Resolving Sets and Split Resolving Sets of Symmetric Designs

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Abstract

We study resolving sets and split resolving sets of the point-block incidence graphs of symmetric designs and we obtain general lower bounds on their cardinality. In some cases, this lower bound is just a constant factor away from the known upper bounds. In particular, we show that for any $\varepsilon > 0$ there exists q_0 and n_0 such that if $q \geqslant q_0$ and $n \geqslant n_0$, then the metric dimension of the point-hyperplane incidence graph of $\mathrm{PG}(n, q)$ is at least $(2 - \varepsilon)nq$. The best known upper bound for the metric dimension of $\mathrm{PG}(n, q)$ is roughly $4nq$. We also prove that the metric dimension of a symmetric (v, k, λ) design, under certain conditions, is at least $\frac{(2-\varepsilon)uv}{k}$ for any $\varepsilon > 0$, where $u = \left\lfloor \frac{\ln v}{\ln v - \ln k + 1} \right\rfloor$.

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1 Introduction

The concept of resolving sets was introduced by Slater in 1975 [15] and independently by Harary and Melter in 1976 [8]. Many results of the topic have been gathered in [3], [5] and [16]. In this paper, we study the resolving sets and split resolving sets of the point-block incidence graphs of symmetric designs. Within this, we deal separately with the point-hyperplane incidence graphs of projective spaces. The study of the metric dimension of incidence graphs was initiated by Bailey [1], [2], while [4] and [17] are devoted exclusively to the case of symmetric designs and projective spaces, respectively.

Definition 1. Let $G = (V, E)$ be a graph, $S \subseteq V$ and let us fix an order of the elements of $S = (s_1, s_2, \ldots s_m)$. The *distance vector of* $v \in V$ *with respect to* S is

$$R(v|S) = (d(v, s_1), d(v, s_2), \ldots, d(v, s_m)).$$

We say that a set $W \subseteq V$ is *resolved by the set* S if $R(v|S) \neq R(u|S)$ for any $v \neq u \in W$, that is, there is a vertex $s \in S$ such that $d(v, s) \neq d(u, s)$.

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S is called a *resolving set* of G if it resolves the set V. A smallest resolving set is called *optimal*, its size is called the *metric dimension of the graph* and it is denoted by $\mu(G)$.

Split resolving sets were introduced by Bailey in [1].

Definition 2. Let $G = (V, E)$ be a bipartite graph with parts A and B. We say that S is a *split resolving set* if $S \cap A$ resolves B and $S \cap B$ resolves A. A smallest split resolving set is called *optimal* and we denote its size by $\mu^*(G)$, which quantity we will call the *split metric dimension* of G. A vertex set S is called a *semi-resolving set* if $S \subset A$ and S resolves B, or if $S \subset B$ and S resolves A.

It is easy to see that a split resolving set is also a resolving set, thus, for any bipartite graph G, we have $\mu(G) \leqslant \mu^*(G)$.

Definition 3. A (v, k, λ) *design* is a triplet $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are disjoint non-empty sets, their elements are called points and blocks, $\mathcal{I}$ is a binary relation between $\mathcal{P}$ and $\mathcal{B}$ called incidence, for which the following hold:

- $v = |\mathcal{P}|$,
- every two different points are incident with exactly λ common blocks,
- every block is incident with exactly k points.

We call the design *symmetric* if $|\mathcal{B}| = v$.

It is well-known that every (v, k, λ) design is r*-regular*, that is, every point is incident with exactly r blocks, where $r = \frac{\lambda(v-1)}{k-1}$.

Theorem 4 ([13] Proposition 2.4.9.)**.** *For any (v, k, λ) design, the following are equivalent:*

- *$|\mathcal{B}| = v$,*
- *$r = k$,*
- *every two different blocks are incident with exactly λ common points.*

Let us remark that for symmetric designs, $\lambda = \frac{k(k-1)}{v-1}$ is determined by v and k, and so is the quantity $k - \lambda = \frac{k(v-k)}{v-1}$, which is usually called the *order* of the design. In many of the known results, the order is involved in the bounds for the metric dimension; in this paper we formulate the results using only v and k.

Definition 5. Let $\mathcal{P}$ and $\mathcal{B}$ be the sets of points and blocks of a (v, k, λ) design. We say that $G = (V, E)$ is the *point-block incidence graph* or the *Levi graph* of the design if the set of vertices is $V = \mathcal{P} \cup \mathcal{B}$, and the set of edges is $E = \{\{p, A\} \ : \ p \in \mathcal{P}, A \in \mathcal{B}, p\mathcal{I}A\}$.

It follows from the definition that for every $p, q \in \mathcal{P}$ and $A, B \in \mathcal{B}$

$$d(p,q) = \begin{cases} 0 & \text{if } p = q \\ 2 & \text{if } p \neq q \end{cases} ; \qquad d(A,B) = \begin{cases} 0 & \text{if } A = B \\ 2 & \text{if } A \neq B \wedge A \cap B \neq \emptyset \\ 4 & \text{if } A \neq B \wedge A \cap B = \emptyset \end{cases} ;$$

$$d(a,P) = \begin{cases} 1 & \text{if } p \in A \\ 3 & \text{if } p \notin A \end{cases} .$$

If the design is symmetric, then $d(A,B) \neq 4$ since every two blocks have common points, therefore the diameter of the point-block incidence graph of a symmetric block design is three.

We mention some known results about the metric dimension of symmetric designs. The following general upper bound is proven by Bailey in [2].

Theorem 6. *If G is the point-block incidence graph of a symmetric (v, k, λ) design and $k - \lambda \geqslant 2$, then*

$$\mu(G) \leqslant \mu^*(G) \leqslant 2 \left\lceil \frac{v \ln v}{k - \lambda} \right\rceil .$$

In [17], the following lower bound was proven.

Theorem 7. *If G is the point-block incidence graph of a symmetric (v, k, λ) design, then*

$$\mu^*(G) \geqslant \mu(G) \geqslant \left\lceil \frac{v}{k + 1 - \lambda} \right\rceil .$$

There is a straightforward general lower bound for any graph G with diameter d on n vertices: $d^{\mu(G)} + \mu(G) \geqslant n$ (see e.g. [5]). For incidence graphs of symmetric designs, this gives

$$2v \leqslant 3^{\mu(G)} + \mu(G) < 3^{\mu(G)} + 2 \cdot 3^{\mu(G)} = 3^{\mu(G)+1}.$$

Applying the logarithm to both sides, we obtain

$$\mu(G) > \frac{\ln(2v)}{\ln 3} - 1.$$

Hence,

$$\mu(G) \geqslant \left\lfloor \frac{\ln(2v)}{\ln 3} \right\rfloor .$$

Bailey pointed out that, if $v = c(k - \lambda)$, then this bound and Theorem 6 give $c_1 \ln v \leqslant \mu(G) \leqslant c_2 \ln v$.

In this paper, we develop a general method which gives new lower bounds for the metric and split metric dimension of the point-block incidence graph G of any symmetric design. Let us summarize our main results. Theorem 18 claims $\mu^*(G) \geqslant \frac{4(v-1)}{k+2}$ and $\mu(G) \geqslant \frac{4(v-1)}{k+6}$. Moreover, Theorem 21 and Theorem 27 roughly say that, under certain conditions, $\mu^*(G) \geqslant \mu(G) \geqslant \frac{2uv(1-o_u(1))}{k}$, where $u = \left\lfloor \frac{\ln v}{\ln v - \ln k + 1} \right\rfloor$.

We will also consider what our method gives for those particular cases that have been studied in the literature in more detail. In these cases, our lower bound is sharp up to a constant factor or a constant additive error term.

Definition 8. We denote the n-dimensional projective space over the finite field $\mathbb{F}_q$ by $\mathrm{PG}(n, q)$.

If $\mathcal{P}$ and $\mathcal{H}$ are the set of points and the set of hyperplanes of $\mathrm{PG}(n, q)$, then $(\mathcal{P}, \mathcal{H}, \mathcal{I})$ is a symmetric design with

$$v = |\mathcal{P}| = |\mathcal{H}| = \frac{q^{n+1} - 1}{q - 1},\ k = \frac{q^n - 1}{q - 1},\ \text{and } \lambda = \frac{q^{n-1} - 1}{q - 1}.$$

We will also call this design a projective space. The point-block incidence graph of this design is also called point-hyperplane graph. We will use the notations $\mu(\mathrm{PG}(n, q))$ and $\mu^*(\mathrm{PG}(n, q))$ for the metric dimension and the split metric dimension of this graph.

For the metric dimension of the incidence graph of projective planes and spaces the following are known:

Theorem 9 ([9],[12])**.** *The metric dimension of a projective plane of order $q \geqslant 8$ is $4q-4$.*

Theorem 10 ([4])**.** *If q is large enough, then*

$$\mu(PG(n, q)) \geqslant 2nq - \frac{2n^n}{n!}.$$

Theorem 11 ([4])**.** *Let p be a prime $p > 3$ and $q = p^r > 36086$. If $n > 3$, then*

$$\mu(PG(n, q)) \leqslant (n^2 + n - 8)q.$$

We will show that (see Theorem 32 and Corollary 37)

$$2(1 - o_n(1))(1 - o_q(1))nq \leqslant \mu(\mathrm{PG}(n, q)) \leqslant 4nq.$$

The upper bound follows from known results for so-called higgledy-piggledy line sets and strong blocking sets; see Section 5 for details. Our lower bound is stated in a more precise form in Theorem 35. Let us remark that, unlike Theorem 10, our bound works when n is large compared to q as well.

Definition 12. A biplane is a symmetric design with $\lambda = 2$; that is a $\left(\frac{q^2+3q+4}{4}, q + 2, 2\right)$ design.

In [17], Tang, Zhou, Chen and Zhang proved that the metric dimension of a biplane is 5 if $q = 2$ and $2q$ if $q \geqslant 3$, where $q = k - 2$.

Throughout this paper, e will denote Euler's number. We will use the following combinatorial inequality:

Proposition 13. *If $0 < u \leqslant k$, then*

$$\sum_{i=0}^{u} \binom{k}{i} < \left(\frac{ek}{u}\right)^u. \tag{1}$$

The proof of this proposition can be found in [14, Proposition 1.4.]. However, it is presented here due to its brevity.

Proof. Let $x = \frac{u}{k} \in (0,1]$. Then

$$\sum_{i=0}^{u} \binom{k}{i} \leqslant \sum_{i=0}^{u} \binom{k}{i} \frac{x^i}{x^u} \leqslant \frac{1}{x^u} \sum_{i=0}^{k} \binom{k}{i} x^i = \frac{(1+x)^k}{x^u} = \frac{\left(1+\frac{u}{k}\right)^k}{\left(\frac{u}{k}\right)^u} < \frac{\left(e^{\frac{u}{k}}\right)^k}{\left(\frac{u}{k}\right)^u}. \qquad \square$$

Notation: In this paper, we deal with bipartite graphs. If X and Y denote the parts of the graph and S is a fixed subset of $V = X \cup Y$, then let $X_S = X \cap S$ and $Y_S = Y \cap S$.

2 Bipartite and semi-regular graphs

We start with giving a lower bound for the split metric dimension and the metric dimension of bipartite and semi-regular graphs.

Definition 14. Let $G = (A, B, E)$ be a bipartite graph with bipartition (A, B). If every vertex in A has degree r and every vertex in B has degree k, then G is called *semi-regular* of bi-degree (r, k).

Clearly, the incidence graphs of (v, k, λ) designs are particular semi-regular graphs of bi-degree (r, k). In case of symmetric designs, $r = k$ so their incidence graphs are in fact regular.

Theorem 15. *Let $G = (A, B, E)$ be a bipartite graph with diameter 3, $v = |A|$ and $b = |B|$. If S is a split resolving set, then*

$$|A_S| \geqslant \log_2 b \ , \quad |B_S| \geqslant \log_2 v.$$

If S is a resolving set, then

$$|A_S| \geqslant \log_2(b - |S|) \ , \quad |B_S| \geqslant \log_2(v - |S|).$$

Proof. It is enough to prove the statements for $|A_S|$. Let $l = |A_S|$. If S is a split resolving set and $w_1 \neq w_2 \in B$, then the sequences $R(w_1|A_S)$ and $R(w_2|A_S)$ have to be different. The number of different sequences is b and the number of sequences is at most 2^l, thus $b \leqslant 2^l$.

If S is a resolving set and $w_1 \neq w_2 \in B \setminus B_S$, then their sequences have to be different. The number of different sequences is $b - |B_S|$, and it is also at most 2^l, hence

$$b - |B_S| \leqslant 2^l,$$

thus

$$|A_S| = l \geqslant \log_2(b - |B_S|) \geqslant \log_2(b - |S|). \qquad \square$$

Proposition 16. *Let $G = (A, B, E)$ be a semi-regular bipartite graph of bi-degree (r, k) with diameter 3, $b = |B|$, let u be a positive integer and let $s \in (0, 1)$. If S is a split resolving set and $u \leqslant |A_S|$, then*

$$|A_S| \geqslant \min \left\{ \frac{sb(u+1)}{r} \ , \ \frac{u\sqrt[u]{(1-s)b}}{e} \right\}. \tag{2}$$

If S is a resolving set and $u \leqslant |A_S|$, then

$$|A_S| \geqslant \min \left\{ \frac{sb(u+1)}{r} \ , \ \frac{u\sqrt[u]{(1-s)b-|S|}}{e} \right\}. \tag{3}$$

Proof. In the first part of the proof, S can be a resolving set or a split resolving set as well.

Case 1: Suppose that there are at least sb vertices in B such that each of them has at least $u+1$ neighbors in A_S. We denote the set of these vertices by B_1. By double counting the set

$$\{(x, y) \in A_S \times B_1 \ : \ \{x, y\} \in E\},$$

we get

$$|A_S| r \geqslant sb(u+1).$$

Case 2: There are less then sb vertices in B which have at least $u+1$ neighbors in A_S. Therefore, there are at least $(1-s)b$ vertices in B which have at most u neighbors in A_S. Let $l = |A_S|$.

If S is a split resolving set, then let B_2 denote the set of vertices in B that have at most u neighbors in A_S, and if S is a resolving set, then let B_2 denote the set of vertices in $B \backslash B_S$ that have at most u neighbors in A_S. If $w_1 \neq w_2 \in B_2$, then $R(w_1|A_S) \neq R(w_2|A_S)$. In both cases, the number of different sequences is $|B_2|$ and the number of sequences is at most $\sum_{i=0}^{u} \binom{l}{i}$, as there can be at most u ones in each sequence. Therefore, if S is a split resolving set, then

$$(1-s)b \leqslant |B_2| \leqslant \sum_{i=0}^{u} \binom{l}{i} \leqslant \left(\frac{el}{u}\right)^u. \tag{4}$$

By rearranging this inequality, we get $|A_S| = l \geqslant \frac{u\sqrt[u]{(1-s)b}}{e}$.
If S is a resolving set, then

$$(1-s)b - |S| \leqslant |B_2| \leqslant \sum_{i=0}^{u} \binom{l}{i} \leqslant \left(\frac{el}{u}\right)^u. \tag{5}$$

Straightforward rearrangements yield $|A_S| = l \geqslant \frac{u\sqrt[u]{(1-s)b-|S|}}{e}$.

As either Case 1 or Case 2 holds, we have proved the assertions. □

The following is trivially implied by Proposition 16.

Proposition 17. *Let $G = (A, B, E)$ be a regular bipartite graph of degree k with diameter 3, $v = |A| = |B|$, let u be a positive integer and let $s \in (0, 1)$. If S is a split resolving set and $u \leqslant |A_S|$ and $u \leqslant |B_S|$, then*

$$|S| \geqslant \min \left\{ \frac{2sv(u+1)}{k} \ , \ \frac{2u \sqrt[u]{(1-s)v}}{e} \right\}.$$

If S is a resolving set and $u \leqslant |A_S|$ and $u \leqslant |B_S|$, then

$$|S| \geqslant \min \left\{ \frac{2sv(u+1)}{k} \ , \ \frac{2u \sqrt[u]{(1-s)v - |S|}}{e} \right\}.$$

Let us derive a simple lower bound.

Theorem 18. *Let $G = (A, B, E)$ a k-regular bipartite graph with diameter 3, $v = |A| = |B|$. Then*

$$\mu^*(G) \geqslant \frac{4(v-1)}{k+2}, \text{ and } \ \mu(G) \geqslant \frac{4(v-1)}{k+6}.$$

Proof. We follow the proof of Proposition 16, setting the parameter $u = 1$. Let us consider split resolving sets first. Without using Formula (1) in inequality (4), we get

$$(1-s)v \leqslant \sum_{i=0}^{1} \binom{l}{i} = 1 + l,$$

hence

$$l \geqslant (1-s)v - 1.$$

Applying this and substituting $u = 1$ into the first formula of (2), we get the lower bound

$$|A_S| \geqslant \min \left\{ \frac{2sv}{k} \ , \ (1-s)v - 1 \right\}.$$

The same lower bound can be given for $|B_S|$. Thus

$$\mu^*(G) \ = |A_S| + |B_S| \geqslant 2 \min \left\{ \frac{2sv}{k} \ , \ (1-s)v - 1 \right\}.$$

Let us choose the parameter s so that the two formulas are equal:

$$\frac{2sv}{k} = (1-s)v - 1.$$

By rearranging, we get $s = \frac{k(v-1)}{v(k+2)}$, therefore

$$\mu^*(G) \geqslant \frac{4sv}{k} = \frac{4(v-1)}{k+2}.$$

Using the same ideas for resolving sets, Formula (5) becomes

$$(1-s)v - |S| \leqslant \sum_{i=0}^{1} \binom{l}{i} = 1 + l,$$

hence

$$l \geqslant (1-s)v - |S| - 1.$$

Thus

$$|A_S| \geqslant \min\left\{\frac{2sv}{k}\ ,\ (1-s)v - |S| - 1\right\}.$$

The same lower bound holds for $|B_S|$, hence

$$|S| = |A_S| + |B_S| \geqslant 2\min\left\{\frac{2sv}{k}\ ,\ (1-s)v - |S| - 1\right\}.$$

If $\frac{2sv}{k} \leqslant (1-s)v - |S| - 1$, then $|S| \geqslant \frac{4sv}{k}$. If $\frac{2sv}{k} \geqslant (1-s)v - |S| - 1$, then $|S| \geqslant 2((1-s)v - |S| - 1)$. By rearranging, we get $|S| \geqslant \frac{2((1-s)v-1)}{3}$. Thus

$$|S| \geqslant 2\min\left\{\frac{2sv}{k}\ ,\ \frac{(1-s)v-1}{3}\right\}.$$

Let us choose the parameter s so that the two formulas are equal, that is

$$\frac{2sv}{k} = \frac{(1-s)v-1}{3}.$$

By rearranging, we get

$$s = \frac{k(v-1)}{v(k+6)},$$

hence

$$|S| \geqslant \frac{4sv}{k} = \frac{4(v-1)}{k+6}. \qquad \square$$

3 Split resolving sets of symmetric designs

In this section, $G = (\mathcal{P}, \mathcal{B}, E)$ will denote the point-block incidence graph of a symmetric (v, k, λ) design. Note hat G is a regular graph of degree k and diameter 3, thus Theorem 15 and Theorem 18 are valid for $\mu^*(G)$, and Proposition 17 can be applied to derive a lower bound for $\mu^*(G)$. To obtain sharper bounds, we will need the following.

Lemma 19. *Let S be a split-resolving set. Then*

$$|\mathcal{P}_S| \geqslant \frac{\ln v}{\ln v - \ln k + 1}\ ,\quad |\mathcal{B}_S| \geqslant \frac{\ln v}{\ln v - \ln k + 1}.$$

Proof. By duality, it is enough to prove the lower bound for $|\mathcal{P}_S|$. By Theorem 15, we have $|\mathcal{P}_S| \geqslant \log_2 v = \frac{\ln v}{\ln 2}$, thus it is enough to prove

$$\frac{\ln v}{\ln 2} \geqslant \frac{\ln v}{\ln v - \ln k + 1}.$$

Rearranging, this is equivalent to

$$\ln v \geqslant \ln k + \ln 2 - 1 = \ln \frac{2k}{e}$$

or, equivalently, $v \geqslant \frac{2k}{e}$, which is trivially true. □

We will also use the following technical lemma.

Lemma 20. *If $m > 1$, then*

$$m^{-\frac{m}{m-1}} + m^{-\frac{1}{m-1}} - 1 \leqslant 0.$$

Proof. Let $f(m) := m^{-\frac{m}{m-1}} + m^{-\frac{1}{m-1}} - 1$. The derivative of f is

$$f'(m) = \frac{2m^{\frac{m}{1-m}} + m^{\frac{m}{1-m}} \ln m + m^{\frac{1}{1-m}} \ln m - 2m^{\frac{1}{1-m}}}{(m-1)^2}.$$

The derivative is positive on the interval $(1, \infty)$, so f is strictly increasing on $(1, \infty)$. Clearly, $\lim_{m\to\infty} f(m) = 0$, thus f is negative on $(1, \infty)$. □

Now we are ready to prove our main theorem.

Theorem 21. *If $v \leqslant \frac{k^2}{e^2}$, then*

$$\mu^*(G) \geqslant \frac{2vu}{k \sqrt[u-1]{u}},$$

where

$$u = \left\lfloor \frac{\ln v}{\ln v - \ln k + 1} \right\rfloor.$$

Proof. The equation $\sqrt[u]{v} = \frac{ev}{k}$ is equivalent to $u = \frac{\ln v}{\ln v - \ln k + 1}$. Let $u = \left\lfloor \frac{\ln v}{\ln v - \ln k + 1} \right\rfloor$. Then $u \geqslant 2$ and $\sqrt[u]{v} \geqslant \frac{ev}{k}$. By Lemma 19, we can use this parameter u in Proposition 17 to obtain

$$\mu^*(G) \geqslant \min \left\{ \frac{2sv(u+1)}{k} \;,\; \frac{2u \sqrt[u]{(1-s)v}}{e} \right\} \geqslant$$

$$\frac{2v}{k} \min \left\{ s(u+1) \;,\; u \sqrt[u]{1-s} \right\} \geqslant \frac{2v}{k} u \min \left\{ s \;,\; (1-s)^{\frac{1}{u}} \right\}.$$

Let us set $s = u^{-\frac{1}{u-1}}$. Then $s \leqslant (1-s)^{\frac{1}{u}}$ holds, because it is equivalent to

$$s^u + s - 1 = u^{-\frac{u}{u-1}} + u^{-\frac{1}{u-1}} - 1 \leqslant 0,$$

and this inequality holds by Lemma 20, since $u \geqslant 2$. □

Let us compare our results to previously known ones. The general lower bound for graphs with diameter 3 yields roughly $\mu^*(G) \geqslant \mu(G) \geqslant \frac{\ln(2v)}{\ln 3}$. If $v \approx k^c$ for some $c > 1$, then this yields $\mu^*(G) \geqslant \frac{c \ln k}{\ln 3}$, while Theorem 18 gives roughly $\mu^*(G) \geqslant 4k^{c-1}$. But also in case $v \leqslant ck$ for some c, such as Hadamard designs of order n, which are symmetric $(4n-1, 2n-1, n-1)$-designs, the general bound gives $\mu^*(G) \geqslant \frac{\ln(4n-1)}{\ln 3}$. The next theorem shows that our methods improve this by a constant factor.

Theorem 22. *Let G be the point-block incidence graph of a Hadamard design of order n. If $n \geqslant 9$, then*

$$\mu^*(G) \geqslant \frac{(8n-2)u}{(2n-1)\sqrt[u-1]{u}},$$

where

$$u = \left\lfloor \frac{\ln(4n-1)}{\ln \frac{4n-1}{2n-1} + 1} \right\rfloor.$$

Moreover, for any positive integer n,

$$\mu^*(G) \geqslant \frac{2\ln(4n-1)}{\ln 2}.$$

Proof. The first formula is a consequence of Theorem 21. The condition $v \leqslant \frac{k^2}{e^2}$ is equivalent to

$$4n - 1 \leqslant \frac{(2n-1)^2}{e^2},$$

which holds if $n \geqslant 9$. The second formula follows from Theorem 15. □

Note that Bailey's general result Theorem 6 applied for Hadamard designs gives

$$\mu^*(G) \leqslant 2\left\lceil \frac{v \ln v}{k-\lambda} \right\rceil = 2\left\lceil \frac{(4n-1)\ln(4n-1)}{n} \right\rceil \leqslant 2\left\lceil 4\ln(4n-1) \right\rceil,$$

so our lower bound is sharp within a factor of 3.

Metric dimension of biplanes have been studied in detail in [17], where $\mu(G) = 2q$ was derived for $q \geqslant 3$, where $q = k-2$. This is a lower bound for the size of split resolving sets as well. Up to our knowledge, there is no other lower bound known for the split metric dimension of biplanes. Since $k = \frac{2(v-1)}{k-1}$, $v = \frac{k(k-1)+2}{2}$. Theorem 18 yields

$$\mu^*(G) \geqslant \frac{4(v-1)}{k+2} = \frac{4k(k-1)}{2(k+2)} = \frac{2(q+2)(q+1)}{q+4} = 2q - 2 + \frac{12}{q+4},$$

which is very close to the former one.

4 Metric dimension of symmetric designs

As in the previous section, $G = (\mathcal{P}, \mathcal{B}, E)$ denotes the point-block incidence graph of a symmetric (v, k, λ) design, thus it is a regular graph of degree k and diameter 3. Therefore, Theorem 15 and Theorem 18 provide lower bounds for $\mu(G)$. In this section, we give a sharper lower bound using Proposition 17.

In the second formula of Proposition 17, the term $|S|$ can obviously be replaced by any known upper bound for $|S|$. Therefore, by using Theorem 6, we obtain the following result.

Proposition 23. *Let S be an optimal resolving set of a symmetric (v, k, λ) design, $\mathcal{P}_S = S \cap \mathcal{P}$ and $\mathcal{B}_S = S \cap \mathcal{B}$. Let u be a positive integer and let $s \in (0, 1)$. If $k - \lambda \geqslant 2$, $u \leqslant |\mathcal{P}_S|$ and $u \leqslant |\mathcal{B}_S|$, then*

$$|S| \geqslant \min \left\{ \frac{2sv(u+1)}{k} \ , \ \frac{2u}{e} \sqrt[u]{(1-s)v - 2\left\lceil \frac{v \ln v}{k - \lambda} \right\rceil} \right\}.$$

The disadvantage of this result is that finding the optimal values for u and s to get the largest lower bound is quite complicated in general. To this end, we need some more technical preparations.

Lemma 24. *For any positive $a \in \mathbb{R}$, if $v \geqslant \dfrac{k(k+a)}{k - a \ln v}$ and $v < \exp \dfrac{k}{a}$, then $\left\lceil \dfrac{v \ln v}{k - \lambda} \right\rceil < \dfrac{v}{a}$.*

Proof. Since our (v, k, λ) design is symmetric, $k = \frac{\lambda(v-1)}{k-1}$, and therefore $k - \lambda = \frac{k(v-k)}{v-1}$. Then

$$\left\lceil \frac{v \ln v}{k - \lambda} \right\rceil < \frac{v \ln v}{k - \lambda} + 1 = \frac{(v-1)v \ln v + k(v-k)}{k(v-k)}$$

$$< \frac{v^2 \ln v + k(v-k)}{k(v-k)} < \frac{v^2 \ln v + kv}{k(v-k)}.$$

We have to prove that

$$\frac{v^2 \ln v + kv}{k(v-k)} \leqslant \frac{v}{a}.$$

By rearranging, this is equivalent to

$$va \ln v - kv \leqslant -k^2 - ka.$$

By dividing by $a \ln v - k$, we get

$$v \geqslant \frac{-k^2 - ka}{a \ln v - k} = \frac{k^2 + ka}{k - a \ln v}.$$

The sign is reversed since $v < \exp \dfrac{k}{a}$, that is, $a \ln v - k < 0$. ☐

Lemma 25. *If* $v \geqslant 769$*, then* $\left\lceil \dfrac{v \ln v}{k-\lambda} \right\rceil < \frac{v}{4}$.

Proof. We use the inequality $k - \lambda \geqslant \sqrt{v} - 1$, which follows from the inequality $v \leqslant (k-\lambda)^2 + (k-\lambda) + 1$ (see [13, Proposition 2.4.12]). With this,

$$\left\lceil \frac{v \ln v}{k-\lambda} \right\rceil < \frac{v \ln v}{k-\lambda} + 1 < \frac{v \ln v}{\sqrt{v}-1} + 1 < \frac{v}{4},$$

where the last inequality holds if $v \geqslant 769$. □

Lemma 26. *Let* S *be a resolving set. If* $k - \lambda \geqslant 2$, $v \geqslant \frac{4k}{e}$ *and*

- v *is at least 769 or*
- $v \geqslant \dfrac{k(k+4)}{k - 4\ln v}$ *and* $v < \exp \dfrac{k}{4}$,

then

$$|\mathcal{P}_S| \geqslant \frac{\ln v}{\ln v - \ln k + 1} \quad , \quad |\mathcal{B}_S| \geqslant \frac{\ln v}{\ln v - \ln k + 1}.$$

Proof. By duality, it is enough to prove the lower bound for $|\mathcal{P}_S|$. We use Theorem 15, Theorem 6 and either Lemma 24 or Lemma 25 to get

$$|\mathcal{P}_S| \geqslant \log_2(v - |S|) \geqslant \log_2\left(v - 2\left\lceil \frac{v \ln v}{k-\lambda} \right\rceil\right) > \log_2\left(v - \frac{2v}{4}\right) = \log_2 \frac{v}{2} = \frac{\ln v - \ln 2}{\ln 2},$$

thus it is enough to prove

$$\frac{\ln v - \ln 2}{\ln 2} \geqslant \frac{\ln v}{\ln v - \ln k + 1}.$$

Rearranging, this is equivalent to

$$\ln^2 v - \ln v(\ln k + 1 - 2\ln 2) + \ln k \ln 2 - \ln 2 \geqslant 0.$$

As $k \geqslant 3$, this follows if

$$\ln^2 v - \ln v(\ln k - 1 + 2\ln 2) \geqslant 0,$$

that is, if

$$\ln v \geqslant \ln k - 1 + 2\ln 2 = \ln \frac{4k}{e},$$

which follows from the assumptions. □

Now we are ready to prove our stronger lower bound.

Theorem 27. *Let* G *be the point-block incidence graph of a* (v, k, λ) *symmetric design. If* $k - \lambda \geqslant 2$, $v \leqslant \frac{k^2}{e^2}$ *and*

- *v is at least 769 or*
- $v \geqslant \dfrac{k(k+4)}{k-4\ln v}$ *and* $v < \exp\dfrac{k}{4}$,

then

$$\mu(G) \geqslant \frac{2uv}{k}\left(u^{-\frac{1}{u-1}} - \frac{2(v-1)\ln v + k}{k(v-k)}\right),$$

where $u = \left\lfloor \dfrac{\ln v}{\ln v - \ln k + 1} \right\rfloor$.

Proof. Let $u = \left\lfloor \dfrac{\ln v}{\ln v - \ln k + 1} \right\rfloor$. Then $\sqrt[u]{v} \geqslant \frac{ev}{k}$. Proposition 23 with this u gives

$$\mu(G) \geqslant \min\left\{\frac{2sv(u+1)}{k}\ ,\ \frac{2u}{e}\sqrt[u]{(1-s)v - 2\left\lceil\frac{v\ln v}{k-\lambda}\right\rceil}\right\} \geqslant$$

$$2u\min\left\{\frac{sv}{k}\ ,\ \frac{1}{e}\sqrt[u]{(1-s)v - \frac{2v((v-1)\ln v + k)}{k(v-k)}}\right\} \geqslant$$

$$2u\min\left\{\frac{sv}{k}\ ,\ \frac{\sqrt[u]{v}}{e}\sqrt[u]{(1-s) - \frac{2((v-1)\ln v + k)}{k(v-k)}}\right\} \geqslant$$

$$\frac{2uv}{k}\min\left\{s\ ,\ \sqrt[u]{(1-s) - \frac{2((v-1)\ln v + k)}{k(v-k)}}\right\}.$$

Let $s = u^{-\frac{1}{u-1}} - \frac{2((v-1)\ln v + k)}{k(v-k)}$. We prove that

$$s \leqslant \sqrt[u]{(1-s) - \frac{2((v-1)\ln v + k)}{k(v-k)}},$$

which means that

$$s^u + s - 1 + \frac{2((v-1)\ln v + k)}{k(v-k)} \leqslant 0.$$

By substituting s, we get

$$\left(u^{-\frac{1}{u-1}} - \frac{2((v-1)\ln v + k)}{k(v-k)}\right)^u + u^{-\frac{1}{u-1}} - 1 \leqslant 0.$$

This follows from the inequality

$$\left(u^{-\frac{1}{u-1}}\right)^u + u^{-\frac{1}{u-1}} - 1 \leqslant 0.$$

By Lemma 20, this holds for $u \geqslant 2$, which follows from $k^2 \geqslant ve^2$ as in the proof of Theorem 21. □

Let us recall that, for the incidence graph of a Hadamard design of order n (that is, a $(4n-1, 2n-1, n-1)$ design), $\mu(G) \leqslant \mu^*(G) \leqslant 2\left\lceil\frac{(4n-1)\ln(4n-1)}{n}\right\rceil \leqslant 2\lceil 4\ln(4n-1)\rceil$ follows from Bailey's general result, Theorem 6. We obtain the following.

Theorem 28. *Let G be the point-block incidence graph of a Hadamard design of order n. Then*

$$\mu(G) \geqslant 2\log_2\left(4n-1-2\left\lceil\frac{(4n-1)\ln(4n-1)}{n}\right\rceil\right).$$

Proof. It is a consequence of Theorem 15 and Theorem 6. □

Let us also recall that the metric dimension of projective planes and biplanes have been determined exactly; it is $4q-4$ for projective planes of order $q \geqslant 8$ and $2q$ for biplanes of order $q \geqslant 3$. Theorem 18 gives $\mu(G) \geqslant 4q - 24 + \frac{168}{q+7}$ for the former and $\mu(G) \geqslant 2q - 10 + \frac{84}{q+8}$ for the latter. Both are just an additive constant away from the exact values.

For the point-hyperplane graph of $\mathrm{PG}(n,q)$, which has parameter $v = \frac{q^{n+1}-1}{q-1}$, $k = \frac{q^n-1}{q-1}$, Theorem 27 can be used to obtain $\mu(\mathrm{PG}(n,q)) \geqslant 2(1-o_n(1))(1-o_q(1))nq$. However, as the case of projective spaces is one of the most studied areas in the topic, we devote a standalone section to elaborate more precise results on it.

5 Projective spaces

In order to specify and enhance the lower bound of Theorem 27 for the metric dimension of projective spaces, we will also need an upper bound on the size of an optimal resolving set better than that of Bailey's general Theorem 6 to plug into Proposition 17. Such a bound can be obtained in a constructive manner. To this end, we will need some definitions.

Definition 29 ([6] Section 3)**.** A set B of points in $\mathrm{PG}(n,q)$ is a *strong blocking set* if every hyperplane H of $\mathrm{PG}(n,q)$ is spanned by $B \cap H$.

It is clear that any strong blocking set is a semi-resolving set (with respect to hyperplanes). An efficient and comfortable way to construct strong blocking sets is to find an appropriate set of lines.

Definition 30 ([11] Section 4)**.** A set S of lines in $\mathrm{PG}(n,q)$ is *higgledy-piggledy* if the union of (the point sets of) the lines in S is a strong blocking set.

Fancsali and Sziklai [7, Theorem 11] proved that if there is no 2-codimensional subspace intersecting every element of a line set S, then S is higgledy-piggledy. Let us call this condition Condition A. They proved that if $q \geqslant 2n-1$, then there exists a set of lines possessing Condition A, and thus higgledy-piggledy. They also noted that Condition A is not always necessary (if, roughly saying, $|S|$ is small and q is larger than $2n-1$, then it is). In [4, Lemma 8], Bartoli et. al. proved that if S is a set of lines satisfying Condition A, then there exists a semi-resolving set of size $|S|q$. In [4], Condition A was used as the definition of higgledy-piggledy line sets, but this is not totally correct. Here we shall give a proof of the same result but relying on the correct definition of higgledy-piggledy lines. This will enable us to use another construction of higgledy-piggledy lines which works for all n and q.

Theorem 31. *If S is a higgledy-piggledy line set in* $\mathrm{PG}(n,q)$*, then there exists a semi-resolving set of* $\mathrm{PG}(n,q)$ *of size* $|S|q$ *with respect to hyperplanes.*

Proof. Let us erase one point from each line of S. Let S' be the set of these punctured lines. It is obvious that the point set of the union of the lines of S is a semi-resolving set. We prove that the union of elements of S' is also a semi-resolving set. Let α and β be two different hyperplanes. We need that there is a point of S' that is in α but not in β, or vice versa. By the definition of higgledy-piggledy line sets, there exists a point $P \in l \in S$ such that $P \in \alpha$ and $P \notin \beta$ (otherwise $S \cap \alpha \subset \alpha \cap \beta$ would contradict that $S \cap \alpha$ generates α). If $P \in S'$, then we are done. Suppose that $P \notin S'$, that is, we erased P from l. If l lies in α, then there exists a point $Q \in l \setminus \{P\}$ such that $Q \in \alpha$ and $Q \notin \beta$. If l is not in α, then $R = \beta \cap l$ is a suitable point, since $R \notin \alpha$ and $R \in \beta$. □

Theorem 32.

$$\mu(PG(n,q)) \leqslant \mu^*(PG(n,q)) \leqslant 4nq.$$

Proof. In [10, Theorem 31.], Héger and Nagy proved that in $\mathrm{PG}(n,q)$, there exists a higgledy-piggledy set of $2n$ lines, thus the union of these lines forms a strong blocking set. By Theorem 31, there is a semi-resolving set with $2nq$ points with respect to hyperplanes. By duality, there exists a set of $2nq$ hyperplanes that resolves the set of points. □

Now we specify and improve our general lower bounds for projective spaces. In this section, S always denotes a (split) resolving set for the point-hyperplane incidence graph of $\mathrm{PG}(n,q)$.

Proposition 33. *Let u be a positive integer and let $s \in (0,1)$. If S is an optimal resolving set such that*

$$u \leqslant |\mathcal{P}_S| \ , \ \ u \leqslant |\mathcal{H}_S|$$

and

$$\frac{(1-s)(q^n-1)}{q-1} \geqslant 4nq,$$

then

$$\mu(\mathrm{PG}(n,q)) \geqslant \min\left\{2(u+1)sq \ , \ \frac{2uq^{\frac{n}{u}}\sqrt[u]{1-s}}{e}\right\}.$$

If S is an optimal split resolving set such that

$$u \leqslant |\mathcal{P}_S| \ \ \textit{and} \ \ u \leqslant |\mathcal{H}_S|,$$

then

$$\mu^*(\mathrm{PG}(n,q)) \geqslant \min\left\{2(u+1)sq \ , \ \frac{2uq^{\frac{n}{u}}\sqrt[u]{1-s}}{e}\right\}.$$

Proof. We use Proposition 16. A lower bound of the first formula:

$$|\mathcal{P}_S| \geqslant \frac{sv(u+1)}{k} = \frac{s(u+1)\frac{q^{n+1}-1}{q-1}}{\frac{q^n-1}{q-1}} = \frac{(u+1)s(q^{n+1}-1)}{q^n-1} \geqslant (u+1)sq.$$

In the second formula, we can use Theorem 32 and the condition $\frac{(1-s)(q^n-1)}{q-1} \geqslant 4nq$ as follows:

$$(1-s)v - |S| \geqslant \frac{(1-s)(q^{n+1}-1)}{q-1} - 4nq \geqslant$$

$$(1-s)\left(\frac{q^{n+1}-1}{q-1} - \frac{q^n-1}{q-1}\right) = (1-s)q^n. \qquad \square$$

Let us first prove simple and general bounds for the (split) metric dimension of $\mathrm{PG}(n,q)$.

Theorem 34. *For any integer $n \geqslant 2$ and any positive prime power q,*

$$\mu^*(\mathrm{PG}(n,q)) \geqslant \frac{2nq\sqrt[n]{\frac{1}{2}}}{e}.$$

If $q^n - 1 \geqslant 8nq(q-1)$, then

$$\mu(\mathrm{PG}(n,q)) \geqslant \frac{2nq\sqrt[n]{\frac{1}{2}}}{e}.$$

Proof. In Proposition 33, let $u = n$ and $s = \frac{1}{2}$. Then

$$\mu(\mathrm{PG}(n,q)) \geqslant \min\left\{(n+1)q\ ,\ \frac{2nq\sqrt[n]{\frac{1}{2}}}{e}\right\}.$$

It is easy to prove that $\frac{2nq\sqrt[n]{\frac{1}{2}}}{e} \leqslant (n+1)q$.

The lower bound for the cardinality of a split-resolving can be proved in the same way. $\square$

Next we give better but more complex lower bounds for the (split) metric dimension of $\mathrm{PG}(n,q)$.

Theorem 35. *If $q \geqslant 3$, $n \geqslant 3$ and $\frac{n}{2} \geqslant \frac{\ln q+1}{\ln q-1}$, then*

$$\mu(\mathrm{PG}(n,q)) \geqslant 2\left\lfloor\frac{n\ln q}{1+\ln q}\right\rfloor q\left(\frac{2}{n}\right)^{\frac{2}{n-2}}.$$

Proof. In Proposition 33, let $u = \left\lfloor \frac{n \ln q}{1+\ln q} \right\rfloor$. Then

$$\frac{q^{\frac{n}{u}}}{e} \geqslant \frac{q^{\frac{1+\ln q}{\ln q}}}{e} = \frac{q\ q^{\frac{1}{\ln q}}}{e} = q, \tag{6}$$

and, moreover, we have

$$\sqrt[u]{1-s} = (1-s)^{\frac{1}{\left\lfloor \frac{n \ln q}{1+\ln q} \right\rfloor}} \geqslant (1-s)^{\frac{1}{\frac{n \ln q}{1+\ln q}-1}} = (1-s)^{\frac{1+\ln q}{(n-1)\ln q - 1}}.$$

We give an upper bound to $\frac{1+\ln q}{(n-1)\ln q-1}$. Due to the assumption

$$\frac{n}{2} \geqslant \frac{\ln q + 1}{\ln q - 1},$$

by rearranging, we obtain

$$n \leqslant n \ln q - 2\ln q - 2,$$

or, equivalently,

$$n + n\ln q \leqslant 2n \ln q - 2\ln q - 2,$$

thus

$$\frac{1+\ln q}{(n-1)\ln q - 1} \leqslant \frac{2}{n}.$$

Using this, we get the lower bound

$$\begin{aligned}
\mu(\mathrm{PG}(n,q)) &\geqslant \min\left\{ 2(u+1)sq\ ,\ \frac{2uq^{\frac{n}{u}}\sqrt[u]{1-s}}{e} \right\} \\
&\geqslant \min\left\{ 2sq\left(\left\lfloor \frac{n\ln q}{1+\ln q} \right\rfloor + 1 \right)\ ,\ 2(1-s)^{\frac{1+\ln q}{(n-1)\ln q-1}} \left\lfloor \frac{n\ln q}{1+\ln q} \right\rfloor q \right\} \\
&\geqslant 2q \left\lfloor \frac{n\ln q}{1+\ln q} \right\rfloor \min\left\{ s, (1-s)^{\frac{2}{n}} \right\}.
\end{aligned} \tag{7}$$

Let $s = \left(\frac{n}{2}\right)^{-\frac{1}{\frac{n}{2}-1}}$. We will soon see that with this choice of s, $s \leqslant (1-s)^{\frac{2}{n}}$, and so the assertion of the Theorem follows with substituting s into (7). Now $s \leqslant (1-s)^{\frac{2}{n}}$ is equivalent to

$$s^{\frac{n}{2}} + s - 1 \leqslant 0.$$

By substituting s we get

$$\left(\frac{n}{2}\right)^{-\frac{\frac{n}{2}}{\frac{n}{2}-1}} + \left(\frac{n}{2}\right)^{-\frac{1}{\frac{n}{2}-1}} - 1 \leqslant 0.$$

Let $m = \frac{n}{2}$. Then we get the inequality

$$m^{-\frac{m}{m-1}} + m^{-\frac{1}{m-1}} - 1 \leqslant 0.$$

By Lemma 20, this holds if $m > 1$, that is, $n > 2$.

We also have to prove that $4nq \leqslant \frac{(1-s)(q^n-1)}{q-1}$. Let us substitute $s = \left(\frac{n}{2}\right)^{-\frac{1}{\frac{n}{2}-1}}$. Then $s \leqslant (1-s)^{\frac{2}{n}}$ or, equivalently, $\left(\frac{n}{2}\right)^{-\frac{n}{n-2}} \leqslant 1-s$. It is enough to prove that

$$\left(\frac{n}{2}\right)^{\frac{n}{n-2}} \frac{q^n-1}{q-1} \geqslant 4nq. \tag{8}$$

If $n = 3$, then the inequality can be reduced to

$$27(q^2+q+1) \geqslant 96q,$$

which holds for every $q \geqslant 3$. Let $n \geqslant 4$. Dividing the inequality (8) by $\frac{n}{2}$, we get

$$\left(\frac{n}{2}\right)^{\frac{2}{n-2}} \frac{q^n-1}{q-1} \geqslant 8q.$$

This holds if

$$\left(\frac{n}{2}\right)^{\frac{2}{n-2}} q^{n-1} \geqslant 8q,$$

which, by rearranging, is equivalent to

$$q \geqslant \sqrt[n-2]{8}\left(\frac{2}{n}\right)^{\frac{2}{(n-2)^2}}.$$

This is implied by

$$q \geqslant \sqrt[n-2]{8}$$

which is obviously true if $n \geqslant 4$ and $q \geqslant 3$. □

Theorem 36. *If $n \geqslant 3$ and $\frac{n}{2} \geqslant \frac{\ln q+1}{\ln q-1}$, then*

$$\mu^*(\mathrm{PG}(n,q)) \geqslant 2\left\lfloor \frac{n \ln q}{1+\ln q} \right\rfloor q \left(\frac{n}{2}\right)^{-\frac{2}{n-2}}.$$

Proof. It can be proven as the previous Theorem. □

Since $\lim_{q\to\infty} \frac{\ln q}{1+\ln q} = 1$ and $\lim_{n\to\infty} \left(\frac{n}{2}\right)^{-\frac{2}{n-2}} = 1$, Theorem 35 yields the following.

Corollary 37. *For any $\varepsilon > 0$, there exist q_0 and n_0 such that if $q > q_0$ and $n > n_0$, then*

$$\mu^*(\mathrm{PG}(n,q)) \geqslant \mu(\mathrm{PG}(n,q)) > (2-\varepsilon)nq.$$

For projective spaces over the binary field, the only known results are $\mu(\mathrm{PG}(2,2)) = 5$ (easy to check manually) and $\mu(\mathrm{PG}(n,2)) = 2(n+1)$ for $n = 3, 4$ (verified by computer search in [4]). We determine $\mu(\mathrm{PG}(n,2))$ and $\mu^*(\mathrm{PG}(n,2))$ for all n.

Theorem 38. $\mu^*(\mathrm{PG}(2,2)) = 6$. *If* $n \geqslant 3$*, then*

$$\mu(\mathrm{PG}(n,2)) = \mu^*(\mathrm{PG}(n,2)) = 2(n+1).$$

Moreover, if $n \geqslant 3$ *and* S *is a resolving set of size* $2(n+1)$*, then* S *consists of* $n+1$ *points not contained in a hyperplane and* $n+1$ *hyperplanes whose intersection is empty.*

Proof. Let $\{P_1, P_2, \ldots, P_{n+1}\}$ be a set of points in general position, that is, not contained in a hyperplane. We prove by induction that this set resolves the set of hyperplanes.

It is obviously true for $n = 2$. Suppose $n \geqslant 3$. Let $\alpha, \beta \in \mathcal{H}$ be two arbitrary, distinct hyperplanes. If α and β intersect the hyperplane spanned by $P_1, P_2, \ldots, P_n$ in distinct $(n-2)$-dimensional hyperplanes α' and β', then by induction, there is a point $P_i, (1 \leqslant i \leqslant n)$ that is incident with either α' or β', but not with both.

If α and β intersect the hyperplane spanned by $P_1, P_2, \ldots, P_n$ in the same $(n-2)$-dimensional hyperplane γ, then there exists a point $P_i \in \{P_1, P_2, \ldots, P_n\}$ that is not on γ. In this case, α and β intersect the line P_iP_{n+1} in different points. This line has three points and P_i lies in neither α nor β, thus P_{n+1} lies in exactly one of them.

If α is the hyperplane spanned by the points $P_1, P_2, \ldots, P_n$, then one of these points does not lie in β.

By duality, there are $n+1$ hyperplanes that resolve the set of points and therefore $\mu(\mathrm{PG}(n,2)) \leqslant \mu^*(\mathrm{PG}(n,2)) \leqslant 2(n+1)$.

We prove that $2(n+1)$ is also a lower bound and, moreover, if $n \geqslant 3$, then an optimal (split) resolving set has to be the same construction that we presented to get an upper bound. Let S be an optimal resolving set. Then $|S| \leqslant 2(n+1)$. If $\mathcal{P}_S$ spans $\mathrm{PG}(n,2)$ and $\mathcal{H}_S$ spans the dual of $\mathrm{PG}(n,2)$, then $|\mathcal{P}_S| \geqslant n+1$ and $|\mathcal{H}_S| \geqslant n+1$, so $|S| \geqslant 2(n+1)$ and we are done.

Suppose now to the contrary that, say, $\mathcal{P}_S$ spans a subspace whose dimension is at most $n-1$. Let γ be a hyperplane that contains the points of $\mathcal{P}_S$. Let δ be an $(n-2)$-dimensional hyperplane in γ. There are three hyperplanes that contain δ, one of them is γ itself. One of the other two has to be in $\mathcal{H}_S$ because they intersect the set $\mathcal{P}_S$ in the same points. The subspace δ can be chosen in $2^n - 1$ ways, thus $|\mathcal{H}_S| \geqslant 2^n - 1$. If $n \geqslant 4$, then $|S| \geqslant |\mathcal{H}_S| \geqslant 2^n - 1 > 2(n+1)$, a contradiction. If $n = 3$, then Theorem 15 and $|S| \leqslant 8$ give $|\mathcal{P}_S| \geqslant \lceil \log_2(15-8) \rceil = 3$. Thus $|S| = |\mathcal{P}_S| + |\mathcal{H}_S| \geqslant 3 + 2^3 - 1 = 10 > 8 = 2(n+1)$, a contradiction. □

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