A Construction that Preserves the Configuration of a Matroid, with Applications to Lattice Path Matroids

Joseph E. Bonin[a] Anna de Mier[b]

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Abstract

The configuration of a matroid M is the abstract lattice of cyclic flats (flats that are unions of circuits) where we record the size and rank of each cyclic flat, but not the set. One can compute the Tutte polynomial of M, and stronger invariants (notably, the $\mathcal{G}$-invariant), from the configuration. Given a matroid M in which certain pairs of cyclic flats are non-modular, we show how to produce a matroid that is not isomorphic to M but has the same configuration as M. We show that this construction applies to a lattice path matroid if and only if it is not a fundamental transversal matroid, and we enumerate the connected lattice path matroids on $[n]$ that are fundamental; these results imply that, asymptotically, almost no lattice path matroids are Tutte unique. We give a sufficient condition for a matroid to be determined, up to isomorphism, by its configuration. We treat constructions that yield matroids with different configurations where each matroid is determined by its configuration and all have the same $\mathcal{G}$-invariant. We also show that for any lattice L other than a chain, there are non-isomorphic transversal matroids that have the same configuration and where the lattices of cyclic flats are isomorphic to L.

Mathematics Subject Classifications: 05B35

1 Introduction

The configuration of a matroid is the abstract lattice that is formed by the cyclic flats (the flats that are unions of circuits) together with the size and rank of each cyclic flat, without recording the sets that are cyclic flats. The configuration is important in part because it contains all of the data that is needed to compute many enumerative matroid invariants. The most well-known of these invariants is the Tutte polynomial (see, e.g.,

[a]Department of Mathematics, The George Washington University, Washington, D.C. 20052, U.S.A. (jbonin@gwu.edu).

[b]Departament de Matemàtiques and IMTECH, Universitat Politècnica de Catalunya, Jordi Girona 1–3, 08034, Barcelona, Spain (anna.de.mier@upc.edu).

[14, 19]). For a matroid M on the set $E(M)$, its *Tutte polynomial* is defined to be

$$T(M;x,y) = \sum_{A \subseteq E(M)} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)},$$

which is a generating function for the pairs $(|A|, r(A))$ as A ranges over all subsets of $E(M)$. Derksen [16] introduced a strictly stronger enumerative invariant, the $\mathcal{G}$-invariant, denoted $\mathcal{G}(M)$, that records, for each permutation $\pi = (e_1, e_2, \dots, e_n)$ of $E(M)$, the $0,1$-vector $(r_1, r_2, \dots, r_n)$ of rank increases when the elements of $E(M)$ are added in the order that π gives, that is, $r_i = r(\{e_1, e_2, \dots, e_i\}) - r(\{e_1, e_2, \dots, e_{i-1}\})$. Bonin and Kung [8] showed that $\mathcal{G}(M)$ is equivalent to recording, for each $(r(M)+1)$-tuple of integers $(d_0, d_1, \dots, d_{r(M)})$, the number of flags $\mathrm{cl}_M(\emptyset) = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r(M)} = E(M)$ of flats of M for which $d_0 = |F_0|$ and $d_i = |F_i - F_{i-1}|$ for $1 \leqslant i \leqslant r(M)$. Eberhardt [17] proved that $T(M;x,y)$ can be computed from the configuration of M, and Bonin and Kung [8] showed that the same holds for $\mathcal{G}(M)$.

This paper develops a new line of inquiry: given a matroid M, under what conditions, and how, can one construct a matroid that is not isomorphic to M and yet has the same configuration as M? Complementary to that, we also consider matroids M for which all matroids that have the same configuration as M are isomorphic to M; such matroids are *configuration unique*. Two related notions also play roles: a matroid M is *Tutte unique* if $T(M;x,y) = T(N;x,y)$ implies that N is isomorphic to M, and M is $\mathcal{G}$ *unique* if $\mathcal{G}(M) = \mathcal{G}(N)$ implies that N is isomorphic to M. Since the Tutte polynomial can be computed from the $\mathcal{G}$-invariant, which can be computed from the configuration, Tutte-unique matroids are $\mathcal{G}$ unique, and $\mathcal{G}$-unique matroids are configuration unique. See [7] for a survey of Tutte uniqueness.

In Theorem 3.1, for a matroid M in which certain pairs of cyclic flats are non-modular, we show how to construct a matroid that has the same configuration but is not isomorphic to M. In Section 4, we prove that that construction applies to a lattice path matroid if and only if it is not a fundamental (or principal) transversal matroid. We characterize lattice path matroids that are fundamental transversal matroids in several ways, and we show that such matroids are configuration unique. In Section 5, we show that, for $n \geqslant 2$, there are $(3^{n-2}+1)/2$ connected lattice path matroids on $[n] = \{1, 2, \dots, n\}$ that are fundamental transversal matroids, and we refine that count by rank; several well-known combinatorial sequences, such as Pell and Delannoy numbers, arise naturally in this work. It follows that, asymptotically, almost no lattice path matroids are configuration unique. Theorem 6.1 gives a sufficient condition for a matroid to be configuration unique. Section 6 also treats several constructions of matroids that are configuration unique but not $\mathcal{G}$ unique. While nested matroids (matroids for which the lattice of cyclic flats is a chain) are known to be Tutte unique, in Theorem 7.1, we show that for every lattice L that is not a chain, there are transversal matroids whose lattices of cyclic flats are isomorphic to L and to which the construction in Theorem 3.1 applies.

2 Background

For general matroid theory background, see Oxley [26]. We adopt the matroid notation used there. All matroids and lattices considered in this paper are finite. We use $[a, b]$ to denote the interval $\{a, a+1, \ldots, b\}$ in the set $\mathbb{Z}$ of integers, and we simplify $[1, n]$ to $[n]$.

2.1 Cyclic flats, configurations, modular pairs, and principal extensions

A *cyclic set* of a matroid M is a (possibly empty) union of circuits; equivalently, a set $X \subseteq E(M)$ is cyclic if $M|X$ has no coloops. A *cyclic flat* is a flat that is cyclic. The set of cyclic flats of M is denoted $\mathcal{Z}(M)$. With inclusion as the order, $\mathcal{Z}(M)$ is a lattice: for $A, B \in \mathcal{Z}(M)$, the join $A \vee B$ is $\mathrm{cl}(A \cup B)$ and the meet $A \wedge B$ is $(A \cap B) - C$ where C is the set of coloops of $M|(A \cap B)$. Routine arguments show that, for all $X \subseteq E(M)$,

$$r(X) = \min\{r(F) + |X - F| \,:\, F \in \mathcal{Z}(M)\}, \tag{2.1}$$

so M is determined by its ground set $E(M)$ along with the cyclic flats of M and their ranks, that is, by the set $\{E(M)\} \cup \{(F, r(F)) \,:\, F \in \mathcal{Z}(M)\}$. One cyclic flat F that yields the minimum in Equation (2.1) is the closure of the union of the circuits of $M|X$. We will use the following result from [28, 6], which characterizes matroids from the perspective of cyclic flats and their ranks.

Theorem 2.1. *For a collection $\mathcal{Z}$ of subsets of a set E and a function $r : \mathcal{Z} \to \mathbb{Z}$, there is a matroid M on E with $\mathcal{Z}(M) = \mathcal{Z}$ and $r_M(X) = r(X)$ for all $X \in \mathcal{Z}$ if and only if*

(Z0) *$(\mathcal{Z}, \subseteq)$ is a lattice,*

(Z1) *$r(0_{\mathcal{Z}}) = 0$, where $0_{\mathcal{Z}}$ is the least set in $\mathcal{Z}$,*

(Z2) *$0 < r(Y) - r(X) < |Y - X|$ for all sets X, Y in $\mathcal{Z}$ with $X \subsetneq Y$, and*

(Z3) *for all pairs of sets X, Y in $\mathcal{Z}$ (or, equivalently, just incomparable sets in $\mathcal{Z}$),*

$$r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)| \leqslant r(X) + r(Y).$$

By Equation (2.1), for matroids M and N, a function $\phi : E(M) \to E(N)$ is an isomorphism of M onto N if and only if ϕ is a bijection, ϕ maps $\mathcal{Z}(M)$ onto $\mathcal{Z}(N)$, and $r_M(A) = r_N(\phi(A))$ for all $A \in \mathcal{Z}(M)$.

A set A is cyclic in a matroid M if and only if $E(M) - A$ is a flat of the dual matroid M^* since A being a union of circuits of M is equivalent to $E(M) - A$ being an intersection of hyperplanes of M^*. Thus, X is a cyclic flat of M if and only if $E(M) - X$ is a cyclic flat of M^*, and so $\mathcal{Z}(M^*)$, the lattice of cyclic flats of M^*, is isomorphic to the order dual of $\mathcal{Z}(M)$.

The *configuration* of a matroid M is a 4-tuple $(L, s, \rho, |E(M)|)$, where L is a lattice and $s : L \to \mathbb{Z}$ and $\rho : L \to \mathbb{Z}$ are functions such that there is an isomorphism $\phi : L \to \mathcal{Z}(M)$ for which $s(x) = |\phi(x)|$ and $\rho(x) = r(\phi(x))$ for all $x \in L$. Many 4-tuples can satisfy

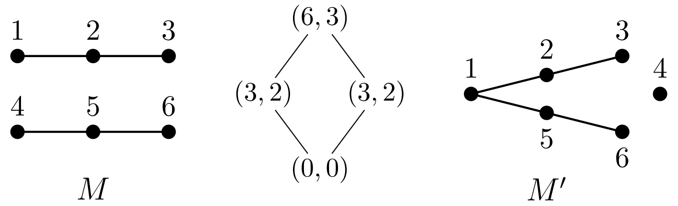

Figure 1: Two non-isomorphic matroids M and M' that have the same configuration. Each pair shown in the lattice gives the size and rank of the corresponding cyclic flat.

these properties, but they all contain the same data, so we view them as the same. Two matroids *have the same configuration* if some 4-tuple (L, s, ρ, n) is the configuration of both. As Figure 1 illustrates, non-isomorphic matroids can have the same configuration.

The next lemma holds since, for any element x of the lattice in the configuration, we can compare the size $s(x)$ and rank $\rho(x)$ to those of pairs y, z for which $y \wedge z = 0$ and $y \vee z = x$.

Lemma 2.2. *For a cyclic flat F of M, whether $M|F$ is connected can be deduced from the corresponding element of the configuration. For each connected component X of M, we can obtain the configuration of the restriction $M|X$ from that of M, so M is configuration unique if and only if all such restrictions $M|X$ are configuration unique.*

A pair (X, Y) of sets in a matroid M is *modular* if $r(X)+r(Y) = r(X\cup Y)+r(X\cap Y)$. Routine calculations with the rank function of the dual give the following two results.

Lemma 2.3. *For subsets X and Y of $E(M)$, the pair (X, Y) is modular in M if and only if the pair $(E(M) - X, E(M) - Y)$ is modular in M^*.*

Lemma 2.4. *For any matroid M, the configuration of M^* can be computed from that of M. Thus, M and N have the same configuration if and only if M^* and N^* have the same configuration, so M is configuration unique if and only if M^* is configuration unique.*

We next define principal extension, which makes precise the idea of adding a point freely to a flat of a matroid. For a matroid M, a subset X of $E(M)$, and an element $e \notin E(M)$, define $r' : 2^{E(M)\cup e} \to \mathbb{Z}$ by, for all $Y \subseteq E(M)$, setting $r'(Y) = r_M(Y)$ and

$$r'(Y \cup e) = \begin{cases} r_M(Y), & \text{if } X \subseteq \mathrm{cl}_M(Y), \\ r_M(Y) + 1, & \text{otherwise.} \end{cases}$$

It is routine to check that r' is the rank function of a matroid on $E(M)\cup e$. This matroid is the *principal extension* of M in which e has been *added freely* to X, and is denoted $M +_X e$. Clearly $M +_X e = M +_{\mathrm{cl}(X)} e$ and $\mathrm{cl}(X) \cup e$ is a flat of $M +_X e$. Also, for $X, Y \subseteq E(M)$ and $e, f \notin E(M)$, we have $(M +_X e) +_Y f = (M +_Y f) +_X e$, so the order in which we apply several principal extensions to subsets of $E(M)$ does not matter. The *free extension* of M is $M +_{E(M)} e$. Note that for $X \subseteq E(M)$, the set $X \cup e$ is a circuit of $M +_{E(M)} e$ if and only if X is a basis of M.

2.2 Transversal matroids

A *set system* on a set E is an indexed family of subsets of E, which we write as $\mathcal{A} = (A_1, A_2, \dots, A_r)$. A set may occur multiple times in $\mathcal{A}$. A *partial transversal* of $\mathcal{A}$ is a subset I of E for which there is an injection $\phi : I \to [r]$ with $x \in A_{\phi(x)}$ for all $x \in I$. *Transversals* of $\mathcal{A}$ are partial transversals of size r. Edmonds and Fulkerson [18] showed that the partial transversals of a set system $\mathcal{A}$ on E are the independent sets of a matroid on E; we say that $\mathcal{A}$ is a *presentation* of this *transversal matroid* $M[\mathcal{A}]$. A transversal matroid is *fundamental* (or *principal*) if for some presentation $(A_1, \dots, A_r)$ and each $i \in [r]$, some element in A_i is in no A_j with $j \in [r] - \{i\}$. The following well-known results, especially Corollary 2.7, are relevant to our work (see [12]).

Lemma 2.5. *Any transversal matroid M has a presentation with $r(M)$ sets. If M has no coloops, then each presentation of M has exactly $r(M)$ nonempty sets.*

Lemma 2.6. *If M is a transversal matroid, then so is $M|X$ for each $X \subseteq E(M)$. If $(A_1, \dots, A_r)$ is a presentation of M, then $(A_1 \cap X, \dots, A_r \cap X)$ is a presentation of $M|X$.*

Corollary 2.7. *If $(A_1, \dots, A_r)$ is a presentation of M and X is any cyclic set of M, then $r(X) = |\{i : X \cap A_i \neq \emptyset\}|$.*

Brylawski [13] gave a useful way to view a transversal matroid M. Let $(A_1, A_2, \dots, A_r)$ be a presentation of M and let the set $V = \{v_1, v_2, \dots, v_r\}$ be disjoint from $E(M)$. View the free matroid on V (i.e., all subsets of V are independent) as having the elements of V at the vertices of an r-vertex simplex, one element at each vertex. For each $e \in E(M)$, add e to this free matroid by taking the principal extension using the set $\{v_i : e \in A_i\}$; that is, put e freely in the face of the simplex that is spanned by the set $\{v_i : e \in A_i\}$ of vertices. Once all elements of $E(M)$ are placed, delete V, and the result is a geometric representation of M. Note that each cyclic set of M spans a face of the simplex. Also, it follows that a transversal matroid is fundamental if and only if it has a representation on a simplex in which, for each vertex of the simplex, at least one element of the matroid is placed there.

The characterization of transversal matroids in the next theorem was first formulated by Mason [24] using sets of cyclic sets; the observation that his result easily implies its streamlined counterpart for sets of cyclic flats was made by Ingleton [23]. For a family $\mathcal{F}$ of sets we shorten $\cap_{A \in \mathcal{F}} A$ to $\cap\mathcal{F}$ and $\cup_{A \in \mathcal{F}} A$ to $\cup\mathcal{F}$.

Theorem 2.8. *A matroid is transversal if and only if for all nonempty sets $\mathcal{F}$ of cyclic flats,*

$$r(\cap\mathcal{F}) \leqslant \sum_{\mathcal{X} \subseteq \mathcal{F}} (-1)^{|\mathcal{X}|+1} r(\cup\mathcal{X}). \tag{2.2}$$

As explained in [9], the condition in Theorem 2.8 is equivalent to having Inequality (2.2) hold for all nonempty antichains $\mathcal{F}$ of cyclic flats (i.e., no set in $\mathcal{F}$ is a subset of another set in $\mathcal{F}$). Inequality (2.2) holds trivially when $|\mathcal{F}| = 1$, and it is the submodular inequality when $|\mathcal{F}| = 2$. Thus, to show that a matroid is transversal, it suffices to check Inequality (2.2) for all antichains $\mathcal{F}$ of cyclic flats with $|\mathcal{F}| \geqslant 3$,

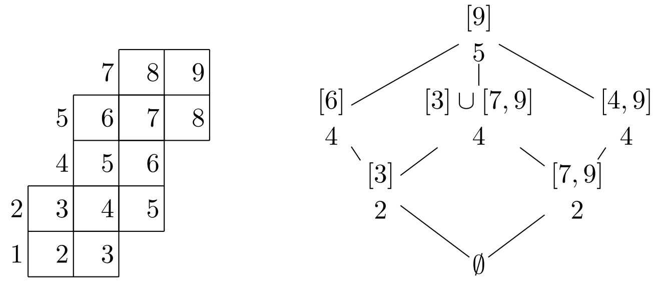

Figure 2: The region bounded by two paths P and Q that give rise to a lattice path matroid, along with its lattice of cyclic flats, with the rank of each cyclic flat shown below the flat. Each north step is labeled with the position it would have in any lattice path from $(0, 0)$ to $(4, 5)$ that uses that step.

2.3 Lattice path matroids

The lattice paths of interest are strings of steps that start at $(0, 0)$, and where each step has unit length and goes either north or east. We write lattice paths as words with the letters N (north) and E (east). Let $P = p_1p_2 \dots p_n$ and $Q = q_1q_2 \dots q_n$ be two lattice paths from $(0, 0)$ to (m, r), so $m + r = n$, with P never going above Q. Let $p_{u_1}, p_{u_2}, \dots, p_{u_r}$ be the north steps of P with $u_1 < u_2 < \cdots < u_r$; let $q_{l_1}, q_{l_2}, \dots, q_{l_r}$ be the north steps of Q with $l_1 < l_2 < \cdots < l_r$. Let N_i be the interval $[l_i, u_i]$ of integers. Let $M[P, Q]$ be the transversal matroid on the ground set $[n]$ that has the presentation $(N_1, N_2, \dots, N_r)$. In the example in Figure 2, the upper path Q is $NNENNENEE$ while P is $EENENNENN$; the set N_1 is $\{1, 2, 3\}$, and, at the top, the set N_5 is $\{7, 8, 9\}$. A *lattice path matroid* is a matroid M that is isomorphic to $M[P, Q]$ for some such pair of lattice paths P and Q. In this paper, we will always take $E(M)$ to be $[n]$ with the usual order, and we will focus on the presentation $(N_1, N_2, \dots, N_r)$, which we call the *path presentation* of M.

The transversals of the path presentation $(N_1, N_2, \dots, N_r)$, i.e., the bases of $M[P, Q]$, are the sets of positions of the north steps in the lattice paths that go from $(0, 0)$ to (m, r) and remain in the region that is bounded by P and Q (see [4, Theorem 3.3]). It is easy to show that the lattice path matroid $M[P, Q]$ is connected if and only if the paths P and Q meet only at their common endpoints, $(0, 0)$ and (m, r) (see [4, Theorem 3.6] and the lattice path interpretation of direct sums discussed before it).

A *nested matroid* is a lattice path matroid where either the lower path P has the form $E^{n-r}N^r$ or the upper path Q has the form N^rE^{n-r}. When P is $E^{n-r}N^r$ and Q is N^rE^{n-r}, the nested matroid is the uniform matroid $U_{r,n}$. A matroid is nested if and only if its lattice of cyclic flats is a chain (see [27, Lemma 2]; this also follows easily from the ideas in the proof of Theorem 4.7 below).

The next result is from [5, Lemma 5.2 and Theorem 5.7]. *Trivial flats* are the flats X with $r(X) = |X|$. *Connected flats* are the flats X for which $M|X$ is connected. (In [5], the term *fundamental flats* refers to the flats that satisfy the first property below.)

Lemma 2.9. *Let M be a connected lattice path matroid on $[n]$ that is not a nested matroid. There are two chains of proper, nontrivial, connected flats, $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_h$ and*

$G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_k$, *in* M *that have the following properties:*

- *the flats in those chains are precisely the proper, nontrivial, connected flats* X *for which, for some spanning circuit* C *of* M*, the set* $C \cap X$ *is a basis of* $M|X$*,*
- *each* F_i *is an interval* $[a]$ *and each* G_j *is an interval* $[b, n]$*, and*
- *the other proper, nontrivial, connected flats of* M *are the intersections* $F_i \cap G_j$ *for which* $\eta(M) < \eta(F_i) + \eta(G_j)$*, where* $\eta(X)$ *is the nullity of* X*, that is,* $|X| - r(X)$*.*

The flats F_i of $M[P,Q]$ above are the intervals $[a]$ for which steps a and $a+1$ of the upper path Q are east and north, respectively. The point at which such steps a and $a+1$ of Q meet is an *EN corner of Q* and we call $F_i = [a]$ an *initial connected flat*. The flats G_j are the intervals $[b,n]$ for which steps $b-1$ and b of the lower path P are north and east, respectively. The point at which such steps $b-1$ and b of P meet is an *NE corner of P* and we call $G_j = [b,n]$ a *final connected flat*. By Lemma 2.9, each connected flat of $M[P,Q]$ is an interval in $[n]$.

Observe that if the lattice path diagram for $M[P,Q]$ is rotated 180° about $(m/2, r/2)$, so step i becomes step $n+1-i$, then the original initial connected flats give rise to the final connected flats after the rotation, and likewise with initial and final switched. The lower path P gives the upper path P^t after the rotation, where the steps in P^t are those of P but in the reverse order; likewise, the upper path Q gives the lower path Q^t after the rotation. The bijection mapping i to $n+1-i$ is an isomorphism of $M[P,Q]$ onto $M[Q^t, P^t]$. If we know the size and rank of each flat in the two chains identified in Theorem 2.9, then we know P and Q up to the 180° rotation, and so we know $M[P,Q]$ up to isomorphism. (See [5, Theorem 5.6] and the discussion before it.)

The dual of a lattice path matroid M is also a lattice path matroid; its diagram is obtained by flipping the diagram for M around the line $y = x$. This holds because, in any lattice path, this flip switches the steps that are not in the corresponding basis (east steps) with those that are in the basis (north steps). A consequence of this is [5, Corollary 5.5], which we state next.

Lemma 2.10. *For a connected lattice path matroid* M *on* $[n]$*, the interval* $[a]$ *is an initial connected flat of* M *if and only if its complement* $[a+1, n]$ *is a final connected flat of the dual* M^**. The same holds with* M *and* M^* *switched.*

In addition to being closed under duality, the class of lattice path matroids is also closed under direct sums and minors (see [5, Theorem 3.1]). We will use just the special case of restriction to an interval, the endpoints of which are not loops. Given the lattice path diagram for $M[P,Q]$, to obtain the diagram for the restriction of $M[P,Q]$ to an interval $[a,b]$, where neither a nor b is a loop of $M[P,Q]$, consider the lowest north step that can be step a in a path and the highest north step that can be step b in a path; if the former is not strictly to the right of the latter, then the restriction of $M[P,Q]$ to $[a,b]$ is represented by the region of the diagram for $M[P,Q]$ that is between the two steps just identified; otherwise the restriction is a free matroid. This is illustrated in Figure 3 and it gives the following lemma.

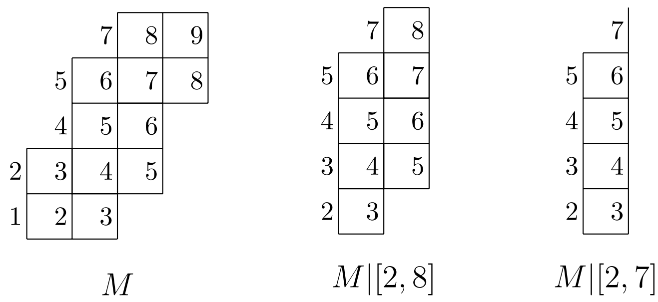

Figure 3: The diagrams representing a lattice path matroid M and its restrictions to $[2,8]$ and $[2,7]$. The restriction to $[3,5]$ is the free matroid $U_{3,3}$ on $[3,5]$, and the restriction to $[2,5]$ is also free.

Lemma 2.11. *Let M be a lattice path matroid on $[n]$ and let the path presentation of M be $(N_1, N_2, \ldots, N_r)$. For an interval $X = [a+1, a+k]$ in $[n]$, if the sets N_i with $N_i \cap X \neq \emptyset$ are $N_{j+1}, N_{j+2}, \ldots, N_{j+t}$, then $r(X) = \min(t, k)$.*

3 A construction to produce matroids having the same configuration as a given matroid

The next theorem gives the central construction of the paper. Throughout the paper, we set $\mathcal{Z}_e = \{A \in \mathcal{Z}(M) : e \in A\}$ for any $e \in E(M)$. To start with an example for motivation, consider the matroid M in Figure 1, and the elements 1 and 4. We have $\mathcal{Z}_1 = \{[3], [6]\}$ and $\mathcal{Z}_4 = \{[4,6], [6]\}$. Both $\mathcal{Z}_1 - \mathcal{Z}_4$ and $\mathcal{Z}_4 - \mathcal{Z}_1$ are nonempty, and the only pair (A, B) with $A \in \mathcal{Z}_1 - \mathcal{Z}_4$ and $B \in \mathcal{Z}_4 - \mathcal{Z}_1$ is non-modular. We get the cyclic flats of the matroid M' in Figure 1, which has the same configuration as M, by taking the cyclic flat in $\mathcal{Z}_4 - \mathcal{Z}_1$, namely, $[4,6]$, and replacing 4 by 1 to get $\{1,5,6\}$.

Theorem 3.1. *Let M be a matroid. Assume that for some $x, y \in E(M)$, (i) both $\mathcal{Z}_x - \mathcal{Z}_y$ and $\mathcal{Z}_y - \mathcal{Z}_x$ are nonempty, and (ii) if $X \in \mathcal{Z}_x - \mathcal{Z}_y$ and $Y \in \mathcal{Z}_y - \mathcal{Z}_x$, then (X, Y) is not a modular pair. For each $Y \in \mathcal{Z}_y - \mathcal{Z}_x$, let $Y_x = (Y - y) \cup x$, let*

$$\mathcal{Z}' = (\mathcal{Z}(M) - (\mathcal{Z}_y - \mathcal{Z}_x)) \cup \{Y_x : Y \in \mathcal{Z}_y - \mathcal{Z}_x\},$$

and let $r' : \mathcal{Z}' \to \mathbb{Z}$ be given by $r'(A) = r_M(A)$ if $A \in \mathcal{Z}(M)$, and $r'(Y_x) = r_M(Y)$ if $Y \in \mathcal{Z}_y - \mathcal{Z}_x$. Then the pair $(\mathcal{Z}', r')$ satisfies properties (Z0)–(Z3) in Theorem 2.1 and so defines a matroid M' on $E(M)$. The matroids M and M' have the same configuration but are not isomorphic. Also, the matroid M' is isomorphic to the matroid that results from the construction above with the roles of x and y switched.

Proof. By the construction, the map $\phi : \mathcal{Z}(M) \to \mathcal{Z}'$ defined by

$$\phi(Y) = \begin{cases} Y_x, & \text{if } Y \in \mathcal{Z}_y - \mathcal{Z}_x, \\ Y, & \text{otherwise} \end{cases}$$

is a bijection. The following properties of ϕ are easy to check: for all $X, Y \in \mathcal{Z}(M)$,

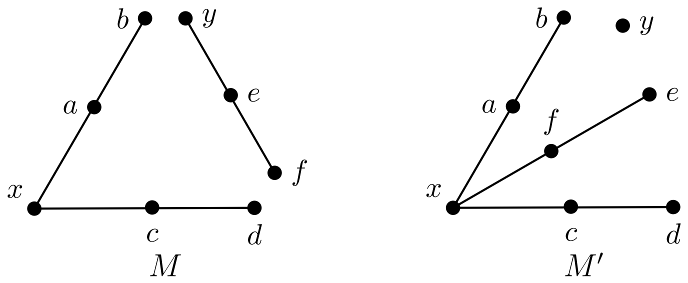

Figure 4: The matroid M is transversal, but the matroid M' that results when x bumps y is not transversal. Both matroids have rank 3.

(i) $X \subseteq Y$ if and only if $\phi(X) \subseteq \phi(Y)$,

(ii) $|\phi(X) \cap \phi(Y)| = |X \cap Y| + 1$ if $X \in \mathcal{Z}_x - \mathcal{Z}_y$ and $Y \in \mathcal{Z}_y - \mathcal{Z}_x$, or vice versa; otherwise $|\phi(X) \cap \phi(Y)| = |X \cap Y|$.

By property (i), ϕ is a lattice isomorphism from $(\mathcal{Z}(M), \subseteq)$ onto $(\mathcal{Z}', \subseteq)$, so property (Z0) holds for the pair $(\mathcal{Z}', r')$. Properties (Z1)–(Z3) are direct to check, with only slightly more care needed for property (Z3) for sets X and Y_x where $X \in \mathcal{Z}_x - \mathcal{Z}_y$ and $Y \in \mathcal{Z}_y - \mathcal{Z}_x$. For that, we have

$$r_M(X \cap Y) = r_M(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|$$

since the elements of $(X \cap Y) - (X \wedge Y)$ are the coloops of $M|(X \cap Y)$, so the assumption that (X, Y) is not a modular pair gives

$$r_M(X \vee Y) + r_M(X \wedge Y) + |(X \cap Y) - (X \wedge Y)| < r_M(X) + r_M(Y).$$

Since $|(X \cap Y_x) - (X \wedge Y_x)| = |(X \cap Y) - (X \wedge Y)| + 1$ by property (ii), we have

$$r'(X \vee Y_x) + r'(X \wedge Y_x) + |(X \cap Y_x) - (X \wedge Y_x)| \leqslant r'(X) + r'(Y_x),$$

so property (Z3) holds for X and Y_x. Thus, the pair $(\mathcal{Z}', r')$ indeed defines a matroid M' on $E(M)$. Clearly the configuration of M' is that of M. Also, M and M' are not isomorphic since the multisets of sizes of intersections of cyclic flats differ by property (ii).

By construction, the cyclic flats of M' that contain just one of x and y must contain x. If we switch the roles of x and y, then the cyclic flats of the resulting matroid M'' that contain just one of x and y must contain y. Thus, the transposition of $E(M)$ that switches x and y and fixes all other elements of $E(M)$ is an isomorphism of M' onto M''. □

We say that the matroid M' in Theorem 3.1 results from x *bumping* y in M. Bumping need not preserve the property of being transversal, as the example in Figure 4 shows. (One can apply either Theorem 2.8 or the geometric view of transversal matroids to verify that M is transversal and M' is not.) Likewise, the example in Figure 1 shows that representability over a given field need not be preserved; in that figure, M is ternary but M' is not.

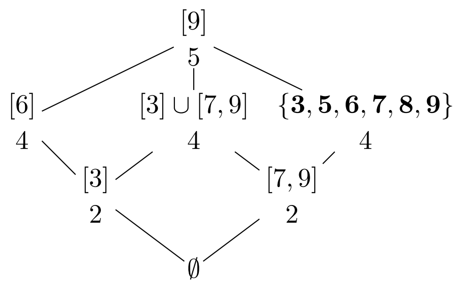

Figure 5: The lattice of cyclic flats obtained when 3 bumps 4 in the matroid shown in Figure 2. In this case, only one set is altered; it is highlighted with boldface.

When q is a prime power that is at least 9 and not prime, there can be non-isomorphic projective planes of order q; many constructions of such planes are known (see, e.g., [21]). All projective planes of order q have the same configuration. The hypotheses of Theorem 3.1 never hold for two elements in a projective plane, so bumping does not apply to such matroids. Also, bumping does not produce projective planes since no two elements x and y in a projective plane have the property noted above, that any cyclic flat that contains one of x or y must contain x. Thus, bumping does not account for all instances of matroids that have the same configuration.

If (X, Y) is a non-modular pair of hyperplanes of a matroid N, then extending N by two principal extensions, one adding an element x freely to X and the other adding an element y freely to Y, where $x, y \notin E(N)$, gives a matroid M to which Theorem 3.1 applies. Thus, such matroids are at most a two-element extension away from a matroid that is not configuration unique.

The next example, using the lattice path matroid in Figure 2, is a preview of what we will see in Section 4 for lattice path matroids that are not fundamental transversal matroids. The modular pairs that consist of an initial connected flat and a final connected flat are $([6], [4, 9])$ and $([3], [7, 9])$; the non-modular pairs of this type are $([6], [7, 9])$ and $([3], [4, 9])$. Consider 3 and 4. We have

$$\mathcal{Z}_3 = \{[3], [6], [3] \cup [7, 9], [9]\} \qquad \text{and} \qquad \mathcal{Z}_4 = \{[6], [4, 9], [9]\}.$$

Both $\mathcal{Z}_3 - \mathcal{Z}_4$ and $\mathcal{Z}_4 - \mathcal{Z}_3$ are nonempty; also, for each $X \in \mathcal{Z}_3 - \mathcal{Z}_4$ and $Y \in \mathcal{Z}_4 - \mathcal{Z}_3$, the pair (X, Y) is not modular, so the hypotheses of Theorem 3.1 hold. The result of replacing 4 by 3 in the sole cyclic flat in $\mathcal{Z}_4 - \mathcal{Z}_3$ is shown in Figure 5.

We end this section with an immediate corollary of Theorem 3.1, Lemma 2.3, and the equality $\mathcal{Z}(M^*) = \{E(M) - A : A \in \mathcal{Z}(M)\}$: up to switching x and y, bumping commutes with taking the dual.

Corollary 3.2. *For elements x and y, the hypotheses of Theorem 3.1 hold in M if and only if they hold in M^*, and the matroid obtained from x bumping y in M is the dual of the matroid obtained from y bumping x in M^*.*

4 An application to lattice path matroids

There are three main results in this section. Theorem 4.3 shows that bumping, the construction in Theorem 3.1, applies to any connected lattice path matroid that has a non-modular pair (A, B) that consists of an initial connected flat A and a final connected flat B. Theorem 4.5 proves the converse by showing that connected lattice path matroids in which all such pairs (A, B) are modular are configuration unique. Corollary 4.8 shows that, for connected lattice path matroids, having all such pairs (A, B) be modular is equivalent to the matroid being fundamental transversal.

We first characterize modular pairs of initial and final connected flats.

Lemma 4.1. *Let M be a connected lattice path matroid on $[n]$, and let $(N_1, N_2, \ldots, N_r)$ be its path presentation. Let $A = [a]$ be an initial connected flat, and $B = [b, n]$ be a final connected flat, of M. The pair (A, B) is modular if and only if the number of $i \in [r]$ with both $A \cap N_i$ and $B \cap N_i$ nonempty is at most $|A \cap B|$.*

Proof. The assertion when $A \cap B = \emptyset$ follows from Corollary 2.7 applied to A, B, and $A \cup B$. Now assume that $A \cap B \neq \emptyset$, so $b \leqslant a$ and $A \cap B = [b, a]$. Since all sets involved are intervals, $A \cap N_i \neq \emptyset$ and $B \cap N_i \neq \emptyset$ if and only if $[b, a] \cap N_i \neq \emptyset$. Thus, we must show that (A, B) is modular if and only if $|\{i : [b, a] \cap N_i \neq \emptyset\}| \leqslant |A \cap B|$. Let $\{i : [b, a] \cap N_i \neq \emptyset\} = [j + 1, j + t]$. Corollary 2.7 gives $r(A) = j + t$ and $r(B) = r - j$, so (A, B) is modular if and only if $r(A \cap B) = t$, which, by Lemma 2.11, is precisely when $t \leqslant |A \cap B|$, as needed. □

We next recast Lemma 4.1 in terms of lattice path diagrams. Recall that the initial connected flats $[a]$ of $M[P, Q]$ arise precisely from the EN corners of the upper path Q, where that east step is the ath step in Q, and the final connected flats $[b, n]$ arise precisely from the NE corners of P, where that east step is the bth step in P. For an initial connected flat $A = [a]$, let (a_1, a_2) be the coordinates of the integer point at the corresponding EN corner of Q. Thus, $a_1 + a_2 = a$. For a final connected flat $B = [b, n]$, let (b_1, b_2) be the coordinates of the integer point at the corresponding NE corner of P. Thus, $b_1 + b_2 = b - 1$. We call the pair (A, B) *mixed* if $a_1 < b_1$ and $a_2 > b_2$. Since P never goes above Q, we cannot have $a_1 > b_1$ and $a_2 < b_2$, so the mixed case is the only option for having the signs of $a_1 - b_1$ and $a_2 - b_2$ differ.

Corollary 4.2. *Let $M = M[P, Q]$ be a connected lattice path matroid on $[n]$. Let A be an initial connected flat of M and let B be a final connected flat of M. The pair (A, B) is modular if and only if it is not mixed.*

Proof. Let A be $[a]$ where a corresponds to the EN corner of Q at (a_1, a_2), and let B be $[b, n]$ where b corresponds to the NE corner of P at (b_1, b_2).

Assume that (A, B) is not mixed. First assume that $a_1 \leqslant b_1$ and $a_2 \leqslant b_2$. Thus, $a < b$, so $A \cap B = \emptyset$. Let N_i be a set in the path presentation of M. If $N_i \cap A \neq \emptyset$, then $i \leqslant a_2$, while if $N_i \cap B \neq \emptyset$, then $i > b_2$. Since $a_2 \leqslant b_2$, no set N_i satisfies both conditions, so (A, B) is a modular pair by Lemma 4.1. Now assume that $a_1 \geqslant b_1$ and $a_2 \geqslant b_2$. In the lattice path diagram for the dual M^*, the upper bounding path has an EN corner at (b_2, b_1) and the lower bounding path has an NE corner at (a_2, a_1). By what we just proved, the pair $([n]-B, [n]-A)$, which consists of an initial and a final connected flat of M^* by Lemma 2.10, is modular in M^*. Thus, (A, B) is a modular pair in M by Lemma 2.3.

Now assume that (A, B) is mixed, so $a_1 < b_1$ and $a_2 > b_2$. First assume that $A \cap B = \emptyset$, so $a < b$. Now $b \in N_{b_2+1}$ and $a \in N_{a_2}$, so $\{a, b\} \subseteq N_i$ for all sets N_i for which $b_2 < i \leqslant a_2$. Thus, (A, B) is not a modular pair. Now assume that $A \cap B \neq \emptyset$. Then $([n]-B, [n]-A)$ is a mixed pair consisting of an initial and a final connected flat of M^*, and the sets are disjoint. By what we just proved, the pair $([n]-B, [n]-A)$ is not modular in M^*, so the pair (A, B) is not modular in M by Lemma 2.3. □

The next theorem is the first main result of this section. The theorem is stated for connected lattice path matroids, but it extends to any lattice path matroid by applying the result to the restrictions to connected components.

Theorem 4.3. *If a connected lattice path matroid M on $[n]$ has an initial connected flat $A = [a]$ and a final connected flat $B = [b, n]$ for which (A, B) is not a modular pair, then some matroid that is not isomorphic to M has the same configuration as M.*

Proof. Let $(N_1, N_2, \ldots, N_r)$ be the path presentation of M. We first consider the case in which $A \cap B = \emptyset$, so $a < b$. Among all non-modular pairs of disjoint initial and final connected flats, choose (A, B) so that $b - a$ is minimal. We claim that $\mathcal{Z}_a$ and $\mathcal{Z}_b$ satisfy the hypotheses of Theorem 3.1. Since $A \in \mathcal{Z}_a - \mathcal{Z}_b$ and $B \in \mathcal{Z}_b - \mathcal{Z}_a$, neither difference is empty. By Lemma 4.1, some set N_i contains both a and b. For any initial connected flat $[c] \in \mathcal{Z}_a - \mathcal{Z}_b$, we have $a \leqslant c < b$, so $([c], B)$ is not a modular pair since $[c] \cap B = \emptyset$ and $\{c, b\} \subseteq N_i$. By having chosen A and B with $b - a$ minimal, it follows that $c = a$, so A is the only initial connected flat in $\mathcal{Z}_a - \mathcal{Z}_b$. Similarly, B is the only final connected flat in $\mathcal{Z}_b - \mathcal{Z}_a$. By those conclusions and Lemma 2.9, no flat in $\mathcal{Z}_a - \mathcal{Z}_b$ or $\mathcal{Z}_b - \mathcal{Z}_a$ contains an element c with $a < c < b$. Consider $F_a \in \mathcal{Z}_a - \mathcal{Z}_b$ and $F_b \in \mathcal{Z}_b - \mathcal{Z}_a$. Each connected component of $M|F_a$ or of $M|F_b$ is a subset of either A or B. The sets F_a, F_b, and $F_a \cup F_b$ are cyclic, so the rank of each set is the number of sets N_j that are not disjoint from it. The set $F_a \cap F_b$ might not be cyclic, so the number of sets N_j that are not disjoint from it is only an upper bound on its rank. If $F_a \cap F_b = \emptyset$, then any set N_i that contains a and b shows that $r(F_a) + r(F_b) - r(F_a \cup F_b) \geqslant 1 > r(F_a \cap F_b)$, so (F_a, F_b) is not a modular pair. Now assume that $F_a \cap F_b \neq \emptyset$. Fix $c \in F_a \cap F_b$. Either $c < a$ or $c > b$. First, assume that $c < a$. Let X_b be the connected component of $M|F_b$ that contains b, and let X_c be the connected component of $M|F_b$ that contains c, so $X_b \subseteq B$ and $X_c \subseteq A$. Since $X_b \cap X_c = \emptyset$ and $r(X_b \cup X_c) = r(X_b) + r(X_c)$, no set N_j in the presentation contains both c and b (otherwise N_j would be counted twice on the right side and only once on the left side). By symmetry, if $c > b$, then no N_j contains both c and a. Thus, the sets N_i that contain both a and b are disjoint from $F_a \cap F_b$ and so do not contribute to $r(F_a \cap F_b)$; however, they contribute to each of $r(F_a)$, $r(F_b)$, and $r(F_a \cup F_b)$. Therefore $r(F_a) + r(F_b) - r(F_a \cup F_b) > r(F_a \cap F_b)$, so (F_a, F_b) is not a modular pair. Thus, $\mathcal{Z}_a$ and $\mathcal{Z}_b$

satisfy the hypotheses of Theorem 3.1, so the matroid that arises from M when a bumps b is not isomorphic to M and has the same configuration as M.

Finally, assume that $A \cap B \neq \emptyset$, so $b \leqslant a$. By Lemma 2.10, the set $[b-1]$ is an initial connected flat of M^*, and $[a+1, n]$ is a final connected flat of M^*. Also, the pair $([b-1], [a+1, n])$ is not modular in M^* by Lemma 2.3, and $[b-1] \cap [a+1, n] = \emptyset$. By the case shown above, bumping applies to some elements a' and b' in M^*, and so it applies to a' and b' in M by Corollary 3.2, as needed. □

The matroid that is constructed in the proof of Theorem 4.3 is not a lattice path matroid by the next result.

Theorem 4.4. *If two lattice path matroids have the same configuration, then they are isomorphic.*

Proof. Let (L, s, ρ, n) be the configuration of a lattice path matroid M on $[n]$. It suffices to show how to obtain M up to isomorphism from (L, s, ρ, n). By Lemma 2.2, we may assume that M is connected. Nested matroids are Tutte unique [15, Theorem 8.12], which is a stronger conclusion, so we can assume that M is not nested. By the comments two paragraphs after Lemma 2.9, it suffices to identify the elements of L that correspond to the connected flats in the two chains identified in that lemma. We do this by showing how to identify all other elements of L. Disconnected cyclic flats of M can be detected from (L, s, ρ, n) by Lemma 2.2. Let $F \in \mathcal{Z}(M)$ be connected but in neither chain identified in Lemma 2.9, so F is an intersection of an initial and a final connected flat. Thus, $[n] - F$ is not an interval, so it is a disconnected cyclic flat of the dual M^*, and this can be detected from the configuration of M^*, which we get from (L, s, ρ, n) by Lemma 2.4. □

The next result, which is another main result of this section, strengthens Theorem 4.4 when each pair that consists of an initial and a final connected flat is modular. It has not yet been shown whether these matroids are $\mathcal{G}$ unique.

Theorem 4.5. *Let $M = M[P, Q]$ be a lattice path matroid. If, for all initial connected flats A and final connected flats B of the restriction to any connected component of M, the pair (A, B) is modular, then M is configuration unique.*

Proof. By Lemma 2.2, we may assume that M is connected. Assume that M and N have the same configuration. Thus, there is a lattice isomorphism $\Phi : \mathcal{Z}(M) \to \mathcal{Z}(N)$ that preserves the size and rank of each cyclic flat. We will shorten $\Phi(A)$ to A'. To show that M and N are isomorphic, it suffices to show that Φ is induced by a bijection $\phi : E(M) \to E(N)$. We first show that $|A \cap B| = |A' \cap B'|$ and $|A \cup B| = |A' \cup B'|$ for any initial connected flat A and final connected flat B of M. Showing one of those equalities suffices since each equality implies the other by inclusion/exclusion.

Let $E(M) = [n]$, let A be $[a]$ where a corresponds to the EN corner of Q at (a_1, a_2), and let B be $[b, n]$ where b corresponds to the NE corner of P at (b_1, b_2). Thus, $a_1 + a_2 = a$ and $b_1 + b_2 = b - 1$. The assumption that (A, B) is modular means that (A, B) is not mixed, which we break into two cases: (i) $a_1 < b_1$ and $a_2 \leqslant b_2$, and (ii) $b_1 \leqslant a_1$ and $b_2 < a_2$.

Case (i) gives $a < b$, so $A \cap B = \emptyset$. Now $r_M(A) + r_M(B) = r_M(A \vee B)$ since (A, B) is modular. By applying Φ we get $r_N(A') + r_N(B') = r_N(A' \vee B')$, which gives $A' \cap B' = \emptyset$ since otherwise the submodular inequality would fail for A' and B'. Thus, $|A \cap B| = 0 = |A' \cap B'|$.

Assume that case (ii) applies to (A, B). Define $\Phi^* : \mathcal{Z}(M^*) \to \mathcal{Z}(N^*)$ by, for all F in $\mathcal{Z}(M)$, setting $\Phi^*([n] - F) = E(N) - F'$. The discussion of duality after Theorem 2.1 shows that Φ^* is a lattice isomorphism; also, it preserves size and rank. By Lemma 2.10, the pair $([n] - B, [n] - A)$ consists of an initial and a final connected flat of M^*; also, the corresponding corners are at (b_2, b_1) and (a_2, a_1) in the lattice path diagram for M^*. Case (i) applies to $([n] - B, [n] - A)$ in M^*, so by what we just proved,

$$([n] - A) \cap ([n] - B) = \emptyset = (E(N) - A') \cap (E(N) - B').$$

Thus, $A \cup B = [n]$ and $A' \cup B' = E(N)$, so $|A \cup B| = n = |A' \cup B'|$, as claimed.

Let $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{s-1}$ be the initial connected flats of M. Set $A_0 = \emptyset$ and $A_s = [n]$. Let $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_{t-1}$ be the final connected flats of M. Set $B_0 = \emptyset$ and $B_t = [n]$. Thus, $A'_0 = B'_0 = \emptyset$ and $A'_s = B'_s = E(N)$. For each $e \in [n]$, we have $e \in (A_i - A_{i-1}) \cap (B_j - B_{j-1})$ for exactly one pair $(i, j) \in [s] \times [t]$. What we showed in the previous two paragraphs also gives

$$|(A_i - A_{i-1}) \cap (B_j - B_{j-1})| = |(A'_i - A'_{i-1}) \cap (B'_j - B'_{j-1})|$$

for all $(i, j) \in [s] \times [t]$. Thus, there is a bijection $\phi : E(M) \to E(N)$ for which

$$\phi\big((A_i - A_{i-1}) \cap (B_j - B_{j-1})\big) = (A'_i - A'_{i-1}) \cap (B'_j - B'_{j-1})$$

for all $(i, j) \in [s] \times [t]$. It follows that $\phi(A_i) = A'_i$, $\phi(B_j) = B'_j$, and $\phi(A_i \cap B_j) = A'_i \cap B'_j$ for each $i \in [s-1]$ and $j \in [t-1]$. By Lemma 2.9, any connected component of any cyclic flat of M and N is among these sets. Thus, ϕ is an isomorphism from M onto N. □

In order to show that the condition on a lattice path matroid M in Theorem 4.3 holds if and only if M is not a fundamental transversal matroid, we will use rook matroids, which Alexandersson and Jal introduced in [1]. For consistency with the convention for lattice path matroids, our description of rook matroids differs superficially from [1]: we switch the roles of rows and columns, and our labeling differs. Given lattice paths P and Q from $(0, 0)$ to (m, r) with P never rising above Q, label the rows of the diagram, from bottom to top, by 1 through r, and the columns, from left to right, by $r + 1$ to $r + m$. (Figure 6 gives an example.) For each $i \in [r]$, let the set A_i consist of i along with the labels of all columns that have a square in row i. The *rook matroid* $R[P, Q]$ is the transversal matroid with the presentation $(A_1, A_2, \ldots, A_r)$. By construction, $R[P, Q]$ is a fundamental transversal matroid; the element $i \in [r]$ is in A_i and in no other A_j. In contrast, lattice path matroids need not be fundamental. In [1], Alexandersson and Jal show that the bases of the rook matroid correspond to non-attacking, non-nesting placements of rooks on the board given by the lattice path diagram. (With our labeling, rook placements amount to bijections $\phi : C \to R$ where $C \subseteq [r + 1, r + m]$, $R \subseteq [r]$, and

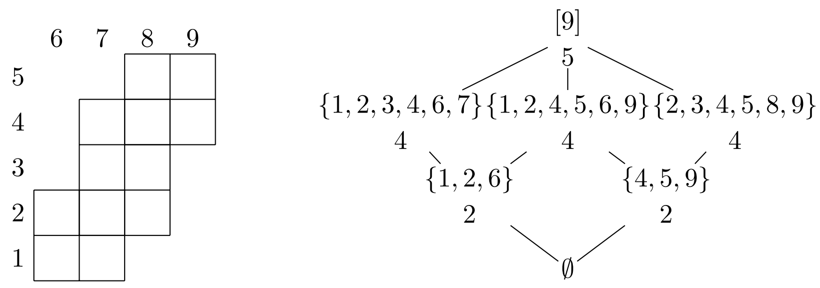

Figure 6: The rook matroid for which the sets in the presentation are $\{1,6,7\}$, $\{2,6,7,8\}$, $\{3,7,8\}$, $\{4,7,8,9\}$, and $\{5,8,9\}$. The lattice of cyclic flats is shown on the right. Note the correspondence with the cyclic flats of the matroid in Figure 2.

$c \in A_{\phi(c)}$ for all $c \in C$; rooks are placed in the squares $(\phi(c), c)$, for $c \in C$, on the board. The non-nesting condition means that for $c, c' \in C$, if $c < c'$, then $\phi(c) < \phi(c')$.)

The interpretation of direct sums via diagrams is the same for rook matroids as for lattice path matroids, so, like lattice path matroids, the class of rook matroids is closed under direct sums; also, a rook matroid is connected if and only if the bounding paths never meet except at the first and last points. (In contrast, as one would expect for a class of fundamental transversal matroids, the class of rook matroids is not closed under minors.) Even if the rook matroid $R[P,Q]$ and the lattice path matroid $M[P,Q]$ are not isomorphic, they have the same configuration, as we show in Theorem 4.7.

We first treat a lemma that applies to both lattice path and rook matroids. Given a presentation $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_r)$ of a transversal matroid N and an element $e \in E(N)$, the *support of* e, denoted $s_{\mathcal{Y}}(e)$, is $\{i \in [r] : e \in Y_i\}$. The *support* $s_{\mathcal{Y}}(X)$ *of* $X \subseteq E(M)$ is the union of all sets $s_{\mathcal{Y}}(e)$ with $e \in X$. By Hall's theorem, $I \subseteq E(N)$ is independent in N if and only if $|X| \leqslant |s_{\mathcal{Y}}(X)|$ for all subsets X of I, so $C \subseteq E(N)$ is a circuit of N if and only if $|s_{\mathcal{Y}}(C)| < |C|$ but $|X| \leqslant |s_{\mathcal{Y}}(X)|$ for all proper subsets X of C.

Lemma 4.6. *Let $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_r)$ be a presentation of a transversal matroid N. If, for each $e \in E(N)$, the support $s_{\mathcal{Y}}(e)$ is an interval in $[r]$, then*

1. *for any circuit C of N, its support $s_{\mathcal{Y}}(C)$ is an interval in $[r]$,*
2. *the support $s_{\mathcal{Y}}(F)$ of any connected flat F of N with $|F| > 1$ is an interval, say I, and $F = \{e : s_{\mathcal{Y}}(e) \subseteq I\}$.*

Proof. Let C be a circuit of N. As noted above, $|C| > |s_{\mathcal{Y}}(C)|$, while $|X| \leqslant |s_{\mathcal{Y}}(X)|$ for all $X \subsetneq C$. Assume, contrary to part (1), that $s_{\mathcal{Y}}(C)$ is not an interval, so there is a partition $\{I, J\}$ of $s_{\mathcal{Y}}(C)$ where I is a maximal interval in $s_{\mathcal{Y}}(C)$ and $J = s_{\mathcal{Y}}(C) - I$. Let $C_I = \{e \in C : s_{\mathcal{Y}}(e) \subseteq I\}$ and $C_J = \{e \in C : s_{\mathcal{Y}}(e) \subseteq J\}$. Since each set $s_{\mathcal{Y}}(e)$ is an interval, $\{C_I, C_J\}$ is a partition of C. The inequality $|s_{\mathcal{Y}}(C)| < |C|$ implies that either $|s_{\mathcal{Y}}(C_I)| < |C_I|$ or $|s_{\mathcal{Y}}(C_J)| < |C_J|$, which contradicts having $|X| \leqslant |s_{\mathcal{Y}}(X)|$ for all $X \subsetneq C$, and so proves assertion (1).

Any two elements in a connected flat F are in a circuit of $N|F$, so it follows from part (1) that $s_{\mathcal{Y}}(F)$ is an interval I in $[r]$. Corollary 2.7 gives $r(F) = |I|$, so if $s_{\mathcal{Y}}(e) \subseteq I$, then

$r(F) = r(F \cup e)$. Thus, $\{e : s_{\mathcal{Y}}(e) \subseteq I\} \subseteq F$. Clearly $F \subseteq \{e : s_{\mathcal{Y}}(e) \subseteq I\}$, so equality holds. □

For example, consider the lattice path matroid M in Figure 2 and the rook matroid R in Figure 6. In M, the connected cyclic flats are (denoting the support of e by $s_{\mathcal{N}}(e)$)

$$\emptyset = \{e : s_{\mathcal{N}}(e) \subseteq \emptyset\}, \qquad [7,9] = \{e : s_{\mathcal{N}}(e) \subseteq [4,5]\},$$
$$[3] = \{e : s_{\mathcal{N}}(e) \subseteq [2]\}, \qquad [4,9] = \{e : s_{\mathcal{N}}(e) \subseteq [2,5]\},$$
$$[6] = \{e : s_{\mathcal{N}}(e) \subseteq [4]\}, \qquad [9] = \{e : s_{\mathcal{N}}(e) \subseteq [5]\}.$$

In R, the connected cyclic flats are (denoting the support of e by $s_{\mathcal{A}}(e)$)

$$\emptyset = \{e : s_{\mathcal{A}}(e) \subseteq \emptyset\}, \qquad \{4,5,9\} = \{e : s_{\mathcal{A}}(e) \subseteq [4,5]\},$$
$$\{1,2,6\} = \{e : s_{\mathcal{A}}(e) \subseteq [2]\}, \qquad \{2,3,4,5,8,9\} = \{e : s_{\mathcal{A}}(e) \subseteq [2,5]\},$$
$$\{1,2,3,4,6,7\} = \{e : s_{\mathcal{A}}(e) \subseteq [4]\}, \qquad [9] = \{e : s_{\mathcal{A}}(e) \subseteq [5]\}.$$

We see that the connected flats in M and R consisting of the elements with support in some interval I of $[5]$ have the same size and rank, and this extends to all cyclic flats. This illustrates the next result, that the lattice path and rook matroids coming from the same lattice path diagram have the same configuration. This result strengthens [1, Theorem 3.38], which shows that $M[P,Q]$ and $R[P,Q]$ have the same Tutte polynomial. It also proves [1, Conjecture 3.39]: any valuative invariant is the same on $M[P,Q]$ and $R[P,Q]$.

Theorem 4.7. *Fix lattice paths P and Q from $(0,0)$ to $(n-r,r)$ with P never rising above Q. The lattice path matroid $M = M[P,Q]$ and the rook matroid $R = R[P,Q]$ have the same configuration.*

Proof. By the observations about direct sums above, it suffices to prove this theorem when M and R are connected, that is, P and Q intersect only at $(0,0)$ and $(n-r,r)$, so we make that assumption. Let $\mathcal{N}$ be the path presentation $(N_1, N_2, \ldots, N_r)$ of M. Let $\mathcal{A}$ be the presentation $(A_1, A_2, \ldots, A_r)$ that we used to define R. For an interval I in $[r]$, let

$$S_I^M = \{e \in [n] : s_{\mathcal{N}}(e) \subseteq I\} \quad \text{and} \quad S_I^R = \{e \in [n] : s_{\mathcal{A}}(e) \subseteq I\}.$$

It is easy to see that S_I^M is a flat of M, as is S_I^R for R. Also, $S_\emptyset^M = S_\emptyset^R = \emptyset$, which is a cyclic flat of both M and R. By Lemma 4.6, each connected flat of M is S_I^M for some interval I in $[r]$, and likewise for R. We have $I \subseteq S_I^R$, so $S_I^R \neq \emptyset$ when $I \neq \emptyset$. The key to the proof is establishing the following claim: for any nonempty interval I in $[r]$, the set S_I^M is a connected flat of M with $|S_I^M| \geqslant 2$ if and only if S_I^R is a connected flat of R with $|S_I^R| \geqslant 2$, and in that case, $|S_I^M| = |S_I^R|$ and $r_M(S_I^M) = r_R(S_I^R)$.

We label each north step in the lattice path diagram with the position it has in each path that contains it, as illustrated in Figure 2. Consider a nonempty interval $I = [s,t]$ in $[r]$. First assume that $S_I^M = \emptyset$. We claim that the flat S_I^R of R is not cyclic. Having $S_I^M = \emptyset$ implies that any label on a north step in a row between rows s and t also labels a north step in either row $s-1$ or row $t+1$. From that, it follows that each column

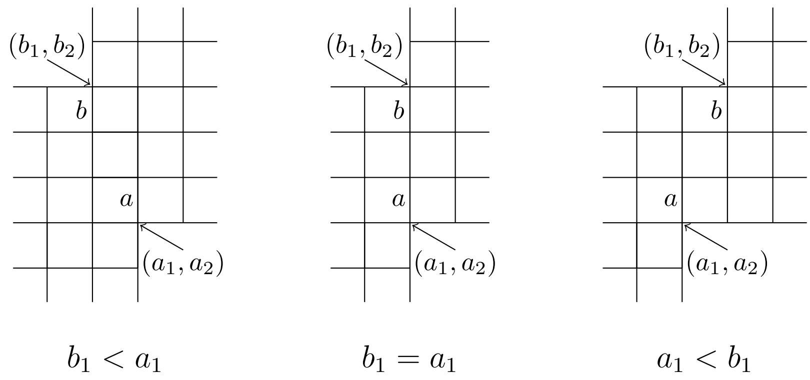

Figure 7: The options for the corners at (a_1, a_2) and (b_1, b_2) in the proof of Theorem 4.7.

extends either below row s or above row t, and so $S_I^R = I$, which, as needed, is either a singleton or disconnected. Now assume that $S_I^M \neq \emptyset$. Since the sets in $\mathcal{N}$ are intervals $[l_i, u_i]$ with $l_1 < l_2 < \cdots < l_r$ and $u_1 < u_2 < \cdots < u_r$, it follows that S_I^M is an interval, say $[a, b]$, in $[n]$. In the lattice path diagram, row s is the first row in which some north step has label a. Note that $s = 1$ if and only if $a = 1$ by our assumption about P and Q. Consider the case with $s > 1$, and so $a > 1$. Row $s - 1$ has a north step labeled $a - 1$ (since $a - 1 \notin S_I^M$), and none labeled a, so the north step labeled $a - 1$ in row $s - 1$ is in P. If the north step labeled a in row s is also in P, then $S_I^M \cap N_s = \{a\}$, so a is a coloop of $M|S_I^M$, and so either $|S_I^M| = 1$ or $M|S_I^M$ is disconnected. In that case, s is the only element of S_I^R that has s in its support, so $R|S_I^R$ has s as a coloop, and so either $|S_I^R| = 1$ or $R|S_I^R$ is disconnected. Thus, we may focus on the case in which the north step labeled a in row s is just above a north-east corner of P. By symmetry, when $t \neq r$ (equivalently, $b \neq n$), the matroids $M|S_I^M$ and $R|S_I^R$ are disconnected or have singleton ground sets unless the north step labeled b in row t has an east-north corner of Q right above it.

Let (a_1, a_2) be the lowest point on the north step labeled a in row s, which is $(0, 0)$ if $a = s = 1$, and otherwise is a north-east corner of P. Let (b_1, b_2) be the highest point on the north step labeled b in row t, which is $(n - r, r)$ if $b = n$ and $t = r$, and otherwise is an east-north corner of Q. If $b_1 \leqslant a_1$, then $M|S_I^M$ is the free matroid on $[a, b]$ and $R|S_I^R$ is the free matroid on I; thus, each is connected if and only if $|I| = 1$. (See Figure 7.) Now assume that $a_1 < b_1$. To get $M|S_I^M$ and $R|S_I^R$, restrict the diagram to the region between these corners and take the resulting lattice path or rook matroid. Both $M|S_I^M$ and $R|S_I^R$ are connected since the bounding paths have no common internal points, and they have the same rank, namely, $|I|$, and the same number of elements (the number of rows plus the number of columns in the restricted diagram). This completes the proof of the claim.

The connected components of the restriction to a cyclic flat of M are connected flats of M with at least two elements, and likewise for R. Thus, the cyclic flats of M have the form $S_{I_1}^M \cup S_{I_2}^M \cup \cdots \cup S_{I_k}^M$ where $I_1, I_2, \ldots, I_k$ are pairwise disjoint intervals in $[r]$, and likewise for R. If $I_1 \cup I_2 \cup \cdots \cup I_k$ is an interval in $[r]$, then $S_{I_1}^M \cup S_{I_2}^M \cup \cdots \cup S_{I_k}^M$ could span a connected flat of M, but in that case, by what we proved above, $S_{I_1}^R \cup S_{I_2}^R \cup \cdots \cup S_{I_k}^R$ would also span a connected flat of R of the same size and rank. When $S^M_{I_1} \cup S^M_{I_2} \cup \cdots \cup S^M_{I_k}$ is a flat of M, its rank is the sum of the ranks of the components, and likewise for the size, and likewise for the counterpart in R. Thus, M and R have the same configuration. □

By the corollary below, the hypothesis of Theorem 4.3 is equivalent to the lattice path matroid not being a fundamental transversal matroid. The proof will use the observation that all pairs of cyclic flats in a fundamental transversal matroid M are modular. To see that, let $\mathcal{A} = (A_1, A_2, \ldots, A_r)$ be a presentation of M and let $b_1, b_2, \ldots, b_r$ be elements for which $s_{\mathcal{A}}(b_i) = \{i\}$ for each $i \in [r]$. Then $B = \{b_1, b_2, \ldots, b_r\}$ is a basis of M and for any $X, Y \in \mathcal{Z}(M)$, we have $r(X) = |X \cap B|$, and likewise for Y, $X \cup Y$, and $X \cap Y$. (That result extends to arbitrary collections of cyclic flats in fundamental transversal matroids; see the proof of [9, Theorem 3.2].)

Corollary 4.8. *For a lattice path matroid M, the following statements are equivalent:*

(1) M is a fundamental transversal matroid,

(2) each pair of cyclic flats of M is modular, and

(3) no restriction of M to any of its connected components has mixed pairs.

Proof. We justified that (1) implies (2) above, and (2) clearly implies (3). To prove that (3) implies (1), note that if M has no mixed pairs, then the rook matroid N defined using the same diagram is fundamental and has the same configuration as M. By Theorem 4.5, the matroids M and N are isomorphic, so M is fundamental. □

While pairs of cyclic flats in fundamental transversal matroids are modular, that also holds in some transversal matroids that are not fundamental. One example is the prism, which is the transversal matroid on [6] that has the presentation $([6], \{1, 2\}, \{3, 4\}, \{5, 6\})$.

Alexandersson and Jal [1, Theorem 3.23] showed that if the lattice path matroid $M[P, Q]$ has no mixed pairs, then it is isomorphic to the rook matroid $R[P, Q]$. The proof that (3) implies (1) in Corollary 4.8 shows that this follows from Theorems 4.5 and 4.7.

5 Enumeration of non-mixed diagrams

In this section we show that exactly $(3^{n-2} + 1)/2$ connected lattice path matroids on $[n]$ have no mixed pairs. We also refine this count by rank and note connections with other enumeration problems. A corollary of this work, along with enumerative results in [4] and the results in the previous section, is that, asymptotically, almost no lattice path matroids are configuration unique.

A *diagram* is the shape formed by the unit squares that lie between two lattice paths P and Q that have the same endpoints, and with P strictly below Q except at the endpoints. Thus, diagrams correspond to connected lattice path matroids on $[n]$ with the usual order. A diagram is *non-mixed* if, in the corresponding lattice path matroid, no pair of initial and final connected flats is mixed. The *size* of a diagram is the length of either bounding path minus one, which is the size of the corresponding lattice path matroid minus one. We use the following recursive description of non-mixed diagrams from [1, Proposition 3.14].

Theorem 5.1. *Non-mixed diagrams are the ones that can be built from a single square by any sequence of the following operations: (R) duplicate the topmost row; (C) duplicate the rightmost column; (S) add a square to the right of the topmost row; (T) add a square above the rightmost column. Moreover, the size of the diagram is the number of operations performed plus one.*

Thus, each non-mixed diagram of size m can be encoded by at least one word of length $m-1$ in the alphabet $\{C, R, S, T\}$ (C stands for *column*, R for *row*, S for *side*, and T for *top*). Let $D(w)$ denote the non-mixed diagram obtained from the word w. Several words can yield the same diagram. For example, SRC, CRC, TCC, RCC, SSR, SCR, CSR, and CCR all give the same diagram. The following lemma refines Theorem 5.1 by identifying the instances of S and T that can be replaced by other letters and the ones that cannot. Recall that by a corner at position (a_1, a_2) we mean an EN corner in the upper path or a NE corner in the lower path, where (a_1, a_2) are the coordinates of the point where the steps of the corner meet.

Lemma 5.2. *Let w be any word in the alphabet $\{C, R, S, T\}$. If $D(w)$ has a corner at (a_1, a_2), then (a) $w_{a_1+a_2} = S$ if the corner is in the lower path, and (b) $w_{a_1+a_2} = T$ if the corner is in the upper path. Moreover, if all other appearances of S and T in w are replaced by C and R, respectively, then the resulting word gives the same diagram $D(w)$.*

Proof. The first assertion holds since C and R never create corners, and S does not create an upper corner and T does not create a lower corner. For the second part, suppose that $w_i = S$ but that this S does not create a corner. Let $\overline{w}$ be the word $w_1 \ldots w_{i-1}$. Note that the rightmost column of the diagram $D(\overline{w})$ must have height one, so $D(\overline{w}S) = D(\overline{w}C)$. Similarly, any T that does not create a corner can be replaced by R. □

It is straightforward to check that replacing any instance of RC by CR yields the same diagram. This observation and Lemma 5.2 allow us to associate a unique word to each non-mixed diagram. For instance, of the eight words that give the same diagram in the example above, only CCR satisfies the conditions in the following result.

Corollary 5.3. *Each non-mixed diagram arises from exactly one word in the alphabet $\{C, R, S, T\}$ in which S occurs precisely to create a corner in the lower path, T occurs precisely to create a corner in the upper path, and RC does not occur as a subword (i.e., a word that occurs as consecutive letters in the word). In particular, this word does not have SS or TT as subwords.*

We introduce a subclass of diagrams that will facilitate the enumeration of non-mixed diagrams. We say that a diagram is *thick* if it is non-mixed and all horizontal or vertical segments joining a point of P with a point of Q through the interior of the diagram have length at least two. A thick diagram has size at least three. An abitrary non-mixed

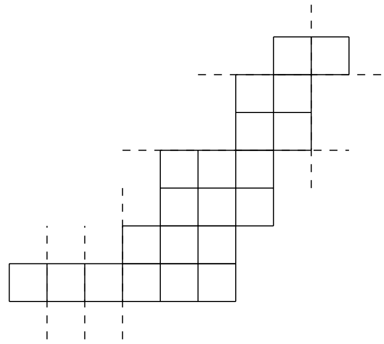

Figure 8: The decomposition of a non-mixed diagram into two thick diagrams and five single squares. This diagram is generated by the word $CCCTCTCSRTSRTS$, and the first thick piece alone is generated by $CRTCSR$.

diagram can be decomposed as a sequence of thick diagrams and single squares as follows: look at all the rows or columns that meet the next row or column in just one square, and cut the diagram along the edges that separate these rows or columns (see Figure 8).

We will use this decomposition of non-mixed diagrams together with the structure of the words describing them to obtain the generating functions and exact counting formulas for the number of non-mixed and thick diagrams. We will use the symbolic method (see Chapter 1 of [20] for a thorough introduction). The main idea is that if $A(z) = \sum_{n\geqslant 0} a_n z^n$ is the generating function for the objects in a collection $\mathcal{A}$ according to some size function (so that a_n is the number of elements of $\mathcal{A}$ that have size n), then the generating function for finite sequences of objects of $\mathcal{A}$ is given by $1/(1 - A(z))$, where the size of a sequence is the sum of the sizes of its components.

The following theorem gives the numbers of thick and arbitrary non-mixed diagrams of fixed size, which turn out to be simple combinatorial expressions. Recall that the *Pell numbers* P_n are given by the recurrence $P_{n+2} - 2P_{n+1} - P_n = 0$ for $n \geqslant 0$ with the initial conditions $P_0 = 0$, $P_1 = 1$ (see sequence A000129 in the OEIS [25]). Their generating function is $P(z) = z/(1 - 2z - z^2)$.

Theorem 5.4. *For $m \geqslant 3$, there are P_{m-2} thick diagrams of size m. For $m \geqslant 1$, there are $(3^{m-1} + 1)/2$ non-mixed diagrams of size m. Thus, for $n \geqslant 2$, there are $(3^{n-2} + 1)/2$ connected lattice path matroids on $[n]$ in which every pair that consists of an initial and a final connected flat is non-mixed.*

Proof. By Corollary 5.3, we can associate a unique word $w = w_1 \dots w_{m-1}$ in the alphabet $\{C, R, S, T\}$ to each non-mixed diagram of size m. For a letter A, we let $A^{\geqslant 1}$ denote a string of As of length at least one. We use $A^{\geqslant 0}$ analogously.

We first consider thick diagrams. A diagram $D(w)$ with no corners is thick if and only if w has at least one C and at least one R, i.e., it has the form $C^{\geqslant 1}R^{\geqslant 1}$. Now suppose that the diagram $D(w)$ has at least one corner, and let $i_1 < i_2 < \cdots < i_c$ be the positions in w that correspond to the corners of $D(w)$. The diagram $D(w)$ is thick if and only if

1. the initial subword $w_1 \dots w_{i_1-1}$ has the form $C^{\geqslant 1}R^{\geqslant 1}$;

2. for all j with $1 \leqslant j \leqslant c$, if $w_{i_j} = S$, then the subword $w_{i_j+1} \dots w_{i_{j+1}-1}$ has the form $C^{\geqslant 0}R^{\geqslant 1}$, and if $w_{i_j} = T$, then the subword $w_{i_j+1} \dots w_{i_{j+1}-1}$ has the form $C^{\geqslant 1}R^{\geqslant 0}$ (where $w_{i_{c+1}-1} = w_{m-1}$ if $j = c$).

Let $\mathrm{Th}(z) = \sum_{m \geqslant 3} t_m z^m$ be the generating function for thick diagrams according to size. From the description above, the word w can be split into subwords $s_0, \dots, s_c$, with $c \geqslant 0$, such that s_0 is of the form $C^{\geqslant 1}R^{\geqslant 1}$ and, when $c > 0$, for each $i \in [c]$, the subword s_i has two possible forms, $SC^{\geqslant 0}R^{\geqslant 1}$ or $TC^{\geqslant 1}R^{\geqslant 0}$. Translating this description into generating functions gives

$$\mathrm{Th}(z) = z\frac{z^2}{(1-z)^2}\frac{1}{1-2\frac{z^2}{(1-z)^2}} = \frac{z^3}{1-2z-z^2} = z^2P(z), \tag{5.1}$$

where the first z accounts for the initial square of the diagram.

We construct an arbitrary non-mixed diagram by gluing consecutive terms in a sequence of thick diagrams and single squares, as explained above. Given consecutive terms D_1 and D_2 (thick diagrams or single squares), there are two ways to glue them along one edge: either the last north step of the bottom path of D_1 is glued to the first north step of the top path of D_2, or the last east step of the top path of D_1 is glued to the first east step of the bottom path of D_2. Thus, a non-mixed diagram is a sequence of diagrams $D_1, \dots, D_k$ such that $k \geqslant 1$, each D_i is either thick or a single square for all $i \in [k]$, and each D_i has a mark on the last step of the top or of the bottom path, for all $i \in [k-1]$. This decomposition gives the following generating function for the number of non-mixed diagrams according to size:

$$\frac{1}{1-(2\mathrm{Th}(z)+2z)}(\mathrm{Th}(z)+z) = \frac{z-2z^2}{(1-z)(1-3z)} = \sum_{m\geqslant 1}\frac{1}{2}(3^{m-1}+1)z^m. \quad \square$$

As shown in [4, Section 4], the number of connected lattice path matroids on $[n]$, with the usual order, is the Catalan number C_{n-1}. (As above, that does not take isomorphism into account; that counts diagrams. The order of magnitude is the same if we count up to matroid isomorphism; see [4, Theorem 4.2].) It is well known that the Catalan numbers C_n grow like $4^n/(n^{3/2}\sqrt{\pi})$, so $\lim_{n\to\infty} 3^{n-2}/C_{n-1} = 0$, which gives the corollary below.

Corollary 5.5. *Asymptotically, almost no lattice path matroids are configuration unique.*

The decompositions in the proof of Theorem 5.4 allow us to refine the enumeration by taking the ranks of the corresponding lattice path matroids into account. The rank of the matroid is the number of rows in the corresponding diagram. In this enumeration we also encounter some known combinatorial numbers, which we review next.

Recall that the *Delannoy numbers* $d_{i,j}$ count the number of paths from $(0,0)$ to (i,j) with steps $(1,0), (0,1)$ and $(1,1)$ (see [2] and sequence A008288 in the OEIS [25]). Their bivariate generating function is

$$\mathrm{Del}(z,y) = \sum_{i,j\geqslant 0} d_{i,j}z^iy^j = \frac{1}{1-(z+y+zy)}.$$

A partition $\{S_1, \ldots, S_r\}$ of the set $[m]$ is *order-consecutive* if there is some permutation $k_1, \ldots, k_r$ of $[r]$ such that the sets $S_{k_1} \cup \cdots \cup S_{k_\ell}$ are intervals for all $\ell \in [r]$ (see [22] and sequence A056241 in the OEIS [25]). The bivariate generating function for the number $\mathrm{oc}_{m,r}$ of order-consecutive partitions of $[m]$ with r parts is

$$\mathrm{OC}(z, y) = \sum_{m \geqslant r \geqslant 1} \mathrm{oc}_{m,r} z^m y^r = zy \frac{1 - z(1 + y)}{1 - 2z(1 + y) + z^2(1 + y + y^2)}.$$

Theorem 5.6. *The number of thick diagrams of size m with r rows is the Delannoy number $d_{m-r-1,r-2}$. The number of non-mixed diagrams of size m with r rows is the number of r-part order-consecutive partitions of $[m]$.*

Proof. Define the bivariate generating function $\mathrm{Th}(z, y) = \sum_{m,r} t_{m,r} y^r z^m$, where $t_{m,r}$ is the number of thick diagrams of size m with r rows. Note that $t_{m,r} > 0$ if and only if $m > r > 1$.

For thick diagrams, the rank increases by one exactly when an operation R or T is performed. With this observation, the analogue of equation (5.1) is

$$\mathrm{Th}(z, y) = zy \frac{z^2 y}{(1 - z)(1 - zy)} \frac{1}{1 - \frac{2z^2 y}{(1-z)(1-zy)}} = \frac{z^3 y^2}{1 - z - zy - z^2 y} = z^3 y^2 \mathrm{Del}(z, zy).$$

The coefficient of $z^i y^j$ in $\mathrm{Del}(z, zy)$ is $d_{i-j,j}$, so the coefficient of $z^m y^r$ in $\mathrm{Th}(z, y)$ is $d_{m-r-1,r-2}$.

We obtain a general non-mixed diagram from a sequence of thick diagrams or single squares, with two ways to glue consecutive diagrams. If we glue two diagrams along east steps, then the rank of the new diagram is the sum of the ranks of the two original diagrams; if we glue along north steps, then we must substract one from this sum. This yields the following generating function

$$\frac{1}{1 - (\mathrm{Th}(z, y) + zy + \frac{\mathrm{Th}(z,y)}{y} + z)} (\mathrm{Th}(z, y) + zy) = zy \frac{1 - z(1 + y)}{1 - 2z(1 + y) + z^2(1 + y + y^2)}. \qquad \square$$

It would be interesting to find bijective proofs of Theorems 5.4 and 5.6.

6 Configuration-unique matroids

To start, we give infinitely many pairs M, M' of matroids that have the same $\mathcal{G}$-invariant but different configurations, and M is a lattice path matroid that is not configuration unique while M' is configuration unique. Fix positive integers b and k. Let M be $M[P, Q]$ where P is $E^{2b+k} N^k E^b N^k E^b N^k$ and Q is $N^k E^b N^k E^b N^k E^{2b+k}$. Figure 9 illustrates this for $b = k = 1$. Figure 10 shows a matroid M' that has the same $\mathcal{G}$-invariant as that in Figure 9, but the configurations differ. Figure 11 shows the lattice of cyclic flats of M in the general case, as well as the lattice of cyclic flats of a matroid M' that generalizes that in Figure 10, that is not a lattice path matroid, and that has the same $\mathcal{G}$-invariant as M. It is easy to verify properties (Z0)–(Z3) in Theorem 2.1 for the sets and ranks given in

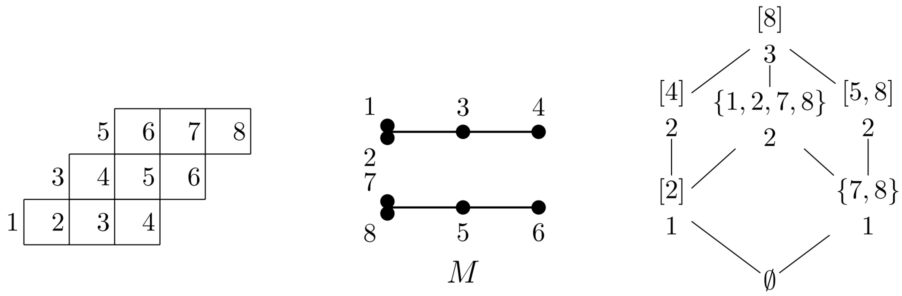

Figure 9: A lattice path matroid M and its lattice of cyclic flats.

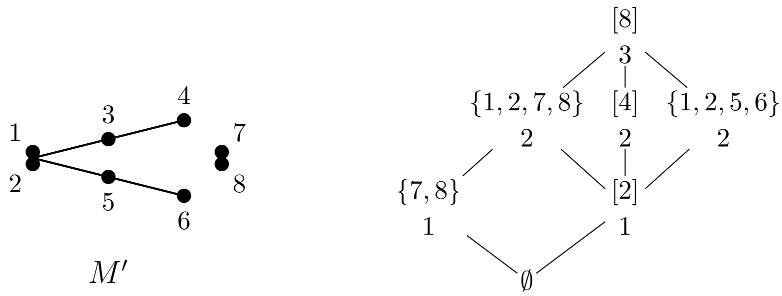

Figure 10: A matroid M' and its lattice of cyclic flats. This matroid has the same $\mathcal{G}$-invariant as that in Figure 9, but the configurations differ.

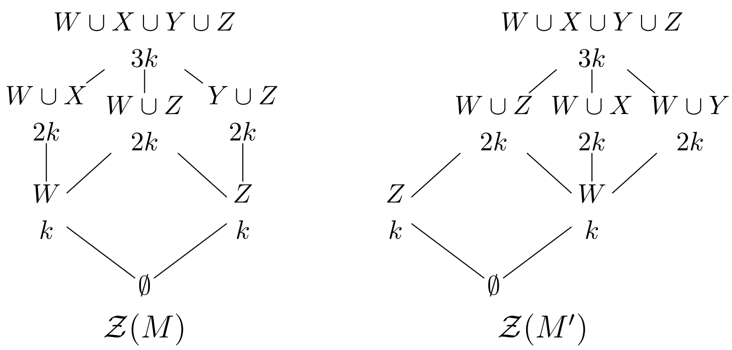

Figure 11: The lattices of cyclic flats of a lattice path matroid M and a matroid M' that have different configurations but the same $\mathcal{G}$-invariant. The sets W, X, Y, and Z are the intervals $[(i-1)(b+k)+1, i(b+k)]$ for $i \in [4]$, respectively; each has $b+k$ elements.

Figure 11. The fact that the configurations yield the same $\mathcal{G}$-invariant is an instance of [3, Theorem 4.1].

The matroid M in Figure 11 is not configuration unique since $(W \cup X, Y \cup Z)$ is a mixed pair. However, the next theorem applies to the matroid M' in Figure 11, which therefore is configuration unique. So, M' is configuration unique but not $\mathcal{G}$ unique.

Theorem 6.1. *For a matroid M, if, for all $A, B \in \mathcal{Z}(M)$, the pair (A, B) is modular and $A \cap B \in \mathcal{Z}(M)$, then M is configuration unique.*

Proof. Without loss of generality, we may assume that M has no coloops. Let N be a matroid that has the same configuration as M, so N has no coloops and there is a lattice isomorphism $\Phi : \mathcal{Z}(M) \to \mathcal{Z}(N)$ that preserves the size and rank of each cyclic flat. To show that M is configuration unique, we construct a bijection $\phi : E(M) \to E(N)$ for which $\Phi(A) = \phi(A)$ for all $A \in \mathcal{Z}(M)$, which therefore is the isomorphism we need.

We claim that $A' \cap B' \in \mathcal{Z}(N)$ for all $A', B' \in \mathcal{Z}(N)$. Take $A, B \in \mathcal{Z}(M)$ with $A' = \Phi(A)$ and $B' = \Phi(B)$. The assumptions that $A \cap B \in \mathcal{Z}(M)$ (so $A \wedge B = A \cap B$) and that (A, B) is a modular pair give

$$r_M(A) + r_M(B) = r_M(A \vee B) + r_M(A \wedge B).$$

From that equality, the properties of Φ give

$$r_N(A') + r_N(B') = r_N(A' \vee B') + r_N(A' \wedge B').$$

The submodular inequality $r_N(A') + r_N(B') \geqslant r_N(A' \cup B') + r_N(A' \cap B')$ along with the inclusion $A' \wedge B' \subseteq A' \cap B'$ force the flat $A' \cap B'$ to be the cyclic flat $A' \wedge B'$, so, as claimed, $A' \cap B' \in \mathcal{Z}(N)$.

For each $A \in \mathcal{Z}(M)$, let its *height in* $\mathcal{Z}(M)$, denoted $h(A)$, be the largest integer h for which there is a chain $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_h \subsetneq A$ with all A_i in $\mathcal{Z}(M)$. Thus, $\mathrm{cl}_M(\emptyset)$ has height 0 and the covers of $\mathrm{cl}_M(\emptyset)$ in $\mathcal{Z}(M)$ have height 1. Let $E(M)$ have height t, which, adapting the definition to N, is also the height of $E(N)$. Note that the height of A in $\mathcal{Z}(M)$ is the height of $\Phi(A)$ in $\mathcal{Z}(N)$. For each element $e \in E(M)$, let the *height* $h(e)$ *of* e be $\min\{h(A) : e \in A \in \mathcal{Z}(M)\}$, the least height of a cyclic flat that contains e. Thus, $h(e) = 0$ if and only if e is a loop, and elements of height t are in no proper cyclic flats. Let $E_i(M) = \{e \in E(M) : h(e) \leqslant i\}$ for i with $0 \leqslant i \leqslant t$, and let $E_i(N)$ be defined similarly. We define ϕ recursively: as i ranges from 0 to t, we define bijections $\phi_i : E_i(M) \to E_i(N)$ for which $\Phi(A) = \phi_i(A)$ for all $A \in \mathcal{Z}(M)$ with $h_M(A) \leqslant i$, and, if $i > 0$, the restriction of ϕ_i to $E_{i-1}(M)$ is ϕ_{i-1}. Thus, the bijection ϕ that we want is ϕ_t. For $i = 0$, since $|\,\mathrm{cl}_M(\emptyset)| = |\,\mathrm{cl}_N(\emptyset)|$, let ϕ_0 be any bijection from $\mathrm{cl}_M(\emptyset)$ onto $\mathrm{cl}_N(\emptyset)$. Now assume that for some $k \in [t]$, the bijection $\phi_{k-1} : E_{k-1}(M) \to E_{k-1}(N)$ has the required properties. Each element $e \in E_k(M) - E_{k-1}(M)$ is in exactly one cyclic flat F with $h(F) = k$, for if e were in two such cyclic flats, say F and F', then $e \in F \cap F'$, and, by the hypotheses of the theorem, $F \cap F'$ is a cyclic flat, and clearly $h(F \cap F') < k$, contrary to having $e \notin E_{k-1}(M)$. Let F be a cyclic flat with $h(F) = k$. Now if $f \in F$ and $h(f) < k$, then f is in a cyclic flat F' with $h(F') < k$, and so f is in the cyclic flat $F \cap F'$, which is properly contained in F and has height less than k. Let F_0 be the union of the cyclic flats that are properly contained in F, so $E_{k-1}(M) \cap F = F_0$. By the principle of inclusion/exclusion, $|F_0|$ can be found from the sizes of the cyclic flats that are properly contained in F, and the sizes of intersections of such cyclic flats. Since (i) such intersections are cyclic flats that are properly contained in F, (ii) $\mathcal{Z}(N)$ is closed under intersections, and (iii) the bijection Φ preserves sizes and inclusions of cyclic flats, we get $|E_{k-1}(N) \cap \Phi(F)| = |F_0|$. Thus, $|F - E_{k-1}(M)| = |\Phi(F) - E_{k-1}(N)|$. Therefore we can extend $\phi_{k-1} : E_{k-1}(M) \to E_{k-1}(N)$ to $\phi_k : E_k(M) \to E_k(N)$ by, for any cyclic flat F of height k, letting the restriction to $F - E_{k-1}(M)$ be any bijection onto $\Phi(F) - E_{k-1}(N)$; such an extension is well defined since each element in $E_k(M) - E_{k-1}(M)$ is in exactly one cyclic flat of height k. As noted above, ϕ_t is the isomorphism of M onto N that we needed. □

By duality, the equality $\mathcal{Z}(M^*) = \{E(M) - A : A \in \mathcal{Z}(M)\}$, and Lemmas 2.3 and 2.4, we get the corollary below.

Corollary 6.2. *For a matroid M, if, for all $A, B \in \mathcal{Z}(M)$, the pair (A, B) is modular and $A \cup B \in \mathcal{Z}(M)$, then M is configuration unique.*

Neither Theorem 6.1 nor Corollary 6.2 has Theorem 4.5 as a corollary.

Another sufficient condition for configuration uniqueness is having all elements of M be in 2-circuits since that implies that all flats of M are cyclic, so the configuration is an unlabeled copy of the lattice of flats, with the size of each parallel class given. That observation is used in the first of two constructions discussed below that produce pairs of non-isomorphic matroids that are configuration unique but have the same $\mathcal{G}$-invariant.

For the first construction, start with any two non-isomorphic matroids M and N with $\mathcal{G}(M) = \mathcal{G}(N)$ and an integer $k \geqslant 1$; for each $e \in E(M)$, let X_e be a set of size k that is disjoint from $E(M)$ and satisfies $X_e \cap X_f = \emptyset$ when $e \neq f$, and add the elements of X_e parallel to e to get a matroid M^k; form N^k similarly. Thus, a flat of h elements in M gives rise to a flat of $(k+1)h$ elements in M^k, and likewise for N^k. By what we just noted, both M^k and N^k are configuration unique. However, the formulation of the $\mathcal{G}$-invariant using sizes of differences in flags of flats shows that $\mathcal{G}(M^k) = \mathcal{G}(N^k)$.

The next construction, discussed after Theorem 6.3, will use the free m-cone, where m is a positive integer. For a matroid M with no loops, the *free m-cone* of M, denoted $Q_m(M)$, is formed by extending M by adding a coloop, say a, and, by taking iterated principal extensions, for each $e \in E(M)$, adding m points freely to the line spanned by a and e. When M and m are understood, we shorten $Q_m(M)$ to Q. (How many elements are on a line through a is determined by the size of the corresponding rank-1 flat of M; a rank-1 flat consisting of h parallel elements in M gives rise to a line of Q with $h(m+1)+1$ elements.) That is one of the two views of free m-cones from [10], where this construction was introduced; the other approach specifies the cyclic flats of Q and their ranks. The cyclic flats of Q are those of M along with all unions of the form $\bigcup_{x \in X} \mathrm{cl}_Q(a, x)$ as X ranges over the nonempty flats of M.

The example in Figure 12 shows why we must assume that $r(M) > 2$ in the next result.

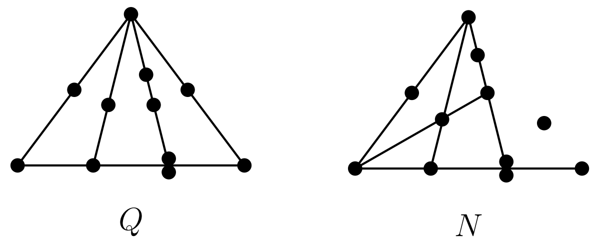

Figure 12: The matroid Q is the free 1-cone of a parallel extension of $U_{2,4}$, while N is not a free cone but has the same configuration.

Theorem 6.3. *The free m-cone Q of a matroid M with no loops and with $r(M) > 2$ is configuration unique.*

Proof. We start from a key step that is established in the proof of [10, Theorem 3.8]: among all sets $\mathcal{C}$ of lines (i.e., rank-2 flats) in $\mathcal{Z}(Q)$ for which

(L1) for each $L \in \mathcal{C}$, at most one proper, nonempty subset of L is in $\mathcal{Z}(Q)$,

(L2) $\bigvee_{L\in\mathcal{C}} L = E(Q)$, and

(L3) if $L, L' \in \mathcal{C}$ with $L \neq L'$, then $r(L \vee L') = 3$,

there is a unique $\mathcal{C}$ for which $|\mathcal{C}|$ is maximal, and that $\mathcal{C}$, which we henceforth call $\mathcal{L}$, is the set of lines of Q that contain a, so $|\mathcal{L}|$ is the number of rank-1 flats of M. Note that whether a set $\mathcal{C}$ of lines in $\mathcal{Z}(Q)$ satisfies properties (L1)–(L3) can be deduced from the configuration alone, so $\mathcal{L}$ can be identified in the configuration.

There is a maximal cyclic flat F of Q that contains no lines in $\mathcal{L}$; indeed, F is the largest cyclic flat in $\mathcal{Z}(M)$. This flat F contains all rank-1 cyclic flats of Q, and for any $L \in \mathcal{L}$, we have $r_Q(F \vee L) = r_Q(F) + 1$ if and only if $F \cap L \neq \emptyset$.

Assume that N has the same configuration as Q. Thus, there is a lattice isomorphism $\Phi : \mathcal{Z}(Q) \to \mathcal{Z}(N)$ that preserves size and matroid rank. As in several other arguments, it suffices to show that Φ is induced by a bijection $\phi : E(Q) \to E(N)$.

We first show that one element of $E(N)$ is in all lines in $\Phi(\mathcal{L})$. To see this, first note that the meet in $\mathcal{Z}(Q)$ of any two lines in $\mathcal{L}$ is $\emptyset$, so the same is true of the meet in $\mathcal{Z}(N)$ of any two lines in $\Phi(\mathcal{L})$. Thus, the intersection of any two lines of $\Phi(\mathcal{L})$ is a singleton or empty. Also, if all lines of $\Phi(\mathcal{L})$ were disjoint, then we would have

$$|E(N)| \geqslant \sum_{L'\in\Phi(\mathcal{L})} |L'| = \sum_{L\in\mathcal{L}} |L| > |E(Q)|,$$

contrary to N and Q having the same configuration. So assume that $L_1 \cap L_2 = \{a'\}$ for some $L_1, L_2 \in \Phi(\mathcal{L})$. Since $r(M) > 2$, there are lines $L' \in \Phi(\mathcal{L})$ not in the plane $L_1 \vee L_2$. For any such line L', the planes $L' \vee L_1$ and $L' \vee L_2$ contain a' and intersect in the line L', so $a' \in L'$. Applying the same argument using L_1 and L' and any other line in $L_1 \vee L_2$ that is in $\Phi(\mathcal{L})$ shows that all such lines contain a', so all lines in $\Phi(\mathcal{L})$ contain a'.

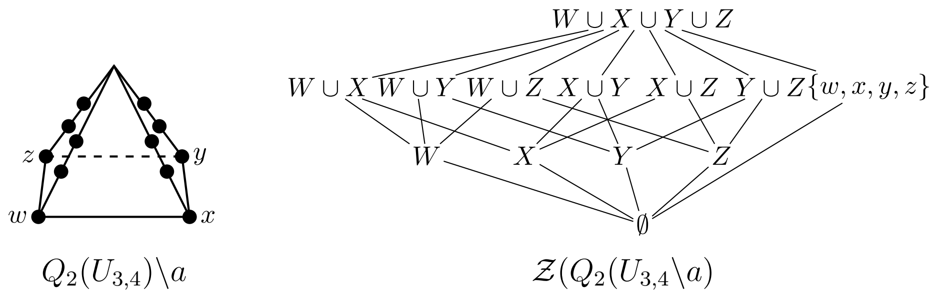

Figure 13: The tipless 2-cone of $U_{3,4}$ and its lattice of cyclic flats. The set W is the cyclic line that contains w, and likewise for X, Y, and Z.

Since all lines in $\Phi(\mathcal{L})$ contain a' and N has the same configuration as Q, counting shows that $E(N)$ is the disjoint union of the sets $\{a'\}$ and $L - \{a'\}$ as L ranges over all lines in $\Phi(\mathcal{L})$. Counting also shows that if $L \in \Phi(\mathcal{L})$ and $r_N(\Phi(F) \vee L) = r_N(\Phi(F)) + 1$, then $L \cap \Phi(F)$ is a rank-1 flat of $N|\Phi(F)$ (which may or may not be cyclic). With this, we can define $\phi : E(Q) \to E(N)$, namely,

- $\phi(a) = a'$;
- for each rank-1 flat A of M where A does not consist of a coloop of M, let $L = \text{cl}_Q(A \cup a)$, so $L \in \mathcal{L}$, and let $A' = \Phi(L) \cap \Phi(F)$; then let ϕ map A onto A' bijectively, and map $L - (a \cup F)$ onto $\Phi(L) - (a' \cup \Phi(F))$ bijectively;
- finally, for a line L in $\mathcal{L}$ that contains no element of F, so $L \cap E(M)$ is a coloop of M, let ϕ map $L - a$ bijectively onto $\Phi(L) - a'$.

It follows that $\Phi(X) = \phi(X)$ for any cyclic flat of Q that either contains a or has rank 1. Also, $\Phi(F) = \phi(F)$. Now consider any other cyclic flat X of M with $r_M(X) > 1$. The closure $\text{cl}_Q(X \cup a)$ is a cyclic flat of Q of rank $r_M(X) + 1$, and $X = \text{cl}_Q(X \cup a) \cap F$. Since $\Phi(\text{cl}_Q(X \cup a))$ covers $\Phi(X)$ in both the lattice of flats of N and the lattice of cyclic flats of N, we must have $\Phi(X) = \Phi(\text{cl}_Q(X \cup a)) \cap \Phi(F)$. Thus, as needed,

$$\Phi(X) = \Phi(\text{cl}_Q(X \cup a)) \cap \Phi(F) = \phi(\text{cl}_Q(X \cup a)) \cap \phi(F) = \phi(X). \quad \square$$

In [10], it is shown that if matroids M and N have the same $\mathcal{G}$-invariant but are not isomorphic, then the free m-cones $Q_m(M)$ and $Q_m(N)$ have the same $\mathcal{G}$-invariant but different configurations. Theorem 6.3 strengthens the conclusion: via free m-cones, we get non-isomorphic configuration-unique matroids that have the same $\mathcal{G}$-invariant. Several variations on free m-cones are also treated in [10]. The proof of Theorem 6.3 easily adapts to show that if $m > 1$ and $r(M) > 2$, then the baseless free m-cone $Q_m(M)\backslash E(M)$ is configuration unique. However, the tipless free m-cone $Q_m(M)\backslash a$ and the tipless/baseless free m-cone $Q_m(M)\backslash(E(M) \cup a)$ need not be configuration unique. We illustrate this with $Q_2(U_{3,4})\backslash a$, which, along with its lattice of cyclic flats, is shown in Figure 13. To get a non-isomorphic matroid N that has the same configuration as $Q_2(U_{3,4})\backslash a$, consider the following sets and ranks:

- $\emptyset$ has rank 0;
- the sets $\{w,w',a\}$, $\{x,x',a\}$, $\{y,y',a\}$, and $\{z,z',a\}$ have rank 2;
- the sets $\{w,w',x,x',a,b\}$, $\{w,w',y,y',a,c\}$, $\{w,w',z,z',a,d\}$, $\{x,x',y,y',a,d\}$, $\{x,x',z,z',a,c\}$, $\{y,y',z,z',a,b\}$, and $\{w,x,y,z\}$ have rank 3;
- the set $\{w,w',x,x',y,y',z,z',a,b,c,d\}$ has rank 4.

It is easy to check that properties (Z0)–(Z3) in Theorem 2.1 hold, so this data defines a matroid N which is clearly not isomorphic to $Q_2(U_{3,4})\backslash a$ but has the same configuration.

7 Which lattices come from non-configuration-unique matroids?

As noted earlier, matroids for which the lattice of cyclic flats is a chain (i.e., nested matroids) are Tutte unique [15]. In this section, we prove the following theorem, which shows that any lattice that is not a chain is isomorphic to the lattice of cyclic flats of a matroid that is not even configuration unique.

Theorem 7.1. *Let L be a lattice that is not a chain. There are pairs of non-isomorphic transversal matroids that have the same configuration and have their lattices of cyclic flats isomorphic to L.*

We will use one of the two constructions that we gave in [6] that, for a lattice L, produce transversal matroids for which the lattices of cyclic flats are isomorphic to L. We start by recalling the construction, which we illustrate in Figure 14.

Let $B = L - \{\hat{1}\}$ where $\hat{1}$ is the greatest element of L. For each $z \in L$, let V_z be the set $\{y \in L : y \not\geqslant z\}$. Observe that $V_x \subseteq V_z$ if and only if $x \leqslant z$. For each $z \in L$, let S_z be a set of $|V_z| + 1$ elements, where $S_z \cap S_y = \emptyset$ whenever $z \neq y$, and $S_z \cap B = \emptyset$. Consider a $|B|$-vertex simplex Δ. Put one element of B at each vertex of Δ. For each $z \in L$ put the points in S_z freely in the face of Δ that is spanned by V_z, and then delete B. The resulting transversal matroid has as cyclic flats the sets $F_z = \cup_{y \leqslant z} S_y$, for each $z \in L$; also, $F_z \cap F_x = F_{z \wedge x}$ for all $z, x \in L$, so the meet in the lattice of cyclic flats is the intersection (as in the lattice of all flats). To give another view, the presentation consists of the sets $A_y = \cup_{z \not\leqslant y} S_z$ for each $y \in B$.

Proof of Theorem 7.1. Observe that the greatest cyclic flat of a matroid M without coloops covers exactly one cyclic flat, say X, in $\mathcal{Z}(M)$ if and only if we obtain M from $M|X$ by adding $r(M) - r(X)$ elements of $E(M) - X$ as coloops and then adding the remaining elements of $E(M) - X$ by free extension. It follows that if we prove the result for lattices where $\hat{1}$ covers at least two elements, then it holds for all lattices that are not chains. So we assume that the greatest element $\hat{1}$ of L covers at least two elements of L, say z and w.

Let M be the matroid constructed in the paragraph before the proof. Let $r = |B|$. Thus, $|V_z| = |V_w| = r - 1$, so $r_M(F_z) = r_M(F_w) = r - 1$. Now $r_M(F_{z\wedge w}) \leqslant r - 3$ since z,

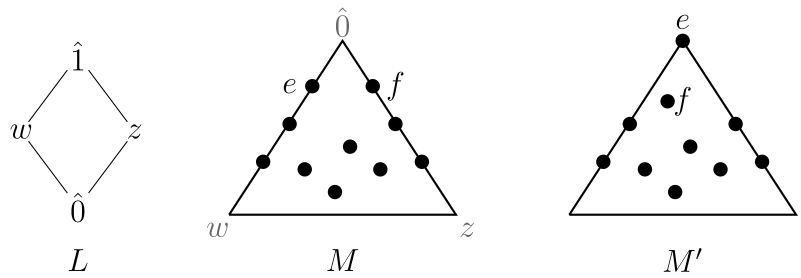

Figure 14: In this example of the construction used to prove Theorem 7.1, we have $V_{\hat{0}} = \emptyset$, $V_w = \{\hat{0}, z\}$, $V_z = \{\hat{0}, w\}$, and $V_{\hat{1}} = \{\hat{0}, w, z\}$. The three points in S_w are on the edge of the simplex labeled by the elements of V_w, namely, $\hat{0}$ and z. We have omitted the loop that $S_\emptyset$ contributes. We get M' by having e bump f in M.

w, and $z \wedge w$ are not in $V_{z\wedge w}$. Since $F_z \cap F_w = F_{z\wedge w}$, the pair (F_z, F_w) is not modular. For any $e \in S_z$ and $f \in S_w$, we have, in the notation of Theorem 3.1, $\mathcal{Z}_e = \{F_z, E(M)\}$ and $\mathcal{Z}_f = \{F_w, E(M)\}$, so that theorem applies; thus, e can bump f in M to obtain a matroid M' that is not isomorphic to M but has the same configuration as M.

We show that M' is transversal by applying Theorem 2.8 and the remarks that follow it. Let $\mathcal{F}'$ be an antichain of cyclic flats of M' with $|\mathcal{F}'| \geqslant 3$. Since $|\mathcal{F}'| \geqslant 3$, the antichain $\mathcal{F}'$ contains a cyclic flat of M' other than $E(M')$, F_z, and $(F_w - f) \cup e$. Thus, since $\mathcal{Z}_e = \{F_z, E(M)\}$ and $\mathcal{Z}_f = \{F_w, E(M)\}$, we have $e, f \notin \cap\mathcal{F}'$. Let $\mathcal{F}$ be the corresponding antichain in $\mathcal{Z}(M)$ (so, replace $(F_w - f) \cup e$ by F_w if it is in $\mathcal{F}'$; otherwise $\mathcal{F} = \mathcal{F}'$). Since $e, f \notin \cap\mathcal{F}'$, we have $\cap\mathcal{F}' = \cap\mathcal{F}$. Also, $r_{M'}(\cap\mathcal{F}') = r_M(\cap\mathcal{F})$ by Equation (2.1) since, with $e, f \notin \cap\mathcal{F}'$, the cyclic flats $F = (F_w - f) \cup e$ and $F = F_w$ in which $\mathcal{Z}(M)$ and $\mathcal{Z}(M')$ differ yield the same sum $r(F) + |(\cap\mathcal{F}') - F|$. Also, the right side of Inequality (2.2) depends only on the configuration, which M and M' share. Thus, Inequality (2.2) holds for $\mathcal{F}'$ in M' since it holds for $\mathcal{F}$ in M, so M' is transversal. □

To close, we note a stronger version of this result. First, delete the loop, the element of $S_{\hat{0}}$, from the matroids M and M' constructed above. The elements of $S_{\hat{1}}$ are in no proper cyclic flats of either M or M', so any circuit of M that contains an element of $S_{\hat{1}}$ spans M, and likewise for M'. Thus, M and M' have spanning circuits but no loops and so are connected. As shown in [11], applying the operation of t-expansion to M and M' yields matroids N and N' that have the same configuration, are transversal, and, by letting t be large enough, have arbitrarily high connectivity. (The operation of t-expansion, introduced in [11], can be seen as magnifying all sets and ranks in the lattice of cyclic flats by a factor of t [11, Definition 3.1] or as the result of taking a matroid and replacing each element by a set of t parallel elements (or t loops if the element is a loop), and taking the matroid union of t copies of this matroid [11, Theorem 3.11]. The fact that t-expansions inherit the properties of interest follows from [11, Lemmas 3.2 and 3.3, and Theorem 3.13].) Similar remarks apply trivially to vertical connectivity and branch-width since the vertical connectivity of M and M' is their rank, and their branch-width is one more than their rank.

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