Fourier Analysis on Distance-Regular Cayley Graphs over Abelian Groups

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Abstract

The problem of constructing or characterizing strongly regular Cayley graphs (or equivalently, regular partial difference sets) has garnered significant attention over the past half-century. A classic result in this area is the complete classification of strongly regular Cayley graphs over cyclic groups, which was established by Bridges and Mena (1979), independently by Ma (1984), and partially by Marušič (1989). Miklavič and Potočnik (2003) extended this work by providing a complete characterization of distance-regular Cayley graphs over cyclic groups through the method of Schur rings. Building on this, Miklavič and Potočnik (2007) formally posed the problem of characterizing distance-regular Cayley graphs for arbitrary classes of groups. Within this framework, abelian groups are of particular significance, as many distance-regular graphs with classical parameters are Cayley graphs over abelian groups. In this paper, we employ Fourier analysis on abelian groups to establish connections between distance-regular Cayley graphs over abelian groups and combinatorial objects in finite geometry. By combining these insights with classical results from finite geometry, we classify all distance-regular Cayley graphs over the group $\mathbb{Z}_n \oplus \mathbb{Z}_p$, where n is a positive integer and p is an odd prime.

Mathematics Subject Classifications: 05E30, 05C50, 05C25

1 Introduction

In graph theory, distance-regular graphs form a class of regular graphs with strong combinatorial symmetry. A connected graph Γ is distance-regular if, for each triple of non-negative integers i, j and k, and for each pair of vertices u and v at distance k in Γ, the number of vertices at distance i from u and distance j from v depends only on i, j and k,

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and not on the particular choice of u and v. While this defining condition is purely combinatorial, the concept of distance-regular graphs holds fundamental importance in design theory and coding theory. Furthermore, it exhibits deep connections with diverse mathematical disciplines including finite group theory, finite geometry, representation theory and association schemes [8, 51].

Within the study of distance-regular graphs, the characterization and construction of graphs with specific types or parameters constitute essential research problems. Cayley graphs — vertex-transitive graphs defined through groups and their subsets — emerge as natural candidates for such investigations. This relevance stems from two observations: most known distance-regular graphs are vertex-transitive [50], and numerous infinite families of strongly regular graphs (the diameter-2 case of distance-regular graphs) arise from Cayley graph constructions [6, 9, 12, 13, 14, 23, 24, 25, 27, 28, 33, 34, 35, 36, 37, 38, 42, 46].

Let G be a finite group with identity e and let S be an inverse closed subset of $G \setminus \{e\}$. The *Cayley graph* $\mathrm{Cay}(G,S)$ is defined as the graph with vertex set G, where two vertices g and h are adjacent if and only if $g^{-1}h \in S$. The set S is referred to as the *connection set* of $\mathrm{Cay}(G,S)$. It is well-known that $\mathrm{Cay}(G,S)$ is connected if and only if $\langle S \rangle = G$, and that G acts regularly on the vertex set of $\mathrm{Cay}(G,S)$ via left multiplication.

In 2007, Miklavič and Potočnik [30] (see also [51, Problem 71]) proposed the problem of characterizing distance-regular Cayley graphs:

Problem 1. For a class of groups $\mathcal{G}$, determine all distance-regular graphs, which are Cayley graphs on a group in $\mathcal{G}$.

Early progress on Problem 1 was made by Miklavič and Potočnik [29], who classified distance-regular Cayley graphs over cyclic groups (known as *circulants*) using the framework of Schur rings.

Theorem 2 ([29, Theorem 1.2, Corollary 3.7])**.** *Let* Γ *be a circulant on* n *vertices. Then* Γ *is distance-regular if and only if it is isomorphic to one of the following graphs:*

(i) *the cycle* C_n*;*

(ii) *the complete graph* K_n*;*

(iii) *the complete multipartite graph* $K_{t\times m}$*, where* $tm = n$*;*

(iv) *the complete bipartite graph without a perfect matching* $K_{m,m} - mK_2$*, where* $2m = n$ *and* m *is odd;*

(v) *the Paley graph* $P(n)$*, where* $n \equiv 1 \pmod 4$ *is a prime.*

In particular, Γ *is a primitive distance-regular graph if and only if* $\Gamma \cong K_n$*, or* n *is a prime and* $\Gamma \cong C_n$ *or* $P(n)$*.*

Subsequently, Miklavič and Potočnik [30] extended their approach by combining Schur rings with Fourier analysis to characterize distance-regular Cayley graphs over dihedral groups through difference sets. Further advancements were achieved by Miklavič and Šparl [31, 32], who employed elementary group theory and structural analysis to classify distance-regular Cayley graphs over abelian groups and generalized dihedral groups under minimality conditions on their connection sets. Significant contributions include the work of Abdollahi, van Dam and Jazaeri [1], who classified distance-regular Cayley graphs of diameter at most 3 with least eigenvalue -2. van Dam and Jazaeri [48, 49] later determined some distance-regular Cayley graphs with small valency and provided some characterizations for bipartite distance-regular Cayley graphs with diameter 3 or 4. For additional results on distance-regular Cayley graphs, including recent developments, we refer to [17, 18, 19, 53].

As is well-known, an effective approach for constructing distance-regular Cayley graphs, particularly strongly regular graphs, involves utilizing Cayley graphs over abelian groups. For instance, numerous infinite families of strongly regular Cayley graphs over the additive group of finite fields have been constructed via methods such as cyclotomic classes [13, 12], Gauss sums with even indices [52], three-valued Gauss periods [35] and p-ary (weakly) regular bent functions [9, 46, 42]. Furthermore, it is known that several important classes of distance-regular graphs with classical parameters — such as Hamming graphs, halved cubes, bilinear forms graphs, alternating forms graphs, Hermitian forms graphs, affine $E_6(q)$ graphs and extended ternary Golay code graphs — are distance-regular Cayley graphs over abelian groups (cf. [7, p. 194]). However, providing a complete solution to Problem 1 for general abelian groups remains challenging.

In this paper, we investigate distance-regular Cayley graphs over abelian groups with small diameters. We establish necessary conditions for their existence, which are closely connected to finite geometry (see Sections 5 and 6 for details). Moreover, we demonstrate that these necessary conditions prove particularly useful for the following significant class of abelian groups.

Problem 3. Let n and m be positive integers with $\gcd(n,m) \neq 1$. Characterize all distance-regular Cayley graphs over the group $\mathbb{Z}_n \oplus \mathbb{Z}_m$.

Thus far, significant progress has been made toward resolving Problem 3. Let n be a positive integer and let p be an odd prime. In 2005, Leifman and Muzychuck [23] classified strongly regular Cayley graphs over $\mathbb{Z}_{p^s} \oplus \mathbb{Z}_{p^s}$. Recently, the authors [53] characterized all distance-regular Cayley graphs over $\mathbb{Z}_{p^s} \oplus \mathbb{Z}_p$ and $\mathbb{Z}_n \oplus \mathbb{Z}_2$. In this work, we extend these results by providing a complete classification of distance-regular Cayley graphs over $\mathbb{Z}_n \oplus \mathbb{Z}_p$. Of course, $\mathbb{Z}_n \oplus \mathbb{Z}_p$ becomes cyclic when $p \nmid n$. Hence, because of Theorem 2, we focus exclusively on the case when $p \mid n$. Our main result is stated as follows.

Theorem 4. *Let p be an odd prime and let Γ be a Cayley graph over $\mathbb{Z}_n \oplus \mathbb{Z}_p$ with $p \mid n$. Then Γ is distance-regular if and only if it is isomorphic to one of the following graphs:*

(i) the complete graph K_{np};

(ii) the complete multipartite graph $K_{t\times m}$ with $tm = np$, which is the complement of the union of t copies of K_m;

(iii) the complete bipartite graph without a 1-factor $K_{\frac{np}{2},\frac{np}{2}} - \frac{np}{2}K_2$, where $n \equiv 2 \pmod 4$;

(iv) the graph $\mathrm{Cay}(\mathbb{Z}_p \oplus \mathbb{Z}_p, S)$ *with* $S = \cup_{i=1}^{r} H_i \setminus \{(0,0)\}$ *for some* $2 \leqslant r \leqslant p-1$*, where* H_i *(*$i = 1, \ldots, r$*) are subgroups of* $\mathbb{Z}_p \oplus \mathbb{Z}_p$ *with order* p*.*

In particular, each graph in (iv) is the line graph of a transversal design $TD(r,p)$*, which is a strongly regular graph with parameters* $(p^2, r(p-1), p + r^2 - 3r, r^2 - r)$*.*

Our approach. The proof of Theorem 4 is inspired by the work of Miklavič and Potočnik [30] but requires substantial new ideas. If the graph Γ is primitive, then the result follows immediately from [53, Lemma 3.1] and the theory of Schur rings. The majority of the work concerns the case where Γ is imprimitive. Here, by [7, Theorem 4.2.1], the analysis is divided into three distinct cases: Γ is antipodal but not bipartite; Γ is antipodal and bipartite; and Γ is bipartite but not antipodal. In each of these cases, we study the antipodal quotient or the halved graph of Γ. This reduction process ultimately leads to three particularly challenging subcases: (i) Γ is an antipodal non-bipartite graph of diameter 3; (ii) Γ is an antipodal bipartite graph of diameter 4; (iii) the halved graph of Γ is isomorphic to the line graph of a transversal design. The core novelty of our proof lies in the resolution of these three subcases in Lemma 32, Lemma 33 and Lemma 25, respectively. To handle subcases (i) and (ii) (see Lemmas 32 and 33), we first employ character equations to derive necessary conditions for distance-regularity. We then develop new computational techniques for Fourier analysis over abelian groups in Sections 5 and 6. Leveraging these computational results, we demonstrate a fundamental connection between the structure of Γ and certain combinatorial objects from finite geometry, specifically relative difference sets and polynomial addition sets. This connection allows us to complete the classification for these subcases. For subcase (iii) (see Lemma 25), a different strategy is required. We translate the graph-theoretic problem into a geometric one by showing that the existence of such a graph Γ implies the existence of a specific configuration within the Desarguesian affine plane. A detailed geometric argument then shows that this configuration cannot exist, thus resolving this subcase.

The paper is organized as follows. In Section 2, we review fundamental results on association schemes and distance-regular graphs. Section 3 presents algebraic characterizations for distance-regular Cayley graphs established by Miklavič and Potočnik. In Section 4, we introduce key combinatorial objects and classical theorems from finite geometry. Sections 5 and 6 utilize Fourier analysis on abelian groups to derive necessary conditions for the existence of distance-regular Cayley graphs over abelian groups with small diameter. Finally, Section 7 provides a complete proof of Theorem 4.

2 Association schemes and distance-regular graphs

In this section, we introduce some notations and properties related to association schemes and distance-regular graphs.

2.1 Association schemes

Suppose that X is a finite set and $\mathfrak{X} = (X, \mathfrak{R} = \{R_0, R_1, \ldots, R_d\})$ is a commutative association scheme of class d on X, where $R_i \subset X \times X$ is the i-th relation (see [4, Section

2.2] for the definition). Let $M_X(\mathbb{C})$ be the full matrix algebra of $|X| \times |X|$-matrices over the complex field $\mathbb{C}$ whose rows and columns are indexed by the elements of X. For each $i \in \{0, 1, \ldots, d\}$, the *adjacency matrix* $A_i \in M_X(\mathbb{C})$ of the relation R_i is defined as:

$$A_i(x, y) = \begin{cases} 1, & \text{if } (x, y) \in R_i, \\ 0, & \text{if } (x, y) \notin R_i. \end{cases}$$

According to [4, Section 2.2], the matrices A_i satisfy the following properties:

(I) $A_0 = I$, where I is the identity matrix of order $|X|$;

(II) $A_0 + A_1 + \cdots + A_d = J$, where J is the all-ones matrix of order $|X|$;

(III) for each $i \in \{0, \ldots, d\}$, there exists some $i' \in \{0, \ldots, d\}$ such that $A_i^T = A_{i'}$;

(IV) for any $i, j \in \{0, \ldots, d\}$, there exist non-negative integers $p_{i,j}^k$ (called *intersection numbers*) with $0 \leqslant k \leqslant d$ such that $A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k$;

(V) for any $i, j \in \{0, \ldots, d\}$, we have $A_i A_j = A_j A_i$.

Let $\mathfrak{A}$ be the linear subspace of $M_X(\mathbb{C})$ spanned by the adjacency matrices $A_0, A_1, \ldots, A_d$ of X. By (IV) and (V), $\mathfrak{A}$ is a $(d+1)$-dimensional commutative subalgebra of $M_X(\mathbb{C})$ under the ordinary multiplication. Moreover, by (II), for any $i, j \in \{0, 1, \ldots, d\}$, we have

$$A_i \circ A_j = \delta_{i,j} A_i,$$

where '$\circ$' denotes the Hadamard product, $\delta_{i,j} = 1$ if $i = j$, and $\delta_{i,j} = 0$ otherwise. This implies that $\mathfrak{A}$ is also a commutative subalgebra of $M_X(\mathbb{C})$ under the Hadamard product. Thus $\mathfrak{A}$ has two algebraic structures, and is called the *Bose–Mesner algebra*. It is known that $\mathfrak{A}$ is semisimple, and so there exists a basis of primitive idempotents $E_0 = \frac{1}{|X|} J, E_1, \ldots, E_d$ in $\mathfrak{A}$. That is, every matrix in $\mathfrak{A}$ can be expressed as a linear combination of $E_0, E_1, \ldots, E_d$, and it holds that $\sum_{i=0}^{d} E_i = I$ and $E_i E_j = \delta_{i,j} E_i$ for all $i, j \in \{0, \ldots, d\}$ (cf. [39, Section 2.3]). This implies the existence of complex numbers $P_i(j) \in \mathbb{C}$ such that $A_i E_j = P_i(j) E_j$ for all $i, j \in \{0, \ldots, d\}$. For any fixed $i \in \{0, 1, \ldots, d\}$, the values $P_i(0), P_i(1), \ldots, P_i(d)$ constitute a complete set of the eigenvalues of A_i (cf. [5, p. 58], [39, Section 2.3]), and furthermore,

$$A_i = \sum_{j=0}^{d} P_i(j) E_j.$$

On the other hand, since $\mathfrak{A}$ is closed under the Hadamard product, for any $i, j \in \{0, 1, \ldots, d\}$, there exist constants $q_{i,j}^k$ $(0 \leqslant k \leqslant d)$ such that

$$E_i \circ E_j = \frac{1}{|X|} \sum_{k=0}^{d} q_{i,j}^k E_k.$$

The structure constants $q_{i,j}^k$ $(0 \leqslant i, j, k \leqslant d)$ of $\mathfrak{A}$ with respect to the Hadamard product are called the *Krein parameters*. According to [4, Chapter II, Theorem 3.8], the Krein parameters $q_{i,j}^k$ are non-negative real numbers.

2.2 Schur ring and its duality

Let G be a finite group and let $\mathbb{Z}G$ denote the group algebra of G over the ring of integers $\mathbb{Z}$. For a subset $S \subseteq G$, let $\underline{S}$ denote the element $\sum_{s \in S} s$ of $\mathbb{Z}G$. In particular, if S contains exactly one element s, we write s instead of $\underline{S}$ for simplicity. For an integer m and an element $\sum_{g \in G} r_g g \in \mathbb{Z}G$, we define

$$\left(\sum_{g \in G} r_g g\right)^{(m)} = \sum_{g \in G} r_g g^m \in \mathbb{Z}G.$$

Suppose that $\{N_0, N_1, \ldots, N_d\}$ is a partition of G satisfying

(i) $N_0 = \{e\}$;

(ii) for any $i \in \{1, \ldots, d\}$, there exists some $j \in \{1, \ldots, d\}$ such that $\underline{N_i}^{(-1)} = \underline{N_j}$;

(iii) for any $i, j \in \{1, \ldots, d\}$, there exist integers $p_{i,j}^k$ $(0 \leqslant k \leqslant d)$ such that

$$\underline{N_i} \cdot \underline{N_j} = \sum_{k=0}^{r} p_{i,j}^k \cdot \underline{N_k}.$$

Then the $\mathbb{Z}$-module $\mathcal{S}(G)$ spanned by $\underline{N_0}, \underline{N_1}, \ldots, \underline{N_d}$ is a subalgebra of $\mathbb{Z}G$, and is called a *Schur ring* over G. In this situation, the basis $\{\underline{N_0}, \underline{N_1}, \ldots, \underline{N_d}\}$ is called the *simple basis* of the Schur ring $\mathcal{S}(G)$. We say that the Schur ring $\mathcal{S}(G)$ is *primitive* if $\langle N_i \rangle = G$ for every $i \in \{1, \ldots, d\}$. In particular, if $N_0 = \{e\}$ and $N_1 = G \setminus \{e\}$, then the Schur ring spanned by $\underline{N_0}$ and $\underline{N_1}$ is called *trivial*. Clearly, a trivial Schur ring is primitive.

Now suppose that G is an abelian group. For convenience, we express G as

$$G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r},$$

where for each $1 \leqslant i \leqslant r$, n_i is a prime power. It is clear that the order of G is the product $|G| = n_1 n_2 \cdots n_r$. We note that every element $g \in G$ can be uniquely represented as a tuple $g = (g_1, g_2, \ldots, g_r)$, with each $g_i \in \mathbb{Z}_{n_i}$ for $1 \leqslant i \leqslant r$. For an element $g = (g_1, g_2, \ldots, g_r) \in G$, we define χ_g as the function from G to $\mathbb{C}$ by letting

$$\chi_g(x) = \prod_{i=1}^{r} \zeta_{n_i}^{g_i x_i}, \text{ for all } x = (x_1, x_2, \ldots, x_r) \in G, \tag{1}$$

where ζ_{n_i} denotes a primitive n_i-th root of unity.

Let $\mathcal{S}(G) = \mathrm{span}\{\underline{N_0}, \underline{N_1}, \ldots, \underline{N_d}\}$ be a Schur ring over the abelian group G. For any $i \in \{0, 1, \ldots, d\}$, we denote by $R_i = \{(g, h) \mid h^{-1}g \in N_i\}$. Then $\mathfrak{X} = (G, \mathfrak{R} = \{R_0, R_1, \ldots, R_d\})$ is a commutative association scheme of class d on G (cf. [4, p. 105]). Moreover, if N_i is inverse closed for each $i \in \{0, 1, \ldots, d\}$, then $\mathfrak{X}$ is a symmetric association scheme (see (cf. [4, Section 2.2]) for the definition). The intersection numbers and Krein parameters of $\mathfrak{X}$ are also called the *intersection numbers* and *Krein parameters* of $\mathcal{S}(G)$, respectively. We have the following classic results about Schur rings over abelian groups.

Lemma 5 ([40, Theorem 3.4]). *Let G be an abelian group of composite order with at least one cyclic Sylow subgroup. Then there is no non-trivial primitive Schur ring over G.*

Lemma 6 ([22, Kochendorfer's theorem]). *Let p be a prime and let a, b be positive integers with $a \neq b$. Then there is no non-trivial primitive Schur ring over $\mathbb{Z}_{p^a} \oplus \mathbb{Z}_{p^b}$.*

Lemma 7 ([4, Chapter II, Theorem 6.3]). *Let G be an abelian group and let $\mathcal{S}(G) = \text{span}\{\underline{N_0}, \underline{N_1}, \ldots, \underline{N_d}\}$ be a Schur ring over G. Let $\mathcal{R}$ be the equivalence relation on G defined by $g\mathcal{R}h$ if and only if $\chi_g(\underline{N_i}) = \chi_h(\underline{N_i})$ for all $i \in \{0, 1, \ldots, d\}$. If $E_0, E_1, \ldots, E_f$ are the equivalence classes of G with respect to $\mathcal{R}$, then $f = d$ and the $\mathbb{Z}$-submodule $\widehat{\mathcal{S}}(G) = \text{span}\{\underline{E_0}, \underline{E_1}, \ldots, \underline{E_d}\}$ of $\mathbb{Z}G$ is a Schur ring over G with intersection numbers $q_{i,j}^k$, where $q_{i,j}^k$ $(0 \leqslant i, j, k \leqslant d)$ are the Krein parameters of $\mathcal{S}(G)$.*

The Schur ring $\widehat{\mathcal{S}}(G)$ defined in Lemma 7 is called the *dual* of $\mathcal{S}(G)$. Note that Lemma 7 implies that the Krein parameters of $\mathcal{S}(G)$ are integers.

2.3 Distance-regular graphs

Let Γ be a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The *distance* $\partial_\Gamma(x, y)$ between two vertices x, y of Γ is the length of a shortest path connecting them in Γ, and the *diameter* d_Γ of Γ is the maximum value of the distances between vertices of Γ. For $x \in V(\Gamma)$, let $S_i^\Gamma(x)$ denote the set of vertices at distance i from x in Γ. In particular, we denote $S^\Gamma(x) = S_1^\Gamma(x)$. When Γ is clear from the context, we use $\partial(x, y)$, d, $S_i(x)$ and $S(x)$ instead of $\partial_\Gamma(x, y)$, d_Γ, $S_i^\Gamma(x)$ and $S^\Gamma(x)$, respectively. For $x, y \in V(\Gamma)$ with $\partial(x, y) = i$ $(0 \leqslant i \leqslant d)$, let

$$c_i(x, y) = |S_{i-1}(x) \cap S(y)|, \quad a_i(x, y) = |S_i(x) \cap S(y)|, \quad b_i(x, y) = |S_{i+1}(x) \cap S(y)|.$$

Here $c_0(x, y) = b_d(x, y) = 0$. The graph Γ is called *distance-regular* if $c_i(x, y)$, $b_i(x, y)$ and $a_i(x, y)$ only depend on the distance i between x and y but not on the choice of x, y.

For a distance-regular graph Γ with diameter d, we denote $c_i = c_i(x, y)$, $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$, where $x, y \in V(\Gamma)$ and $\partial(x, y) = i$. Note that $c_0 = b_d = 0$, $a_0 = 0$ and $c_1 = 1$. Also, we set $k_i = |S_i(x)|$, where $x \in V(\Gamma)$. Clearly, k_i is independent of the choice of x. By definition, Γ is a regular graph with valency $k = b_0$, and $a_i + b_i + c_i = k$ for $0 \leqslant i \leqslant d$. The array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is called the *intersection array* of Γ. In particular, $\lambda = a_1$ is the number of common neighbors of two adjacent vertices in Γ and $\mu = c_2$ is the number of common neighbors of two vertices at distance 2 in Γ. A distance-regular graph on n vertices with valency k and diameter 2 is called a *strongly regular graph* with parameters $(n, k, \lambda = a_1, \mu = c_2)$.

Suppose that Γ is a distance-regular graph of diameter d with vertex set $X = V(\Gamma)$ and edge set $R = E(\Gamma)$. For $0 \leqslant i \leqslant d$, we define

$$R_i = \{(x, y) \in X \times X \mid \partial(x, y) = i\}.$$

Then $\mathfrak{X} = (X, \mathfrak{R} = \{R_0, R_1, \ldots, R_d\})$ is a symmetric (necessarily commutative) association scheme of class d on X (cf. [4, Section 2.2, Section 3.1]). In this context, the intersection numbers $p^k_{i,j}$ and Krein parameters $q^k_{i,j}$ of $\mathfrak{X}$ are also called the *intersection numbers* and *Krein parameters* of Γ, respectively. Note that $p^i_{1,i+1} = b_i$, $p^i_{1,i} = a_i$ and $p^i_{1,i-1} = c_i$ for $0 \leqslant i \leqslant d$. Additionally, for $i, j, k \in \{0, 1, \ldots, d\}$, if $p^k_{i,j} \neq 0$ then $k \leqslant i + j$, and moreover, $p^{i+j}_{i,j} \neq 0$ (cf. [4, Section 3.1]).

A symmetric association scheme together with an ordering of relations is called *P-polynomial* if $p^k_{i,j} \neq 0$ implies $k \leqslant i+j$ for all $i, j, k \in \{0, 1, \ldots, d\}$, and also $p^{i+j}_{i,j} \neq 0$ for all $i, j \in \{0, 1, \ldots, d\}$ (cf. [7, Section 2.7]). By definition, the symmetric association scheme derived from a distance-regular graph is P-polynomial. Conversely, every P-polynomial association scheme is derived from a distance-regular graph. Therefore, a distance-regular graph is equivalent to a P-polynomial association scheme.

Lemma 8 ([7, Proposition 2.7.1])**.** *Let $\mathfrak{X} = (X, \mathfrak{R})$ be a symmetric association scheme with an ordering of relations $R_0, R_1, \ldots, R_d$. Then $\mathfrak{X}$ is P-polynomial if and only if (X, R_1) is a distance-regular graph.*

Analogously, a symmetric association scheme together with an ordering of primitive idempotents is called *Q-polynomial* if $q^k_{i,j} \neq 0$ implies $k \leqslant i+j$ for all $i, j, k \in \{0, 1, \ldots, d\}$, and also $q^{i+j}_{i,j} \neq 0$ for all $i, j \in \{0, 1, \ldots, d\}$ (cf. [7, Section 2.7]). In particular, we say that a distance-regular graph is *Q-polynomial* if the symmetric association scheme derived from it is Q-polynomial.

A P-polynomial (resp. Q-polynomial) association scheme is called *bipartite* (resp. *Q-bipartite*) if $p^k_{i,j} = 0$ (resp. $q^k_{i,j} = 0$) whenever $i + j + k$ is odd. A P-polynomial (resp. Q-polynomial) association scheme is called *antipodal* (resp. *Q-antipodal*) if $p^k_{d,d} = 0$ (resp. $q^k_{d,d} = 0$) whenever $k \notin \{0, d\}$.

Lemma 9 ([7, p. 241])**.** *Let Γ be a Q-polynomial distance-regular graph. Then Γ is bipartite (resp. antipodal) if and only if Γ is Q-antipodal (resp. Q-bipartite), that is, the symmetric association scheme derived from Γ is Q-antipodal (resp. Q-bipartite).*

2.4 Primitivity of distance-regular graphs

Let Γ be a graph and let $\mathcal{B} = \{B_1, \ldots, B_\ell\}$ be a partition of $V(\Gamma)$ (here B_i are called *blocks*). The *quotient graph* of Γ with respect to $\mathcal{B}$, denoted by $\Gamma_{\mathcal{B}}$, is the graph with vertex set $\mathcal{B}$, and with B_i, B_j $(i \neq j)$ adjacent if and only if there exists at least one edge between B_i and B_j in Γ. Moreover, we say that $\mathcal{B}$ is an *equitable partition* of Γ if there are integers $b_{i,j}$ $(1 \leqslant i, j \leqslant \ell)$ such that every vertex in B_i has exactly $b_{i,j}$ neighbors in B_j. In particular, if every block of $\mathcal{B}$ is an independent set, and between any two blocks there are either no edges or there is a perfect matching, then $\mathcal{B}$ is an equitable partition of Γ. In this situation, Γ is called a *cover* of its quotient graph $\Gamma_{\mathcal{B}}$, and the blocks are called *fibres*. If $\Gamma_{\mathcal{B}}$ is connected, then all fibres have the same size, say r, which is called the *covering index* [21].

A graph Γ with diameter d is *antipodal* if the relation $\mathcal{R}$ on $V(\Gamma)$ defined by $x\mathcal{R}y \Leftrightarrow \partial(x, y) \in \{0, d\}$ is an equivalence relation. Under this equivalence relation, the corresponding equivalence classes are called *antipodal classes*. A cover of index r, in which the fibres are antipodal classes, is called *an r-fold antipodal cover* of its quotient. In particular, if Γ is an antipodal distance-regular graph with diameter d, then all antipodal classes have the same size, say r, and form an equitable partition $\mathcal{B}^*$ of Γ (cf. [30, Section 2.2]). In this case, we define the *antipodal quotient* of Γ as the quotient graph $\overline{\Gamma} := \Gamma_{\mathcal{B}^*}$. If $d = 2$, then Γ is a complete multipartite graph. If $d \geqslant 3$, then the edges between two distinct antipodal classes of Γ form an empty set or a perfect matching (cf. [30, Section 2.2]). Thus Γ is an r-fold antipodal cover of $\overline{\Gamma}$ with the antipodal classes as its fibres. Moreover, it is known that a distance-regular graph Γ with diameter d is antipodal if and only if $b_i = c_{d-i}$ for every $i \neq \lfloor \frac{d}{2} \rfloor$ (cf. [7, Proposition 4.2.2]).

Let Γ be a distance-regular graph with diameter d. For $i \in \{1, \ldots, d\}$, the *i-th distance graph* Γ_i is the graph with vertex set $V(\Gamma)$ in which two distinct vertices are adjacent if and only if they are at distance i in Γ. If, for each $1 \leqslant i \leqslant d$, Γ_i is connected, then Γ is *primitive*. Otherwise, Γ is *imprimitive*. It is known that an imprimitive distance-regular graph with valency at least 3 is either bipartite, antipodal, or both (cf. [7, Theorem 4.2.1]). Moreover, if Γ is bipartite, then Γ_2 has two connected components (not necessarily isomorphic), which are called the *halved graphs* of Γ and denoted by Γ^+ and Γ^-. For convenience, we assume that $\frac{1}{2}\Gamma$ is one of these two graphs.

For distance-regular Cayley graphs over abelian groups, we have the following result about antipodal quotients and halved graphs.

Lemma 10. *Let G be an abelian group and let Γ be a distance-regular Cayley graph over G. Then the following two statements hold.*

(i) If Γ is antipodal and H is the antipodal class containing the identity vertex e, then H is a subgroup of G, and $\overline{\Gamma}$ is distance-regular and isomorphic to $\mathrm{Cay}(G/H, S/H)$*, where $S/H = \{sH \mid s \in S\}$;*

(ii) If Γ is bipartite and H is the bipartition set containing the identity vertex e, then H is an index 2 subgroup of G, and the halved graphs of Γ are distance-regular and isomorphic to $\mathrm{Cay}(H, S_2(e))$.

Proof. (i) By [7, pp. 140–141], $\overline{\Gamma}$ is distance-regular. Thus it suffices to prove that H is a subgroup of G and that $\overline{\Gamma} \cong \mathrm{Cay}(G/H, S/H)$. Since Γ is antipodal, the relation $\mathcal{R}$ on $V(\Gamma)$ defined by $x\mathcal{R}y \Leftrightarrow \partial(x, y) \in \{0, d\} \Leftrightarrow \partial(y^{-1}x, e) \in \{0, d\} \Leftrightarrow y^{-1}x \in H$ is an equivalence relation. For any $h_1, h_2 \in H$, we have $h_1\mathcal{R}e$ and $e\mathcal{R}h_2$, and hence $h_1\mathcal{R}h_2$, or equivalently, $h_2^{-1}h_1 \in H$. Thus H is a subgroup of G, and the antipodal classes of Γ coincide with the cosets of H in G. For any two vertices xH and yH of $\overline{\Gamma}$, we have that xH and yH are adjacent if and only if there exists some edge between xH and yH in Γ, which is the case if and only if there exist some $h_1, h_2 \in H$ such that $(xh_1)^{-1}yh_2 \in S$, which is the case if and only if $(xH)^{-1}yH \in S/H$. Therefore, we conclude that $\overline{\Gamma} \cong \mathrm{Cay}(G/H, S/H)$, and the result follows.

(ii) By [7, pp. 140–141], the halved graphs Γ^+ and Γ^- are distance-regular. Suppose that $V(\Gamma^+) = H$. In a similar way as in (i), we can prove that H is an index 2 subgroup of G. Thus it remains to show that $\Gamma^+ \cong \Gamma^- \cong \mathrm{Cay}(H, S_2(e))$. For any two vertices $x, y \in V(\Gamma^+) = H$, we have that x, y are adjacent if and only if $\partial(x, y) = 2$, which is the case if and only if $\partial(e, x^{-1}y) = 2$, or equivalently, $x^{-1}y \in S_2(e)$. Therefore, we conclude that $\Gamma^+ \cong \mathrm{Cay}(H, S_2(e))$. Furthermore, as Γ is vertex-transitive, we have $\Gamma^- \cong \Gamma^+$, and the result follows. □

Let $\mathbb{F}_q$ denote the finite field of order q where q is a prime power and $q \equiv 1 \pmod 4$. The *Paley graph* $P(q)$ is defined as the graph with vertex set $\mathbb{F}_q$ in which two distinct vertices u, v are adjacent if and only if $u - v$ is a square in the multiplicative group of $\mathbb{F}_q$. It is known that $P(q)$ is a strongly regular graph with parameters $(q, (q-1)/2, (q-5)/4, (q-1)/4)$ [11].

Lemma 11 ([7, p. 180]). *Let Γ be a Paley graph. Then Γ has no distance-regular r-fold antipodal covers for $r > 1$, except for the pentagon $C_5 \cong P(5)$, which is covered by the decagon C_{10}. Moreover, Γ cannot be a halved graph of a bipartite distance-regular graph.*

The *Hamming graph* $H(n, q)$ is the graph having as vertex set the collection of all n-tuples with entries in a fixed set of size q, where two n-tuples are adjacent when they differ in only one coordinate. Note that $H(2, v)$ is just the lattice graph $K_v \square K_v$, which is the Cartesian product of two copies of K_v.

Lemma 12 ([47, Proposition 5.1]). *Let $n, q \geqslant 2$. Then $H(n, q)$ has no distance-regular r-fold antipodal covers for $r > 1$, except for $H(2, 2)$.*

3 The algebraic characterizations of distance-regular Cayley graphs

In this section, we present several algebraic characterizations for distance-regular Cayley graphs, which were established by Miklavič and Potočnik in [29, 30].

3.1 Schur ring and distance-regular Cayley graphs

Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley graph with diameter d. For $i \in \{0, 1, \dots, d\}$, we denote by $S_i := S_i(e)$. The $\mathbb{Z}$-submodule of $\mathbb{Z}G$ spanned by $\underline{S_0}, \underline{S_1}, \dots, \underline{S_d}$ is called the *distance module* of Γ, and is denoted by $\mathcal{D}_{\mathbb{Z}}(G, S)$.

In [29], Miklavič and Potočnik provided an algebraic characterization for distance-regular Cayley graphs in terms of Schur rings and distance modules.

Lemma 13 ([29, Proposition 3.6]). *Let $\Gamma = \mathrm{Cay}(G, S)$ be a distance-regular Cayley graph and let $\mathcal{D} = \mathcal{D}_{\mathbb{Z}}(G, S)$ denote the distance module of Γ. Then:*

(i) $\mathcal{D}$ is a (primitive) Schur ring over G if and only if Γ is a (primitive) distance-regular graph;

(ii) $\mathcal{D}$ is the trivial Schur ring over G if and only if Γ is isomorphic to the complete graph.

Let $n > 1$ and let p be a prime such that $p \neq n$. If Γ is a primitive distance-regular Cayley graph over $\mathbb{Z}_n \oplus \mathbb{Z}_p$, then its distance module is a primitive Schur ring over $\mathbb{Z}_n \oplus \mathbb{Z}_p$ by Lemma 13 (i), and hence must be the trivial Schur ring by Lemma 5 and Lemma 6. Therefore, by Lemma 13 (ii), we obtain the following result.

Corollary 14. *Let $n > 1$ and let p be a prime such that $p \neq n$. If Γ is a primitive distance-regular Cayley graph over $\mathbb{Z}_n \oplus \mathbb{Z}_p$, then Γ is isomorphic to the complete graph K_{np}.*

As every Cayley graph is vertex-transitive, by the definitions of Cayley graphs and distance-regular graphs, we immediately deduce the following characterization for distance-regular Cayley graphs.

Lemma 15. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph with diameter d. Then Γ is distance-regular with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ if and only if*

$$\begin{cases} \underline{S_1} \cdot \underline{S_1} = b_0 \cdot e + a_1 \underline{S_1} + c_2 \underline{S_2}, \\ \underline{S_2} \cdot \underline{S_1} = b_1 \underline{S_1} + a_2 \underline{S_2} + c_3 \underline{S_3}, \\ \qquad \vdots \\ \underline{S_d} \cdot \underline{S_1} = b_{d-1} \underline{S_{d-1}} + a_d \underline{S_d}. \end{cases} \tag{2}$$

Since $e + \underline{S_1} + \underline{S_2} + \cdots + \underline{S_d} = \underline{G}$, we see that the conclusion of Lemma 15 still holds if we remove an arbitrary equation from (2). Recall that a distance-regular graph with diameter d is antipodal if and only if $b_i = c_{d-i}$ for every $i \neq \lfloor \frac{d}{2} \rfloor$. Additionally, the intersection array of an r-antipodal distance-regular graph with diameter 3 must be of the form $\{k, k - \lambda - 1 = \mu(r-1), 1; 1, \mu, k\}$. Thus, by Lemma 15, we can deduce the following result immediately.

Corollary 16. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph with diameter 3. Then Γ is an antipodal distance-regular graph with intersection array $\{k, k-\lambda-1 = \mu(r-1), 1; 1, \mu, k\}$ if and only if*

$$\begin{cases} \underline{S}^2 = k \cdot e + (\lambda - \mu)\underline{S} + \mu(\underline{G} - \underline{S_3} - e), \\ (\underline{S_3} + e) \cdot (\underline{S} + e) = \underline{G}. \end{cases}$$

3.2 Distance-regular Cayley graphs over abelian groups

Let G be a finite abelian group and let $\mathbb{C}^*$ be the multiplicative group of the complex field $\mathbb{C}$. A *character* of G is a group homomorphism χ from G to $\mathbb{C}^*$. It is known that the set of all characters of G is given by (cf. [44, Sections 4.4–4.5])

$$\widehat{G} = \{\chi_g \mid g \in G\},$$

where χ_g is the function defined in (1).

Let $\mathbb{C}G$ denote the group algebra of G over $\mathbb{C}$. For any $\mathcal{K} = \sum_{g\in G} a_g \cdot g \in \mathbb{C}G$ and $\chi \in \widehat{G}$, we denote by $a_g(\mathcal{K}) := a_g$ the *coefficient* of g in $\mathcal{K}$, and define $\chi(\mathcal{K}) = \sum_{g\in G} a_g(\mathcal{K})\chi(g)$. Then the *Fourier inversion formula* (cf. [44, Theorem 5.3.6]) establishes that

$$a_g(\mathcal{K}) = \frac{1}{|G|}\sum_{h\in G}\chi_h(\mathcal{K})\cdot\chi_h(g^{-1}). \tag{3}$$

For any $\mathcal{K}, \mathcal{L} \in \mathbb{C}G$, it follows from the Fourier inversion formula (3) that

$$\mathcal{K} = \mathcal{L} \text{ if and only if } \chi_g(\mathcal{K}) = \chi_g(\mathcal{L}) \text{ for all } g \in G.$$

Combining this with Lemma 15, we obtain the following characterization for distance-regular Cayley graphs over abelian groups.

Lemma 17. *Let G be an abelian group and let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph with diameter d. Then Γ is distance-regular with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ if and only if for every $g \in G$, the following system of equations holds:*

$$\begin{cases} \chi_g(\underline{S_1})\cdot\chi_g(\underline{S_1}) = b_0 + a_1\chi_g(\underline{S_1}) + c_2\chi_g(\underline{S_2}),\\ \chi_g(\underline{S_2})\cdot\chi_g(\underline{S_1}) = b_1\chi_g(\underline{S_1}) + a_2\chi_g(\underline{S_2}) + c_3\chi_g(\underline{S_3}),\\ \qquad\qquad\vdots\\ \chi_g(\underline{S_d})\cdot\chi_g(\underline{S_1}) = b_{d-1}\chi_g(\underline{S_{d-1}}) + a_d\chi_g(\underline{S_d}). \end{cases}$$

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph over the abelian group G. According to [3], the eigenvalues of $\Gamma = \mathrm{Cay}(G, S)$ are given by

$$\chi_g(\underline{S}) = \sum_{s\in S}\chi_g(s), \quad \text{for all } g \in G.$$

Suppose further that Γ is distance-regular and has diameter d. Then Γ has exactly $d+1$ distinct eigenvalues, denoted as $\theta_0 > \theta_1 > \cdots > \theta_d$. Let $\mathfrak{A}$ be the Bose-Mesner algebra corresponding to the symmetric association scheme derived from Γ and let $E_0 = \frac{1}{|G|}J, E_1, \ldots, E_d$ be the primitive idempotents of $\mathfrak{A}$ such that $A(\Gamma)E_i = A_1E_i = \theta_iE_i$ for all $i \in \{0, 1, \ldots, d\}$. We denote by $\widehat{S}_i = \{g \in G \mid \chi_g(\underline{S}) = \theta_i\}$. Clearly, there is a one-to-one correspondence between $\widehat{S}_i$ and E_i. Let τ be a permutation on $\{0, 1, \ldots, d\}$ that fixes 0. We say that Γ has a *Q-polynomial ordering* $\widehat{S}_{\tau(0)}, \widehat{S}_{\tau(1)}, \ldots, \widehat{S}_{\tau(d)}$ if it is Q-polynomial with respect to the ordering of primitive idempotents $E_{\tau(0)}, E_{\tau(1)}, \ldots, E_{\tau(d)}$.

Lemma 18. *Let G be an abelian group and let $\Gamma = \mathrm{Cay}(G, S)$ be a distance-regular Cayley graph with diameter d over G. Let $\theta_0 > \theta_1 > \cdots > \theta_d$ be all the distinct eigenvalues of Γ and let $\widehat{S}_i = \{g \in G \mid \chi_g(\underline{S}) = \theta_i\}$ for $0 \leqslant i \leqslant d$. If Γ has a Q-polynomial ordering $\widehat{S}_{\tau(0)}, \widehat{S}_{\tau(1)}, \ldots, \widehat{S}_{\tau(d)}$, where τ is a permutation on the set $\{0, 1, \ldots, d\}$ that fixes 0, then the Cayley graph $\widehat{\Gamma} = \mathrm{Cay}(G, \widehat{S}_{\tau(1)})$ is a distance-regular graph of diameter d with intersection numbers $q_{i,j}^k$, where $q_{i,j}^k$ are the Krein parameters of Γ.*

Proof. By Lemma 13, $\mathcal{S}(G) = \mathrm{span}\{\underline{S_0}, \underline{S_1}, \ldots, \underline{S_d}\}$ is a Schur ring over G. Note that $\widehat{S}_i$ is inverse closed for all $i \in \{0, 1, \ldots, d\}$. For any $g, h \in G$, we have $g, h \in \widehat{S}_i$ if and only if $\chi_g(\underline{S}) = \chi_h(\underline{S}) = \theta_i$, which is the case if and only if $\chi_g(\underline{S_j}) = \chi_h(\underline{S_j})$ for all $j \in \{0, 1, \ldots, d\}$ by Lemma 17. Then Lemma 7 indicates that $\widehat{\mathcal{S}}(G) = \mathrm{span}\{\underline{\widehat{S}_0}, \underline{\widehat{S}_1}, \ldots, \underline{\widehat{S}_d}\}$ is a Schur ring over G with intersection numbers $q^k_{i,j}$. Since Γ has a Q-polynomial ordering $\widehat{S}_{\tau(0)}, \widehat{S}_{\tau(1)}, \ldots, \widehat{S}_{\tau(d)}$, we claim that the (symmetric) association scheme derived from the Schur ring $\widehat{\mathcal{S}}(G)$ is P-polynomial with an ordering of relations $\widehat{S}_{\tau(0)}, \widehat{S}_{\tau(1)}, \ldots, \widehat{S}_{\tau(d)}$. Therefore, by Lemma 8, the Cayley graph $\widehat{\Gamma} = \mathrm{Cay}(G, \widehat{S}_{\tau(1)})$ is a distance-regular graph of diameter d with intersection numbers $q^k_{i,j}$. □

The Cayley graph $\widehat{\Gamma} = \mathrm{Cay}(G, \widehat{S}_{\tau(1)})$ in Lemma 18 is called the *dual graph* of the Q-polynomial distance-regular graph $\Gamma = \mathrm{Cay}(G, S)$.

A graph is called *integral* if all its eigenvalues are integers. Let $\mathcal{F}_G$ be the set of all subgroups of G. The *Boolean algebra* $\mathbb{B}(\mathcal{F}_G)$ is the set whose elements are obtained by arbitrary finite intersections, unions and complements of the elements in $\mathcal{F}_G$. The minimal non-empty elements of $\mathbb{B}(\mathcal{F}_G)$ are called *atoms*. It is known that each element of $\mathbb{B}(\mathcal{F}_G)$ is the union of some atoms, and the atoms for $\mathbb{B}(\mathcal{F}_G)$ are the sets $[g] = \{x \in G \mid \langle x \rangle = \langle g \rangle\}$, $g \in G$ (see [2]). The following lemma provides a characterization for integral Cayley graphs over abelian groups.

Lemma 19 ([2]). *Let G be an abelian group and let S be an inverse closed subset of G with $e \notin S$. Then the Cayley graph $\mathrm{Cay}(G, S)$ is integral if and only if $S \in \mathbb{B}(\mathcal{F}_G)$.*

4 Finite geometry

In this section, we introduce some classic results in finite geometry, which play a key role in the proof of our main result.

Let G be a finite group and let N be a proper subgroup of G with order $|N| = r$ and index $[G : N] = m$. A k-subset D of G is called an (m, r, k, μ)*-relative difference set* relative to N (the forbidden subgroup) if and only if

$$\underline{D} \cdot \underline{D}^{(-1)} = k \cdot e + \mu \cdot \underline{G \setminus N}.$$

Lemma 20 ([41, Theorem 4.1.1]). *Let D be a (nm, n, nm, m)-relative difference set relative to N in an abelian group G. Let g be an element in G. Then the order of g divides nm, or $n = 2$, $m = 1$ and $G \cong \mathbb{Z}_4$.*

A subset D of G is called a *polynomial addition set* if there exists a polynomial $f(x) \in \mathbb{Z}[x]$ with degree $\deg f \geqslant 1$ such that $f(\underline{D}) = m\underline{G}$ for some integer m. In this context, we also describe D as a $(v, k, f(x))$-polynomial addition set, where $|G| = v$ and $|D| = k$. If G is cyclic, then D is called a $(v, k, f(x))$*-cyclic polynomial addition set.*

Lemma 21 ([26, Corollary 5.4.5]). *There is no $(v, k, x^n - b)$-cyclic polynomial addition set with $1 < k < v - 1$ and $n \geqslant 1$.*

The proof of Lemma 21 relies on the following crucial lemma from [26], which is also useful in the proof of our main result.

Lemma 22 ([43, Lemma 1.5.1], [26, Lemma 3.2.3])**.** *Let p be a prime and let G be an abelian group with a cyclic Sylow p-subgroup S. If $\underline{Y} \in \mathbb{Z}G$ satisfies $\chi(\underline{Y}) \equiv 0 \pmod{p^a}$ for all characters $\chi \in \widehat{G}$ of order divisible by $|S|$, then there exist $\underline{X_1}, \underline{X_2} \in \mathbb{Z}G$ such that $\underline{Y} = p^a \underline{X_1} + \underline{P} \cdot \underline{X_2}$, where P is the unique subgroup of order p of G. Furthermore, if $\underline{Y}$ has non-negative coefficients only, then X_1 and X_2 also can be chosen to have non-negative coefficients only.*

A *transversal design* $TD(r, v)$ of order v with line size r ($r \leqslant v$) is a triple $(\mathcal{P}, \mathcal{G}, \mathcal{L})$ such that (see [15])

(i) $\mathcal{P}$ is a set of rv elements (called *points*);

(ii) $\mathcal{G}$ is a partition of $\mathcal{P}$ into r classes, each of size v (called *groups*);

(iii) $\mathcal{L}$ is a collection of subsets of $\mathcal{P}$ (called *lines*);

(iv) $|G \cap L| = 1$ for every $G \in \mathcal{G}$ and every $L \in \mathcal{L}$;

(v) every unordered pair of points from distinct groups is contained in exactly one line.

It follows immediately that $|L| = r$ for every $L \in \mathcal{L}$, and $|\mathcal{L}| = v^2$. The *line graph* of a transversal design $TD(r, v)$ is the graph with lines as vertices and two of them being adjacent whenever there is a point incident to both lines. It is known that the line graph Γ of a transversal design $TD(r, v)$ is a strongly regular graph with parameters $(v^2, r(v-1), v + r^2 - 3r, r^2 - r)$ (cf. [20, p. 122]), and so has exactly three distinct eigenvalues, namely $r(v-1)$, $v - r$ and $-r$. For $r = 2$, Γ is the lattice graph $K_v \square K_v$, and for $r = v$, Γ is the complete multipartite graph $K_{v \times v}$.

The following lemma provides certain restrictions for the antipodal cover of the line graph of a transversal design $TD(r, v)$ with $r \leqslant v$.

Lemma 23 ([21, Proposition 2.4])**.** *An antipodal cover of the line graph of a transversal design $TD(r, v)$, $r \leqslant v$, has diameter four when $r = 2$ and diameter three otherwise.*

Let p be an odd prime. In [53], it was shown that every distance-regular Cayley graph over $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is the line graph of a transversal design.

Lemma 24 ([53, Lemma 3.1])**.** *Let p be an odd prime and let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \oplus \mathbb{Z}_p, S)$ be a Cayley graph over $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Then Γ is distance-regular if and only if $S = \cup_{i=1}^{r} H_i \setminus \{(0,0)\}$, where $2 \leqslant r \leqslant p+1$, and H_i ($i = 1, \ldots, r$) are subgroups of order p in $\mathbb{Z}_p \oplus \mathbb{Z}_p$. In this situation, Γ is isomorphic to the line graph of a transversal design $TD(r, p)$ when $r \leqslant p$, and to a complete graph when $r = p + 1$. In particular, Γ is primitive if and only if $2 \leqslant r \leqslant p - 1$ or $r = p + 1$, and Γ is imprimitive if and only if $r = p$, in which case Γ is the complete multipartite graph $K_{p \times p}$.*

A *clique* in a graph Γ is a subgraph in which every pair of vertices are adjacent. A *maximal clique* is a clique that cannot be extended by including an additional vertex that is adjacent to all its vertices. The *clique number* of Γ is the cardinality of a clique of maximum size in Γ.

Let p be an odd prime and let $\mathbb{F}_p$ denote the finite field of order p. It is known that the Desarguesian affine plane $AG(2,p)$ can be identified with $\mathbb{F}_p^2$. Let $U = \{(a_j, b_j) \mid 1 \leqslant j \leqslant \ell\}$ be an ℓ-subset of $\mathbb{F}_p^2$. We define

$$\mathrm{Dir}(U) = \left\{ \frac{b_j - b_k}{a_j - a_k} \mid 1 \leqslant j \neq k \leqslant \ell \right\}.$$

Then the elements of $\mathrm{Dir}(U)$ are called the *directions* determined by U. Let W be a subset of $AG(2,p)$ with $1 < |W| \leqslant p$. According to [45, Theorem 5.2], the set W is either contained in a line, or satisfies the inequality

$$|\mathrm{Dir}(W)| \geqslant \frac{|W| + 3}{2}. \tag{4}$$

Note that $\mathbb{Z}_p \oplus \mathbb{Z}_p$ coincides with $\mathbb{F}_p^2$ as sets. For any subset B of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ with $(0,0) \in B$, we see that B is contained in a line if and only if B is contained in some subgroup of order p in $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Moreover, if K_1 and K_2 are two distinct subgroups of order p in $\mathbb{Z}_p \oplus \mathbb{Z}_p$, then $\mathrm{Dir}(K_1) \neq \mathrm{Dir}(K_2)$.

Lemma 25. *Let p be an odd prime. Suppose that $\Gamma = \mathrm{Cay}(\mathbb{Z}_p \oplus \mathbb{Z}_p, S)$ with $S = \cup_{i=1}^r H_i \setminus \{(0,0)\}$, where $2 \leqslant r \leqslant p-1$, and H_i $(i = 1, \ldots, r)$ are subgroups of order p in $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Then the following statements hold.*

(i) The clique number of Γ is equal to p.

(ii) If C is clique of Γ that is not contained in any line, then $|C| \leqslant 2r - 3$.

(iii) If Γ is the halved graph of a bipartite distance-regular graph Γ', then the number of common neighbors of two vertices at distance 2 in Γ' is exactly one.

Proof. (i) Clearly, H_i is a clique of order p in Γ. Since Γ has eigenvalues $r(p-1)$, $p-r$ and $-r$, the Delsarte bound (cf. [10, Section 3.3.2]) implies that the clique number of Γ is at most $1 - \frac{r(p-1)}{-r} = p$. Therefore, the clique number of Γ is exactly p.

(ii) Since any two vertices in C are adjacent, we assert that the directions determined by C are contained in the set $\{\mathrm{Dir}(H_i) \mid 1 \leqslant i \leqslant r\}$, and hence $r \geqslant |\mathrm{Dir}(C)|$. As C is not contained in any line, from (4) we obtain $\mathrm{Dir}(C) \geqslant (|C| + 3)/2$. Therefore, $|C| \leqslant 2r - 3$.

(iii) Suppose that Γ is the halved graph of a bipartite distance-regular graph Γ'. Let k denote the valency of Γ' and let μ denote the number of common neighbors of two vertices at distance 2 in Γ'. By contradiction, assume that $\mu \geqslant 2$. By [7, Proposition 4.2.2], we have

$$\frac{k^2 - k}{\mu} = r(p-1). \tag{5}$$

For any $v \in V(\Gamma')$, let $S(v)$ be the neighborhood of v in Γ'. By [16, Lemma 2], $S(v)$ is a maximal clique in Γ'^{+} or Γ'^{-}. Clearly, the maximal cliques $S(v)$ for $v \in V(\Gamma')$ must cover Γ'^{+} and Γ'^{-}. Thus there exists some vertex $v \in V(\Gamma')$ such that $C := S(v)$ is a maximal clique in Γ containing the vertex $(0,0)$. According to (i), the clique number of Γ is p, and $k = |S(v)| = |C| \leqslant p$. Thus we have $r\mu \leqslant p$ by (5). If C is contained in a line, then $k = |C| = p$ because C is a maximal clique. By (5), we have $r\mu = p$, and so $r = 1$ or p, contrary to our assumption. If C is not contained in any line, then (ii) indicates that $k = |C| \leqslant 2r - 3 < 2r$. Combining this with (5) and $\mu \geqslant 2$, we obtain $2r > p$, which is impossible because $p \geqslant \mu r \geqslant 2r$. □

5 Imprimitive distance-regular Cayley graphs with diameter three over abelian groups

In this section, we present some properties of imprimitive distance-regular Cayley graphs with diameter 3 over abelian groups.

It is known that an antipodal bipartite distance-regular graph with diameter 3 is a complete bipartite graph without a perfect matching. Also, by [7, Corollary 8.2.2], every non-antipodal bipartite distance-regular graph with diameter 3 is Q-polynomial. Therefore, by Lemma 9 and Lemma 18, we can deduce the following result immediately.

Proposition 26. *Let G be an abelian group. If Γ is a non-antipodal bipartite distance-regular Cayley graph with diameter 3 over G, then its dual graph $\widehat{\Gamma}$ is an antipodal non-bipartite distance-regular Cayley graph with diameter 3 over G.*

By Proposition 26 and the above arguments, in order to study distance-regular Cayley graphs with diameter 3 over abelian groups, the primary task is to consider those that are antipodal and non-bipartite.

For the sake of convenience, we maintain the following notation throughout the remainder of this paper.

Notation. Let G and H be finite abelian groups under addition and let $G \oplus H$ denote the direct product of G and H. For subsets $A \subseteq G$, $B \subseteq H$ and elements $g \in G$, $h \in H$, we define $g + A = \{g + a \mid a \in A\}$, $(g, B) = \{(g, b) \mid b \in B\}$, $(A, h) = \{(a, h) \mid a \in A\}$, and $(A, B) = \{(a, b) \mid a \in A, b \in B\}$. If $\Gamma = \mathrm{Cay}(G \oplus H, S)$ is a Cayley graph over $G \oplus H$, then the connection set S can be expressed as

$$S = \cup_{h \in H}(R_h, h) = \cup_{g \in G}(g, L_g),$$

where R_h is a subset of G such that $0_G \notin R_{0_H}$ and $R_h = -R_{-h}$ for all $h \in H$, and L_g is a subset of H such that $0_H \notin L_{0_G}$ and $L_g = -L_{-g}$ for all $g \in G$. Let $R = \cup_{h \in H} R_h$ and $L = \cup_{g \in G} L_g$. Furthermore, if Γ is distance-regular, then we denote by k, λ, μ and d the valency, the number of common neighbors of two adjacent vertices, the number of common neighbors of two vertices at distance 2 and the diameter of Γ, respectively.

Recall that the set of characters of $G \oplus H$ can be represented as $\widehat{G \oplus H} = \{(\chi, \psi) \mid \chi \in \widehat{G}, \psi \in \widehat{H}\}$, where the pair (χ, ψ) is defined such that $(\chi, \psi)((g, h)) = \chi(g)\psi(h)$ for every $(g, h) \in G \oplus H$ (cf. [44, Proposition 4.5.1]).

Lemma 27. *Let G and H be finite abelian groups under addition and let $S = \cup_{h\in H}(R_h, h) = \cup_{g\in G}(g, L_g)$ be a subset of $G \oplus H$, where $R_h \subseteq G$ for all $h \in H$ and $L_g \subseteq H$ for all $g \in G$. Then*

$$\chi(\underline{R_{0_H}}) = \frac{1}{|H|} \sum_{\psi\in\widehat{H}} (\chi, \psi)(\underline{S})$$

for every $\chi \in \widehat{G}$, and

$$\psi(\underline{L_{0_G}}) = \frac{1}{|G|} \sum_{\chi\in\widehat{G}} (\chi, \psi)(\underline{S})$$

for every $\psi \in \widehat{H}$.

Proof. By symmetry, we only need to prove the first part of the lemma. Note that

$$\sum_{\psi\in\widehat{H}} \psi(h) = \begin{cases} |H|, & \text{if } h = 0_H; \\ 0, & \text{otherwise.} \end{cases}$$

For every $\chi \in \widehat{G}$, we have

$$\begin{aligned} \sum_{\psi\in\widehat{H}} (\chi, \psi)(\underline{S}) &= \sum_{\psi\in\widehat{H}} (\chi, \psi) \left(\sum_{h\in H} \underline{(R_h, h)} \right) = \sum_{\psi\in\widehat{H}} \left(\sum_{h\in H} \chi(\underline{R_h})\psi(h) \right) \\ &= \sum_{h\in H} \left(\sum_{\psi\in\widehat{H}} \psi(h) \right) \chi(\underline{R_h}) = |H| \cdot \chi(\underline{R_{0_H}}), \end{aligned}$$

and the result follows. □

Proposition 28. *Let G be an abelian group and let Γ be an antipodal non-bipartite distance-regular Cayley graph with diameter 3 over G. Let r be the common size of antipodal classes of Γ. If r is a prime, then $G \cong M \oplus \mathbb{Z}_r$ for some abelian group M of order $|G|/r$, and Γ is isomorphic to a Cayley graph over $M \oplus \mathbb{Z}_r$ in which the antipodal class containing the identity vertex is $S_0 \cup S_3 = (0_M, \mathbb{Z}_r)$.*

Proof. Since r is a prime divisor of $|G|$, we can express G as

$$G = K \oplus \mathbb{Z}_{r^{s_1}} \oplus \mathbb{Z}_{r^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{r^{s_t}},$$

where $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_t$ and $r \nmid |K|$. Let H denote the antipodal class of G containing the identity vertex, that is, $H = S_0 \cup S_3$. Since $|H| = r$ is a prime and K contains no elements of order r, we claim that every element of S_3 is of the form $(0_K, a_1, \ldots, a_t)$ with

$a_i \in \mathbb{Z}_{r^{s_i}}$ and $r^{s_i-1} \mid a_i$ for $1 \leqslant i \leqslant t$. Thus there exists some $l \in \{1, \ldots, t\}$ such that $b = (0_K, b_1, \ldots, b_l = r^{s_l-1}, 0, \ldots, 0) \in S_3$ with $r^{s_i-1} \mid b_i$ for $1 \leqslant i \leqslant l$. Then $H = \langle b \rangle$ because $|H| = r$ is a prime. Let σ be the mapping on G defined by letting

$$\sigma((k, i_1, \ldots, i_t)) = \left(k, i_1 - \frac{b_1}{r^{s_l-1}} i_l, \ldots, i_{l-1} - \frac{b_{l-1}}{r^{s_l-1}} i_l, i_l, \ldots, i_t\right)$$

for all $(k, i_1, \ldots, i_t) \in G = K \oplus \mathbb{Z}_{r^{s_1}} \oplus \mathbb{Z}_{r^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{r^{s_t}}$. Clearly, σ is an automorphism of G, and $\sigma(b) = (0_K, 0, \ldots, 0, r^{s_l-1}, 0, \ldots, 0)$. Then $G \cong M \oplus \mathbb{Z}_m$, where $m = r^{s_l}$ and $M = K \oplus (\oplus_{i=1, i \neq l}^{t} \mathbb{Z}_{r^{s_i}})$, and Γ is isomorphic to a Cayley graph over $M \oplus \mathbb{Z}_m$ in which the antipodal class containing the identity vertex is $S_0 \cup S_3 = (0_M, \frac{m}{r}\mathbb{Z}_m)$. Therefore, it suffices to prove $m = r$. We consider the following three cases.

Case A. $r = 2$.

In this situation, $S_3 = \{(0_M, \frac{m}{2})\}$. Assume that $4 \mid m$. If $(0_M, \frac{m}{4}) \in S_1$, then $(0_M, -\frac{m}{4}) = (0_M, \frac{m}{2}) + (0_M, \frac{m}{4}) \in S_2$, which is impossible due to $(0_M, -\frac{m}{4}) \in -S_1 = S_1$. If $(0_M, \frac{m}{4}) \in S_2$, then $(0_M, \frac{m}{4})$ and $(0_M, \frac{m}{2})$ are adjacent, and hence $(0_M, \frac{m}{4}) = (0_M, \frac{m}{2}) - (0_M, \frac{m}{4}) \in S_1$, a contradiction. Thus we have $(0_M, \frac{m}{4}) \notin S_1 \cup S_2 \cup S_3$, which is impossible. Therefore, we conclude that $m = 2 = r$, as desired.

Case B. $r \neq 2$ and $\lambda = \mu$.

By [7, p. 431], we have $\frac{m|M|}{r} - 1 = k = 1 + r\mu$, and hence $\frac{m|M|}{r} \equiv 2 \pmod r$. As $r \neq 2$, we assert that $m = r$, as required.

Case C. $r \neq 2$ and $\lambda \neq \mu$.

In this case, by [7, p. 431], Γ is integral, and so $(\chi, \psi)(\underline{S}) \in \mathbb{Z}$ for all $(\chi, \psi) \in \widehat{M \oplus \mathbb{Z}_m}$. By Lemma 27, for any $\psi \in \widehat{\mathbb{Z}_m}$, we have

$$\psi(\underline{L_{0_M}}) = \frac{1}{|M|} \sum_{\chi \in \widehat{M}} (\chi, \psi)(\underline{S}) \in \mathbb{Q},$$

and hence $\psi(\underline{L_{0_M}}) \in \mathbb{Z}$ because it is an algebraic integer. Note that $\{\psi(\underline{L_{0_M}}) \mid \psi \in \widehat{\mathbb{Z}_m}\}$ gives a complete set of eigenvalues of the Cayley graph $\mathrm{Cay}(\mathbb{Z}_m, L_{0_M})$. Hence, by Lemma 19, L_{0_M} is a union of some atoms for $\mathbb{B}(\mathcal{F}_{\mathbb{Z}_m})$. On the other hand, by Corollary 16,

$$\underline{(0_M, \frac{m}{r}\mathbb{Z}_m)} \cdot \underline{S \cup \{(0_M, 0)\}} = \underline{M \oplus \mathbb{Z}_m},$$

and it follows that

$$\underline{(0_M, L_{0_M} \cup \{0\})} \cdot \underline{(0_M, \frac{m}{r}\mathbb{Z}_m)} = \underline{(0_M, \mathbb{Z}_m)},$$

or equivalently,

$$\underline{L_{0_M} \cup \{0\}} \cdot \underline{\frac{m}{r}\mathbb{Z}_m} = \underline{\mathbb{Z}_m}.$$

This implies that $L_{0_M} \cup \{0\}$ contains exactly one element from each coset of $\frac{m}{r}\mathbb{Z}_m$ in $\mathbb{Z}_m$. Then there exists an element $a \in L_{0_M}$ such that $(1 + \frac{m}{r}\mathbb{Z}_m) \cap L_{0_M} = \{a\}$. If $1 + \frac{m}{r}\mathbb{Z}_m \subseteq \mathbb{Z}_m^*$, then $a \in \mathbb{Z}_m^*$. Since L_{0_M} is a union of some atoms for $\mathbb{B}(\mathcal{F}_{\mathbb{Z}_m})$ and $\mathbb{Z}_m^*$ is exactly an atom, we assert that $1 + \frac{m}{r}\mathbb{Z}_m \subseteq \mathbb{Z}_m^* \subseteq L_{0_M}$. Thus $1 + \frac{m}{r}\mathbb{Z}_m = \{a\}$, and it follows that $m = r$, as desired. If $1 + \frac{m}{r}\mathbb{Z}_m \not\subseteq \mathbb{Z}_m^*$, then there exists some $i \in \mathbb{Z}_m$ such that $1 + i\frac{m}{r} \notin \mathbb{Z}_m^*$. Combining this with $\gcd(1 + i\frac{m}{r}, \frac{m}{r}) = 1$, we obtain $\gcd(1 + i\frac{m}{r}, m) = r$, which gives that $m = r$ because m is a power of r. The result follows. □

Proposition 29. *Let G be an abelian group and let p be a prime. Assume that $\Gamma = \mathrm{Cay}(G \oplus \mathbb{Z}_p, S)$ is an antipodal non-bipartite distance-regular Cayley graph with diameter 3 in which the antipodal class containing the identity vertex is $S_0 \cup S_3 = (0_G, \mathbb{Z}_p)$. Let $k \geqslant \theta_1 \geqslant -1 \geqslant \theta_3$ be all distinct eigenvalues of Γ and $2\delta = \theta_1 - \theta_3$. Then the following statements hold.*

(i) The sets R_i, for $i \in \mathbb{Z}_p$, form a partition of $G \setminus \{0_G\}$.

(ii) If $p > 2$, then $\frac{|G|}{2\delta}$, θ_1 and $-\theta_3$ are positive integers. Moreover, for every non-trivial character $\psi \in \widehat{\mathbb{Z}_p}$, the set $B = \{g \in G \mid (\chi_g, \psi)(\underline{S}) = \theta_1\}$ is a $(|G|, -\frac{|G|}{2\delta}\theta_3, x^p - \frac{|G|^p}{(2\delta)^p})$-polynomial addition set such that

$$\chi_l(\underline{B}) = \frac{|G|}{2\delta}\left(\sum_{i \in \mathbb{Z}_p} \psi(i) a_{-l}(\underline{R_i}) - \theta_3 \cdot a_{-l}(0_G)\right) \text{ for all } l \in G.$$

(iii) If $p = 2$, then there exists a strongly regular Cayley graph over G with parameters $(|G|, \frac{|G|}{2\delta}\theta, \frac{|G|}{4\delta^2}(\theta^2 - 1), \frac{|G|}{4\delta^2}(\theta^2 - 1))$, where $\theta = \theta_1$ or $-\theta_3$.

Proof. (i) By Corollary 16, we have

$$\underline{G \oplus \mathbb{Z}_p} = \underline{(0_G, \mathbb{Z}_p)} \cdot \left(\sum_{i \in \mathbb{Z}_p} \underline{(R_i, i)} + e\right) = \sum_{i \in \mathbb{Z}_p} \underline{(R_i, \mathbb{Z}_p)} + \underline{(0_G, \mathbb{Z}_p)}.$$

Therefore, the sets R_i ($i \in \mathbb{Z}_p$) form a partition of $G \setminus \{0_G\}$.

(ii) Again by Corollary 16,

$$\underline{S}^2 = k \cdot e + (\lambda - \mu)\underline{S} + \mu(\underline{G \oplus \mathbb{Z}_p} - \underline{(0_G, \mathbb{Z}_p)}). \tag{6}$$

Let $\psi \in \widehat{\mathbb{Z}_p}$ be a non-trivial character of $\mathbb{Z}_p$. We have $\psi(\underline{\mathbb{Z}_p}) = 0$. For any $g \in G$, let $\chi_g \in \widehat{G}$ be the character of G defined in (1). Then (χ_g, ψ) is a non-trivial character of $\widehat{G \oplus \mathbb{Z}_p}$, and so $(\chi_g, \psi)(\underline{\widehat{G \oplus \mathbb{Z}_p}}) = \chi_g(\underline{G}) \cdot \psi(\underline{\mathbb{Z}_p}) = 0$. By applying (χ_g, ψ) on both sides of (6), we obtain

$$((\chi_g, \psi)(\underline{S}))^2 = k + (\lambda - \mu)(\chi_g, \psi)(\underline{S}),$$

which implies that $(\chi_g, \psi)(\underline{S}) = \theta_1$ or θ_3 for all $g \in G$ according to [7, p. 431]. Then

$$\sum_{g \in G} (\chi_g, \psi)(\underline{S}) \cdot g = \theta_1 \underline{B} + \theta_3 \underline{G \setminus B} = 2\delta \underline{B} + \theta_3 \underline{G}.$$

Since $\underline{S} = \sum_{i\in\mathbb{Z}_p} \underline{(R_i, i)}$, we have

$$2\delta\underline{B} + \theta_3\underline{G} = \sum_{g\in G}\left(\sum_{i\in\mathbb{Z}_p}\psi(i)\chi_g(\underline{R_i})\right) g. \tag{7}$$

Let $l \in G$. By applying the character $\chi_l \in \widehat{G}$ on both sides of (7), we get

$$\begin{aligned} 2\delta\chi_l(\underline{B}) + \theta_3\chi_l(\underline{G}) &= \sum_{g\in G}\left(\sum_{i\in\mathbb{Z}_p}\psi(i)\chi_g(\underline{R_i})\right)\chi_l(g) = \sum_{g\in G}\left(\sum_{i\in\mathbb{Z}_p}\psi(i)\chi_g(\underline{R_i})\right)\chi_g(l) \\ &= \sum_{i\in\mathbb{Z}_p}\psi(i)\left(\sum_{g\in G}\chi_g(\underline{R_i})\chi_g(l)\right) = \sum_{i\in\mathbb{Z}_p}\psi(i)\cdot|G|a_{-l}(\underline{R_i}), \end{aligned} \tag{8}$$

where the last equality follows from the Fourier inversion formula (3). Note that $\chi_l(\underline{G}) = |G|$ if $l = 0_G$, and $\chi_l(\underline{G}) = 0$ otherwise. By (8), we obtain

$$\chi_l(\underline{B}) = \frac{|G|}{2\delta}\left(\sum_{i\in\mathbb{Z}_p}\psi(i)a_{-l}(\underline{R_i}) - \theta_3\cdot a_{-l}(0_G)\right), \tag{9}$$

and it follows that $|B| = -\frac{|G|}{2\delta}\theta_3 > 0$. Recall that the sets R_i, for $i \in \mathbb{Z}_p$, form a partition of $G \setminus \{0_G\}$. Then from (9) we can deduce that

$$\begin{aligned} \chi_l(\underline{B}^p) = \chi_l(\underline{B})^p &= \frac{|G|^p}{(2\delta)^p}\left(\sum_{i\in\mathbb{Z}_p}a_{-l}(\underline{R_i}) + (-\theta_3)^p\cdot a_{-l}(0_G)\right) \\ &= \frac{|G|^p}{(2\delta)^p}\left(1 + ((-\theta_3)^p - 1)\cdot a_{-l}(0_G)\right). \end{aligned}$$

Using the Fourier inversion formula (3), we get

$$\underline{B}^p = \frac{|G|^{p-1}}{(2\delta)^p}(((-\theta_3)^p - 1)\cdot\underline{G} + |G|\cdot 0_G). \tag{10}$$

By [7, p. 431], $(2\delta)^2 = 4k + (\lambda-\mu)^2 \in \mathbb{Z}$. As p is odd and $\underline{B}^p \in \mathbb{Z}G$, from (10) and $(2\delta)^2 \in \mathbb{Z}$ we can deduce that $\frac{|G|}{2\delta}$ and 2δ are integers, and so are θ_1 and θ_3. Moreover, again by (10), we assert that B is a $(|G|, -\frac{|G|}{2\delta}\theta_3, x^p - \frac{|G|^p}{(2\delta)^p})$-polynomial addition set in G.

(iii) If $p = 2$, then $\mathbb{Z}_2$ has only one non-trivial character, namely ψ_1, where $\psi_1(0) = 1$ and $\psi_1(1) = -1$. By substituting $\psi = \psi_1$ in (9), we obtain

$$\chi_l(\underline{B}) = \frac{|G|}{2\delta}\left(a_{-l}(\underline{R_0}) - a_{-l}(\underline{R_1}) - \theta_3\cdot a_{-l}(0_G)\right) \text{ for all } l \in G,$$

which implies $-B = B$. Moreover, by (10), we have

$$\underline{B}^2 = \frac{|G|}{(2\delta)^2}((\theta_3^2 - 1) \cdot \underline{G} + |G| \cdot 0_G). \tag{11}$$

If $0_G \notin B$, then (11) indicates that the Cayley graph $\mathrm{Cay}(G, B)$ is with diameter $d \leqslant 2$. If $d = 1$, then $B = G \setminus \{0_G\}$, and hence $|G| - 1 = |B| = -\frac{|G|}{2\delta}\theta_3$. On the other hand, by [7, p. 431], we have $\theta_3 = \frac{\lambda-\mu}{2} - \delta$, $\delta = \sqrt{k + (\frac{\lambda-\mu}{2})^2}$, $|G| = k+1$ and $k = \mu + \lambda + 1$. Thus, we obtain $k - 1 = \lambda - \mu$ or $\mu - \lambda$, implying that $\mu = 0$ or $\lambda = 0$, which is a contradiction. Therefore, $d = 2$. Again by (11), we assert that $\mathrm{Cay}(G, B)$ is a strongly regular Cayley graph with parameters $(|G|, -\frac{|G|}{2\delta}\theta_3, \frac{|G|}{4\delta^2}(\theta_3^2 - 1), \frac{|G|}{4\delta^2}(\theta_3^2 - 1))$. If $0_G \in B$, then $0_G \notin G \setminus B$. Combining (11) with $\theta_1 - \theta_3 = 2\delta$ and $|B| = -\frac{|G|}{2\delta}\theta_3$ yields that

$$(\underline{G \setminus B})^2 = \frac{|G|}{(2\delta)^2}((\theta_1^2 - 1) \cdot \underline{G} + |G| \cdot 0_G). \tag{12}$$

By a similar analysis, we can deduce from (12) that $\mathrm{Cay}(G, G \setminus B)$ is a strongly regular Cayley graph with parameters $(|G|, \frac{|G|}{2\delta}\theta_1, \frac{|G|}{4\delta^2}(\theta_1^2 - 1), \frac{|G|}{4\delta^2}(\theta_1^2 - 1))$. □

6 Imprimitive distance-regular Cayley graphs with diameter four over abelian groups

In this section, we present some properties of antipodal bipartite distance-regular Cayley graphs with diameter 4 over abelian groups.

Proposition 30. *Let G be an abelian group and let Γ be an antipodal bipartite distance-regular Cayley graph with diameter 4 over G. Let r be the common size of antipodal classes of Γ. If r is an odd prime, then $G \cong M \oplus \mathbb{Z}_r$ for some abelian group M of order $|G|/r$, and Γ is isomorphic to a Cayley graph over $M \oplus \mathbb{Z}_r$ in which the antipodal class and the bipartition set containing the identity vertex are $S_0 \cup S_4 = (0_M, \mathbb{Z}_r)$ and $S_0 \cup S_2 \cup S_4 = M_1 \oplus \mathbb{Z}_r$, respectively, where M_1 is an index 2 subgroup of M.*

Proof. Since r is a prime divisor of $|G|$, as in Proposition 28, we assert that $G \cong M \oplus \mathbb{Z}_m$ with M being an abelian group of order $|G|/m$ and $m = r^\ell$ for some $\ell \geqslant 1$, and that Γ is isomorphic to a Cayley graph Γ' over $M \oplus \mathbb{Z}_m$ in which the antipodal class containing the identity vertex is $S_0 \cup S_4 = (0_M, \frac{m}{r}\mathbb{Z}_m)$. Furthermore, since Γ' is bipartite and $m = r^\ell$ is odd, we have $2 \mid |M|$. Let $H = S_0 \cup S_2 \cup S_4$ be the bipartition set in Γ'. Then H is an index 2 subgroup of $M \oplus \mathbb{Z}_m$, and so $H = M_1 \oplus \mathbb{Z}_m$, where M_1 is an index 2 subgroup of M. Therefore, it remains to prove $m = r$.

Since M is an abelian group of even order, we can assume that $M = K \oplus \mathbb{Z}_{2^{s_1}} \oplus \mathbb{Z}_{2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{2^{s_t}}$, where $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_t \geqslant 1$ ($t \geqslant 1$) and $2 \nmid |K|$. Then $M_1 = K \oplus \mathbb{Z}_{2^{s_1}} \oplus \cdots \oplus 2\mathbb{Z}_{2^{s_i}} \oplus \cdots \oplus \mathbb{Z}_{2^{s_t}}$ for some $i \in \{1, \ldots, t\}$. Let $m' = 2^{s_i}m$. As r is odd and $m = r^\ell$, we have $\mathbb{Z}_{2^{s_i}} \oplus \mathbb{Z}_m \cong \mathbb{Z}_{m'}$. Let $M' = K \oplus (\oplus_{j=1,j\neq i}^{t} \mathbb{Z}_{2^{s_i}})$. Then $M \oplus \mathbb{Z}_m \cong M' \oplus \mathbb{Z}_{m'}$, and it is easy to check that Γ' is isomorphic to a Cayley graph over $M' \oplus \mathbb{Z}_{m'}$ in which the antipodal class and the bipartition set containing the identity vertex are $S_0 \cup S_4 = (0_{M'}, \frac{m'}{r}\mathbb{Z}_{m'})$ and $S_0 \cup S_2 \cup S_4 = M' \oplus 2\mathbb{Z}_{m'}$, respectively. By Lemma 15,

$$\underline{\left(0_{M'}, \frac{m'}{r}\mathbb{Z}_{m'}\right)} \cdot \underline{S_1} = (\underline{S_0} + \underline{S_4}) \cdot \underline{S_1} = \underline{S_1} + \underline{S_3} = \underline{(M', 1 + 2\mathbb{Z}_{m'})},$$

which implies that $\underline{\frac{m'}{r}\mathbb{Z}_{m'}} \cdot \underline{L_{0_{M'}}} = \underline{1 + 2\mathbb{Z}_{m'}}$. Thus $|L_{0_{M'}}| = \frac{m'}{2r}$, and $L_{0_{M'}}$ contains exactly one element from each coset in the set $\{i + \frac{m'}{r}\mathbb{Z}_{m'} \mid i \in 1+2\mathbb{Z}_{m'}\}$. Let $\widehat{\mathbb{Z}_{m'}} = \{\psi_g \mid g \in \mathbb{Z}_{m'}\}$ be the set of all irreducible characters of $\mathbb{Z}_{m'}$. In what follows, we shall determine the value of $\psi_g(\underline{L_{0_{M'}}})$ for all $g \in \mathbb{Z}_{m'}$. Clearly, $\psi_g(\underline{L_{0_{M'}}}) = |L_{0_{M'}}| = \frac{m'}{2r}$ if $g = 0$, and $\psi_g(\underline{L_{0_{M'}}}) = -|L_{0_{M'}}| = -\frac{m'}{2r}$ if $g = \frac{m'}{2}$ due to $L_{0_{M'}} \subseteq 1 + 2\mathbb{Z}_{m'}$. Again by Lemma 15, we have

$$\underline{S}^2 = k \cdot e + \mu \underline{S_2} = k \cdot e + \mu \left(\underline{M' \oplus 2\mathbb{Z}_{m'}} - \underline{\left(0_{M'}, \frac{m'}{r}\mathbb{Z}_{m'}\right)} \right). \tag{13}$$

If $g \in r\mathbb{Z}_{m'} \setminus \frac{m'}{2}\mathbb{Z}_{m'}$, then from (13) and [7, p. 425] we obtain $(\chi, \psi_g)(\underline{S})^2 = k - \mu r = 0$, and hence $(\chi, \psi_g)(\underline{S}) = 0$ for all $\chi \in \widehat{M'}$. Therefore, by Lemma 27,

$$\psi_g(\underline{L_{0_{M'}}}) = \frac{1}{|M'|} \sum_{\chi \in \widehat{M'}} (\chi, \psi_g)(\underline{S}) = 0.$$

If $g \notin r\mathbb{Z}_{m'}$, then from (13) we get $((\chi, \psi_g)(\underline{S}))^2 = k$, and hence $(\psi, \chi_g)(\underline{S}) \in \{-\sqrt{k}, \sqrt{k}\}$ for all $\chi \in \widehat{M'}$. Again by Lemma 27,

$$\psi_g(\underline{L_{0_{M'}}}) = \frac{1}{|M'|} \sum_{\chi \in \widehat{M'}} (\chi, \psi_g)(\underline{S}) \in \left\{ \pm \frac{i\sqrt{k}}{|M'|} \mid i = 0, 1, \ldots, |M'| \right\}. \tag{14}$$

We consider the following two cases.

Case A. $\sqrt{k} \in \mathbb{Q}$.

In this situation, for any $g \in \mathbb{Z}_{m'}$, $\psi_g(\underline{L_{0_{M'}}}) \in \mathbb{Z}$ because it is a rational algebraic integer. Thus $L_{0_{M'}}$ is a union of some atoms for $\mathbb{B}(\mathcal{F}_{\mathbb{Z}_{m'}})$. Furthermore, since $L_{0_{M'}}$ is a subset of $1 + 2\mathbb{Z}_{m'}$ that contains exactly one element from each coset in the set $\{i + \frac{m'}{r}\mathbb{Z}_{m'} \mid i \in 1+2\mathbb{Z}_{m'}\}$, there exists some $a \in L_{0_{M'}}$ such that $(1 + \frac{m'}{r}\mathbb{Z}_{m'}) \cap L_{0_{M'}} = \{a\}$. If $1 + \frac{m'}{r}\mathbb{Z}_{m'} \subseteq \mathbb{Z}_{m'}^*$, then $a \in \mathbb{Z}_{m'}^*$. As $\mathbb{Z}_{m'}^*$ is exactly an atom, we assert that $1 + \frac{m'}{r}\mathbb{Z}_{m'} \subseteq \mathbb{Z}_{m'}^* \subseteq L_{0_{M'}}$. Thus $1 + \frac{m'}{r}\mathbb{Z}_{m'} = \{a\}$, and it follows that $m' = r$, which is impossible. If $1 + \frac{m'}{r}\mathbb{Z}_{m'} \nsubseteq \mathbb{Z}_{m'}^*$, then there exists some $i \in \mathbb{Z}_{m'}$ such that $1 + i\frac{m'}{r} \notin \mathbb{Z}_{m'}^*$. Combining this with $m' = 2^{s_i}m = 2^{s_i}r^\ell$ and $\gcd(1 + i\frac{m'}{r}, \frac{m'}{r}) = 1$, we obtain $\gcd(1 + i\frac{m'}{r}, m') = r$, which implies that $r^2 \nmid m'$. Therefore, we have $m = r$, and the result follows.

Case B. $\sqrt{k} \notin \mathbb{Q}$.

In this situation, $\psi_g(\underline{L_{0_{M'}}}) \in \mathbb{Q}$ for any $g \in r\mathbb{Z}_{m'}$, and $\psi_g(\underline{L_{0_{M'}}}) \in \{\pm \frac{i\sqrt{k}}{|M'|} \mid i = 0, 1, \ldots, |M'|\} \subseteq \mathbb{Q}(\omega) \setminus \mathbb{Q}$ for any $g \notin r\mathbb{Z}_{m'}$, where ω is a primitive m'-th root of unity.

Then there exists an element σ_{c_0} with $c_0 \in \mathbb{Z}_{m'}^*$ in the Galois group $\mathrm{Gal}[\mathbb{Q}(\omega) : \mathbb{Q}] = \{\sigma_c : \omega \mapsto \omega^c \mid c \in \mathbb{Z}_{m'}^*\}$ such that $\sigma_{c_0}(\sqrt{k}) = -\sqrt{k}$. By applying σ_{c_0} on both sides of (14), we obtain

$$\psi_g(\underline{c_0 L_{0_{M'}}}) = -\frac{1}{|M'|} \sum_{\chi \in \widehat{M'}} (\chi, \psi_g)(\underline{S}) = -\psi_g(\underline{L_{0_{M'}}}) \text{ for } g \notin r\mathbb{Z}_{m'}.$$

Recall that $\psi_g(\underline{L_{0_{M'}}}) = |L_{0_{M'}}| = \frac{m'}{2r}$ if $g = 0$, $\psi_g(\underline{L_{0_{M'}}}) = -|L_{0_{M'}}| = -\frac{m'}{2r}$ if $g = \frac{m'}{2}$, and $\psi_g(\underline{L_{0_{M'}}}) = 0$ if $g \in r\mathbb{Z}_{m'} \setminus \frac{m'}{2}\mathbb{Z}_{m'}$. Thus $\psi_g(\underline{c_0 L_{0_{M'}}}) = \psi_g(\underline{L_{0_{M'}}}) \in \mathbb{Q}$ for all $g \in r\mathbb{Z}_{m'}$. According to the Fourier inversion formula (3), for each $g \in \mathbb{Z}_{m'}$, we have

$$\begin{aligned} a_g(\underline{L_{0_{M'}}}) + a_g(\underline{c_0 L_{0_{M'}}}) =& \frac{1}{m'} \sum_{h \in \mathbb{Z}_{m'}} \left(\psi_h(\underline{L_{0_{M'}}}) + \psi_h(\underline{c_0 L_{0_{M'}}})\right) \psi_h(g^{-1}) \\ =& \frac{1}{m'} \sum_{h \in \frac{m'}{2}\mathbb{Z}_{m'}} \left(\psi_h(\underline{L_{0_{M'}}}) + \psi_h(\underline{c_0 L_{0_{M'}}})\right) \psi_h(g^{-1}) \\ =& \frac{2}{m'} \sum_{h \in \frac{m'}{2}\mathbb{Z}_{m'}} \psi_h(\underline{L_{0_{M'}}}) \psi_h(g^{-1}). \end{aligned}$$

Therefore,

$$\max_{g \in \mathbb{Z}_{m'}} \left(a_g(\underline{L_{0_{M'}}}) + a_g(\underline{c_0 L_{0_{M'}}})\right) = \max_{g \in \mathbb{Z}_{m'}} \frac{2}{m'} \sum_{h \in \frac{m'}{2}\mathbb{Z}_{m'}} \psi_h(\underline{L_{0_{M'}}}) \psi_h(g^{-1}) \leqslant \frac{1}{m'} \cdot 4 \cdot \frac{m'}{2r} = \frac{2}{r},$$

which is impossible because $L_{0_{M'}} \neq \varnothing$ and $r \geqslant 3$.

We complete the proof. □

Proposition 31. *Let G be an abelian group and let p be an odd prime. Assume that $\Gamma = \mathrm{Cay}(G \oplus \mathbb{Z}_p, S)$ is an antipodal bipartite distance-regular Cayley graph with diameter 4 in which the antipodal class and the bipartition set containing the identity vertex are $S_0 \cup S_4 = (0_G, \mathbb{Z}_p)$ and $S_0 \cup S_2 \cup S_4 = H \oplus \mathbb{Z}_p$, respectively, where H is an index 2 subgroup of G. Then the following statements hold.*

(i) The sets R_i, for $i \in \mathbb{Z}_p$, form a partition of $G \setminus H$.

(ii) For every non-trivial character $\psi \in \widehat{\mathbb{Z}_p}$, $B = \{g \in G \mid (\chi_g, \psi)(\underline{S}) = \sqrt{k}\}$ is a non-empty set such that

$$\chi_l(\underline{B}) = \frac{|G|}{2\sqrt{k}} \left(\sum_{i \in \mathbb{Z}_p} \psi(i) a_{-l}(\underline{R_i}) + \sqrt{k} \cdot a_{-l}(0_G) \right) \text{ for all } l \in G.$$

(iii) $\frac{|G|}{2\sqrt{k}}$ is an integer.

Proof. (i) As in Proposition 29, from Lemma 15 we can deduce that $\sum_{i\in\mathbb{Z}_p} \underline{R_i} = \underline{G \setminus H}$. Thus the sets R_i, for $i \in \mathbb{Z}_p$, form a partition of $G \setminus H$.

(ii) Again by Lemma 15, we have

$$\underline{S}^2 = k \cdot e + \mu \underline{S_2} = k \cdot e + \mu \left(\underline{H \oplus \mathbb{Z}_p} - \underline{(0_G, \mathbb{Z}_p)} \right). \tag{15}$$

Let $\psi \in \widehat{\mathbb{Z}_p}$ be a non-trivial character of $\mathbb{Z}_p$ and let $\chi \in \widehat{G}$. By applying the character $(\chi, \psi) \in \widehat{G \oplus \mathbb{Z}_p}$ on both sides of (15), we obtain

$$((\chi, \psi)(\underline{S}))^2 = k,$$

implying that $(\chi, \psi)(\underline{S}) = \sqrt{k}$ or $-\sqrt{k}$. Let $B = \{g \in G \mid (\chi_g, \psi)(\underline{S}) = \sqrt{k}\}$. By a similar analysis as in Proposition 29, we can deduce that

$$\chi_l(\underline{B}) = \frac{|G|}{2\sqrt{k}} \left(\sum_{i\in\mathbb{Z}_p} \psi(i) a_{-l}(\underline{R_i}) + \sqrt{k} \cdot a_{-l}(0_G) \right) \text{ for } l \in G.$$

In particular, $|B| = \chi_0(\underline{B}) = \frac{|G|}{2}$, and so B is non-empty.

(iii) Combining (i) and (ii), we get

$$\chi_l(\underline{B^p}) = \left(\frac{|G|}{2\sqrt{k}} \right)^p \left(a_l(\underline{G \setminus H}) + \sqrt{k^p} \cdot a_l(0_G) \right) \text{ for } l \in G.$$

Let $\sigma : G \to \mathbb{C}$ be the mapping defined by

$$\sigma(g) = \begin{cases} 1, & \text{if } g \in H; \\ -1, & \text{if } g \in G \setminus H. \end{cases}$$

As H is an index 2 subgroup of G, the mapping σ is exactly an irreducible representation of G, and so $\sigma \in \widehat{G}$ because G is abelian. Thus we assert that there exists some involution $a \in G$ such that $\sigma = \chi_a \in \widehat{G}$. Then from the Fourier inversion formula (3) we obtain

$$\underline{B^p} = \left(\frac{|G|}{2\sqrt{k}} \right)^p \left(\frac{1}{2} \cdot 0_G - \frac{1}{2} \cdot a + \frac{\sqrt{k^p}}{|G|} \underline{G} \right).$$

Therefore, $\frac{|G|}{2\sqrt{k}}$ is an integer because p is odd and $\underline{B^p} \in \mathbb{Z}G$. □

7 Distance-regular Cayley graphs over $\mathbb{Z}_n \oplus \mathbb{Z}_p$

In this section, we shall prove Theorem 4, which determines all distance-regular Cayley graphs over the group $\mathbb{Z}_n \oplus \mathbb{Z}_p$. To achieve this goal, we need the following two lemmas.

Lemma 32. *Let p be an odd prime and let n be a positive integer such that $p \mid n$. Then there are no antipodal non-bipartite distance-regular Cayley graphs with diameter 3 over $\mathbb{Z}_n \oplus \mathbb{Z}_p$.*

Proof. By contradiction, assume that $\Gamma = \mathrm{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_p, S)$ is an antipodal non-bipartite distance-regular Cayley graph of diameter 3 over $\mathbb{Z}_n \oplus \mathbb{Z}_p$ with n as small as possible (with respect to p). Let k and r ($r \geqslant 2$) denote the valency and the common size of antipodal classes (or fibres) of Γ, respectively. According to [7, p. 431], $k+1 = \frac{np}{r}$, $k = \mu(r-1)+\lambda+1$, and Γ has the intersection array $\{k, \mu(r-1), 1; 1, \mu, k\}$ and eigenvalues k, θ_1, $\theta_2 = -1$, θ_3, where

$$\theta_1 = \frac{\lambda - \mu}{2} + \delta, \quad \theta_3 = \frac{\lambda - \mu}{2} - \delta \text{ and } \delta = \sqrt{k + \left(\frac{\lambda - \mu}{2}\right)^2}. \tag{16}$$

Let $H = S_3 \cup \{(0,0)\}$ denote the antipodal class containing the identity vertex. Then $|H| = r$. By Lemma 10, H is a subgroup of $\mathbb{Z}_n \oplus \mathbb{Z}_p$. If r is not a prime, then H has a non-trivial subgroup K. Let $\mathcal{B}$ denote the partition of $\mathbb{Z}_n \oplus \mathbb{Z}_p$ consisting of all cosets of K in $\mathbb{Z}_n \oplus \mathbb{Z}_p$ and let $\Gamma_{\mathcal{B}}$ be the quotient graph of Γ with respect to $\mathcal{B}$. Then, in a similar way as in Lemma 10, we can verify that $\Gamma_{\mathcal{B}} \cong \mathrm{Cay}((\mathbb{Z}_n \oplus \mathbb{Z}_p)/K, S/K)$, where $S/K = \{sK \mid s \in S\}$. Since $K \cap (S_1 \cup S_2) = \varnothing$, for any two distinct $s_1, s_2 \in S$, we have $s_1K \neq s_2K$. Also, by Corollary 16,

$$\begin{cases} \underline{S}^2 = k \cdot 0_G + (\lambda - \mu)\underline{S} + \mu(\underline{\mathbb{Z}_n \oplus \mathbb{Z}_p} - \underline{H}), \\ \underline{H} \cdot (\underline{S} + e) = \underline{\mathbb{Z}_n \oplus \mathbb{Z}_p}. \end{cases} \tag{17}$$

Let f be the mapping from the group algebra $\mathbb{Z} \cdot (\mathbb{Z}_n \oplus \mathbb{Z}_p)$ to the group algebra $\mathbb{Z} \cdot ((\mathbb{Z}_n \oplus \mathbb{Z}_p)/K)$ defined by

$$f\left(\sum_{x \in \mathbb{Z}_n \oplus \mathbb{Z}_p} a_x x\right) = \sum_{x \in \mathbb{Z}_n \oplus \mathbb{Z}_p} a_x \cdot xK.$$

By applying f on both sides of the two equations in (17), we obtain

$$\begin{cases} (\underline{S/K})^2 = k \cdot K + (\lambda - \mu)\underline{S/K} + \mu|K|(\underline{(\mathbb{Z}_n \oplus \mathbb{Z}_p)/K} - \underline{H/K}), \\ |K|\underline{H/K} \cdot (\underline{S/K} + K) = |K|\underline{(\mathbb{Z}_n \oplus \mathbb{Z}_p)/K}, \end{cases}$$

or equivalently,

$$\begin{cases} (\underline{S/K})^2 = k \cdot K + ((\lambda - \mu + \mu|K|) - \mu|K|)\underline{S/K} + \mu|K|(\underline{(\mathbb{Z}_n \oplus \mathbb{Z}_p)/K} - \underline{H/K}), \\ \underline{H/K} \cdot (\underline{S/K} + K) = \underline{(\mathbb{Z}_n \oplus \mathbb{Z}_p)/K}. \end{cases} \tag{18}$$

Then from (18) and Corollary 16 we can deduce that $\Gamma_{\mathcal{B}}$ is an $(r/|K|)$-antipodal distance-regular graph of diameter 3 with intersection array $\{k, k-(\lambda-\mu+\mu|K|)-1 = \mu|K|(r/|K|-1), 1; 1, \mu|K|, k\}$. If $\Gamma_{\mathcal{B}}$ is bipartite, then Γ is also bipartite, a contradiction. Hence, $\Gamma_{\mathcal{B}}$ is an antipodal non-bipartite distance-regular Cayley graph of diameter 3 over the cyclic group or the group $\mathbb{Z}_{n'} \oplus \mathbb{Z}_p$ with $n' \mid n$. By Theorem 2, we assert that the former case cannot occur. For the later case, this violates the minimality of n. Therefore, r is a prime. Then, by Proposition 28, $\mathbb{Z}_n \oplus \mathbb{Z}_p \cong M \oplus \mathbb{Z}_r$ and Γ is isomorphic to a Cayley graph over $M \oplus \mathbb{Z}_r$ in which the antipodal class containing the identity vertex is $S_3 \cup \{(0_M, 0)\} = (0_M, \mathbb{Z}_r)$, where M is an abelian group of order $|G|/r$. Thus we only need to consider the following two cases.

Case A. $M = \mathbb{Z}_n$, $r = p$ and $S_0 \cup S_3 = (0_M, \mathbb{Z}_p)$.

In this situation, $r = p$ is odd. By Proposition 29 (ii), there exists a non-empty $(n, -\frac{n}{2\delta}\theta_3, x^p - \frac{n^p}{(2\delta)^p})$-polynomial addition set B in $\mathbb{Z}_n$. Note that $|B| = -\frac{n}{2\delta}\theta_3$. On the other hand, by Lemma 21, we assert that $|B| \in \{1, n-1, n\}$. If $|B| = -\frac{n}{2\delta}\theta_3 = 1$, then from (16) and $k = n-1$ we can deduce that $\lambda - \mu = n-2 = k-1$, which is impossible because $k = \mu(p-1) + \lambda + 1 \geqslant 2\mu + \lambda + 1$ and $\mu \geqslant 1$. Similarly, if $|B| = -\frac{n}{2\delta}\theta_3 = n-1$ then $\mu - \lambda = n-2 = k-1$, and if $|B| = -\frac{n}{2\delta}\theta_3 = n$ then $k = 0$, which are also impossible.

Case B. $M = \mathbb{Z}_{\frac{n}{r}} \oplus \mathbb{Z}_p$ and $S_0 \cup S_3 = (0_M, \mathbb{Z}_r)$.

In this situation, we must have $\gcd(r, \frac{n}{r}) = 1$.

Subcase B.1. $r = 2$.

Since p is odd and $\gcd(2, \frac{n}{2}) = 1$, we see that $|M| = \frac{np}{2}$ is odd. By Proposition 29 (iii), there exists a strongly regular Cayley graph Γ' over M with parameters $(|M|, k' = \frac{|M|}{2\delta}\theta, \lambda' = \frac{|M|}{4\delta^2}(\theta^2 - 1), \mu' = \frac{|M|}{4\delta^2}(\theta^2 - 1))$, where $\theta = \theta_1$ or $-\theta_3$. Clearly, Γ' is non-bipartite because it is of odd order. If $k' = 2$, then Γ' is a cycle, and hence $\Gamma' \cong C_4$, which is impossible. Now suppose $k' \geqslant 3$. We see that Γ' must be primitive or antipodal. If Γ' is antipodal, then it is a complete multipartite graph, which is impossible because $\lambda' = \mu'$. If Γ' is primitive, then Corollary 14 indicates that $\frac{n}{2} = p$ and $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$ because Γ' cannot be a complete graph. Thus, by Lemma 24, Γ' is isomorphic to the line graph of a transversal design $TD(r', p)$ with $2 \leqslant r' \leqslant p-1$. However, this is also impossible because $\lambda' = \mu'$.

Subcase B.2. $r \neq 2$.

If $r = p$, then we are done by Case A. Now suppose $r \neq p$. Recall that $S = S_1 = \cup_{i \in \mathbb{Z}_r}(R_i, i)$. By Proposition 29 (i), the sets R_i, for $i \in \mathbb{Z}_r$, form a partition of $M \setminus \{0_M\}$. Furthermore, by Proposition 29 (ii), both $\frac{|M|}{2\delta}$ and θ_3 are integers, and for every non-trivial character $\psi \in \widehat{\mathbb{Z}_r}$, there exists a non-empty polynomial addition set $B \subseteq M$ such that

$$\chi_l(\underline{B}) = \frac{|M|}{2\delta}\left\{\sum_{i \in \mathbb{Z}_r} \psi(i) a_{-l}(\underline{R_i}) - \theta_3 \cdot a_{-l}(0_M)\right\} \text{ for all } l \in M. \tag{19}$$

Let $l_0 \in M \setminus \{0_M\}$. Then there exists some $i_0 \in \mathbb{Z}_r \setminus \{0\}$ such that $-l_0 \in R_i$, and (19) implies that

$$\chi_{l_0}(\underline{B}) = \frac{|M|}{2\delta}\psi(i_0). \tag{20}$$

Since r is an odd prime and $\psi \in \widehat{\mathbb{Z}_r}$ is non-trivial, we assert that $\psi(i_0) \in \mathbb{Q}(\omega_r) \setminus \mathbb{Q}$, where ω_r is a primitive r-th root of unity. Thus it follows from (20) and $\frac{|M|}{2\delta} \in \mathbb{Z}$ that $\chi_{l_0}(B) \in \mathbb{Q}(\omega_r) \setminus \mathbb{Q}$. On the other hand, we have $\chi_{l_0}(B) \in \mathbb{Q}(\omega_{\frac{n}{r}})$ because $\chi_{l_0} \in \widehat{M} = \widehat{\mathbb{Z}_{\frac{n}{r}} \oplus \mathbb{Z}_p}$ and $p \mid \frac{n}{r}$ due to $p \mid n$ and $p \neq r$, where $\omega_{\frac{n}{r}}$ is a primitive $\frac{n}{r}$-th root of unity. Hence, $\chi_{l_0}(B) \in (\mathbb{Q}(\omega_{\frac{n}{r}}) \cap \mathbb{Q}(\omega_r)) \setminus \mathbb{Q}$. However, this is impossible because $\mathbb{Q}(\omega_r) \cap \mathbb{Q}(\omega_{\frac{n}{r}}) = \mathbb{Q}$ due to $\gcd(r, \frac{n}{r}) = 1$.

Therefore, we conclude that there are no antipodal non-bipartite distance-regular graphs with diameter 3 over $\mathbb{Z}_n \oplus \mathbb{Z}_p$. □

Lemma 33. *Let p be an odd prime and let n be a positive integer such that $p \mid n$. Then there are no antipodal bipartite distance-regular Cayley graphs with diameter* 4 *over* $\mathbb{Z}_n \oplus \mathbb{Z}_p$.

Proof. By contradiction, assume that $\Gamma = \mathrm{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_p, S)$ is an antipodal bipartite distance-regular Cayley graph with diameter 4. Then n is even and the bipartition set of Γ containing the identity vertex is $S_0 \cup S_2 \cup S_4 = 2\mathbb{Z}_n \oplus \mathbb{Z}_p$. Let k and r ($r \geqslant 2$) denote the valency and the common size of antipodal classes (or fibres) of Γ, respectively. By [7, p. 425],

$$np = 2r^2\mu \quad \text{and} \quad k = r\mu, \tag{21}$$

and Γ has the intersection array $\{r\mu, r\mu - 1, (r-1)\mu, 1; 1, \mu, r\mu - 1, r\mu\}$. Moreover, by Lemma 15,

$$\underline{S}^2 = k \cdot e + \mu\underline{S_2} = k \cdot 0 + \mu(\underline{2\mathbb{Z}_n \oplus \mathbb{Z}_p} - \underline{S_0 \cup S_4}). \tag{22}$$

Note that $S_0 \cup S_4$ is the antipodal class of Γ containing the identity vertex, and so is a subgroup of $S_0 \cup S_2 \cup S_4 = 2\mathbb{Z}_n \oplus \mathbb{Z}_p$. Since S is inverse closed, from (21) and (22) we see that S is exactly an $(r\mu, r, r\mu, \mu)$-relative difference set relative to $S_0 \cup S_4$ in $S_0 \cup S_2 \cup S_4 = 2\mathbb{Z}_n \oplus \mathbb{Z}_p$. Then from Lemma 20 we can deduce that $\frac{n}{2} = \frac{r^2\mu}{p}$ is a divisor of $r\mu$, that is, $r = p$. Furthermore, by Proposition 30, we may assume that $S_0 \cup S_4 = (0, \mathbb{Z}_p)$. In this context, by Proposition 31, the sets R_i, for $i \in \mathbb{Z}_p$, form a partition of $\mathbb{Z}_n \setminus 2\mathbb{Z}_n = 1 + 2\mathbb{Z}_n$, and for every non-trivial character $\psi \in \widehat{\mathbb{Z}_p}$, there exists a non-empty set B in $\mathbb{Z}_n$ such that

$$\chi_l(\underline{B}) = \frac{n}{2\sqrt{k}} \left(\sum_{i \in \mathbb{Z}_p} \psi(i) a_{-l}(\underline{R_i}) + \sqrt{k} \cdot a_{-l}(0) \right) \text{ for all } l \in \mathbb{Z}_n. \tag{23}$$

Moreover, we have $\frac{n}{2\sqrt{k}} \in \mathbb{Z}$. Clearly, $\frac{n}{2\sqrt{k}} \neq 1$ by (21). Let q be a prime dividor of $\frac{n}{2\sqrt{k}}$. Since the sets R_i, for $i \in \mathbb{Z}_p$, form a partition of $1 + 2\mathbb{Z}_n$, we can deduce from (23) that $\chi_l(\underline{B}) \equiv 0 \pmod{q}$ for all $l \in \mathbb{Z}_n$. Then, by Lemma 22, there exist some $\underline{X_1}, \underline{X_2} \in \mathbb{Z} \cdot \mathbb{Z}_n$ with non-negative cofficients only such that

$$\underline{B} = q\underline{X_1} + \underline{\frac{n}{q}\mathbb{Z}_n} \cdot \underline{X_2}.$$

Since $q > 1$ and the cofficients of $\underline{B}$ in $\mathbb{Z} \cdot \mathbb{Z}_n$ is either 0 or 1, we assert that

$$\underline{B} = \underline{\frac{n}{q}\mathbb{Z}_n} \cdot \underline{X_2}.$$

Note that $(1 + 2\mathbb{Z}_n) \setminus q\mathbb{Z}_n \neq \varnothing$. Taking $l_0 \in (1 + 2\mathbb{Z}_n) \setminus q\mathbb{Z}_n$, we have $\chi_{l_0}(\underline{B}) = \chi_{l_0}(\underline{\frac{n}{q}\mathbb{Z}_n}) \cdot \chi_{l_0}(\underline{X_2}) = 0$. Let $i_0 \in \mathbb{Z}_p$ be such that $-l_0 \in R_{i_0}$. Then (23) gives that $\chi_{l_0}(\underline{B}) = \frac{n}{2\sqrt{k}}\overline{\psi(i_0)}$, and hence $\psi(i_0) = 0$, which is impossible.

Therefore, we conclude that there are no antipodal bipartite distance-regular Cayley graphs with diameter 4 over $\mathbb{Z}_n \oplus \mathbb{Z}_p$. □

Now we are in a position to give the proof of Theorem 4.

Proof of Theorem 4. First of all, it is easy to verify that the graphs listed in (i)-(iii) are distance-regular Cayley graphs over $\mathbb{Z}_n \oplus \mathbb{Z}_p$. Furthermore, by Lemma 24, the graph listed in (iv) is a distance-regular graph with diameter 2.

Conversely, suppose that $\Gamma = \mathrm{Cay}(\mathbb{Z}_n \oplus \mathbb{Z}_p, S)$ is a distance-regular Cayley graph over $\mathbb{Z}_n \oplus \mathbb{Z}_p$. Let k be the valency of Γ and let μ denote the number of common neighbors of two vertices at distance 2 in Γ. If $n = p$, then from Lemma 24 we see that Γ is isomorphic to one of the graphs listed in (i), (ii) and (iv). Thus we may assume that $n \neq p$. If Γ is primitive, by Corollary 14, Γ is isomorphic to the complete graph K_{np}, as desired. Now suppose that Γ is imprimitive. Clearly, Γ cannot be isomorphic to a cycle because it is a Cayley graph over $\mathbb{Z}_n \oplus \mathbb{Z}_p$. Thus $k \geqslant 3$, and it suffices to consider the following three situations.

Case A. Γ is antipodal but not bipartite.

By [7, pp. 140–141] and Lemma 10, the antipodal quotient $\overline{\Gamma}$ of Γ is a primitive distance-regular Cayley graph over a cyclic group or a group of the form $\mathbb{Z}_{n'} \oplus \mathbb{Z}_p$ for some $n' \mid n$. Then it follows from Theorem 2, Corollary 14 and Lemma 24 that $\overline{\Gamma}$ is a complete graph, a cycle of prime order, a Payley graph of prime order, or the line graph of a transversal design $TD(r,p)$ with $2 \leqslant r \leqslant p-1$. If $\overline{\Gamma}$ is a cycle of prime order, then Γ would be a cycle, which is impossible. If $\overline{\Gamma}$ is a Payley graph of prime order, by Lemma 11, we also deduce a contradiction. If $\overline{\Gamma}$ is the line graph of a transversal design $TD(r,p)$ with $2 \leqslant r \leqslant p-1$, then [7, pp. 140–141] implies that $d = 4$ or 5. By Lemma 23, we assert that $d = 4$ and $r = 2$, and hence $\overline{\Gamma}$ is the Hamming graph $H(2,p)$. However, by Lemma 12, $H(2,p)$ has no distance-regular antipodal covers for $p > 2$, and we obtain a contradiction. Therefore, $\overline{\Gamma}$ is a complete graph, and so $d = 2$ or 3. By Lemma 32, $d \neq 3$, whence $d = 2$. Since complete multipartite graphs are the only antipodal distance-regular graphs with diameter 2, we conclude that Γ is a complete multipartite graph with at least three parts.

Case B. Γ is antipodal and bipartite.

In this situation, n is even. If d is odd, by [7, pp. 140–141], $\overline{\Gamma}$ is primitive. Also, by Lemma 10, $\overline{\Gamma}$ is a distance-regular Cayley graph over a cyclic group or a group of the form $\mathbb{Z}'_n \oplus \mathbb{Z}_p$ for some $n' \mid n$. As in Case A, we assert that $\overline{\Gamma}$ is a complete graph. Hence, $d = 3$. Since Γ is antipodal and bipartite, we obtain $\Gamma \cong K_{\frac{np}{2},\frac{np}{2}} - \frac{np}{2}K_2$. Moreover, we assert that $n/2$ must be odd, i.e., $n \equiv 2 \pmod 4$, since Γ is a Cayley graph over $\mathbb{Z}_n \oplus \mathbb{Z}_p$. Now suppose that d is even. Then [7, pp. 140–141] and Lemma 10 imply that $\frac{1}{2}\Gamma$ is an antipodal non-bipartite distance-regular Cayley graph over $\mathbb{Z}_{\frac{n}{2}} \oplus \mathbb{Z}_p$ with diameter $d_{\frac{1}{2}\Gamma} = d/2$. Clearly, $d \neq 2$. By Lemma 33, $d \neq 4$. Thus $d \geqslant 6$ and $d_{\frac{1}{2}\Gamma} = d/2 \geqslant 3$. However, this is impossible by Case A.

Case C. Γ is bipartite but not antipodal.

In this situation, n is even. By [7, pp. 140–141] and Lemma 10, $\frac{1}{2}\Gamma$ is a primitive distance-regular Cayley graph over $\mathbb{Z}_{\frac{n}{2}} \oplus \mathbb{Z}_p$. As in Case A, $\frac{1}{2}\Gamma$ is a complete graph or the line graph of a transversal design $TD(r,p)$ with $2 \leqslant r \leqslant p-1$.

First suppose that $\frac{1}{2}\Gamma$ is a complete graph. Then we have $d = 2$ or 3. If $d = 2$, then Γ is a complete bipartite graph, which is impossible because Γ is not antipodal. If $d = 3$, then Γ is a non-antipodal bipartite distance-regular graph with diameter 3 over the abelian group $\mathbb{Z}_n \oplus \mathbb{Z}_p$. By Proposition 26, the dual graph $\widehat{\Gamma}$ of Γ is an antipodal non-bipartite distance-regular graph with diameter 3 over $\mathbb{Z}_n \oplus \mathbb{Z}_p$. However, there are no such graphs by Lemma 32, and we obtain a contradiction.

Now suppose that $\frac{1}{2}\Gamma$ is the line graph of a transversal design $TD(r, p)$ with $2 \leqslant r \leqslant p - 1$. Then we have $\mu = 1$ by Lemma 25. If there exist two distinct elements $a, b \in S$ such that $-a \neq b$, then by the bipartiteness of Γ, we have $\partial(0, a + b) = 2$, and hence the vertices $0, a, a + b, b$ form a cycle of length 4 in Γ. This implies that $\mu \geqslant 2$, a contradiction. Thus $S = \{a, -a\}$ for some $a \in \mathbb{Z}_n \oplus \mathbb{Z}_p$, and we see that Γ is a cycle, which is also impossible. □

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