$ satisfies $T \circ T_W = T_U \circ T$. It is also clear that T is surjective and preserves the L^2-norm. □

4 Cospectral inapproximability

As we mentioned in Section 1, the main motivation for our second main result is the following theorem of Hladký and Hng [6].

Theorem 4. *Suppose that U and W are fractionally isomorphic graphons. Then there exist sequences $(G_n)_n$ and $(H_n)_n$ of graphs such that $(G_n)_n$ converges to U, $(H_n)_n$ converges to W, and H_n is fractionally isomorphic to G_n for each n.*

Here, we show that a counterpart of this theorem does not hold for cospectral graphons.

Theorem 5. *Consider graphons $U(x,y) = \frac{1}{2}$ and $W(x,y) = \mathbf{1}_{x\in[0,\frac{1}{2}]} \cdot \mathbf{1}_{y\in[0,\frac{1}{2}]}$ for $(x,y) \in [0,1]^2$. Then U and W are cospectral with Spec(U) = Spec(W) = $\{\frac{1}{2}\}$. If we have sequences $(G_n)_n$ and $(H_n)_n$ of graphs such that $(G_n)_n$ converges to U, $(H_n)_n$ converges to W, then G_n and H_n are not cospectral for each n sufficiently large.*

We now prove Theorem 5. The statement about the spectra of U and W is easy to verify, with the only eigenvector of U being the constant-1 and the only eigenvector of W being $\mathbf{1}_{x\in[0,\frac{1}{2}]}$. The particular property on which our proof depends is $\|U\|_1 \neq \|W\|_1$. That is, we prove that any two graphons with different L^1-norms (whether they are cospectral or not) cannot be cospectrally approximated, not only by finite graphs, but also by the more general class of $\{0,1\}$-valued graphons.

Proposition 6. *Let U and W be graphons with $\|U\|_1 > \|W\|_1$. Suppose that U' and W' are two $\{0,1\}$-valued graphons with $\delta_\square(U,U'), \delta_\square(W,W') < (\|U\|_1 - \|W\|_1)/2$. Then U' and W' are not cospectral.*

Proof. As a preparatory step, we claim that

$$\int U(x,y) - U'(x,y)dxdy \leqslant \delta_\square(U,U') \; . \tag{9}$$

To verify this, we need to check that for every measure preserving bijection $\varphi : [0,1] \to [0,1]$ we have

$$\int U(x,y) - U'(x,y)dxdy = \int U^\varphi(x,y) - U'(x,y)dxdy \leqslant d_\square(U^\varphi, U') \; .$$

This becomes obvious when we consider $S = T = [0,1]$ in (1).

Thus,

$$\|U'\|_1 = \int U'(x,y)dxdy = \int U(x,y)dxdy - \int U(x,y) - U'(x,y)dxdy \overset{(9)}{\geqslant} \|U\|_1 - \delta_\square(U,U') \; .$$

Since U' is $\{0,1\}$-valued, and since $0^2 = 0$ and $1^2 = 1$, we have $\|U'\|_2^2 = \|U'\|_1$. We conclude that $\|U'\|_2^2 \geqslant \|U\|_1 - \delta_\square(U,U')$.

Similarly, $\|W'\|_2^2 \leqslant \|W\|_1 + \delta_\square(W,W')$. Combining with the main assumption of the proposition, we get $\|U'\|_2^2 \neq \|W'\|_2^2$. By Parseval's Theorem (3) we conclude that U' and W' are not cospectral. □

We pose as an open problem, whether we can remove the assumption $\|U\|_1 \neq \|W\|_1$.

Problem 7. Do there exist two cospectral graphons U and W with $\|U\|_1 = \|W\|_1$ that cannot be cospectrally approximated?

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