A Local Framework for Proving Combinatorial Matrix Inversion Theorems

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Abstract

Combinatorial transition matrices arise frequently in the theory of symmetric functions and their generalizations. The entries of such matrices often count signed, weighted combinatorial structures such as semistandard tableaux, rim-hook tableaux, or brick tabloids. Bijective proofs that two such matrices are inverses of each other may be difficult to find. This paper presents a general framework for proving such inversion results in the case where the combinatorial objects are built up recursively by successively adding some incremental structure such as a single horizontal strip or rim-hook. In this setting, we show that a sequence of matrix inversion results $A_nB_n = I$ can be reduced to a certain "local" identity involving the incremental structures. Here, A_n and B_n are matrices that might be non-square, and the columns of A_n and the rows of B_n are indexed by compositions of n. We illustrate the general theory with four classical applications involving the Kostka matrices, the character tables of the symmetric group, incidence matrices for composition posets, and matrices counting brick tabloids. We obtain a new, canonical bijective proof of an inversion result for rectangular Kostka matrices, which complements the proof for the square case due to Eğecioğlu and Remmel. We also give a new bijective proof of the orthogonality result for the irreducible S_n-characters that is shorter than the original version due to White.

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1 Introduction

Combinatorial matrices occur ubiquitously in algebraic combinatorics, especially in the theory of symmetric functions and their generalizations. Transition matrices connecting two bases of symmetric functions (or quasisymmetric functions, etc.) very often have combinatorial interpretations where the matrix entries count signed, weighted structures such as semistandard tableaux, rim-hook tableaux, or brick tabloids. For example, the

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classical Kostka matrix $\mathbf{K}$ gives the monomial expansion of the Schur symmetric functions. The matrix entry $K_{\lambda,\mu}$ identifies the coefficient of m_μ in s_λ as the number of semistandard Young tableaux of shape λ and content μ. Eğecioğlu and Remmel [3] gave a combinatorial formula expressing each entry in $\mathbf{K}^{-1}$ as a signed sum of special rim-hook tableaux. (See Section 3 for detailed definitions of the objects mentioned here.)

As another example, the character table of the symmetric group S_n can be viewed as a matrix $\mathbf{X} = [\chi^\lambda_\mu]$ with rows and columns indexed by integer partitions of n. Here, χ^λ_μ is the value of the irreducible character χ^λ indexed by λ on the conjugacy class of S_n consisting of permutations of cycle type μ. Remarkably, χ^λ_μ is also the coefficient of the Schur function s_λ when the power-sum symmetric function p_μ is written in the Schur basis. The entry χ^λ_μ has a combinatorial interpretation as the signed sum of rim-hook tableaux of shape λ and content μ. The inverse of $\mathbf{X}$ has entries $\chi^\mu_\lambda/z_\lambda$, where $n!/z_\lambda$ is the number of permutations in S_n with cycle type λ. The monograph [10] gives a very clear exposition of the results stated in this paragraph.

Bijective proofs that two combinatorial matrices are inverses of each other may be difficult to find. In the case of the Kostka matrix, Eğecioğlu and Remmel [3] gave an ingenious proof based on a sign-reversing involution on pairs consisting of a semistandard tableau and a special rim-hook tableau with compatible content. In the case of the character matrix $\mathbf{X}$, the formula for $\mathbf{X}^{-1}$ follows from the orthogonality relations for irreducible characters. Finding a bijective proof based on rim-hook tableaux is very challenging; Dennis White gave such a proof via an intricate algorithmic construction [12]. There are many other instances of combinatorial transition matrices where inversion results require elaborate algebraic manipulations, subtle bijective arguments, or some combination of these.

Our goal here is to develop a general framework for proving inversion results for certain combinatorial matrices. We often have not just one matrix and its inverse, but a whole family of matrices. For example, what we have called the Kostka matrix is really a sequence of matrices $\mathbf{K}_n$ for $n \geqslant 0$, where the rows and columns of $\mathbf{K}_n$ are indexed by integer partitions of n. A simple recursion (based on removing the largest value from a semistandard tableau) relates the entries of $\mathbf{K}_n$ to entries in various smaller matrices $\mathbf{K}_m$.

Our general theory, presented in Section 2, considers two sequences of matrices $(A_n : n \geqslant 0)$ and $(B_n : n \geqslant 0)$ where the columns of A_n and the rows of B_n are indexed by compositions of n. These matrices are not $n \times n$ and may be rectangular rather than square. We assume that the entries in A_n can be computed recursively from certain entries in $A_0, \ldots, A_{n-1}$, and similarly for B_n. This setup applies to many combinatorial matrices (such as suitably adjusted versions of the Kostka matrices and their inverses) where the combinatorial objects are built up recursively by successively adding some incremental structure such as a single horizontal strip, special rim-hook, etc. In this setting, we show that the sequence of (one-sided) matrix inversion results $A_nB_n = I$ is equivalent to a certain "local" identity involving the incremental structures. Intuitively, this local identity contains the combinatorial essence of why each B_n is a right-inverse of A_n. Section 7 explains how a bijective proof of the local identity can often be leveraged to create a bijective proof of the matrix inversion result itself.

We illustrate our general theory with four classical applications that have relevance to transition matrices for bases of Sym (the space of symmetric functions), QSym (the space of quasisymmetric functions), and NSym (the space of non-commutative symmetric functions). Section 3 studies rectangular versions of the Kostka matrix and its inverse. Section 7.1 uses this analysis to build a new bijective proof of the inversion result for rectangular Kostka matrices. This bijection is canonical (in a certain precise technical sense) and differs from the bijection for the square case due to Eğecioğlu and Remmel [3]. Section 4 studies rectangular versions of the matrix $\mathbf{X}$ and its inverse. Here, the local identity has an "almost canonical" bijective proof that can be expressed elegantly in terms of abaci. Section 7.2 lifts this to a bijective proof of the matrix inversion result that is substantially shorter than Dennis White's original version [12]. The proof uses novel combinatorial interpretations of $n!$ parametrized by partitions λ (see Remark 41). Section 5 uses our general theory to invert the incidence matrices for posets of compositions of n ordered by refinement. Although this classical result is easy enough to prove algebraically, the combinatorial proof of the local identity is still illuminating. The latter proof also generalizes readily to prove a more daunting weighted version of the classical result (Section 5.4) with relevance to certain transition matrices for NSym. Section 6 gives a fourth application involving matrices counting brick tabloids. These matrices connect the power-sum and monomial bases of symmetric functions.

Most of this paper (except for a few isolated remarks) can be read without any prior knowledge of Sym, QSym, NSym, or the various combinatorial structures mentioned in this introduction. The four application sections are almost entirely independent of one another, except that Lemma 20 and some definitions in Section 5 are needed in Section 6.

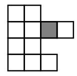
2 General Framework for Matrix Inversion

This section presents the general theory for proving combinatorial matrix inversion results via reduction to local identities. We begin by defining the needed notation and terminology.

2.1 Notation and Definitions

For each integer $n \geqslant 0$, a *composition of* n is a list $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ of positive integers with $\alpha_1 + \cdots + \alpha_s = n$. The *size* of α is $|\alpha| = n$, and the *length* of α is $\ell(\alpha) = s$. The entries α_i are the *parts* of α. Write $L(\alpha) = \alpha_s$ for the *last part* of α. If $s \geqslant 1$, write $\alpha^* = (\alpha_1, \alpha_2, \ldots, \alpha_{s-1})$ for the *truncation of* α, which is the composition of $n - \alpha_s$ obtained by deleting the last part of α. An *integer partition* is a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s$.

Write $C(n)$ for the set of compositions of n. Write $P(n)$ for the set of partitions of n. The empty sequence is the unique element in $C(0)$ and $P(0)$. Write $[n]$ for the set $\{1, 2, \ldots, n\}$. Define sort : $C(n) \to P(n)$ by letting sort(α) be the weakly decreasing rearrangement of the list $\alpha \in C(n)$. Given $\alpha \in C(n)$, the *diagram* $\mathrm{dg}(\alpha)$ of α is the set $\{(i, j) \in \mathbb{Z}^2 : 1 \leqslant i \leqslant \ell(\alpha), 1 \leqslant j \leqslant \alpha_i\}$. We visualize $\mathrm{dg}(\alpha)$ as a left-justified array of unit boxes with α_i boxes in the ith row from the top. The box in row i, column j is identified with the pair $(i,j) \in \mathrm{dg}(\alpha)$.

Example 1. For $\alpha = (2,4,2,3)$ we have $\mathrm{dg}(\alpha) =$  and the marked box has the label $(2,3)$.

For any finite sets R and C and field $\mathbb{F}$, an $R \times C$ *matrix* is a function $A : R \times C \to \mathbb{F}$. For $r \in R$ and $c \in C$, we call $A(r,c)$ the *entry in row* r*, column* c *of* A. For any statement Q, let $\chi(Q) = 1$ if Q is true and $\chi(Q) = 0$ if Q is false. For any finite set R, let I_R be the *identity matrix* with rows and columns indexed by R, which satisfies $I_R(\lambda,\mu) = \chi(\lambda = \mu)$ for all $\lambda, \mu \in R$. We may omit the subscript R if it is clear from context.

2.2 Setup for the Matrix Inversion Framework

Next we describe the assumed setup for our matrix inversion framework. The ingredients in the setup consist of two sequences of matrices, with rows and columns indexed by certain combinatorial objects, and two recursions that describe how the matrices in each sequence are built from earlier matrices in that sequence.

We begin by describing the matrices. Let $\mathbb{F}$ be a field (which in our applications will be $\mathbb{Q}$). Suppose, for each integer $n \geqslant 0$, we have a finite set of objects $R(n)$, where $R(0)$ consists of a single element. (In all applications considered in this paper, $R(n)$ will be either $P(n)$ or $C(n)$.) Suppose we are given a sequence of matrices $A_0, A_1, A_2, \ldots, A_n, \ldots$ where each A_n is an $R(n) \times C(n)$ matrix with values in $\mathbb{F}$. Thus, the rows of A_n are indexed by objects in $R(n)$, while the columns of A_n are indexed by compositions of n. We may abbreviate $A_n(\lambda,\beta)$ as $A(\lambda,\beta)$, since n must be $|\beta|$. Also suppose we are given a sequence of matrices $B_0, B_1, B_2, \ldots, B_n, \ldots$, where each B_n is a $C(n) \times R(n)$ matrix with values in $\mathbb{F}$. Again we write $B(\beta,\mu) = B_n(\beta,\mu)$ where $n = |\beta|$.

We continue by describing the recursions assumed as part of the setup. For each $n > 0$, $\lambda \in R(n)$, and $L \in [n]$, assume there is a finite set $S(\lambda, L) \subseteq R(n-L)$ and an $\mathbb{F}$-valued weight function wt_A such that the matrices A_n satisfy the recursion

$$A(\lambda,\beta) = \sum_{\gamma \in S(\lambda, L(\beta))} \mathrm{wt}_A(\lambda,\gamma) A(\gamma, \beta^*) \tag{1}$$

and initial condition $A_0 = [1]$. We interpret this recursion informally as follows. The entry $A(\lambda,\beta)$ counts signed, weighted objects with *shape* $\lambda \in R(n)$ and *content* $\beta = (\beta_1, \ldots, \beta_s) = (\beta^*, L(\beta))$. The recursion asserts that each such object can be built uniquely by starting with some smaller shape $\gamma \in R(n - L(\beta))$, drawn from some set $S(\lambda, L(\beta))$ of allowable shapes depending on λ and $L(\beta)$, choosing any object of shape γ and content β^* counted by $A(\gamma, \beta^*)$, and augmenting that object with some incremental structure. The effect of the augmenting step is reflected by the factor $\mathrm{wt}_A(\lambda,\gamma)$. The initial condition amounts to the assumption that there is a unique "empty object" with empty shape in $R(0)$ and empty content in $C(0)$.

We make the analogous assumption for the sequence of matrices B_n. For each $n > 0$, $\mu \in R(n)$, and $L \in [n]$, assume there is a finite set $T(\mu, L) \subseteq R(n-L)$ and an $\mathbb{F}$-valued weight function wt_B such that the matrices B_n satisfy the recursion

$$B(\beta, \mu) = \sum_{\delta \in T(\mu, L(\beta))} \mathrm{wt}_B(\mu, \delta) B(\beta^*, \delta) \tag{2}$$

and initial condition $B_0 = [1]$. This recursion has a similar interpretation as before, but here the column index μ gives the shape and the row index β gives the content of the objects counted by the entries in B_n. In some applications, the roles of shape and content in recursions (1) or (2) may be interchanged.

2.3 Matrix Inversion via the Local Identity

We now come to the first main result of the general theory.

Theorem 2. *Assume the setup in §2.2. The family of matrix identities*

$$A_n B_n = I_{R(n)} \quad \textit{for all } n \geqslant 0 \tag{3}$$

is equivalent to the family of local identities

$$\sum_{L=1}^{n} \sum_{\gamma \in S(\lambda, L) \cap T(\mu, L)} \mathrm{wt}_A(\lambda, \gamma)\, \mathrm{wt}_B(\mu, \gamma) = \chi(\lambda = \mu) \quad \textit{for all } n > 0 \textit{ and all } \lambda, \mu \in R(n). \tag{4}$$

Proof. For $n > 0$ and $\lambda, \mu \in R(n)$, we compute the entry in row λ, column μ of $A_n B_n$ as follows:

$$\begin{aligned}(A_n B_n)(\lambda, \mu) &= \sum_{\beta \in C(n)} A(\lambda, \beta) B(\beta, \mu) \\ &= \sum_{\beta \in C(n)} \left(\sum_{\gamma \in S(\lambda, L(\beta))} \mathrm{wt}_A(\lambda, \gamma) A(\gamma, \beta^*) \right) \cdot \left(\sum_{\delta \in T(\mu, L(\beta))} \mathrm{wt}_B(\mu, \delta) B(\beta^*, \delta) \right).\end{aligned}$$

Each $\beta \in C(n)$ has the form $\beta = (\beta^*, L)$ for a unique $L = L(\beta) \in [n]$ and a unique $\beta^* \in C(n-L)$. So we can rewrite the sum over β as a sum over β^* and L. We get:

$$\begin{aligned}(A_n B_n)(\lambda, \mu) &= \sum_{L=1}^{n} \sum_{\beta^* \in C(n-L)} \left(\sum_{\gamma \in S(\lambda, L)} \mathrm{wt}_A(\lambda, \gamma) A(\gamma, \beta^*) \right) \cdot \left(\sum_{\delta \in T(\mu, L)} \mathrm{wt}_B(\mu, \delta) B(\beta^*, \delta) \right) \\ &= \sum_{L=1}^{n} \sum_{\gamma \in S(\lambda, L)} \sum_{\delta \in T(\mu, L)} \mathrm{wt}_A(\lambda, \gamma)\, \mathrm{wt}_B(\mu, \delta) \sum_{\beta^* \in C(n-L)} A(\gamma, \beta^*) B(\beta^*, \delta).\end{aligned}$$

We recognize the innermost sum as the definition of a matrix product entry, so

$$(A_nB_n)(\lambda,\mu)=\sum_{L=1}^{n}\sum_{\gamma\in S(\lambda,L)}\sum_{\delta\in T(\mu,L)}\mathrm{wt}_A(\lambda,\gamma)\,\mathrm{wt}_B(\mu,\delta)(A_{n-L}B_{n-L})(\gamma,\delta). \tag{5}$$

On one hand, assume (3) holds for all $n\geqslant 0$. Then, in particular, $(A_nB_n)(\lambda,\mu)=\chi(\lambda=\mu)$ and $(A_{n-L}B_{n-L})(\gamma,\delta)=\chi(\gamma=\delta)$. Inserting these expressions into (5), the double sum over γ and δ simplifies to a single sum over $\gamma=\delta\in S(\lambda,L)\cap T(\mu,L)$. So (5) reduces to (4), as needed.

On the other hand, assume instead that (4) holds. We prove (3) by induction on n. For $n=0$, the formula is true since $A_0=B_0=[1]$. Fix $n>0$ and assume $A_mB_m=I_{R(m)}$ for all $m<n$. Taking $m=n-L$ for $L=1,2,\ldots,n$, Equation (5) becomes

$$(A_nB_n)(\lambda,\mu)=\sum_{L=1}^{n}\sum_{\gamma\in S(\lambda,L)}\sum_{\delta\in T(\mu,L)}\mathrm{wt}_A(\lambda,\gamma)\,\mathrm{wt}_B(\mu,\delta)\chi(\gamma=\delta).$$

As before, the right side here simplifies to the left side of the local identity (4). Thus, $(A_nB_n)(\lambda,\mu)=\chi(\lambda=\mu)$ for all $\lambda,\mu\in R(n)$, proving that $A_nB_n=I_{R(n)}$. □

Remark 3. The theorem does not assert that $B_nA_n=I_{C(n)}$. In most of our applications, $R(n)$ will be $P(n)$, and hence $B_nA_n=I_{C(n)}$ cannot possibly hold because the left side has rank at most $|P(n)|$. However, we can often convert the one-sided matrix inverse formula $A_nB_n=I_{R(n)}$ (for rectangular matrices A_n,B_n) to a related formula involving square matrices. In the square case, $AB=I$ automatically implies $BA=I$, as is well known.

In many applications, A_n is a $P(n)\times C(n)$ matrix satisfying the following *sorting condition*: for all $\lambda\in P(n)$ and $\alpha,\beta\in C(n)$, if $\mathrm{sort}(\alpha)=\mathrm{sort}(\beta)$ then $A(\lambda,\alpha)=A(\lambda,\beta)$. In this case, we can convert from rectangular to square matrices as follows. Define $A'_n: P(n)\times P(n)\to\mathbb{F}$ to be the restriction of A_n to the columns indexed by partitions, so A'_n is formed from A_n by deleting some columns of A_n that are duplicates of other columns. Define $B'_n: P(n)\times P(n)\to\mathbb{F}$ by adding together those rows indexed by compositions that sort to the same partition, that is,

$$B'_n(\nu,\mu)=\sum_{\substack{\beta\in C(n):\\ \mathrm{sort}(\beta)=\nu}}B_n(\beta,\mu)\qquad\text{for }\nu,\mu\in P(n). \tag{6}$$

If we know $A_nB_n=I_{P(n)}$, then we can deduce $A'_nB'_n=I_{P(n)}$ as follows. For $\lambda,\mu\in P(n)$,

$$\begin{aligned}(A'_nB'_n)(\lambda,\mu)&=\sum_{\nu\in P(n)}A'_n(\lambda,\nu)B'_n(\nu,\mu)=\sum_{\nu\in P(n)}A_n(\lambda,\nu)\sum_{\substack{\beta\in C(n):\\ \mathrm{sort}(\beta)=\nu}}B_n(\beta,\mu)\\&=\sum_{\nu\in P(n)}\sum_{\substack{\beta\in C(n):\\ \mathrm{sort}(\beta)=\nu}}A_n(\lambda,\beta)B_n(\beta,\mu)=\sum_{\beta\in C(n)}A_n(\lambda,\beta)B_n(\beta,\mu)=(A_nB_n)(\lambda,\mu).\end{aligned}$$

Remark 4. Here is an informal combinatorial interpretation of the local identities (4). Fix $\lambda, \mu \in R(n)$. We consider all possible ways of transforming λ into μ via the following two-step process. First, go from λ to some intermediate object γ by removing some differential structure of size L (determined by the recursion for the A matrix). Second, go from this γ to μ by adding another differential structure of size L (determined by the recursion for the B matrix). For fixed γ, the transition from λ to μ via γ contributes a term $\mathrm{wt}_A(\lambda,\gamma)\,\mathrm{wt}_B(\mu,\gamma)$ that could be negative. The sum of such terms over all feasible γ is required to be 1 if $\lambda = \mu$ and 0 if $\lambda \neq \mu$. In later applications, we use the notation $G(\lambda,\mu) = \bigcup_{L=1}^{n} S(\lambda, L) \cap T(\mu, L)$ for the set of γ that are viable intermediate shapes in the passage from λ to μ. The local identities (4) take the form

$$\sum_{\gamma \in G(\lambda,\mu)} \mathrm{wt}_A(\lambda,\gamma)\,\mathrm{wt}_B(\mu,\gamma) = \chi(\lambda = \mu).$$

Remark 5. The proof of Theorem 2 highlights why compositions are so effective as indexing objects for the columns of A_n and the rows of B_n. In particular, we could reduce to smaller instances of A_n and B_n thanks to the bijection from the disjoint union $\bigcup_{L=1}^{n} C(n-L)$ to $C(n)$ sending $\beta^* \in C(n-L)$ to (β^*, L). More generally, suppose we have a set of objects $X(n)$ and a set $\phi(n) \subseteq \{1, 2, \ldots, n\}$ for each integer $n \geqslant 0$, such that there is a bijection from $X(n)$ to the disjoint union $\bigcup_{L \in \phi(n)} X(n-L)$ for each $n > 0$. Theorem 2 still works with the indexing sets $C(n)$ replaced by $X(n)$, with the new local condition being

$$\sum_{L \in \phi(n)} \sum_{\gamma \in S(\lambda,L) \cap T(\mu,L)} \mathrm{wt}_A(\lambda,\gamma)\,\mathrm{wt}_B(\mu,\gamma) = \chi(\lambda = \mu) \quad \text{for all } n > 0 \text{ and all } \lambda, \mu \in R(n).$$

3 Application 1: Rectangular Kostka Matrices

As the first application of the general theory, we prove an inversion result for a rectangular version of the Kostka matrices. In this case, the local identities (4) can be informally summarized by the slogan: “A local inverse of a horizontal strip is a signed special rim-hook.”

3.1 Semistandard Tableaux

Given $\lambda \in P(n)$ and $\beta \in C(n)$, a *semistandard Young tableau (SSYT)* of *shape* λ and *content* β is a filling of the diagram of λ using β_k copies of k for each k, such that values weakly increase reading left to right along each row, and values strictly increase reading top to bottom along each column. Let $\mathrm{SSYT}(\lambda, \beta)$ be the set of all such SSYT. In this first application of the general setup of §2.2, we let A_n be the $P(n) \times C(n)$ *rectangular Kostka matrix* with entries $A_n(\lambda, \beta) = |\,\mathrm{SSYT}(\lambda, \beta)|$.

Example 6. Let $\beta = (2, 3, 2)$ and $\lambda = (4, 3)$. The entry $A_n(\lambda, \beta) = 2$ counts the following tableaux in $\mathrm{SSYT}(\lambda, \beta)$:

1	1	2	2
2	3	3	

1	1	2	3
2	2	3	

A *horizontal strip of size* L is a set of L unit boxes occupying distinct columns. Given $\lambda \in P(n)$ and $\beta \in C(n)$, suppose $T \in \mathrm{SSYT}(\lambda, \beta)$ and write $\beta = (\beta_1, \ldots, \beta_s) = (\beta^*, L(\beta))$. For $1 \leqslant i \leqslant s$, the cells containing value i in T form a horizontal strip of size β_i, as is readily checked. Also, the cells containing values $1, 2, \ldots, i$ form the diagram of some partition $\lambda^{(i)}$ where $\mathrm{dg}(\lambda^{(i)}) \subseteq \mathrm{dg}(\lambda)$. Let $S(\lambda, L)$ be the set of partitions $\gamma \in P(n-L)$ such that $\mathrm{dg}(\gamma) \subseteq \mathrm{dg}(\lambda)$ and the set difference $\lambda/\gamma = \mathrm{dg}(\lambda) \setminus \mathrm{dg}(\gamma)$ is a horizontal strip of size L. There is exactly one way to construct any given $T \in \mathrm{SSYT}(\lambda, \beta)$ as follows: choose $\gamma \in S(\lambda, L(\beta))$; choose any $T^* \in \mathrm{SSYT}(\gamma, \beta^*)$; and fill the $L(\beta)$ boxes in $\mathrm{dg}(\lambda) \setminus \mathrm{dg}(\gamma)$ with the value $s = \ell(\beta)$. This proves the recursion

$$A(\lambda, \beta) = \sum_{\gamma \in S(\lambda, L(\beta))} A(\gamma, \beta^*), \tag{7}$$

which is an instance of (1) with $\mathrm{wt}_A(\lambda, \gamma) = 1$. In fact, we have a bijection $F : \mathrm{SSYT}(\lambda, \beta) \to \bigcup_{\gamma \in S(\lambda, L(\beta))} \mathrm{SSYT}(\gamma, \beta^*)$, where $F(T)$ is the SSYT obtained by removing the $L(\beta)$ boxes containing the value $s = \ell(\beta)$ from T. The construction prior to (7) describes how to compute F^{-1}.

3.2 Special Rim-Hook Tableaux

Eğecioğlu and Remmel [3] discovered a combinatorial interpretation for the inverse of the (square) $P(n) \times P(n)$ Kostka matrix involving special rim-hook tableaux. We define a rectangular analogue of their inverse Kostka matrix, which also appeared in [4]. A *special rim-hook* (SRH) of size L is a set of L unit boxes that can be traversed by starting with a box in column 1 (the leftmost column) and successively taking one step to the right or up to go from one box to the next. The *sign* of a SRH σ occupying r rows is $\mathrm{sgn}(\sigma) = (-1)^{r-1}$.

Given $\mu \in P(n)$ and $\beta \in C(n)$, a *special rim-hook tableau (SRHT) of shape* μ and *content* β is a filling S of the diagram of μ such that for all k, the cells of S containing k form a special rim-hook of size β_k, and the values in column 1 weakly increase from top to bottom. Let $\mathrm{SRHT}(\mu, \beta)$ be the set of all such SRHT. The *sign* of an SRHT S is the product of the signs of all special rim-hooks in S. Let B_n be the $C(n) \times P(n)$ matrix defined by $B_n(\beta, \mu) = \sum_{S \in \mathrm{SRHT}(\mu, \beta)} \mathrm{sgn}(S)$.

Example 7. For $\beta = (3, 2, 4)$ and $\mu = (3, 3, 3)$, $\mathrm{SRHT}(\mu, \beta)$ is the set containing this object

1	1	1
2	2	3
3	3	3

which has sign $(-1)^{1-1}(-1)^{1-1}(-1)^{2-1} = -1$. If $\beta = (2, 4, 3)$ and $\mu = (3, 3, 3)$, we have this SRHT with sign -1:

1	1	2
2	2	2
3	3	3

For $\beta = (4, 2, 3)$ and $\mu = (3, 3, 3)$, we see that $\mathrm{SRHT}(\mu, \beta) = \varnothing$.

Let $S \in \mathrm{SRHT}(\mu, \beta)$ where $\beta = (\beta_1, \ldots, \beta_s) = (\beta^*, L(\beta))$. Removing the special rim-hook σ in S of size $L(\beta)$ containing the value $s = \ell(\beta)$ leaves a smaller SRHT S^* of content β^* and some partition shape δ, as is readily checked. Moreover, $\mathrm{sgn}(S) = \mathrm{sgn}(S^*)\,\mathrm{sgn}(\sigma)$. Let $T(\mu, L)$ be the set of partitions $\delta \in P(n-L)$ such that $\mathrm{dg}(\delta) \subseteq \mathrm{dg}(\mu)$ and $\mu/\delta = \mathrm{dg}(\mu) \setminus \mathrm{dg}(\delta)$ is a special rim-hook of size L. Removing the last special rim-hook gives a bijection $G : \mathrm{SRHT}(\mu, \beta) \to \bigcup_{\delta \in T(\mu, L(\beta))} \mathrm{SRHT}(\delta, \beta^*)$. Taking signs into account, we deduce the recursion

$$B(\beta, \mu) = \sum_{\delta \in T(\mu, L(\beta))} \mathrm{sgn}(\mu/\delta) B(\beta^*, \delta), \tag{8}$$

which is an instance of (2) with $\mathrm{wt}_B(\mu, \delta) = \mathrm{sgn}(\mu/\delta)$. A version of (8) appears in [4, Sec. 4].

Remark 8. For given $\mu \in P(n)$ and $L \in [n]$, there is at most one way to remove a special rim-hook of size L from the southeast rim of $\mathrm{dg}(\mu)$ to leave a partition shape. In other words, $|T(\mu, L)| \leqslant 1$. Iterating this reasoning shows that $|\mathrm{SRHT}(\mu, \beta)| \leqslant 1$ for all $\mu \in P(n)$ and all $\beta \in C(n)$.

Example 9. For $n = 4$, the rectangular matrices are as follows.

A_4 :

	4	31	22	211	13	121	112	1111
4	1	1	1	1	1	1	1	1
31	0	1	1	2	1	2	2	3
22	0	0	1	1	0	1	1	2
211	0	0	0	1	0	1	1	3
1111	0	0	0	0	0	0	0	1

B_4 :

	4	31	22	211	1111
4	1	−1	0	1	−1
31	0	1	0	−1	1
22	0	0	1	−1	1
211	0	0	0	1	−1
13	0	0	−1	0	1
121	0	0	0	0	−1
112	0	0	0	0	−1
1111	0	0	0	0	1

3.3 Proof of the Local Identity

In this first application, the local identity (4) takes the following form.

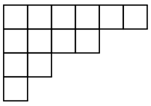
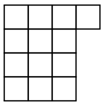
Theorem 10. *Given $n > 0$ and $\lambda, \mu \in P(n)$, let $G(\lambda, \mu)$ be the set of partitions γ such that $\mathrm{dg}(\gamma)$ can be obtained either by removing a nonempty horizontal strip from $\mathrm{dg}(\lambda)$ or by removing a nonempty special rim-hook from $\mathrm{dg}(\mu)$. Then*

$$\sum_{\gamma \in G(\lambda, \mu)} \mathrm{sgn}(\mu/\gamma) = \chi(\lambda = \mu). \tag{9}$$

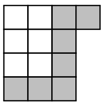
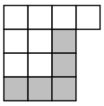
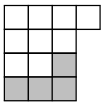
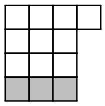
Therefore (by Theorem 2) the rectangular Kostka matrix A_n has a right-inverse B_n whose entries count signed SRHT.

Proof. First consider the case $\lambda = \mu$. The set $G(\lambda, \mu)$ consists of the single partition γ obtained by deleting the last part of λ. This is because the only removable horizontal strip in $\mathrm{dg}(\lambda)$ that is also a removable special rim-hook of $\mathrm{dg}(\lambda)$ is the set of all boxes in the lowest row of $\mathrm{dg}(\lambda)$. Since this set occupies one row, $\mathrm{sgn}(\mu/\gamma) = +1$ in this case, and (9) holds.

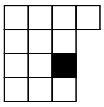
Next consider the case $\lambda \neq \mu$. We accompany the general proof by a running example where $\lambda = (6, 4, 2, 1)$ and $\mu = (4, 3, 3, 3)$. The diagrams are shown here:

$\mathrm{dg}(\lambda) =$  $\mathrm{dg}(\mu) =$ 

We prove that $G(\lambda, \mu)$ is either empty or consists of exactly two elements where the corresponding special rim-hooks μ/γ have opposite signs. This suffices to prove (9). Write $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$. There are exactly s special rim-hooks that can be removed from $\mathrm{dg}(\mu)$ to leave another partition diagram γ. Namely, for $i = 1, 2, \ldots, s$, we can remove the SRH σ_i that starts in the lowest box of column 1 and moves along the southeast rim of $\mathrm{dg}(\mu)$ until it ends at the rightmost box in row i of the diagram. Let γ^i be the partition obtained by removing σ_i from $\mathrm{dg}(\mu)$. These are the only possible elements of $G(\lambda, \mu)$. For our running example, the diagrams of γ^i are drawn below using white boxes, with the corresponding SRH σ_i depicted in gray.

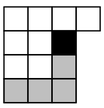
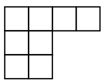
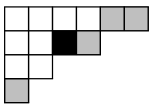
$\mathrm{dg}(\gamma^1) =$  $\mathrm{dg}(\gamma^2) =$  $\mathrm{dg}(\gamma^3) =$  $\mathrm{dg}(\gamma^4) =$ 

Step 1. We show $G(\lambda, \mu)$ cannot contain both γ^i and γ^j when $j > i + 1$. For if this happened, consider the rightmost box c in row $i + 1$ of $\mathrm{dg}(\mu)$. In our running example, taking $i = 2$ and $j = 4$, the box c is marked in black below:



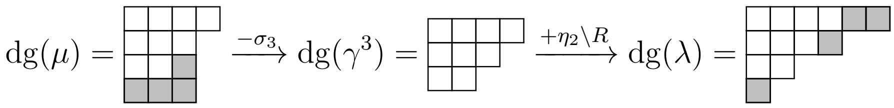
Box c and the box just above c both belong to σ_i and so do not belong to $\mathrm{dg}(\gamma^i)$. Since λ/γ^i is a horizontal strip, c cannot belong to $\mathrm{dg}(\lambda)$. On the other hand, because $j > i+1$, c does not belong to σ_j and so does belong to $\mathrm{dg}(\gamma^j)$, which is contained in $\mathrm{dg}(\lambda)$. Thus c does belong to $\mathrm{dg}(\lambda)$, which gives a contradiction.

Step 2. Suppose $G(\lambda, \mu)$ is nonempty, say with $\gamma^i \in G(\lambda, \mu)$. We prove $G(\lambda, \mu) = \{\gamma^i, \gamma^{i+1}\}$ or $G(\lambda, \mu) = \{\gamma^i, \gamma^{i-1}\}$. Let $\eta_i = \lambda/\gamma^i$, which must be a horizontal strip. We can pass from $\mathrm{dg}(\mu)$ to $\mathrm{dg}(\lambda)$ by first removing σ_i to produce $\mathrm{dg}(\gamma^i)$, then adding η_i to reach $\mathrm{dg}(\lambda)$. Let c be the rightmost box in row i of $\mathrm{dg}(\mu)$, which is in σ_i and thus not in $\mathrm{dg}(\gamma^i)$. Let R be the set of boxes of σ_i lying in row i of $\mathrm{dg}(\mu)$. In our running example, we may take $i = 2$. The black box in the next figure is c, the rest of σ_i is shown in gray, and in this instance, $R = \{c\}$.

$\mathrm{dg}(\mu) =$  $\xrightarrow{-\sigma_2}$ $\mathrm{dg}(\gamma^2) =$  $\xrightarrow{+\eta_2}$ $\mathrm{dg}(\lambda) =$ 

Case 1: Assume c is in $\mathrm{dg}(\lambda)$. Then we must have $c \in \eta_i$, hence $R \subseteq \eta_i \subseteq \mathrm{dg}(\lambda)$.

- Case 1a: Assume $i < s$, so γ^{i+1} is defined. In this situation, $\sigma_{i+1} = \sigma_i \setminus R$, $\gamma^{i+1} = \gamma^i \cup R$, $\mathrm{dg}(\gamma^{i+1}) \subseteq \mathrm{dg}(\lambda)$, and (critically) $\lambda/\gamma^{i+1} = \eta_i \setminus R$ is a horizontal strip. Thus $\gamma^{i+1} \in G(\lambda, \mu)$, and Step 1 applies to show $G(\lambda, \mu) = \{\gamma^i, \gamma^{i+1}\}$. Note $\mathrm{sgn}(\sigma_{i+1}) = -\,\mathrm{sgn}(\sigma_i)$. In our running example, the next figure shows how to go from μ to λ by removing σ_3 and then adding $\eta_2 \setminus R$.

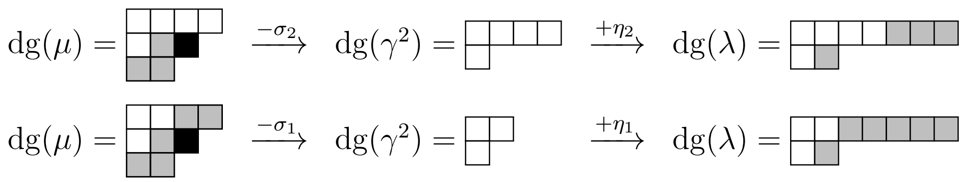


- Case 1b: Assume $i = s$; we rule out this case by deducing a contradiction. Here, σ_s and R must both be the set of all cells in the lowest row of $\mathrm{dg}(\mu)$. But since R is a subset of η_s, where η_s has the same size as σ_s, we must have $\eta_s = R$. This in turn forces $\lambda = \mu$, contradicting the assumption $\lambda \neq \mu$.

Case 2: Assume c is not in $\mathrm{dg}(\lambda)$.

- Case 2a: Assume $i > 1$, so γ^{i-1} is defined. Let R' be the set of cells of σ_{i-1} lying in row $i-1$ of $\mathrm{dg}(\mu)$. By assumption on c, the horizontal strip η_i cannot contain any cell of row i directly below a cell of R' (although η_i might contain some cells in row $i-1$ to the right of R'). We now see that $\sigma_{i-1} = \sigma_i \cup R'$, $\gamma^{i-1} = \gamma^i \setminus R'$, $\mathrm{dg}(\gamma^{i-1}) \subseteq \mathrm{dg}(\gamma^i) \subseteq \mathrm{dg}(\lambda)$, and (critically) $\lambda/\gamma^{i-1} = \eta_i \cup R'$ is a horizontal strip. Thus $\gamma^{i-1} \in G(\lambda, \mu)$, and Step 1 applies to show $G(\lambda, \mu) = \{\gamma^i, \gamma^{i-1}\}$. Note $\mathrm{sgn}(\sigma_{i-1}) = -\,\mathrm{sgn}(\sigma_i)$.

  We illustrate this case with a new example. Let $\mu = (4, 3, 2)$ and $\lambda = (7, 2)$. We label the special rim-hooks and horizontal strips in gray and label the cell $c = (2, 3)$ in black. On one hand, we remove σ_2 from $\mathrm{dg}(\mu)$ to reach the diagram of $\gamma^2 = (4, 1)$, then add η_2 to $\mathrm{dg}(\gamma^2)$ to obtain $\mathrm{dg}(\lambda)$. On the other hand, we remove $\sigma_1 = \sigma_2 \cup R'$ from $\mathrm{dg}(\mu)$ to reach the diagram of $\gamma^1 = (2, 1)$, then add η_1 to $\mathrm{dg}(\gamma^1)$ to obtain $\mathrm{dg}(\lambda)$.



- Case 2b: Assume $i = 1$; we rule out this case by deducing a contradiction. The special rim-hook σ_1 starts in column 1 and ends at cell c in row 1 and column μ_1, so σ_1 contains at least μ_1 boxes. So the horizontal strip η_1, which has the same size as σ_1, must have at least μ_1 boxes, all of which occupy at least μ_1 columns. Since η_1 is added to $\mathrm{dg}(\gamma^1) \subseteq \mathrm{dg}(\mu)$, cell c must belong to $\eta_1 \subseteq \mathrm{dg}(\lambda)$, contrary to the assumption of Case 2. □

Remark 11. Bender and Knuth [1] gave a bijective proof that the rectangular Kostka matrix satisfies the sorting condition of Remark 3; see also [8, Thm. 9.27]. By Remark 3, we can deduce Eğecioğlu and Remmel's inversion result [3, Thm. 1] for the square $P(n) \times P(n)$ Kostka matrix from our rectangular version. For $S \in \mathrm{SRHT}(\mu, \beta)$, those authors refer to $\nu = \mathrm{sort}(\beta)$ as the *type* of S. Their square inverse Kostka matrix has entries $K^{-1}(\nu, \mu) = \sum_{\beta:\,\mathrm{sort}(\beta)=\nu} \sum_{S \in \mathrm{SRHT}(\mu,\beta)} \mathrm{sgn}(S)$, in accordance with (6).

Remark 12. By the proof of Theorem 10, there is a bijection $\gamma^i \mapsto \gamma^{i\pm1}$ that matches the two oppositely-signed objects contributing to the sum, in the case where $\lambda \neq \mu$ and the sum is not already empty. Since there is only one positive object and one negative object, this is a *canonical* bijection (not relying on arbitrary choices). We build this up to a canonical bijective proof of $A_n B_n = I_{P(n)}$ in Section 7.1.

4 Application 2: Rim-Hook Tableaux

As the second application of the general theory, we prove an inversion result for a rectangular version of the transition matrices between power-sum symmetric functions and Schur functions. In this case, the local identities (4) can be informally summarized by the slogan: "A local inverse of a signed rim-hook is a rescaled signed rim-hook."

4.1 Rim-Hook Tableaux

A *rim-hook* of size L is a set of L unit boxes that can be traversed by starting at some box and successively moving one step to the right or one step up to go from one box to the next. The *sign* of a rim-hook σ occupying r rows is $\mathrm{sgn}(\sigma) = (-1)^{r-1}$. Given $\lambda \in P(n)$ and $\beta \in C(n)$, a *rim-hook tableau (RHT)* of *shape* λ and *content* β is a filling S of the diagram of λ such that for all k, the cells of S containing k form a rim-hook of size β_k, and the cells of S containing $1, 2, \ldots, k$ form a partition diagram. Let $\mathrm{RHT}(\lambda, \beta)$ be the set of all such RHT. The *sign* of an RHT S is the product of the signs of all rim-hooks in S. In this second application of the general setup of §2.2, we let A_n be the $P(n) \times C(n)$ matrix with entries $A_n(\lambda, \beta) = \sum_{S \in \mathrm{RHT}(\lambda,\beta)} \mathrm{sgn}(S)$. The symmetric function literature often uses χ^λ_β to denote $A_n(\lambda, \beta)$.

Example 13. Let $\lambda = (4, 3, 3, 1)$ and $\beta = (3, 4, 4)$. We find $A(\lambda, \beta) = -2$ based on the two RHT shown below, which both have sign -1.

1	1	3	3
1	2	3	
2	2	3	
2			

1	1	2	2
1	2	2	
3	3	3	
3			

Let $U \in \mathrm{RHT}(\lambda, \beta)$ where $\beta = (\beta_1, \ldots, \beta_s) = (\beta^*, L(\beta))$. Removing the rim-hook σ in U of size $L(\beta)$ containing the value $s = \ell(\beta)$ leaves a smaller RHT U^* of content β^* and some shape γ, by definition. Moreover, $\mathrm{sgn}(U) = \mathrm{sgn}(U^*)\,\mathrm{sgn}(\sigma)$. Let $S(\lambda, L)$ be the set of partitions $\gamma \in P(n - L)$ such that $\mathrm{dg}(\gamma) \subsetneq \mathrm{dg}(\lambda)$ and $\lambda/\gamma = \mathrm{dg}(\lambda) \setminus \mathrm{dg}(\gamma)$ is a rim-hook of size L. Removing the last rim-hook gives a bijection $G : \mathrm{RHT}(\lambda, \beta) \to$

$\bigcup_{\gamma \in S(\lambda, L(\beta))} \mathrm{RHT}(\gamma, \beta^*)$. Taking signs into account, we deduce the recursion

$$A(\lambda, \beta) = \sum_{\gamma \in S(\lambda, L(\beta))} \mathrm{sgn}(\lambda/\gamma) A(\gamma, \beta^*), \tag{10}$$

which is an instance of (1) with $\mathrm{wt}_A(\lambda, \gamma) = \mathrm{sgn}(\lambda/\gamma)$.

4.2 The Scaling Factors Z_β

Given a composition $\beta = (\beta_1, \beta_2, \ldots, \beta_s) \in C(n)$, define the integer

$$Z_\beta = \beta_1(\beta_1 + \beta_2)(\beta_1 + \beta_2 + \beta_3) \cdots (\beta_1 + \beta_2 + \cdots + \beta_s). \tag{11}$$

For instance, $Z_{(3,4,4)} = 3 \cdot 7 \cdot 11 = 231$ but $Z_{(4,3,4)} = 4 \cdot 7 \cdot 11 = 308$. Since the last factor is $|\beta| = n$, we have $Z_\beta = n \cdot Z_{\beta^*}$. Let B_n be the $C(n) \times P(n)$ matrix defined by $B_n(\beta, \mu) = \sum_{S \in \mathrm{RHT}(\mu, \beta)} Z_\beta^{-1} \,\mathrm{sgn}(S)$ for $\beta \in C(n)$ and $\mu \in P(n)$. Observe that B_n is obtained from A_n by rescaling all entries in each column β by Z_β^{-1} and then transposing the matrix.

Define $T(\mu, L)$ to be the set of partitions $\delta \in P(n-L)$ such that $\mathrm{dg}(\delta) \subseteq \mathrm{dg}(\mu)$ and $\mu/\delta = \mathrm{dg}(\mu) \setminus \mathrm{dg}(\delta)$ is a rim-hook of size L (so $T(\mu, L)$ is the same as $S(\mu, L)$ in this application). Removing the last rim-hook σ of $U \in \mathrm{RHT}(\mu, \beta)$ produces $U^* \in \mathrm{RHT}(\delta, \beta^*)$ for a unique $\delta \in T(\mu, L)$. Note that $Z_\beta^{-1} \,\mathrm{sgn}(U) = n^{-1} Z_{\beta^*}^{-1} \,\mathrm{sgn}(U^*) \,\mathrm{sgn}(\sigma)$ where $n = |\beta| = |\mu|$ and $\sigma = \mu/\delta$. So we get the recursion

$$B(\beta, \mu) = \sum_{\delta \in T(\mu, L(\beta))} |\mu|^{-1} \,\mathrm{sgn}(\mu/\delta) B(\beta^*, \delta), \tag{12}$$

which is an instance of (2) with $\mathrm{wt}_B(\mu, \delta) = |\mu|^{-1} \,\mathrm{sgn}(\mu/\delta)$.

Example 14. For $n = 4$, the rectangular matrices are as follows.

$$A_4 : \begin{array}{r} \\ 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{array} \begin{array}{c} \begin{array}{cccccccc} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 & 0 & 1 & 1 & 3 \\ 0 & -1 & 2 & 0 & -1 & 0 & 0 & 2 \\ 1 & 0 & -1 & -1 & 0 & -1 & -1 & 3 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

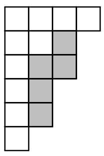
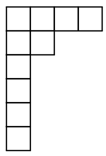
$$B_4 : \begin{array}{r} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 31 & 22 & 211 & 1111 \end{array} \\ \begin{bmatrix} 1/4 & -1/4 & 0 & 1/4 & -1/4 \\ 1/12 & 0 & -1/12 & 0 & 1/12 \\ 1/8 & -1/8 & 2/8 & -1/8 & 1/8 \\ 1/24 & 1/24 & 0 & -1/24 & -1/24 \\ 1/4 & 0 & -1/4 & 0 & 1/4 \\ 1/12 & 1/12 & 0 & -1/12 & -1/12 \\ 1/8 & 1/8 & 0 & -1/8 & -1/8 \\ 1/24 & 3/24 & 2/24 & 3/24 & 1/24 \end{bmatrix} \end{array}$$

4.3 Abacus Notation for Partitions

It is often useful to represent integer partitions by abaci, especially when performing operations that affect the southeast rim of the partition diagram. Given $\lambda \in P(n)$ and any $N \geqslant \ell(\lambda)$, the *$N$-bead abacus representing* λ is the binary word $\mathrm{ab}^N(\lambda) = w_1 w_2 w_3 \cdots$ defined as follows. The word begins with $N - \ell(\lambda)$ copies of 1 (each 1 represents a bead on an abacus). To continue building the word, move along the southeast rim of $\mathrm{dg}(\lambda)$, from the southwest corner to the northeast corner, by a succession of unit-length east steps and north steps. Record a 0 (gap) in the abacus word for each east step, and record a 1 (bead) in the abacus word for each north step. The word terminates with an infinite sequence of 0s (gaps). The value of N is usually irrelevant as long as it is large enough to accommodate any operations performed on the partition and its associated abacus.

For instance, consider the operation of removing a rim-hook of size L from the rim of $\mathrm{dg}(\lambda)$. It is not hard to check that this operation corresponds to making a bead "jump" from some position i on $\mathrm{ab}^N(\lambda)$ to a position $i - L$ that contains a gap. In other words, removal of the rim-hook σ of size L modifies the abacus by changing some w_i from 1 to 0 and w_{i-L} from 0 to 1. Furthermore, if there are b beads in positions $i - L + 1, \ldots, i - 1$ on $\mathrm{ab}^N(\lambda)$, then $\mathrm{sgn}(\sigma) = (-1)^b$. (For a more detailed verification, see [7, Sec. 3.3] or [8, Thm. 10.13].) On the other hand, adding a rim-hook of size L to $\mathrm{dg}(\lambda)$ is accomplished on the abacus by making a bead jump from position i to some position $i + L$ that contains a gap. Here, the number of beads on the abacus must satisfy $N \geqslant \ell(\lambda) + L$ to ensure that there are enough beads to accommodate all possible rim-hooks that might be added.

Example 15. Starting with the partition $\lambda = (4, 3, 3, 2, 2, 1)$, we can remove a rim-hook of size $L = 5$ to obtain $\mu = (4, 2, 1, 1, 1, 1)$ as shown in the following partition diagrams:

$\mathrm{dg}(\lambda) =$  $\to$ $\mathrm{dg}(\mu) =$ 

The corresponding operation on the 9-bead abaci representing μ and ν is shown here:

$$\mathrm{ab}^9(\lambda) = 11101\underline{0}11101\underline{1}01000\cdots \quad \to \quad \mathrm{ab}^9(\mu) = 11101\underline{1}11101\underline{0}01000\cdots$$

The sign associated with this rim-hook removal is $(-1)^3 = -1$.

4.4 Proof of the Local Identity

In this second application, the local identity (4) takes the following form.

Theorem 16. *Given $n > 0$ and $\lambda, \mu \in P(n)$, let $G(\lambda, \mu)$ be the set of partitions γ that can be obtained either by removing a nonempty rim-hook from λ or by removing a nonempty rim-hook from μ. Then*

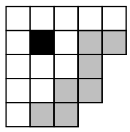
$$\sum_{\gamma \in G(\lambda,\mu)} n^{-1}\,\mathrm{sgn}(\lambda/\gamma)\,\mathrm{sgn}(\mu/\gamma) = \chi(\lambda = \mu). \tag{13}$$

Therefore (by Theorem 2) the matrix A_n whose entries count signed RHT has a right-inverse B_n, a rescaled version of the transpose of A_n.

Proof. First consider the case $\lambda = \mu$. The left side of (13) becomes

$$\sum_{\gamma \in G(\lambda,\lambda)} n^{-1} \operatorname{sgn}(\lambda/\gamma)^2 = |G(\lambda, \lambda)|/n,$$

so it suffices to show that $|G(\lambda, \lambda)| = n = |\lambda|$. This amounts to showing that there are exactly n nonempty rim-hooks that can be removed from the border of $\operatorname{dg}(\lambda)$. We give a bijection from $\operatorname{dg}(\lambda)$ (a set of n unit boxes) onto this set of rim-hooks. Given a box $c \in \operatorname{dg}(\lambda)$, move down (due south) from c to reach a box c' on the southern border of $\operatorname{dg}(\lambda)$, and move right (due east) from c to reach a box c'' on the eastern border of $\operatorname{dg}(\lambda)$. The bijection maps c to the rim-hook that moves northeast along the border starting at c' and ending at c''. The inverse bijection takes a rim-hook on the border of $\operatorname{dg}(\lambda)$, say going northeast from a box c_1 to a box c_2, and maps this rim-hook to the box of $\operatorname{dg}(\lambda)$ in the same row as c_2 and the same column as c_1. In the diagram of $\lambda = (5, 5, 4, 4, 3)$ shown below, we label in gray the rim-hook corresponding to the black cell c.



To handle the case $\lambda \neq \mu$, it suffices to prove

$$\sum_{\gamma \in G(\lambda,\mu)} \operatorname{sgn}(\lambda/\gamma) \operatorname{sgn}(\mu/\gamma) = 0.$$

We do this by showing that $G(\lambda, \mu)$ is either empty or contains exactly two objects contributing oppositely-signed terms to the sum.

Consider the N-bead abaci $\operatorname{ab}^N(\lambda)$ and $\operatorname{ab}^N(\mu)$, where $N = \max(\ell(\lambda), \ell(\mu))$. Assuming $G(\lambda, \mu) \neq \emptyset$, there must be a way go from $\operatorname{dg}(\lambda)$ to $\operatorname{dg}(\mu)$ by removing some rim-hook of size L from $\operatorname{dg}(\lambda)$ to get $\operatorname{dg}(\gamma)$ for some $\gamma \in G(\lambda, \mu)$, then adding some different rim-hook of size L to $\operatorname{dg}(\gamma)$ to get $\operatorname{dg}(\mu)$. Translating to abaci, there must be a way to go from $\operatorname{ab}^N(\lambda)$ to $\operatorname{ab}^N(\mu)$ by the following two-step process. First, a bead in $\operatorname{ab}^N(\lambda)$ jumps from some position i to a gap in some position $i - L$, producing $\operatorname{ab}^N(\gamma)$ for some γ. Second, a different bead in $\operatorname{ab}^N(\gamma)$ jumps from some position $j \neq i$ to a gap in position $j + L$, producing $\operatorname{ab}^N(\mu)$. We must have $j \neq i - L$ since $\lambda \neq \mu$.

The key observation is that there is exactly one other way to execute this two-step process (with new choices of i, j, L, γ) to convert $\operatorname{ab}^N(\lambda)$ to $\operatorname{ab}^N(\mu)$. In more detail, first note that:

- positions i and j contain beads in $\operatorname{ab}^N(\lambda)$ and gaps in $\operatorname{ab}^N(\mu)$;
- positions $i - L$ and $j + L$ contain gaps in $\operatorname{ab}^N(\lambda)$ and beads in $\operatorname{ab}^N(\mu)$;
- each other position contains the same thing (bead or gap) in $\operatorname{ab}^N(\lambda)$ and in $\operatorname{ab}^N(\mu)$.

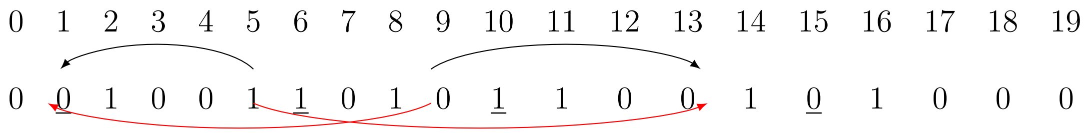
Consider the case $i < j$, so $i - L < i < j < j + L$. The only other way to go from $\mathrm{ab}^N(\lambda)$ to $\mathrm{ab}^N(\mu)$ is to move a bead from position j to position $i - L$ (jumping down by $L' = j - i + L > 0$ positions) and then move a bead from position i to position $j + L$ (jumping up L' positions). To compare the signs of the two ways, let $\mathrm{ab}^N(\lambda)$ have a beads strictly between positions $i - L$ and i, b beads strictly between positions i and j, and c beads strictly between positions j and $j + L$. For the first bead motion, the bead jump from i to $i-L$ contributes $(-1)^a$, and the bead jump from j to $j+L$ contributes $(-1)^c$. For the second bead motion, the bead jump from j to $i-L$ contributes $(-1)^{a+b+1}$ (noting that this bead jumps over a bead at position i), while the bead jump from i to $j+L$ contributes $(-1)^{b+c}$ (noting there is no longer a bead at position j). Since $(-1)^{a+2b+c+1} = -(-1)^{a+c}$, the two terms have opposite signs.

Consider the case $j < i$ and $i - j < L$. Then $i - L < j < i < j + L$. As in the previous case, the other way to go from $\mathrm{ab}^N(\lambda)$ to $\mathrm{ab}^N(\mu)$ is to move the bead in position j to position $i - L$, then move the bead in position i to position $j + L$. A sign analysis (similar to the first case) shows that the two ways lead to terms with opposite signs.

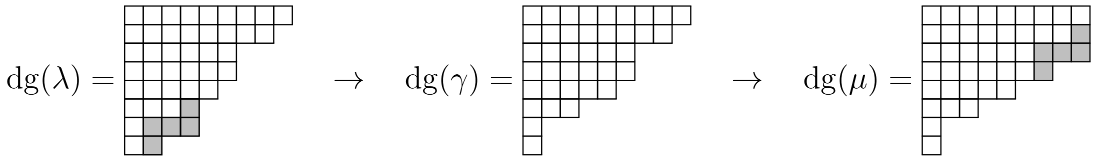
Consider the case $j < i$ and $i - j \geqslant L$. We cannot have $i - j = L$ since $j \neq i - L$ as noted earlier. We cannot have $i - L = j + L$ since otherwise we could not move a bead from j to $j+L$ on $\mathrm{ab}^N(\gamma)$. The possible orderings in this case are therefore $j < j+L < i-L < i$ or $j < i - L < j + L < i$. In both orderings, the second way to go from $\mathrm{ab}^N(\lambda)$ to $\mathrm{ab}^N(\mu)$ first moves a bead from i to $j + L$, then moves a bead from j to $i - L$. As before, we can check that the two ways lead to terms of opposite signs. □

Remark 17. When $\lambda \neq \mu$ and the sum on the left side of (13) is not vacuous, the proof just given shows there is exactly one positive object and one negative object contributing to this sum. Thus we have a *canonical* bijection (not relying on arbitrary choices) proving this identity. The bijection when $\lambda = \mu$, while not canonical in this technical sense, is still quite natural. We build this up to an "almost canonical" bijective proof of $A_n(n!B_n) = n!I_{P(n)}$ in Section 7.2.

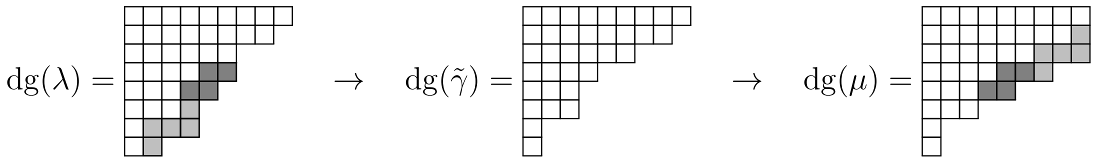
Example 18. Let $\lambda = (9, 8, 6, 6, 5, 4, 4, 2)$ and $\mu = (9, 9, 9, 7, 5, 3, 1, 1)$, so $N = 8$. The first way to go from $\mathrm{ab}^N(\lambda)$ to $\mathrm{ab}^N(\mu)$ takes $L = 5$ and moves a bead from position $i = 6$ to $i - L = 1$, then moves a bead from position $j = 10$ to $j + L = 15$, as shown by the upper arrows in the diagram below. The second way takes $L = 10 - 6 + 5 = 9$ instead and moves a bead from position 10 to 1, then moves a bead from position 6 to 15, as shown by the lower arrows in the diagram. The signs are $(-1)^{2+2} = +1$ for the first way and $(-1)^{4+3} = -1$ for the second way.



Going back to partition diagrams, let $\gamma = (9, 8, 6, 6, 5, 3, 1, 1)$ and $\tilde{\gamma} = (9, 8, 6, 4, 3, 3, 1, 1)$. The first way to go from λ to μ removes a rim-hook $R = \lambda/\gamma$ from $\mathrm{dg}(\lambda)$ to reach $\mathrm{dg}(\gamma)$, then adds a rim-hook $S = \mu/\gamma$ to $\mathrm{dg}(\gamma)$ to reach $\mathrm{dg}(\mu)$, as shown here:



The second way to go from λ to μ removes a rim-hook $\tilde{R} = \lambda/\tilde{\gamma}$ from $\mathrm{dg}(\lambda)$ to reach $\mathrm{dg}(\tilde{\gamma})$, then adds a rim-hook $\tilde{S} = \mu/\tilde{\gamma}$ to $\mathrm{dg}(\tilde{\gamma})$ to reach $\mathrm{dg}(\mu)$, as shown here:



On one hand, $\mathrm{dg}(\gamma) = \mathrm{dg}(\lambda) \cap \mathrm{dg}(\mu)$, and R and S are disjoint rim-hooks that can be added outside the rim of $\mathrm{dg}(\gamma)$ to reach $\mathrm{dg}(\lambda)$ or $\mathrm{dg}(\mu)$, respectively. On the other hand, we get the rim-hook $\tilde{R}$ by adding to R the cells on the inside border of $\mathrm{dg}(\gamma)$ that lead from the northeastern-most cell of R to the southwestern-most cell of S (these cells are shaded dark gray above). Similarly, $\tilde{S}$ consists of S along with these same cells, while $\mathrm{dg}(\tilde{\gamma})$ is $\mathrm{dg}(\gamma)$ with these cells removed. By translating the cases in the abacus-based proof back to partition diagrams, one may check that these descriptions of γ, R, S, $\tilde{\gamma}$, $\tilde{R}$, and $\tilde{S}$ are valid whenever $G(\lambda, \mu) = \{\gamma, \tilde{\gamma}\}$ is nonempty.

Remark 19. Stanton and White [11, Thm. 4] gave a bijective proof that our matrix A_n (counting signed RHT) satisfies the sorting condition of Remark 3. This can also be deduced algebraically from the observation that when $\mathrm{sort}(\alpha) = \mathrm{sort}(\beta)$, the power-sum symmetric functions p_α and p_β are equal and have the same Schur expansion. By Remark 3, the restriction of A_n to a square $P(n) \times P(n)$ matrix has inverse B'_n with entries

$$B'_n(\lambda, \mu) = \sum_{\substack{\beta \in C(n): \\ \mathrm{sort}(\beta) = \lambda}} B_n(\beta, \mu) = \sum_{\substack{\beta \in C(n): \\ \mathrm{sort}(\beta) = \lambda}} A_n(\mu, \beta)/Z_\beta = A_n(\mu, \lambda) \sum_{\substack{\beta \in C(n): \\ \mathrm{sort}(\beta) = \lambda}} Z_\beta^{-1}. \qquad (14)$$

This classical result is usually stated as follows. For a partition λ containing m_k copies of k for each k, define the integer $z_\lambda = \prod_{k \geqslant 1} m_k! k^{m_k}$. Then the square matrix A with λ, μ-entry χ^λ_μ has inverse with λ, μ-entry $\chi^\mu_\lambda / z_\lambda$. A is known to be the character table for the symmetric group S_n, and this result is a special case of the orthogonality of irreducible characters of a finite group. Part (c) of Lemma 20 (below) shows why (14) agrees with the classical formulation.

To state Lemma 20, we need the following definitions. Any $\sigma \in S_n$ can be written as a product of disjoint cycles. Define the *cycle partition* $\mathrm{cycP}(\sigma) \in P(n)$ to be the list of the lengths of these cycles (including 1-cycles) written in weakly decreasing order. Most permutations have several different cycle notations, obtained by reordering cycles or starting each cycle at a different position. Define the *canonical cycle notation* for σ by requiring that each cycle start at its minimum element and writing cycles with their minimum elements in decreasing order.

Define the *cycle composition* $\mathrm{cycC}(\sigma) \in C(n)$ to be the list of lengths of the cycles in this canonical cycle notation for σ. For example, $\sigma = (5,2,1)(6,4,7)(9,3)(8) \in S_9$ has canonical cycle notation $(8)(4,7,6)(3,9)(1,5,2)$, $\mathrm{cycP}(\sigma) = (3,3,2,1)$, and $\mathrm{cycC}(\sigma) = (1,3,2,3)$.

Lemma 20. *(a) For all $\lambda \in P(n)$, $n!/z_\lambda$ is the number of $\sigma \in S_n$ with* $\mathrm{cycP}(\sigma) = \lambda$.
(b) For all $\beta \in C(n)$, $n!/Z_\beta$ is the number of $\sigma \in S_n$ with $\mathrm{cycC}(\sigma) = \beta$.
(c) For all $\lambda \in P(n)$, $\displaystyle\sum_{\substack{\beta \in C(n):\\ \mathrm{sort}(\beta)=\lambda}} Z_\beta^{-1} = z_\lambda^{-1}$. In other words, z_λ is the harmonic mean of the Z_β as β ranges over all rearrangements of λ.

Proof. Part (a) is a standard result giving the size of the conjugacy class of S_n indexed by λ; see, for example, [8, Thm. 7.115]. To prove part (b), we fix $\beta \in C(n)$ and give the following construction that builds all $\sigma \in S_n$ with $\mathrm{cycC}(\sigma) = \beta$. We use $\beta = (1,3,2,3)$ as a running example to illustrate the construction. For convenience, write $k = \ell(\beta)$ and $B_i = \beta_1 + \cdots + \beta_i$ for $i = 1, 2, \ldots, k$.

Begin with a list of cycles with β_1 blanks in the first cycle, β_2 blanks in the second cycle, and so on. Our example begins with $(\_\_)(\_\_,\_\_,\_\_)(\_\_,\_\_)(\_\_,\_\_,\_\_)$. Fill the blanks in the rightmost cycle first. The first blank must be 1; there are $n-1$ choices for the next value, $n-2$ choices for the value after that, and so on; there are $n-(\beta_k-1) = B_{k-1}+1$ choices for the last value in this cycle. In our example, we might pick 1 (forced) then 5 then 2 to get $(\_\_)(\_\_,\_\_,\_\_)(\_\_,\_\_)(1,5,2)$. Move left to the next empty cycle. The first blank must be the minimum value not used already. Then there are $n-\beta_k-1 = B_{k-1}-1$ choices for the next value, $B_{k-1}-2$ choices for the next value, and so on; there are $n-\beta_k-(\beta_{k-1}-1) = B_{k-2}+1$ choices for the last value. In our example, we might pick 3 (forced) then 9 to get $(\_\_)(\_\_,\_\_,\_\_)(3,9)(1,5,2)$. Continue filling the cycles from right to left, noting that the first element in each new cycle must be the minimum element in $[n]$ not already chosen.

The number of ways to execute this construction is the product of all integers in the list $n, n-1, n-2, \ldots, 3, 2, 1$ excluding $n = B_k, B_{k-1}, B_{k-2}, \ldots, B_1 = \beta_1$. The excluded integers correspond to the forced choices of the minimum element in each new cycle. We see that the number of $\sigma \in S_n$ with $\mathrm{cycC}(\sigma) = \beta$ is $\frac{n!}{B_1 B_2 \cdots B_k} = \frac{n!}{Z_\beta}$, as needed.

To prove part (c), fix $\lambda \in P(n)$. Since the canonical cycle notation of a permutation is unique, the set $\{\sigma \in S_n : \mathrm{cycP}(\sigma) = \lambda\}$ is the disjoint union of the sets $\{\sigma \in S_n : \mathrm{cycC}(\sigma) = \beta\}$ as β ranges over all compositions in $C(n)$ that sort to λ. By parts (a) and (b), we deduce

$$n!/z_\lambda = \sum_{\substack{\beta \in C(n):\\ \mathrm{sort}(\beta)=\lambda}} n!/Z_\beta.$$

Dividing by $n!$ gives the result. ∎

Example 21. For $\lambda = (3,2,2) \in P(7)$, we have $z_\lambda = 24$, $Z_{(3,2,2)} = 105$, $Z_{(2,3,2)} = 70$, $Z_{(2,2,3)} = 56$, and $\frac{1}{105}+\frac{1}{70}+\frac{1}{56} = \frac{1}{24}$. With the 7! factor included, this says $48+72+90 = 210$.

5 Application 3: Composition Refinement Matrices

For our third application, we use our local technique to prove a well-known inversion result for matrices whose entries record when one composition refines another. These matrices, along with their weighted generalizations considered in Section 5.4, are transition matrices between certain bases of quasisymmetric functions and non-commutative symmetric functions (see Remark 24). In this application, the local identities (4) can be informally summarized by the slogan: "A local inverse of a composition of L is a signed brick of length L."

5.1 Refinement Ordering on Compositions

Let α, β be compositions of n. Define $\alpha \leqslant \beta$ to mean: there exist i_k such that $0 = i_0 < i_1 < i_2 < \ldots < i_{\ell(\beta)} = \ell(\alpha)$ and $\beta_k = \alpha_{i_{k-1}+1} + \ldots + \alpha_{i_k}$ for $k = 1, 2, \ldots, \ell(\beta)$. When this holds, we say α *refines* β and β *coarsens* α. For example, $(2,1,1,2,1,3,1,1) \leqslant (3,4,3,2)$. In general, we go from β to a composition refining β by replacing each part β_k by a composition of β_k. We go from α to a composition coarser than α by picking zero or more blocks of consecutive parts in α and replacing each block of parts by their sum. The set $C(n)$ is partially ordered by the refinement relation.

We use tableau-like structures called *compositional brick tabloids (CBTs)* to visualize the refinement relation between compositions. These objects resemble the brick tabloids used in [2] to study transition matrices between homogeneous symmetric functions and elementary symmetric functions, but CBTs have simpler combinatorial structure. See also [6] where similar objects are studied in connection with the combinatorics of non-commutative symmetric functions. Given $\alpha, \beta \in C(n)$, a *CBT of shape α and content β* is a filling of the diagram of α with β_k copies of k, for $1 \leqslant k \leqslant \ell(\beta)$, such that values weakly increase from left to right in each row, and each value in any row is strictly larger than the values appearing in all higher rows. Let $\mathrm{CBT}(\alpha,\beta)$ be the set of such objects. We may interpret an element in $\mathrm{CBT}(\alpha,\beta)$ as a tiling of $\mathrm{dg}(\alpha)$ by labeled bricks, where the ith brick has length β_i, satisfying the stated conditions on brick labels.

Example 22. For $\alpha = (4,5,5,3)$ and $\beta = (3,1,3,2,5,1,2)$, the set $\mathrm{CBT}(\alpha,\beta)$ consists of a single object shown here.

1	1	1	2	
3	3	3	4	4
5	5	5	5	5
6	7	7		

In general, given $\alpha, \beta \in C(n)$, the only possible tiling in $\mathrm{CBT}(\alpha,\beta)$ satisfying the ordering conditions is formed by laying down the bricks in order from 1 to $\ell(\beta)$, working through $\mathrm{dg}(\alpha)$ from the top row down, and working left to right within each row. This tiling process succeeds if and only if β refines α. Thus, $|\,\mathrm{CBT}(\alpha,\beta)| = \chi(\beta \leqslant \alpha)$. In the case where $\mathrm{CBT}(\alpha, \beta)$ is nonempty, define the *sign* of the unique CBT in this set to be $(-1)^{\ell(\beta)-\ell(\alpha)}$. We may view the sign combinatorially by attaching a sign of $+1$ to the first brick in each row and -1 to all other bricks, and letting the sign of a tiling be the product of the signs of all its bricks. The CBT in Example 22 has sign $(-1)^3 = -1$.

In this third application of the general setup of §2.2, we let A_n be the $C(n) \times C(n)$ *refinement poset incidence matrix* with entries

$$A_n(\lambda, \beta) = \chi(\lambda \leqslant \beta) = \sum_{T \in \mathrm{CBT}(\beta,\lambda)} 1 \qquad \text{for } \lambda, \beta \in C(n). \tag{15}$$

We will prove that A_n has inverse B_n given by

$$B_n(\beta, \mu) = (-1)^{\ell(\beta)-\ell(\mu)} \chi(\beta \leqslant \mu) = \sum_{T \in \mathrm{CBT}(\mu,\beta)} \mathrm{sgn}(T) \qquad \text{for } \beta, \mu \in C(n). \tag{16}$$

This is a well-known result giving the Möbius function for the refinement poset on $C(n)$ (or equivalently, the poset obtained by ordering the set of subsets of $[n-1]$ by set inclusion). Our point here is to showcase how this inversion result follows from a simple combinatorial argument based on our local inversion technique. This same technique proves a more subtle variation where matrix entries count weighted compositions (see §5.4).

Example 23. For $n = 4$, the incidence matrices are shown here.

$$A_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} \begin{array}{c} \begin{matrix} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$B_4 = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 13 \\ 121 \\ 112 \\ 1111 \end{array} \begin{array}{c} \begin{matrix} 4 & 31 & 22 & 211 & 13 & 121 & 112 & 1111 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

Remark 24. Several classical transition matrices for bases of QSym (the space of quasisymmetric functions), and NSym (the space of non-commutative symmetric functions) can be inverted using this result. We briefly state the results here, referring to [5, 6, 9] for more details and definitions. First, the monomial quasisymmetric basis (M_α) and Gessel's fundamental quasisymmetric basis (F_α) of QSym satisfy

$$F_\lambda = \sum_{\beta\in C(n)} \chi(\beta \leqslant \lambda)M_\beta = \sum_{\beta\in C(n)} A_n(\beta,\lambda)M_\beta \qquad \text{for all } \lambda \in C(n).$$

The (transposed) inversion result for A_n and B_n therefore gives

$$M_\beta = \sum_{\mu\in C(n)} B_n(\mu,\beta)F_\mu = \sum_{\mu\in C(n)} (-1)^{\ell(\mu)-\ell(\beta)}\chi(\mu \leqslant \beta)F_\mu \qquad \text{for all } \beta \in C(n).$$

Second, the ribbon Schur basis $(\mathbf{r}_\alpha)$ and the non-commutative complete basis $(\mathbf{h}_\alpha)$ of NSym satisfy

$$\mathbf{h}_\beta = \sum_{\alpha\in C(n)} \chi(\beta \leqslant \alpha)\mathbf{r}_\alpha \text{ and } \mathbf{r}_\beta = \sum_{\alpha\in C(n)} (-1)^{\ell(\beta)-\ell(\alpha)}\chi(\beta \leqslant \alpha)\mathbf{h}_\alpha \quad \text{for all } \beta \in C(n).$$

Third, by redistributing signs between the matrices A_n and B_n, it is routine to check that the $C(n) \times C(n)$ matrix given by $\overline{A_n}(\lambda,\beta) = (-1)^{n-\ell(\lambda)}\chi(\lambda \leqslant \beta)$ is its own inverse. This matrix is the transition matrix (in both directions) between the NSym bases $(\mathbf{h}_\alpha)$ and $(\mathbf{e}_\alpha)$:

$$\mathbf{h}_\beta = \sum_{\mu\in C(n)} \overline{A_n}(\lambda,\beta)\mathbf{e}_\lambda \quad \text{and} \quad \mathbf{e}_\beta = \sum_{\mu\in C(n)} \overline{A_n}(\lambda,\beta)\mathbf{h}_\lambda \quad \text{for all } \beta \in C(n). \tag{17}$$

Fourth, the weighted variation in Section 5.4 (with a sign adjustment) gives the transition matrices between the NSym bases $(\mathbf{h}_\alpha)$ and $(\boldsymbol{\psi}_\alpha)$, namely:

$$\mathbf{h}_\beta = \sum_{\lambda\in C(n)} \frac{\chi(\lambda \leqslant \beta)}{Z_{\beta,\lambda}}\boldsymbol{\psi}_\lambda, \quad \boldsymbol{\psi}_\mu = \sum_{\beta\in C(n)} (-1)^{\ell(\mu)-\ell(\beta)}\chi(\beta \leqslant \mu)L_{\mu,\beta}\,\mathbf{h}_\beta \;\; \text{for all } \beta,\mu \in C(n), \tag{18}$$

where $Z_{\beta,\lambda}$ and $L_{\mu,\beta}$ are defined in §5.4.

5.2 Recursions for A_n and B_n

Given $\lambda, \beta \in C(n)$ with $\beta = (\beta^*, L(\beta))$, a CBT of shape β and content λ (if it exists) consists of a CBT of shape β^* and content γ (where γ is some prefix of λ), followed by the bottom row of length $L(\beta)$ filled with bricks consisting of the suffix of λ not in γ. For any $\lambda \in C(n)$ and $L \in \mathbb{Z}_{>0}$, define $S(\lambda, L)$ as follows. If λ has a suffix $\lambda_{k+1}, \ldots, \lambda_{\ell(\lambda)}$ with sum L, let $S(\lambda, L) = \{\gamma\}$ where $\gamma = (\lambda_1, \ldots, \lambda_k)$; otherwise let $S(\lambda, L) = \emptyset$. The preceding discussion of CBT proves that

$$A(\lambda,\beta) = \sum_{\gamma\in S(\lambda,L(\beta))} A(\gamma,\beta^*), \tag{19}$$

which is an instance of (1) with $\mathrm{wt}_A(\lambda,\gamma)=1$. Note that if γ exists but does not refine β^*, then λ does not refine β and both sides of (19) are zero. In Example 22, renaming the partitions as $\lambda=(3,1,3,2,5,1,2)$ and $\beta=(4,5,5,3)$, we have $L(\beta)=3$, $\beta^*=(4,5,5)$, $\gamma=(3,1,3,2,5)$, and $A(\lambda,\beta)=1=A(\gamma,\beta^*)$.

Next, given $\beta,\mu\in C(n)$ with $\beta=(\beta^*,L(\beta))$, we may pass from a signed CBT of shape μ and content β (if it exists) to a smaller signed CBT of some shape δ and content β^* by removing the final brick (with length $L(\beta)$ and label $\ell(\beta)$) along with the cells in $\mathrm{dg}(\mu)$ occupied by that brick. We obtain δ from μ by subtracting $L(\beta)$ from the last part of μ and deleting the new part if it is zero. In the case where the last row of $\mathrm{dg}(\mu)$ has more than one brick, the sign changes since we removed a brick of sign -1 from the last row. In the case where the last row of $\mathrm{dg}(\mu)$ has a single brick, the sign does not change since we removed the first (and only) brick from the last row. For $\mu=(\mu_1,\ldots,\mu_k)\in C(n)$ and $L>0$, use three cases to define the set $T(\mu,L)$. If $L>\mu_k$, let $T(\mu,L)=\emptyset$. If $L=\mu_k$, let $T(\mu,L)=\{\delta\}$ with $\delta=(\mu_1,\ldots,\mu_{k-1})$ and $\mathrm{sgn}(\mu,\delta)=+1$. If $L<\mu_k$, let $T(\mu,L)=\{\delta\}$ with $\delta=(\mu_1,\ldots,\mu_{k-1},\mu_k-L)$ with $\mathrm{sgn}(\mu,\delta)=-1$. The preceding discussion of CBT proves the recursion

$$B(\beta,\mu)=\sum_{\delta\in T(\mu,L(\beta))}\mathrm{sgn}(\mu,\delta)B(\beta^*,\delta), \tag{20}$$

which is an instance of (2) with $\mathrm{wt}_B(\mu,\delta)=\mathrm{sgn}(\mu,\delta)$. In Example 22, renaming the partitions as $\beta=(3,1,3,2,5,1,2)$ and $\mu=(4,5,5,3)$, we have $L(\beta)=2<\mu_4$, $\beta^*=(3,1,3,2,5,1)$, $\delta=(4,5,5,1)$, $\mathrm{sgn}(\mu,\delta)=-1$, and $B(\beta,\mu)=-1=-B(\beta^*,\delta)$.

5.3 Proof of the Local Identity

In this application, the local identity (4) takes the following form.

Theorem 25. *Given $n>0$ and compositions $\lambda,\mu\in C(n)$, let $G(\lambda,\mu)$ be the set of compositions γ such that γ is a prefix of λ and γ can be obtained from μ by decreasing the last part of μ by some positive amount L. Then*

$$\sum_{\gamma\in G(\lambda,\mu)}\mathrm{sgn}(\mu,\gamma)=\chi(\lambda=\mu). \tag{21}$$

Therefore (by Theorem 2), the matrices A_n and B_n in (15) and (16) are inverses of each other.

Proof. The left side of (21) can be interpreted as the signed sum of all possible ways of transforming $\mu=(\mu_1,\ldots,\mu_k)$ into λ by the following two-step procedure. First, decrease the last part of μ by some amount $L>0$, using the sign $+1$ if the entire last part is removed and -1 otherwise, to reach some γ. Second, append some composition of L to γ to reach λ. The intermediate object γ must be either $\mu^*=(\mu_1,\ldots,\mu_{k-1})$ or $(\mu_1,\ldots,\mu_{k-1},c)$ where $0<c<\mu_k$. The first object has sign $+1$ and may be used if and only if μ^* is a prefix of λ. The second object has sign -1 and may be used if and only if μ^* is a prefix of λ and $\lambda_k=c$.

In the case where $\lambda = \mu$, we cannot use the second object since $c < \mu_k = \lambda_k$. But the first object works since $\mu^* = \lambda^*$ is a prefix of λ. In this case, the left side of (21) is 1, as needed. In the case where $\lambda \neq \mu$ and μ^* is not a prefix of λ, the sum on the left side of (21) is vacuous and both sides are 0. Finally, consider the case where $\lambda \neq \mu$ and μ^* is a prefix of λ. Here $c = \lambda_k$ must satisfy $0 < c < \mu_k$, since $\lambda_k = \mu_k$ would contradict $\lambda \neq \mu$, while $c = 0$ or $\lambda_k > \mu_k$ would contradict $\lambda, \mu \in C(n)$. In this case, $G(\lambda, \mu) = \{\mu^*, (\mu^*, \lambda_k)\}$. So both sides of (21) are 0, as needed. □

To illustrate the final case of the proof, take $\lambda = (4, 1, 3, 2, 1, 3)$ and $\mu = (4, 1, 3, 6)$. Then $G(\lambda, \mu) = \{(4, 1, 3), (4, 1, 3, 2)\}$. On the other hand, $G(\mu, \mu) = \{(4, 1, 3)\}$.

5.4 Weighted Variant of the Inversion Result

This subsection proves a weighted variation of the inversion result in Theorem 25. After rearranging some signs, we obtain the transition matrices between the bases $(\mathbf{h}_\alpha)$ and $(\boldsymbol{\psi}_\alpha)$ of NSym, as stated in (18) of Remark 24.

Suppose $\alpha, \beta \in C(n)$ and $\beta \leqslant \alpha$, so CBT(α, β) contains a unique object T. For $1 \leqslant i \leqslant \ell(\alpha)$, let $\beta^{(i)} \in C(\alpha_i)$ be the list of consecutive parts in β whose associated bricks appear in T in row i of dg(α). For any composition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$, define $Z_\gamma = \gamma_1(\gamma_1 + \gamma_2) \cdots (\gamma_1 + \gamma_2 + \cdots + \gamma_s)$ as in §4.2. When $\beta \leqslant \alpha$, define $Z_{\alpha,\beta} = \prod_{i=1}^{\ell(\alpha)} Z_{\beta^{(i)}}$ and $L_{\alpha,\beta} = \prod_{i=1}^{\ell(\alpha)} L(\beta^{(i)})$. Thus $Z_{\alpha,\beta}$ is the product of the Z-factors arising from the compositions in each row of T, while $L_{\alpha,\beta}$ is the product of the lengths of the last bricks in each row of T. In Example 22, where $\alpha = (4, 5, 5, 3)$ and $\beta = (3, 1, 3, 2, 5, 1, 2)$, we compute

$$Z_{\alpha,\beta} = Z_{(3,1)} Z_{(3,2)} Z_{(5)} Z_{(1,2)} = (3 \cdot 4) \cdot (3 \cdot 5) \cdot (5) \cdot (1 \cdot 3) = 2700$$

and $L_{\alpha,\beta} = 1 \cdot 2 \cdot 5 \cdot 2 = 20$.

Theorem 26. *For $\lambda, \beta, \mu \in C(n)$, define*

$$A_n(\lambda, \beta) = \chi(\lambda \leqslant \beta) L_{\beta,\lambda} \quad \text{and} \quad B_n(\beta, \mu) = (-1)^{\ell(\beta)-\ell(\mu)} \chi(\beta \leqslant \mu) Z_{\mu,\beta}^{-1}.$$

The matrices A_n and B_n are inverses of each other.

Proof. We adapt the proof of Theorem 25 by incorporating the extra weight factors in the entries of A_n and B_n. First consider the recursion (19). Say we remove the last row of a CBT T of shape β and content λ to get a CBT T^* of shape β^* and content γ. The last bricks of T and T^* are the same except T has a last brick of size $L(\lambda)$ in its final row of length $L(\beta)$. Therefore, $L_{\beta,\lambda} = L_{\beta^*,\gamma} L(\lambda)$. So recursion (19) becomes

$$A(\lambda, \beta) = \sum_{\gamma \in S(\lambda, L(\beta))} L(\lambda) A(\gamma, \beta^*), \tag{22}$$

which is an instance of (1) with $\mathrm{wt}_A(\lambda, \gamma) = L(\lambda)$.

Next consider the recursion (20). Say we remove the last brick in the last row of a CBT T of shape μ and content β to get a CBT T^* of shape δ and content β^*. All the factors in $Z_{\mu,\beta}$ and Z_{δ,β^*} are the same except the final row of T contributes an extra factor $L(\mu)$ compared to T^*. In other words, $Z_{\mu,\beta} = Z_{\delta,\beta^*}L(\mu)$. The sign analysis is the same as before, so we get

$$B(\beta,\mu) = \sum_{\delta\in T(\mu,L(\beta))} \mathrm{sgn}(\mu,\delta)L(\mu)^{-1}B(\beta^*,\delta), \tag{23}$$

which is an instance of (2) with $\mathrm{wt}_B(\mu,\delta) = \mathrm{sgn}(\mu,\delta)L(\mu)^{-1}$.

Finally, taking the new weights into account, the local identity (21) becomes

$$\sum_{\gamma\in G(\lambda,\mu)} \frac{L(\lambda)}{L(\mu)}\mathrm{sgn}(\mu,\gamma) = \chi(\lambda=\mu). \tag{24}$$

This reduces to (21) when $\lambda = \mu$. When $\lambda \neq \mu$, the factor $L(\lambda)/L(\mu)$ can be brought outside the sum, and the result again follows from (21). Theorem 26 follows from (24) and Theorem 2. □

6 Application 4: Brick Tabloids

For our fourth application, we will prove inversion results for rectangular matrices that count certain weighted brick tabloids. These are relevant to the transition matrices between the monomial basis and power-sum basis of symmetric functions. In this case, the local identities (4) can be informally summarized by the slogan: "A local inverse of a brick of size L is a partition of L (with suitable signs and weights)."

6.1 Background on Brick Tabloids

In this application, we often view integer partitions as multisets of parts. For each partition λ and $i > 0$, define the *multiplicity* $m_i(\lambda)$ to be the number of times i appears as a part in λ. For partitions λ and μ, let $\lambda \uplus \mu$, $\lambda \cap \mu$, and $\lambda \setminus \mu$ be the partitions obtained by applying the standard set operations to λ and μ viewed as multisets. More precisely, for each i, we have $m_i(\lambda \uplus \mu) = m_i(\lambda) + m_i(\mu)$, $m_i(\lambda \cap \mu) = \min(m_i(\lambda), m_i(\mu))$, and $m_i(\lambda \setminus \mu) = \max(m_i(\lambda) - m_i(\mu), 0)$. Let $\lambda \subseteq \mu$ mean $m_i(\lambda) \leqslant m_i(\mu)$ for all $i > 0$. Let $i \in \mu$ mean $m_i(\mu) > 0$; that is, i appears as a part of μ.

Example 27. For $\lambda = (4,2,2,1,1,1)$ and $\mu = (3,2,1,1)$, $\lambda \uplus \mu = (4,3,2,2,2,1,1,1,1,1)$, $\lambda \cap \mu = (2,1,1)$, $\lambda \setminus \mu = (4,2,1)$ and $\mu \setminus \lambda = (3)$. Here, and in general, we have $\lambda = (\lambda \setminus \mu) \uplus (\lambda \cap \mu)$ and $\mu = (\mu \setminus \lambda) \uplus (\lambda \cap \mu)$.

Following [2], we define an *ordered brick tabloid* (OBT) of shape $\lambda \in P(n)$ and content $\beta \in C(n)$ to be a filling of $\mathrm{dg}(\lambda)$ such that each label i appears in β_i cells, all in the same row, and labels in each row weakly increase from left to right. Let $\mathrm{OBT}(\lambda,\beta)$ be the set of such objects, which may be viewed as tilings of the diagram of λ using labeled horizontal bricks of lengths $\beta_1, \beta_2, \ldots$.

Example 28. For $\beta = (2, 1, 1, 3)$ and $\lambda = (4, 3)$, the three ordered brick tabloids in $\mathrm{OBT}(\lambda, \beta)$ are shown here.

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 4&4&4\\ \cline{1-3}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 2&4&4&4\\ \hline 1&1&3\\ \cline{1-3}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 3&4&4&4\\ \hline 1&1&2\\ \cline{1-3}\end{array}$$

Recall the set $\mathrm{CBT}(\beta, \alpha)$ of compositional brick tabloids of shape $\beta \in C(n)$ and content $\alpha \in C(n)$, which we defined in §5.1. For $\beta \in C(n)$ and $\mu \in P(n)$, define the set $\mathrm{BT}(\beta, \mu)$ of *brick tabloids of shape β and type μ* to consist of all CBTs of shape β and content α where $\mathrm{sort}(\alpha) = \mu$. For $T \in \mathrm{CBT}(\beta, \alpha)$, define $\mathrm{wt}(T) = L_{\beta,\alpha}$, which is the product of the lengths of the last (rightmost) brick in each row. For $\beta \in C(n)$ and $\mu \in P(n)$, define $w_{\beta,\mu} = \sum_{T\in\mathrm{BT}(\beta,\mu)} \mathrm{wt}(T)$. In the special case $\beta = (n)$, define $W_\mu = w_{(n),\mu} = \sum_\alpha L(\alpha)$, where we sum over all $\alpha \in C(n)$ with $\mathrm{sort}(\alpha) = \mu$. Note that $W_{(n)} = n$.

Example 29. Let $\beta = (3, 1, 2)$ and $\mu = (2, 2, 1, 1)$. The set $\mathrm{BT}(\beta, \mu)$ contains the two objects shown here.

$$\begin{array}{|c|c|c|}\hline 1&2&2\\ \hline 3\\ \cline{1-1} 4&4\\ \cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 3\\ \cline{1-1} 4&4\\ \cline{1-2}\end{array}$$

The objects have weights $2 \cdot 1 \cdot 2 = 4$ and $1 \cdot 1 \cdot 2 = 2$, respectively, so $w_{\beta,\mu} = 6$.

Lemma 30. *(a) For all $\mu \in P(n)$ such that $\mu \neq (n)$, $W_\mu = \sum\limits_{i\in\mu} W_{\mu\setminus(i)}$.*

(b) For all partitions μ, $W_\mu = \frac{|\mu|}{\ell(\mu)}\binom{\ell(\mu)}{m_1(\mu),m_2(\mu),\ldots}$.

Proof. To prove (a), fix $\mu \in P(n)$ with $\mu \neq (n)$. Let S be the set of $\alpha \in C(n)$ with $\mathrm{sort}(\alpha) = \mu$. For each i appearing as a part of μ, let $S_i = \{\alpha \in S : \alpha_1 = i\}$, and let T_i be the set of $\gamma \in C(n - i)$ with $\mathrm{sort}(\gamma) = \mu \setminus (i)$. The map $\gamma \mapsto (i, \gamma)$ is a bijection from T_i onto S_i that preserves the last part. Also S is the disjoint union of the sets S_i. Therefore,

$$W_\mu = \sum_{\alpha\in S} L(\alpha) = \sum_{i\in\mu}\sum_{\alpha\in S_i} L(\alpha) = \sum_{i\in\mu}\sum_{\gamma\in T_i} L(\gamma) = \sum_{i\in\mu} W_{\mu\setminus(i)}.$$

We give a bijective proof of (b) in the rearranged form $\ell(\mu)W_\mu = |\mu|\binom{\ell(\mu)}{m_1(\mu),m_2(\mu),\ldots}$. Write $n = |\mu|$. The left side counts the set S of fillings of a row (n) with a rearrangement of the parts of μ (the bricks) where one brick has been marked (accounting for $\ell(\mu)$) and one cell in the rightmost brick has been marked (accounting for the way brick tabloids are weighted in W_μ). The right side counts the set T of fillings of a row (n) with a rearrangement of the parts of μ where one of the n cells has been marked. Define a bijection $g : T \to S$ as follows. For $t \in T$, suppose the marked cell of t lies in brick b. Interchange b with the last brick in t, moving the marked cell along with b, and also mark the brick (formerly the last brick) that got switched with t. This defines $g(t)$. To get $g^{-1}(s)$ where $s \in S$, find the marked brick in s and interchange it with the last brick in s (which has one of its cells marked). In the following example with $\mu = (3, 3, 2, 2, 1)$, we mark the cell in t and $g(t)$ by $*$ and mark the switched brick in $g(t)$ by gray shading.

$$g\left(\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2^*&2&3&4&4&5&5&5\\ \hline\end{array}\right) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&\colorbox{lightgray}{2}&\colorbox{lightgray}{2}&\colorbox{lightgray}{2}&3&4&4&5^*&5\\ \hline\end{array}$$ □

In this fourth application of the general setup of §2.2, we let A_n be the $P(n) \times C(n)$ matrix with entries

$$A_n(\lambda, \beta) = |\,\mathrm{OBT}(\lambda, \beta)| = \sum_{T \in \mathrm{OBT}(\lambda,\beta)} 1 \qquad \text{for } \lambda \in P(n),\ \beta \in C(n). \tag{25}$$

For $\beta = (\beta_1, \ldots, \beta_s) \in C(n)$, let $Z_\beta = \beta_1(\beta_1+\beta_2)(\beta_1+\beta_2+\beta_3)\cdots(\beta_1+\beta_2+\cdots+\beta_s)$ as in §4.2. We will prove that A_n has inverse B_n with entries given by

$$B_n(\beta, \mu) = (-1)^{\ell(\mu)-\ell(\beta)} \frac{w_{\beta,\mu}}{Z_\beta} \qquad \text{for } \beta \in C(n),\ \mu \in P(n). \tag{26}$$

Remark 31. It can be shown (cf. [2]) that $A_n(\lambda, \beta)$ is the coefficient of m_λ in the monomial expansion of the power-sum symmetric function p_β. Since $\mathrm{sort}(\alpha) = \mathrm{sort}(\beta)$ implies $p_\alpha = p_\beta$, the matrix A_n satisfies the sorting condition of Remark 3. Thus we get a formula for the power-sum expansion of monomial symmetric functions by converting B_n in (26) to the square matrix B'_n defined in (6). If $\beta, \gamma \in C(n)$ both sort to a given $\nu \in P(n)$, then $w_{\beta,\mu} = w_{\gamma,\mu}$, as can be seen by permuting the rows of brick tabloids of shape β to get brick tabloids of shape γ. Using this and Lemma 20(c), we obtain

$$B'_n(\nu, \mu) = (-1)^{\ell(\mu)-\ell(\nu)} \frac{w_{\nu,\mu}}{z_\nu} \qquad \text{for all } \nu, \mu \in P(n).$$

Example 32. For $n = 4$, the matrices A_n and B_n are as follows.

A_4:

	4	31	22	211	13	121	112	1111
4	1	1	1	1	1	1	1	1
31	0	1	0	2	1	2	2	4
22	0	0	2	2	0	2	2	6
211	0	0	0	2	0	2	2	12
1111	0	0	0	0	0	0	0	24

B_4:

	4	31	22	211	1111
4	1	−1	−1/2	1	−1/4
31	0	1/4	0	−1/4	1/12
22	0	0	1/2	−1/2	1/8
211	0	0	0	1/12	−1/24
13	0	3/4	0	−3/4	1/4
121	0	0	0	1/6	−1/12
112	0	0	0	1/4	−1/8
1111	0	0	0	0	1/24

The square versions are shown here.

A'_4:

	4	31	22	211	1111
4	1	1	1	1	1
31	0	1	0	2	4
22	0	0	2	2	6
211	0	0	0	2	12
1111	0	0	0	0	24

B'_4:

	4	31	22	211	1111
4	1	−1	−1/2	1	−1/4
31	0	1	0	−1	1/3
22	0	0	1/2	−1/2	1/8
211	0	0	0	1/2	−1/4
1111	0	0	0	0	1/24

6.2 Recursion for A_n

Fix a partition $\lambda \in P(n)$ and a positive integer L. Let $S(\lambda, L)$ be the set of partitions γ that can be obtained by replacing one part $i \geqslant L$ in λ by $i - L$ and sorting the parts. We can identify i given λ and γ as the sole member of the multiset $\lambda \setminus \gamma$. Define $\mathrm{wt}_A(\lambda, \gamma) = m_i(\lambda) = m_{\lambda\setminus\gamma}(\lambda)$ to be the number of times the part value i that got replaced appears in λ. In this setting, the general recursion (1) takes the following form.

Lemma 33. *For all $\lambda \in P(n)$ and $\beta \in C(n)$,*

$$|\operatorname{OBT}(\lambda, \beta)| = \sum_{\gamma \in S(\lambda, L(\beta))} m_{\lambda\setminus\gamma}(\lambda)\, |\operatorname{OBT}(\gamma, \beta^*)|\,. \tag{27}$$

Proof. Informally, we obtain the recursion by removing the largest-labeled brick in an OBT of shape λ and content β. However, there is some extra complexity since we must sort the new diagram so it still has partition shape. Formally, we proceed by defining a bijection $F : \operatorname{OBT}(\lambda, \beta) \to \bigcup_{\gamma \in S(\lambda, L(\beta))} [\mathrm{wt}_A(\lambda, \gamma)] \times \operatorname{OBT}(\gamma, \beta^*)$. Write $\beta = (\beta_1, \ldots, \beta_s) = (\beta^*, L)$ with $L = L(\beta)$. Given an input $T \in \operatorname{OBT}(\lambda, \beta)$, the brick with largest label s and size L must be at the end of some row of $\operatorname{dg}(\lambda)$ of some length $i \geqslant L$. Suppose the brick appears in the kth highest row of length i in the diagram, where $1 \leqslant k \leqslant m_i(\lambda)$. Delete this brick and its associated cells, and move the truncated row (along with the remaining bricks in it) to become the highest row of length $i - L$ in the new diagram. The new diagram is an OBT T^* of some shape $\gamma \in S(\lambda, L)$ such that $\mathrm{wt}_A(\lambda, \gamma) = m_i(\lambda)$. Define $F(T) = (k, T^*)$.

To invert F, start with any (k, T^*) in the codomain of F, where T^* has shape $\gamma \in S(\lambda, L)$. Here, i must be the unique part in $\lambda \setminus \gamma$, and γ must have at least one part of size $i - L$. Put a brick of label s and size L at the end of the highest row of size $i - L$ in T^*. Move this enlarged row up to become the kth highest row among the rows of length i in a new OBT that has shape λ and content β. It is routine to check that these steps define the two-sided inverse of F. ☐

Example 34. For $\lambda = (3, 3, 2)$ and $\beta = (2, 1, 3, 2)$, the set $S(\lambda, L(\beta))$ contains two partitions: $\gamma = (3, 2, 1)$ with $\mathrm{wt}_A(\lambda, \gamma) = m_3(\lambda) = 2$, and $\delta = (3, 3)$ with $\mathrm{wt}_A(\lambda, \delta) = m_2(\lambda) = 1$. The four objects in $\operatorname{OBT}(\lambda, \beta)$ are shown here.

$$T_1 = \begin{array}{lll} 2 & 4 & 4 \\ 3 & 3 & 3 \\ 1 & 1 & \end{array}, \quad T_2 = \begin{array}{lll} 3 & 3 & 3 \\ 2 & 4 & 4 \\ 1 & 1 & \end{array}, \quad T_3 = \begin{array}{lll} 1 & 1 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & \end{array}, \quad T_4 = \begin{array}{lll} 3 & 3 & 3 \\ 1 & 1 & 2 \\ 4 & 4 & \end{array}$$

On the other hand, $\operatorname{OBT}(\gamma, \beta^*) = \{T_5\}$ and $\operatorname{OBT}(\delta, \beta^*) = \{T_6, T_7\}$, where

$$T_5 = \begin{array}{lll} 3 & 3 & 3 \\ 1 & 1 & \\ 2 & & \end{array}, \quad T_6 = \begin{array}{lll} 1 & 1 & 2 \\ 3 & 3 & 3 \end{array}, \quad T_7 = \begin{array}{lll} 3 & 3 & 3 \\ 1 & 1 & 2 \end{array}.$$

In this example, the bijection F acts as follows:

$$F(T_1) = (1, T_5), \quad F(T_2) = (2, T_5), \quad F(T_3) = (1, T_6), \quad F(T_4) = (1, T_7).$$

6.3 Recursion for $\boldsymbol{B_n}$

For $\mu \in P(n)$ and $L > 0$, define $T(\mu, L) = \{\delta \in P(n - L) : \delta \subseteq \mu\}$, which is the set of partitions of $n - L$ that can be obtained from μ by removing some parts that sum to L.

In other words, for each $\delta \in T(\mu, L)$, $\epsilon = \mu \setminus \delta$ is a partition of L with $\epsilon \subseteq \mu$. For such δ, define

$$\mathrm{wt}_B(\mu, \delta) = \frac{(-1)^{\ell(\mu)-\ell(\delta)-1} W_{\mu\setminus\delta}}{|\mu|}. \tag{28}$$

In this setting, to verify the general recursion (2), we must prove: for $\beta \in C(n)$ and $\mu \in P(n)$,

$$(-1)^{\ell(\mu)-\ell(\beta)} \frac{w_{\beta,\mu}}{Z_\beta} = \sum_{\delta \in T(\mu, L(\beta))} (-1)^{\ell(\mu)-\ell(\delta)-1+\ell(\delta)-\ell(\beta^*)} \frac{W_{\mu\setminus\delta} w_{\beta^*,\delta}}{|\mu| Z_{\beta^*}}. \tag{29}$$

The power of -1 is the same on both sides since $\ell(\beta) = \ell(\beta^*) + 1$. The denominators match as well (independent of δ) since $|\mu| Z_{\beta^*} = n Z_{\beta^*} = |\beta| Z_{\beta^*} = Z_\beta$. Since $\mu \setminus \delta$ is a partition of $L(\beta)$, we are reduced to checking

$$w_{\beta,\mu} = \sum_{\delta \in T(\mu, L)} w_{(L),\mu\setminus\delta} w_{\beta^*,\delta} \quad \text{where } L = L(\beta). \tag{30}$$

This follows by mapping a brick tabloid T counted by $w_{\beta,\mu}$ to the pair of brick tabloids (T', T^*), where T' is the last row of T and T^* consists of all higher rows of T. Note that T^* is a brick tabloid of shape β^* and type δ for some $\delta \in T(\mu, L)$, whereas T' must then be a brick tabloid of shape (L) and type $\mu \setminus \delta$. Taking products of the lengths of the rightmost bricks, we have $\mathrm{wt}(T) = \mathrm{wt}(T')\,\mathrm{wt}(T^*)$. Thus the map $T \mapsto (T', T^*)$ is a weight-preserving bijection, and (30) follows.

6.4 Proof of the Local Identity

In this application, the local identity (4) takes the following form.

Theorem 35. *Given $n > 0$ and partitions $\lambda, \mu \in P(n)$, let $G(\lambda, \mu)$ be the set of partitions γ that can be obtained by decreasing one part of λ by some $L > 0$ (then sorting) and by removing one or more parts from μ. Then*

$$\sum_{\gamma \in G(\lambda,\mu)} \frac{m_{\lambda\setminus\gamma}(\lambda)(-1)^{\ell(\mu)-\ell(\gamma)-1} W_{\mu\setminus\gamma}}{n} = \chi(\lambda = \mu). \tag{31}$$

Therefore (by Theorem 2), the rectangular matrix A_n of OBT counts has a right-inverse B_n defined by (26).

Proof. First consider the case $\lambda = \mu$. For each part $i \in \lambda$, let $\gamma^i = \lambda \setminus (i)$ be λ with a single copy of the part i deleted. Evidently $\gamma^i \in G(\lambda, \lambda)$ for all $i \in \lambda$, and we claim these are the only elements in $G(\lambda, \lambda)$. To see why, fix $\gamma \in G(\lambda, \lambda)$. On one hand, we obtain γ from λ by decreasing some $\lambda_j = i$ by some $L > 0$. On the other hand, $\ell(\gamma) < \ell(\mu) = \ell(\lambda)$. If $L < i$, then we would have $\ell(\gamma) = \ell(\lambda)$, which is impossible. So $L = i$ and $\gamma = \gamma^i$. The left side of (31) becomes $\sum_{i\in\lambda} \frac{m_i(\lambda)(-1)^{1-1} W_{(i)}}{n} = \sum_{i\in\lambda} \frac{i m_i(\lambda)}{n} = 1$, since $|\lambda| = n$.

For the case $\lambda \neq \mu$, identity (31) holds vacuously when $G(\lambda, \mu) = \emptyset$, so assume $G(\lambda, \mu)$ is nonempty. This means there is a way to go from λ to μ by first subtracting some $L > 0$ from a single part in λ to get γ, then merging some partition ν of L with γ to reach μ. In this situation, we claim $\lambda \setminus \mu$ must equal the single-part partition (i) for some i. On one hand, $\lambda \setminus \mu$ cannot be empty since $|\lambda| = |\mu|$ and $\lambda \neq \mu$. On the other hand, suppose the multiset $\lambda \setminus \mu$ contained two parts a and b (where $a = b$ could occur). Decrementing the part a by L and merging with ν would produce a partition with too many copies of b (compared to μ). Decrementing the part b by L and merging with ν would produce a partition with too many copies of a (compared to μ). This proves the claim.

With the claim $\lambda \setminus \mu = (i)$ established, we can describe all elements of $G(\lambda, \mu)$. Write $\mu \setminus \lambda = \rho$, which must be a partition of i different from (i) since $|\lambda| = |\mu|$ but $\lambda \neq \mu$. To get an element of $G(\lambda, \mu)$, we must decrease one of the copies of i in λ by some amount L since that is the only way to get rid of the extra copy of i. We could choose $L = i$, leading to $\gamma = \lambda \setminus (i) = \lambda \cap \mu \in G(\lambda, \mu)$, then merge γ with ρ to reach μ. Or we could choose any $L < i$ such that $i - L \in \rho$, leading to $\gamma = (\lambda \cap \mu) \uplus (i - L) \in G(\lambda, \mu)$, then merge γ with the remaining parts in ρ to reach μ. Putting all of this information into the left side of (31), the numerator becomes

$$m_i(\lambda)(-1)^{\ell(\rho)-1} W_\rho + \sum_{L:\ i-L \in \rho} m_i(\lambda)(-1)^{\ell(\rho)-2} W_{\rho \setminus (i-L)}$$

$$= m_i(\lambda)(-1)^{\ell(\rho)-1} \left[W_\rho - \sum_{j \in \rho} W_{\rho \setminus (j)} \right].$$

By Lemma 30(a), the term in brackets is zero, so (31) holds in this case. □

Example 36. For $\lambda = (5, 2, 2, 1)$ and $\mu = (3, 2, 2, 1, 1, 1)$, we have $\lambda \setminus \mu = (5)$, $\rho = \mu \setminus \lambda = (3, 1, 1)$, and $G(\lambda, \mu) = \{(2, 2, 1), (3, 2, 2, 1), (2, 2, 1, 1)\}$. The left side of (31) is

$$\frac{1(-1)^2 W_{(3,1,1)} + 1(-1)^1 W_{(1,1)} + 1(-1)^1 W_{(3,1)}}{10} = \frac{5 - 1 - 4}{10} = 0.$$

7 Automatically Constructing Bijective Proofs of $AB = I$

Returning to our general framework, it is often the case that the entries in the matrices A_n and B_n count signed, weighted collections of combinatorial objects. Suppose we have bijective proofs of recursion (1) for the entries of A_n, recursion (2) for the entries of B_n, and the local identities (4). Then we can convert the algebraic proof of Theorem 2 to a bijective proof of the matrix identities $A_n B_n = I_{R(n)}$. We sketch the general approach here and provide two concrete examples (revisiting Applications 1 and 2) in the following subsections.

Suppose $A(\lambda, \beta)$ is the sum of signed, weighted objects in a set $\mathcal{A}_{\lambda,\beta}$ for each $\lambda \in R(n)$ and $\beta \in C(n)$. Suppose $B(\beta, \mu)$ is the sum of signed, weighted objects in a set $\mathcal{B}_{\beta,\mu}$ for each $\beta \in C(n)$ and $\mu \in R(n)$. By definition of matrix multiplication, the λ, μ-entry of AB

is the signed, weighted sum of all ordered pairs (S, T) where $S \in \mathcal{A}_{\lambda,\beta}$ for some $\beta \in C(n)$, and $T \in \mathcal{B}_{\beta,\mu}$ for the same choice of β. Let $\mathcal{P}_{\lambda,\mu}$ be the set of all such pairs. We seek to construct sign-reversing, weight-preserving involutions $\mathcal{I}_{\lambda,\mu} : \mathcal{P}_{\lambda,\mu} \to \mathcal{P}_{\lambda,\mu}$ where $\mathcal{I}_{\lambda,\mu}$ has no fixed points for all $\lambda \neq \mu$ in $R(n)$, and $\mathcal{I}_{\lambda,\lambda}$ has exactly one fixed point (with signed weight $+1$) for all $\lambda \in R(n)$.

Tracing through the argument leading to (5) produces the following informal recursive description of $\mathcal{I}_{\lambda,\mu}$. Given $(S, T) \in \mathcal{P}_{\lambda,\mu}$ with $S \in \mathcal{A}_{\lambda,\beta}$ and $T \in \mathcal{B}_{\beta,\mu}$ for some $\beta = (\beta^*, L(\beta))$, use the known bijective proofs of (1) and (2) to remove some local structures from S and T to get smaller objects $S^* \in \mathcal{A}_{\gamma,\beta^*}$ and $T^* \in \mathcal{B}_{\beta^*,\delta}$ for some $\gamma, \delta \in R(n-L(\beta))$. In the case $\gamma \neq \delta$, recursively apply $\mathcal{I}_{\gamma,\delta}$ to (S^*, T^*) to find the matching object (S', T') of opposite sign. In the new object, β^* may be replaced by some new composition $\beta' \in C(n - L(\beta))$. Next, restore to S' and T' the same local structures that were removed from S and T to reach the final output $\mathcal{I}_{\lambda,\mu}(S, T)$.

In the case $\gamma = \delta$, we may sometimes still be able to recursively apply $\mathcal{I}_{\gamma,\delta}$ to obtain a cancellation as in the first case. However, it may be that (S^*, T^*) is the fixed point for $\mathcal{I}_{\gamma,\delta}$. There are two subcases here. If $\lambda \neq \mu$, then we use the given bijective proof of the local identity to obtain a matching object of opposite sign that will cancel with (S, T). If $\lambda = \mu$, the local bijection either yields another cancellation or tells us that (S, T) is the fixed point for $\mathcal{I}_{\lambda,\lambda}$.

Frequently, we can unroll the recursive description of $\mathcal{I}_{\lambda,\mu}$ to obtain an iterative algorithm to compute this map. Suppose we can explicitly describe the unique fixed point for each $\mathcal{I}_{\lambda,\lambda}$; call this fixed point the *survivor of shape* λ. Given a non-survivor $(S, T) \in \mathcal{P}_{\lambda,\mu}$, repeatedly strip off the last local structures from S and T until reaching an intermediate object that is the survivor for some shape $\gamma \in R(k)$. Apply the bijective proof of the local identity, using the most recently removed local structure to determine what λ and μ to use in (4). This bijection replaces the survivor of shape γ (and the next local structure just outside it) by the survivor of some shape $\tilde{\gamma} \in R(k)$ (and suitable new local structure just outside it). To finish, act on this object by restoring the rest of the local structures that were removed from S and T to get the object $\mathcal{I}_{\lambda,\mu}(S, T)$ that cancels with (S, T) in $\mathcal{P}_{\lambda,\mu}$.

Sometimes (as in Application 2 below), this approach can be generalized to the case where there is more than one surviving fixed point in $\mathcal{I}_{\lambda,\lambda}$. This situation might occur if we need to rescale one of the matrices A_n or B_n to ensure all entries are integers.

7.1 Application 1: Bijective Inversion of the Kostka Matrix

We illustrate the general construction by developing a bijective proof of $A_n B_n = I_{P(n)}$, where A_n is the rectangular Kostka matrix from Section 3. In this application, $\mathcal{A}_{\lambda,\beta}$ is the (unsigned, unweighted) set $\mathrm{SSYT}(\lambda, \beta)$ of semistandard Young tableaux of shape λ and content β; and $\mathcal{B}_{\beta,\mu}$ is the (signed, unweighted) set $\mathrm{SRHT}(\mu, \beta)$ of special rim-hook tableaux of shape μ and content β. The set $\mathcal{P}_{\lambda,\mu}$ consists of all pairs (S, T) where $S \in \mathrm{SSYT}(\lambda, \beta)$ and $T \in \mathrm{SRHT}(\mu, \beta)$ for some composition β. For each partition λ, let S_λ be the filling of $\mathrm{dg}(\lambda)$ where every cell in row i contains the value i. The object S_λ belongs to both $\mathrm{SSYT}(\lambda, \lambda)$ and $\mathrm{SRHT}(\lambda, \lambda)$ and has positive sign. It is routine to check that $\mathcal{P}_{\lambda,\lambda}$ consists of the sole object (S_λ, S_λ), which is the survivor for shape λ in this application. For example, the survivor for $\lambda = (5, 3, 3, 2)$ is

$$\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} \end{array} \; , \; \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} \end{array} \right).$$

The involution $\mathcal{I}_{\lambda,\mu}$ acts on input $(S, T) \in \mathcal{P}_{\lambda,\mu}$ as follows. If $\lambda = \mu$ and (S, T) is the survivor for λ, then this is a fixed point. Otherwise let $\beta = (\beta_1, \ldots, \beta_\ell)$ be the content composition for S and T. Remove the outermost horizontal strip from S (meaning the β_ℓ cells in the filling of $\mathrm{dg}(\lambda)$ labeled ℓ), and remove the last special rim-hook from T (meaning the β_ℓ cells in the filling of $\mathrm{dg}(\mu)$ labeled ℓ). Do this repeatedly until reaching a survivor object of some shape γ, where γ might be empty. Suppose this survivor was reached after removing the cells labeled k from S (these cells forming a horizontal strip η of size β_k) and removing the cells labeled k from T (these cells forming a signed special rim-hook ρ of size β_k). Let $\overline{\lambda}$ be the partition consisting of the cells in γ and η, and let $\overline{\mu}$ be the partition consisting of the cells in γ and ρ. Then the set $G(\overline{\lambda}, \overline{\mu})$ (defined in Theorem 10) is nonempty. The proof of that theorem shows that $G(\overline{\lambda}, \overline{\mu})$ consists of exactly two oppositely-signed objects and describes how to go from one of these objects to the other. Let $\tilde{\gamma}$ be the other object appearing with γ in $G(\overline{\lambda}, \overline{\mu})$. To make $(\tilde{S}, \tilde{T}) = \mathcal{I}_{\lambda,\mu}(S, T)$, start with the unique survivor of shape $\tilde{\gamma}$. Using the next label $\tilde{k}$ not appearing in $\tilde{\gamma}$, add a horizontal strip to $\tilde{S}$ to reach $\overline{\lambda}$ and add a special rim-hook to $\tilde{T}$ to reach $\overline{\mu}$ (which is possible by definition of $G(\overline{\lambda}, \overline{\mu})$). Then restore the previously removed horizontal strips and special rim-hooks of lengths $\beta_{k+1}, \ldots, \beta_\ell$ to the fillings $\tilde{S}$ and $\tilde{T}$ (respectively) in the same positions they occupied in S and T. Here the labels (originally $k+1, k+2, \ldots$) of the restored structures get renumbered to be $\tilde{k}+1, \tilde{k}+2, \ldots$.

Example 37. Here are three examples of the action of $\mathcal{I}_{\lambda,\mu}$. First,

$$\left(\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline 4 & 4 \\ \cline{1-2} \end{array} \; , \; \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 4 \\ \hline 4 & 4 \\ \hline \end{array} \right) \overset{\mathcal{I}_{\lambda,\mu}}{\longmapsto} \left(\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2} \end{array} \; , \; \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 2 & 3 \\ \hline 3 & 3 \\ \hline \end{array} \right). \tag{32}$$

In this example, we remove the 4s from the input objects and get a non-survivor. Then we remove the 3s from both objects and reach the survivor for shape $\gamma = (2, 2)$. We compute $\overline{\lambda} = (3, 2)$, $\overline{\mu} = (2, 2, 1)$, $\tilde{\gamma} = (2)$ from the proof of Theorem 10. Starting with the survivor for $\tilde{\gamma}$, we fill $\mathrm{dg}(\overline{\lambda}) \setminus \mathrm{dg}(\tilde{\gamma})$ with a horizontal strip of three 2s, and we fill $\mathrm{dg}(\overline{\mu}) \setminus \mathrm{dg}(\tilde{\gamma})$ with a special rim-hook of three 2s. We then restore the last horizontal strip and special rim-hook, renumbered to contain 3s instead of 4s. The original content $\beta = (2, 2, 1, 3)$ changes to new content $\tilde{\beta} = (2, 3, 3)$.

For our second example, we compute

$$\left(\begin{array}{ccccccc} 1&1&1&1&1&4&4\\ 2&2&2&4&4&6&6\\ 3&3&3&5&&&\\ 4&4&6&&&& \end{array}, \begin{array}{ccccc} 1&1&1&1&1\\ 2&2&2&4&\\ 3&3&3&4&\\ 4&4&4&4&\\ 5&6&&&\\ 6&6&&& \end{array} \right) \stackrel{\mathcal{I}_{\lambda,\mu}}{\longmapsto} \left(\begin{array}{ccccccc} 1&1&1&1&1&4&4\\ 2&2&2&2&4&6&6\\ 3&3&3&5&&&\\ 4&4&6&&&& \end{array}, \begin{array}{ccccc} 1&1&1&1&1\\ 2&2&2&2&\\ 3&3&3&4&\\ 4&4&4&4&\\ 5&6&&&\\ 6&6&&& \end{array} \right). \tag{33}$$

Here, $\beta = (5,3,3,6,1,3)$, $\gamma = (5,3,3)$, $\overline{\lambda} = (7,5,3,2)$, $\overline{\mu} = (5,4,4,4)$, $\tilde{\gamma} = (5,4,3)$, and $\tilde{\beta} = (5,4,3,5,1,3)$.

For our third example, we compute

$$\left(\begin{array}{ccccccc} 1&1&1&1&1&3&4\\ 2&2&2&3&3&6&6\\ 3&3&4&5&&&\\ 4&4&6&&&& \end{array}, \begin{array}{ccccc} 1&1&1&1&1\\ 2&2&2&3&\\ 3&3&3&3&\\ 4&4&4&4&\\ 5&6&&&\\ 6&6&&& \end{array} \right) \stackrel{\mathcal{I}_{\lambda,\mu}}{\longmapsto} \left(\begin{array}{ccccccc} 1&1&1&1&1&3&4\\ 2&2&2&2&3&6&6\\ 3&3&4&5&&&\\ 4&4&6&&&& \end{array}, \begin{array}{ccccc} 1&1&1&1&1\\ 2&2&2&2&\\ 3&3&3&3&\\ 4&4&4&4&\\ 5&6&&&\\ 6&6&&& \end{array} \right). \tag{34}$$

Here, $\beta = (5,3,5,4,1,3)$, $\gamma = (5,3)$, $\overline{\lambda} = (6,5,2)$, $\overline{\mu} = (5,4,4)$, $\tilde{\gamma} = (5,4)$, and $\tilde{\beta} = (5,4,4,4,1,3)$.

Remark 38. The involutions $\mathcal{I}_{\lambda,\mu}$ constructed here are *canonical* in the technical sense that they do not rely on arbitrary choices. On one hand, each set $\mathcal{P}_{\lambda,\lambda}$ contains a unique survivor with no cancellation needed. On the other hand, for each $\lambda \neq \mu$ such that the set $G(\lambda,\mu)$ is nonempty, this set contains exactly one positive object and exactly one negative object. Thus the involution $\gamma \leftrightarrow \tilde{\gamma}$ on this set is uniquely determined. Our general framework assembles the global maps $\mathcal{I}_{\lambda,\mu}$ from these canonical local ingredients without requiring any further choices.

Remark 39. Eğecioğlu and Remmel [3] gave a bijective proof of $K_nK_n^{-1} = I_{P(n)}$ for the square $P(n) \times P(n)$ Kostka matrices K_n. Their bijections differ from ours in several respects. First, as a notational matter, they write partition diagrams in French notation (longest row at the bottom) and use line segments rather than numbered cells to display special rim-hooks. Second, their proof uses a slightly different combinatorial model for the λ,μ-entry of $K_nK_n^{-1}$ compared to our set $\mathcal{P}_{\lambda,\mu}$. Specifically, they consider pairs (S,T) where S is a SSYT of shape λ and T is an SRHT of shape μ such that, for each $k \geqslant 1$, the horizontal strip in S filled with the label k has size equal to the length of the special rim-hook in T starting in the kth row of $\mathrm{dg}(\mu)$; this length is zero if there is no such rim-hook. In particular, the content of S is a weak composition that might have some parts equal to zero. The validity of their model rests on the fact that $|\,\mathrm{SSYT}(\lambda,\alpha)| = |\,\mathrm{SSYT}(\lambda,\beta)|$ for any weak compositions α,β that are rearrangements of each other. Their involution requires (as a subroutine) a bijection that interchanges the frequencies of i and $i+1$ in a SSYT. In contrast, our bijections for the rectangular Kostka matrices do not rely on these ingredients.

Eğecioğlu and Remmel [3, pp. 72–73] give the following example of their involution (denoted here by $\mathcal{ER}_{\lambda,\mu}$; we use our diagram conventions to facilitate comparison):

$$\left(\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline 4 & 4 \\ \cline{1-2} \end{array}\, , \; \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 4 \\ \hline 4 & 4 \\ \hline \end{array} \right) \quad \overset{\mathcal{ER}_{\lambda,\mu}}{\longleftrightarrow} \quad \left(\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 3 & 3 & 4 \\ \hline 4 & 4 \\ \cline{1-2} \end{array}\, , \; \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & 3 \\ \hline 3 & 4 \\ \hline 4 & 4 \\ \hline \end{array} \right). \tag{35}$$

We use the label k for the special rim-hook starting in row k from the top. Their output has content $(2, 0, 3, 3)$, which we do not allow. But comparing to (32), we observe that the actions of the maps $\mathcal{I}_{\lambda,\mu}$ and $\mathcal{ER}_{\lambda,\mu}$ on this input essentially agree, after we adjust for different conventions regarding the content. However, there are inputs on which the two involutions do not agree even in this extended sense. For example, $\mathcal{ER}_{\lambda,\mu}$ matches the two objects on the left sides of (33) and (34) with each other, as well as matching the two objects on the right sides.

7.2 Bijective Matrix Inversion in Application 2

Getting a bijective proof of $A_n B_n = I_{P(n)}$ in Application 2 is more challenging, since the B_n matrix defined in §4 has fractional entries. As a first step, we introduce the rescaled matrix $\overline{B}_n = n! B_n$ and prove $A_n \overline{B}_n = n! I_{P(n)}$ instead. In this case, $\mathcal{A}_n(\lambda, \beta)$ is the signed set $\mathrm{RHT}(\lambda, \beta)$ of rim-hook tableaux of shape λ and content β. Note that $\overline{B}_n(\beta, \mu) = \sum_{T \in \mathrm{RHT}(\mu,\beta)} \frac{n!}{Z_\beta} \operatorname{sgn}(T)$. Using Lemma 20(b), we may take $\mathcal{B}_n(\beta, \mu)$ to be the signed set consisting of pairs (T, σ) such that $T \in \mathrm{RHT}(\mu, \beta)$ and $\sigma \in S_n$ has $\mathrm{cycC}(\sigma) = \beta$. For $\lambda, \mu \in P(n)$, $\mathcal{P}_{\lambda,\mu}$ is the set of triples (S, T, σ) where, for some composition β, $S \in \mathrm{RHT}(\lambda, \beta)$, $T \in \mathrm{RHT}(\mu, \beta)$, $\sigma \in S_n$, and $\mathrm{cycC}(\sigma) = \beta$. The sign of (S, T, σ) is $\operatorname{sgn}(S) \operatorname{sgn}(T)$.

The involutions $\mathcal{I}_{\lambda,\mu}$ must have no fixed points when $\lambda \neq \mu$, but $\mathcal{I}_{\lambda,\lambda}$ should have $n!$ positive fixed points for all $\lambda \in P(n)$. The natural set of fixed points for $\mathcal{I}_{\lambda,\lambda}$ turns out to be the set of objects in $\mathcal{P}_{\lambda,\lambda}$ of the form (S, S, σ); call this set Surv_λ. The next theorem proves there really are $n!$ such objects. In fact, we require a bijective strengthening of this enumerative result and a variation arising in connection with the local identity (13) (rescaled by $n!$). For $n \geqslant 1$, define

$$\mathrm{CS}_n = [n] \times [n-1] \times \cdots \times [3] \times [2] \times [1] = \{(c_n, \ldots, c_2, c_1) : 1 \leqslant c_k \leqslant k \text{ for } n \geqslant k \geqslant 1\}.$$

Clearly, $|\,\mathrm{CS}_n\,| = n!$. Elements of CS_n are called *choice sequences.*

Theorem 40. *(a) For each $\lambda \in P(n)$, there is a bijection $F_\lambda : \mathrm{CS}_n \to \mathrm{Surv}_\lambda$.*
(b) Suppose we are given $\mu \in P(n)$ and a fixed (unlabeled) rim-hook ρ that may be removed from the border of $\mathrm{dg}(\mu)$ to leave a smaller partition shape. Let $\mathrm{Surv}_{\mu,\rho}$ be the set of pairs (T, σ) where T is a RHT of shape μ whose last rim hook is ρ, and $\mathrm{cycC}(\sigma)$ equals the content of T. There is a bijection $F_{\mu,\rho} : \mathrm{CS}_{n-1} \to \mathrm{Surv}_{\mu,\rho}$.

Remark 41. Part (a) gives novel combinatorial interpretations of $n!$ parametrized by the integer partitions of n. Since $\operatorname{sgn}(S) \operatorname{sgn}(S) = +1$, we can say that there are $n!$ pairs

(S,σ) such that: S is an *unsigned* RHT of shape λ and some content $\beta \in C(n)$; and $\mathrm{cycC}(\sigma) = \beta$. If we specify in advance the cells occupied by the last rim-hook in S, then part (b) says there are $(n-1)!$ such objects.

Proof. As a preliminary step, we define a numbering of the rim-hooks that can be removed from the border of the diagram of a partition $\nu \in P(k)$. We number the cells in $\mathrm{dg}(\nu)$ with $1, 2, \ldots, k$ in the reading order, working from the top row down and reading left to right in each row. In other words, the cell in row i, column j of $\mathrm{dg}(\nu)$ receives the number $\nu_1 + \cdots + \nu_{i-1} + j$. Each cell in $\mathrm{dg}(\nu)$ corresponds to a unique removable rim-hook on the border of $\mathrm{dg}(\nu)$, as explained at the start of the proof of Theorem 16. For $1 \leqslant c \leqslant k$, define the *cth border rim-hook of* ν to be that rim-hook corresponding to the cth cell of $\mathrm{dg}(\nu)$. This numbering system is a bijection from $[k]$ to the set of such border rim-hooks for ν.

Next, we construct the bijection F_λ by modifying the counting argument in the proof of Lemma 20(b). Given a choice sequence $\mathbf{c} = (c_n, \ldots, c_1) \in \mathrm{CS}_n$, we create $F_\lambda(\mathbf{c}) = (S, S, \sigma) \in \mathrm{Surv}_\lambda$ by choosing the rim-hooks in S working from the outside border inward and simultaneously filling the cycles in the canonical cycle notation for σ from right to left. To begin, remove the c_nth border rim-hook from $\mathrm{dg}(\lambda)$. This rim-hook will be the largest-labeled rim-hook in S. Say L cells were removed; then we are left with the diagram of a smaller partition $\lambda^{(1)} \in P(n-L)$. Build the rightmost cycle of σ by starting with 1, then using entries $c_{n-1}, c_{n-2}, \ldots, c_{n-(L-1)}$ from $\mathbf{c}$ to choose the remaining values in the cycle. Here and below, an entry c in the choice sequence means "fill the next position in the cycle with the cth smallest element not yet appearing in σ." To continue, remove the c_{n-L}th border rim-hook from $\mathrm{dg}(\lambda^{(1)})$, say containing L' cells, leaving the diagram of some $\lambda^{(2)}$. Build the next-rightmost cycle of σ by starting with the minimum element not used so far, then using entries $c_{n-L-1}, c_{n-L-2}, \ldots, c_{n-L-(L'-1)}$ to choose the remaining values in the cycle. Then remove the $c_{n-L-L'}$th border rim-hook from $\mathrm{dg}(\lambda^{(2)})$, and continue similarly. The passage from the choice sequence to the triple (S, S, σ) is reversible, as illustrated in the example below. So F_λ is a bijection.

Part (b) is proved in the same manner. Here we do not need $c_n \in [n]$ to choose the outermost rim-hook to remove from $\mathrm{dg}(\mu)$, since that choice has been fixed in advance to be ρ. Thus, choice sequences in CS_{n-1} suffice to construct uniquely each object in $\mathrm{Surv}_{\mu,\rho}$. □

Example 42. In this example, we compute $F_\lambda^{-1}(S, S, \sigma) = (c_{18}, c_{17}, \ldots, c_1)$, where $\lambda = (5,4,4,3,2) \in P(18)$, and S and σ are shown here:

$$S = \begin{array}{ccccc} 1 & 1 & 3 & 4 & 4 \\ 1 & 3 & 3 & 4 & \\ 2 & 3 & 4 & 4 & \\ 2 & 5 & 5 & & \\ 2 & 5 & & & \end{array}, \qquad \sigma = (9,16,14)(7,13,15)(6,11,18,12)(2,17,3,10,8)(1,4,5).$$

To recover c_{18}, we use the numbering of the cells of $\mathrm{dg}(\lambda)$ shown here:

$$\mathrm{dg}(\lambda) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 \\ \cline{1-4} 10 & 11 & 12 & 13 \\ \cline{1-4} 14 & \cellcolor{lightgray}15 & \cellcolor{lightgray}16 \\ \cline{1-3} 17 & \cellcolor{lightgray}18 \\ \cline{1-2} \end{array}$$

We see that the highest-numbered rim-hook in S (namely, the set of cells containing 5) is the 15th border-rim hook of $\mathrm{dg}(\lambda)$, so $c_{18} = 15$. To find c_{17}, we look at the second element in the rightmost cycle of σ, namely 4. The value 4 is the third-smallest available element (since 1 has already been used), so $c_{17} = 3$. The next value in this cycle, namely 5, is again the third-smallest available element (since 1 and 4 have already been used), so $c_{16} = 3$. We abbreviate the reasoning in the last two sentences as follows:

$$c_{17} = 3 \text{ via } (2, 3, \underline{4}, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18);$$
$$c_{16} = 3 \text{ via } (2, 3, \underline{5}, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18).$$

To continue, we remove the rim-hook labeled 5 to get a new partition shape $\lambda^{(1)} = (5, 4, 4, 1, 1)$ with cells numbered as follows:

$$\mathrm{dg}(\lambda^{(1)}) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & \cellcolor{lightgray}4 & \cellcolor{lightgray}5 \\ \hline 6 & 7 & 8 & \cellcolor{lightgray}9 \\ \cline{1-4} 10 & 11 & \cellcolor{lightgray}12 & \cellcolor{lightgray}13 \\ \cline{1-4} 14 \\ \cline{1-1} 15 \\ \cline{1-1} \end{array} \quad .$$

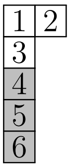
The next highest rim-hook in S (the set of cells containing 4, shaded gray in the preceding diagram) is the third border rim-hook of $\lambda^{(1)}$, so $c_{15} = 3$. We proceed by examining the cycle $(2, 17, 3, 10, 8)$ in σ, which necessarily begins with 2. Reasoning as above, we deduce:

$$c_{14} = 13 \text{ via } (3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \underline{17}, 18);$$
$$c_{13} = 1 \text{ via } (\underline{3}, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18);$$
$$c_{12} = 5 \text{ via } (6, 7, 8, 9, \underline{10}, 11, 12, 13, 14, 15, 16, 18);$$
$$c_{11} = 3 \text{ via } (6, 7, \underline{8}, 9, 11, 12, 13, 14, 15, 16, 18).$$

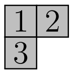
The computation continues similarly, as shown in the next figures.

$$\mathrm{dg}(\lambda^{(2)}) = \begin{array}{|c|c|c|} \hline 1 & 2 & \cellcolor{lightgray}3 \\ \hline 4 & \cellcolor{lightgray}5 & \cellcolor{lightgray}6 \\ \hline 7 & \cellcolor{lightgray}8 \\ \cline{1-2} 9 \\ \cline{1-1} 10 \\ \cline{1-1} \end{array}$$

$c_{10} = 2$ and 6 begins the next cycle of σ;
$c_9 = 3$ via $(7, 9, \underline{11}, 12, 13, 14, 15, 16, 18)$;
$c_8 = 8$ via $(7, 9, 12, 13, 14, 15, 16, \underline{18})$;
$c_7 = 3$ via $(7, 9, \underline{12}, 13, 14, 15, 16)$.

$\mathrm{dg}(\lambda^{(3)}) =$ 

$c_6 = 4$ and 7 begins the next cycle of σ;
$c_5 = 2$ via $(9, \underline{13}, 14, 15, 16)$;
$c_4 = 3$ via $(9, 14, \underline{15}, 16)$.

$\mathrm{dg}(\lambda^{(4)}) =$ 

$c_3 = 1$ and 9 begins the next cycle of σ;
$c_2 = 2$ via $(14, \underline{16})$;
$c_1 = 1$ via $(\underline{14})$.

In summary, $F_\lambda^{-1}(S, S, \sigma) = (15, 3, 3, 3, 13, 1, 5, 3, 2, 3, 8, 3, 4, 2, 3, 1, 2, 1)$. Letting ρ be the rim-hook of S marked with 5s, we also have

$$F_{\lambda,\rho}^{-1}(S, \sigma) = (3, 3, 3, 13, 1, 5, 3, 2, 3, 8, 3, 4, 2, 3, 1, 2, 1).$$

Remark 43. So far we have used permutations $\sigma \in S_n$ that permute the standard set $\{1, 2, \ldots, n\}$. The definition of $\mathrm{cycC}(\sigma)$ and all related constructions still work if we replace this set by any given set of n integers. This remark is needed below, when we operate on sub-permutations of σ where some of the cycles have been temporarily deleted.

The involution $\mathcal{I}_{\lambda,\mu}$ acts on input $(S, T, \sigma) \in \mathcal{P}_{\lambda,\mu}$ as follows. If $\lambda = \mu$ and the input is in Surv_λ, then this is a fixed point. Otherwise let $\beta = (\beta_1, \ldots, \beta_\ell)$ be the content composition for S and T. Remove the outermost rim-hook from S (meaning the β_ℓ cells in the filling of $\mathrm{dg}(\lambda)$ labeled ℓ), remove the outermost rim-hook from T (meaning the β_ℓ cells in the filling of $\mathrm{dg}(\mu)$ labeled ℓ), and remove the rightmost cycle in the canonical cycle notation of σ. Do this repeatedly until reaching a survivor object (S^0, S^0, σ^0) for some shape γ, where γ might be empty. Let (S', T', σ') be the immediately preceding object. We go from S^0 to S' by adding a rim-hook η of size β_k to get an object of some shape $\overline{\lambda}$. We go from S^0 to T' by adding a rim-hook ρ of size β_k to get an object of some shape $\overline{\mu}$. Here, σ^0 consists of the leftmost $k-1$ cycles in σ, while σ' consists of the leftmost k cycles in σ. By construction, $(T', \sigma') \in \mathrm{Surv}_{\overline{\mu},\rho}$, and the set $G(\overline{\lambda}, \overline{\mu})$ (defined in Theorem 16) is nonempty. The proof of that theorem shows that $G(\overline{\lambda}, \overline{\mu})$ consists of exactly two oppositely-signed partitions and describes how to go from one of these objects to the other. Let $\tilde{\gamma}$ be the other object appearing with γ in $G(\overline{\lambda}, \overline{\mu})$, so that $\tilde{\eta} = \overline{\lambda} \setminus \tilde{\gamma}$ and $\tilde{\rho} = \overline{\mu} \setminus \tilde{\gamma}$ are rim-hooks of the same size.

To build $\mathcal{I}_{\lambda,\mu}(S, T, \sigma)$, first compute $(T'', \sigma'') = F_{\overline{\mu},\tilde{\rho}} \circ F_{\overline{\mu},\rho}^{-1}(T', \sigma') \in \mathrm{Surv}_{\overline{\mu},\tilde{\rho}}$ using the bijections from Theorem 40(b), as modified in Remark 43. Get S'' by copying all values in T'' in the sub-shape $\tilde{\gamma}$, then filling the rim-hook $\tilde{\eta}$ with the next unused value $\tilde{k}$. To finish, restore the previously removed rim-hooks of lengths $\beta_{k+1}, \ldots, \beta_\ell$ to the fillings S'' and T'' (respectively) in the same positions they occupied in S and T. Here the labels (originally $k+1, k+2, \ldots$) of the restored rim-hooks get renumbered to be $\tilde{k}+1, \tilde{k}+2, \ldots$. Append to the right end of σ'' the cycles previously removed from the right end of σ. The resulting triple is $\mathcal{I}_{\lambda,\mu}(S, T, \sigma)$.

Acting by $\mathcal{I}_{\lambda,\mu}$ again reverses all the steps and returns us to (S, T, σ), so $\mathcal{I}_{\lambda,\mu}$ is an involution on $\mathcal{P}_{\lambda,\mu}$. The fixed point set of $\mathcal{I}_{\lambda,\lambda}$ in $\mathcal{P}_{\lambda,\lambda}$ has size $n!$, by Theorem 40(a). This completes our bijective proof of $A_n \overline{B}_n = n! I_{P(n)}$.

Example 44. Let $\lambda = (4,3,2,1)$, $\mu = (4,3,3)$. We compute $\mathcal{I}_{\lambda,\mu}(S,T,\sigma)$ for the following objects:

$$S = \begin{ytableau} 1 & 1 & 3 & 3 \\ 1 & 2 & 3 \\ 2 & 2 \\ 2 \end{ytableau}, \quad T = \begin{ytableau} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 \\ 3 & 3 & 3 \end{ytableau}, \quad \sigma = (5,8,6)(2,10,9,4)(1,3,7).$$

Here, $\beta = (3,4,3)$, $\operatorname{sgn}(S) = -1 \cdot 1 \cdot -1 = 1$, and $\operatorname{sgn}(T) = -1 \cdot -1 \cdot 1 = 1$. To reach a survivor (S^0, S^0, σ^0), we remove the rim-hooks labeled 3 and 2 from S and T, and we remove the two rightmost cycles of σ. This produces $\sigma^0 = (5,8,6)$ and $S^0 = \begin{ytableau} 1 & 1 \\ 1 \end{ytableau}$ of shape $\gamma = (2,1)$. We construct S' and T' by restoring the rim-hooks labeled 2 as shown here:

$$S^0 = \begin{ytableau} 1 & 1 \\ 1 \end{ytableau} \xrightarrow{\text{add } \eta} S' = \begin{ytableau} 1 & 1 \\ 1 & *(gray) 2 \\ *(gray) 2 & *(gray) 2 \\ *(gray) 2 \end{ytableau}; \qquad S^0 = \begin{ytableau} 1 & 1 \\ 1 \end{ytableau} \xrightarrow{\text{add } \rho} T' = \begin{ytableau} 1 & 1 & *(gray) 2 & *(gray) 2 \\ 1 & *(gray) 2 & *(gray) 2 \end{ytableau}.$$

We find that $\sigma' = (5,8,6)(2,10,9,4)$, S' has shape $\overline{\lambda} = (2,2,2,1)$ and T' has shape $\overline{\mu} = (4,3)$. By using the bijection in the proof of Theorem 16 (cf. Example 18), we find $\tilde{\gamma} = (2,2)$. This defines $\tilde{\eta}$ and $\tilde{\rho}$, which are shaded in gray here:

$$\tilde{\eta} = \overline{\lambda} \setminus \tilde{\gamma} = \begin{ytableau} {} & {} \\ {} & {} \\ *(gray) & *(gray) \\ *(gray) \end{ytableau}, \qquad \tilde{\rho} = \overline{\mu} \setminus \tilde{\gamma} = \begin{ytableau} {} & {} & *(gray) & *(gray) \\ {} & {} & *(gray) \end{ytableau}.$$

Next we compute the choice sequence $F^{-1}_{\overline{\mu},\rho}(T', \sigma')$, which is the sequence $F^{-1}_{\overline{\mu}}(T', \sigma')$ with its first entry c_7 deleted. One can check that $c_7 = 2$ since ρ is the second border rim-hook of $\overline{\mu}$, but we do not need this value. Note that σ' permutes the set $\{2,4,5,6,8,9,10\}$, and the rightmost cycle of σ' begins with the smallest value in this set, namely 2. Looking at the remaining entries in this cycle, we recover c_6, c_5, c_4 as follows (using the same notation from Example 42):

$$\begin{aligned} c_6 &= 6 \quad \text{via } (4,5,6,8,9,\underline{10}); \\ c_5 &= 5 \quad \text{via } (4,5,6,8,\underline{9}); \\ c_4 &= 1 \quad \text{via } (\underline{4},5,6,8). \end{aligned}$$

Removing the cells containing 2 from T' leaves the shape $(2,1)$. The rim-hook labeled 1 in T' is the first border rim-hook of this shape, so $c_3 = 1$. We deduce the rest of the choice sequence as follows:

$$\begin{aligned} c_3 &= 1 \quad \text{and 5 begins the next cycle of } \sigma'; \\ c_2 &= 2 \quad \text{via } (6,\underline{8}); \\ c_1 &= 1 \quad \text{via } (\underline{6}). \end{aligned}$$

So far, we have $F^{-1}_{\overline{\mu}}(T', \sigma') = (2,6,5,1,1,2,1)$ and $\mathbf{c} = F^{-1}_{\overline{\mu},\rho}(T', \sigma') = (6,5,1,1,2,1)$. The numbered diagram $\begin{ytableau} 1 & 2 & *(gray) 3 & *(gray) 4 \\ 5 & 6 & *(gray) 7 \end{ytableau}$ shows that $\tilde{\rho}$ is the third border rim-hook of $\overline{\mu}$. Therefore, setting $\mathbf{c}' = (3,6,5,1,1,2,1)$, we have $(T'', \sigma'') = F_{\overline{\mu},\tilde{\rho}}(\mathbf{c}) = F_{\overline{\mu}}(\mathbf{c}')$. We compute T''

and σ'' as follows. Like σ', σ'' permutes the set $\{2,4,5,6,8,9,10\}$ and the rightmost cycle of σ'' begins with 2. This cycle contains $|\tilde{\rho}| = 3$ values. We use $c'_6 = 6$ and $c'_5 = 5$ to determine these values, as follows:

$$\begin{array}{ll} (2,10,\_\_) & \text{via } (4,5,6,8,9,\underline{10}) \text{ and } c'_6 = 6; \\ (2,10,9) & \text{via } (4,5,6,8,\underline{9}) \text{ and } c'_5 = 5. \end{array}$$

The next entry $c'_4 = 1$ tells us which rim-hook to remove from the remaining shape $(2,2)$, as shown by the gray cells in the numbered diagram $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$. We begin the next cycle of σ'' with the least available value 4. The remaining two values in this cycle are found as follows:

$$\begin{array}{ll} (4,5,\_\_) & \text{via } (\underline{5},6,8) \text{ and } c'_3 = 1; \\ (4,5,8) & \text{via } (6,\underline{8}) \text{ and } c'_2 = 2. \end{array}$$

We conclude by using $c'_1 = 1$ to remove the first border rim-hook of the remaining shape (1), as shown here: $\begin{array}{|c|}\hline 1 \\ \hline \end{array}$. We use the least remaining value 6 to make the leftmost cycle (6) in σ''. We find T'' by labeling the rim-hooks removed from $\overline{\mu}$ using labels 3, 2, 1. We find S'' by copying the rim-hooks of T'' in the shape $\tilde{\gamma}$ and labeling the cells of $\tilde{\eta}$ with 3. In summary, we have found

$$S'' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}, \quad T'' = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} \end{array}, \quad \sigma'' = (6)(4,5,8)(2,10,9).$$

Finally, we compute $\mathcal{I}_{\lambda,\mu}(S,T,\sigma)$ by restoring to S'' and T'' the rim-hooks originally labeled 3 in S and T (using 4 as their new label) and appending the removed cycle $(1,3,7)$ to σ''. This produces

$$(S^*,T^*,\sigma^*) = \mathcal{I}_{\lambda,\mu}(S,T,\sigma) = \left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 4 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}, \ \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 4 & 4 & 4 \\ \cline{1-3} \end{array}, \ (6)(4,5,8)(2,10,9)(1,3,7) \right).$$

Note that $\operatorname{sgn}(S^*) = 1 \cdot -1 \cdot -1 \cdot -1 = -1$ and $\operatorname{sgn}(T^*) = 1 \cdot -1 \cdot -1 \cdot 1 = 1$, so $\operatorname{sgn}(S^*)\operatorname{sgn}(T^*) = -\operatorname{sgn}(S)\operatorname{sgn}(T)$ as needed. Acting on (S^*,T^*,σ^*) by $\mathcal{I}_{\lambda,\mu}$ reverses all the preceding steps and recreates the original triple (S,T,σ).

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