# Maximum Size $t$-Intersecting Families and Anticodes

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**Abstract**

The maximum size of $t$-intersecting families is one of the most celebrated topics in combinatorics, and its size is known as the Erdős-Ko-Rado theorem. Such intersecting families, also known as constant-weight anticodes in coding theory, were considered in a generalization of the well-known sphere-packing bound. In this work we consider the maximum size of $t$-intersecting families and their associated maximum size constant-weight anticodes over alphabet of size $q > 2$. It is proved that the structure of the maximum size constant-weight anticodes with the same length, weight, and diameter, depends on the alphabet size. This structure implies some hierarchy of constant-weight anticodes.

**Mathematics Subject Classifications:** 05D05

## 1 Introduction

A system $\mathcal{A}$ of $k$-subsets of an $n$-set is ***t-intersecting*** if

$$|A_1 \cap A_2| \geqslant t \quad \text{for all} \quad A_1, A_2 \in \mathcal{A}.$$

Such a system is called a ***t-intersecting family***.

Finding the size of the largest system among all the $\binom{n}{k}$ $k$-subsets is one of the most intriguing combinatorial problems, initiated by Erdős-Ko-Rado [8]. During the years, the problem for $t \geqslant 1$ was considered and many interesting results were found, e.g. [13, 14]. The intersecting problem is also considered for an $n$-set, but with unrestricted subsets instead of $k$-subsets. Wilson [18] gave an exact bound for the Erdős-Ko-Rado theorem.

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**Theorem 1.** *If $n \geqslant (t+1)(k-t+1)$ then any $t$-intersecting family of $k$-subsets from an $n$-set contains at most $\binom{n-t}{k-t}$ subsets. The bound is attained by all $k$-subsets (of an $n$-set) that contain a fixed $t$-subset. If $n > (t+1)(k-t+1)$ this family is unique and if $n = (t+1)(k-t+1)$ there is another family with the same parameters and the same size.*

The bound of Theorem 1 was already proved in [8] for $t = 1$. A survey for the large amount of research associated with this problem for the first 22 years was given by Deza and Frankl [7].

A $t$-intersecting family in which all $k$-subsets intersect in a fixed $t$-subset is called ***trivial***. A $t$-intersecting family in which the intersection of all $k$-subsets is of size smaller than $t$ is called ***non-trivial***. The maximum size of a non-trivial $t$-intersecting family was found by Ahlswede and Khachatrian [2].

**Theorem 2.** *Let $\mathcal{F}$ be a $t$-intersecting family. If $n > (t+1)(k-t+1)$ and*

$$\left|\bigcap_{F \in \mathcal{F}} F\right| < t,$$

*then*

$$|\mathcal{F}| \leqslant \begin{cases} |\mathcal{V}_1(n,k,t)| & \text{if } t+1 \leqslant k \leqslant 2t+1, \\ \max\{|\mathcal{V}_1(n,k,t)|, |\mathcal{V}_2(n,k,t)|\} & \text{if } k > 2t+1, \end{cases}$$

*where*

$$\mathcal{V}_1(n,k,t) = \mathcal{F}_1 = \left\{ V \in \binom{[n]}{k} : |[1,t+2] \cap V| \geqslant t+1 \right\}$$

*and*

$$\mathcal{V}_2(n,k,t) = \left\{ V \in \binom{[n]}{k} : [1,t] \subset V, V \cap [1+t,k+1] \neq \emptyset \right\} \cup \{[1,k+1] \setminus \{i\}, i \in [1,t]\}.$$

It is easy to verify that

$$|\mathcal{V}_1(n,k,t)| = |\mathcal{F}_1| = (t+2)\binom{n-t-2}{k-t-1} + \binom{n-t-2}{k-t-2}$$

and

$$|\mathcal{V}_2(n,k,t)| = \binom{n-t}{k-t} - \binom{n-k-1}{k-t} + t.$$

Hence, $|\mathcal{V}_2(n,k,t)|$ is a polynomial of degree at most $k-t-1$ in $n$, since the leading coefficients of $n^{k-t}$ in $\binom{n-t}{k-t}$ and $\binom{n-k-1}{k-t}$ are both $1/(k-t)!$.

The intersection problem was completely solved by Ahlswede and Khachatrian [3], when they considered $n < (t+1)(k-t+1)$. Later Ahlswede and Khachatrian [4] observed that the intersection problem is strongly connected to the diametric problem in the Hamming spaces. The diametric problem is an important problem in coding theory and it was considered in various metric spaces in many papers [5, 9, 10, 12, 16, 17]. The problem has found an application also in coding theory as was first observed by Delsarte [6]. A ***code*** of length $n$ is a set of words over some alphabet whose length is $n$. A $t$-intersecting family can be called an anticode (a type of code) and it was used to improve the well-known sphere-packing bound with a bound called the code-anticode bound. This bound was used later by Roos [15] to eliminate the possible existence of perfect codes in the Johnson scheme for various parameters. Ahlswede, Aydinian, and Khachatrian [1] defined the concept of diameter perfect code based on this bound. They have discussed diameter perfect codes in the Hamming and the Johnson scheme, and mentioned also the Grassmann scheme.

**Definition 3.** For two words $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ and $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n)$, over an alphabet $\Sigma_q$ with $q \geqslant 2$ letters, the Hamming distance $\mathsf{d}(\mathbf{x}, \mathbf{y})$ is the number of coordinates in which $\mathbf{x}$ and $\mathbf{y}$ differ, i.e.,

$$\mathsf{d}(\mathbf{x}, \mathbf{y}) \triangleq |\{i \ : \ \mathbf{x}_i \neq \mathbf{y}_i, \ 1 \leqslant i \leqslant n\}| \, .$$

W.l.o.g. (without loss of generality) we assume that our alphabet $\Sigma_q$ is $\mathbb{Z}_q$, which contains the integers in the set $\{i \ : \ 0 \leqslant i \leqslant q-1\}$.

The ***weight***, $\mathsf{wt}(\mathbf{x})$, of a word $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$, over an alphabet $\mathbb{Z}_q$ with $q \geqslant 2$ letters, is the number of nonzero entries in $\mathbf{x}$, i.e.,

$$\mathsf{wt}(\mathbf{x}) \triangleq |\{i \ : \ \mathbf{x}_i \neq 0, \ 1 \leqslant i \leqslant n\}| \, .$$

In other words, the weight of $\mathbf{x}$ is its distance from the allzero word, $\mathbf{0}$, i.e., $\mathsf{wt}(\mathbf{x}) = \mathsf{d}(\mathbf{x}, \mathbf{0})$.

Let $\mathcal{F}$ be a set of $k$-subsets of an $n$-set. When the $k$-subsets from $\mathcal{F}$ are represented by words of length $n$, the outcome is a binary code $\mathcal{C}$ whose codewords have constant-weight $k$.

The minimum distance of a code $\mathcal{C}$ is the least distance between any two distinct codewords of $\mathcal{C}$. A constant-weight code $\mathcal{C}$ is referred to as an $(n, d, k)_q$ code, where $n$ is the length of the codewords, $k$ is their weight, and $d$ is the minimum distance of $\mathcal{C}$. If $\mathcal{C}$ is a code associated with a $t$-intersecting family, then we are interested in the maximum distance $D$ between any codewords in $\mathcal{C}$. The maximum distance of the code is called the ***diameter*** of $\mathcal{C}$. A constant-weight code with diameter $D$, codewords of length $n$ and weight $k$ will be referred to as an $(n, D, k)_q$ ***anticode***. A maximum size intersecting family is an anticode of maximum size, but an anticode of maximum size might not be an intersecting family of maximum size (see also the remarks after Theorem 13)

The code-anticode bound was given first by Delsarte [6] and was further discussed in [1] for schemes based on distance-regular graphs. In this paper we are interested in constant-weight codes over $\mathbb{Z}_q$, where $q > 2$. In this case the metric is not based on a distance-regular graph. The proof for this version is presented in [10, 11].

**Theorem 4.** *If $\mathcal{C}$ is an $(n, d, k)_q$ code and $\mathcal{A}$ is an $(n, d-1, k)_q$ anticode, then*

$$|\mathcal{C}| \cdot |\mathcal{A}| \leqslant \binom{n}{k} (q-1)^k. \tag{1}$$

**Definition 5.** The bound in Inequality (1) is the ***code-anticode bound***. A code $\mathcal{C}$ which satisfies Inequality (1) with equality is called a ***diameter perfect constant-weight code***. An anticode $\mathcal{A}$ which satisfies Inequality (1) with equality is called a ***maximum size anticode***.

Inequality (1) is the motivation to consider constant-weight anticodes (as well as their combinatorial interest) over $\mathbb{Z}_q$, $q > 2$. The goal of the current work is to consider $t$-intersecting families and anticodes over a non-binary alphabet, where words have length $n$ and weight $k$.

**Definition 6.** A $\boldsymbol{(t,k)_q}$***-intersecting family*** $\mathcal{F}$ is a set of words of length $n$ and weight $k$ over $\mathbb{Z}_q$ such that for each two words $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ and $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$ in $\mathcal{F}$ the number of equal nonzero entries is at least $t$, i.e.,

$$|\{i \ : \ \mathbf{x}_i = \mathbf{y}_i \ \text{ and } \ \mathbf{x}_i \neq 0, \ 1 \leqslant i \leqslant n\}| \geqslant t.$$

There is another definition for $(t,k)_q$-intersecting family, different from Definition 6. In this definition, e.g. [1, 4], the requirement $\mathbf{x}_i \neq 0$ is dropped, i.e., the intersection includes coordinates where the two words have *zeros*. We use Definition 6 since it is also a generalization of $k$-intersecting family in the binary case. Moreover optimal $(t,k)_q$-intersecting families defined in Section 2 are optimal constant-weight anticodes with small diameter. Definition 6 yields small diameter when $t$ and $k$ are fixed, while $n$ is large as it is required here. In addition, it is interesting to note that by using Definition 6 a $(t,k)_q$-intersecting family is equivalent to a $t$-intersecting family in the Johnson scheme $J(n(q-1),k)$ where we can choose only the transversal $k$-subsets, with respect to the fixed partition of the base $n(q-1)$-set into $n$ subsets of size $q-1$. These $t$-intersecting families in $J(n(q-1),k)$ are anticodes with diameter at most $k-t$ with the Johnson distance (which is half of the Hamming distance). However, when $k = n$, i.e., there are no *zeros* in the words, the two definitions coincide and the following results were obtained for this case.

The first important result for $k = n$ was proved by Frankl and Füredi [13].

**Theorem 7.** *If $\mathcal{F}$ is a maximum size $(t,k)_q$-intersecting family with words of length $n$, where $k = n$ and $t \geqslant 15$, then*

$$|\mathcal{F}| = (q-1)^{n-t} \quad \textit{if and only if} \quad q \geqslant t+1.$$

The result of Theorem 7 was improved later by Frankl and Tokushige [14].

**Theorem 8.** *If $\mathcal{F}$ is a maximum size $(t,k)_q$-intersecting family with words of length $n$, where $k = n$ and $q \geqslant t+1$, then*

$$|\mathcal{F}| = (q-1)^{n-t}.$$

The result of Theorem 8 was proved in parallel by Ahlswede and Khachatrian [4], who also solved the remaining cases when $q < t+1$.

**Corollary 9.** *If $q \geqslant t$, then the maximum size of an anticode of length $n$, and diameter $n-t$ over $\mathbb{Z}_q$ is $q^{n-t}$.*

Similarly to Corollary 9, the diameter of an anticode by the definition of [1, 4], i.e., when the *zeros* are considered in the intersection of size $t$, is $n-t$. This is not the case if we consider Definition 6 which is the definition used in our work.

The rest of the paper is organized as follows. In Section 2, we define one intersecting family and two anticodes, prove asymptotic optimality of these sets and the uniqueness of their structure. The defined sets of anticodes are generalized in Section 3, where two sequences of such constant-weight anticodes are defined. This generalization induces a hierarchy between the anticodes in each sequence. The hierarchy for anticodes is based on their size, where each one is larger for different range of alphabet size and length. Two anticodes in this hierarchy are compared only when they have the same alphabet size, length, weight, and diameter. This hierarchy is analyzed in Section 3. Finally, in Section 4 conclusion, comparison with other maximum size anticodes, and directions for future research are presented and discussed.

## 2 Maximum Size $t$-Intersecting Families and Anticodes

In this section we show asymptotic maximality of one $(t,k)_q$-intersecting family for any admissible triple $(t,k,q)$ and two families of anticodes for any admissible triple $(D,k,q)$, where $D$ is the diameter of the anticode, $q$ is the alphabet size, and $k$ is the constant weight of the anticode. We also prove the uniqueness of the intersecting families and the anticodes. The two families of the anticodes differ in the parity of the diameter, even or odd.

We start with a $(t,k)_q$-intersecting family defined by

$$\mathcal{F}_q(t,k,n) \triangleq \{(\overbrace{1\cdots 1}^{t \text{ times}}\mathbf{b}_1\cdots\mathbf{b}_{n-t}) \;:\; \mathbf{b}_i \in \mathbb{Z}_q,\ \mathsf{wt}(\mathbf{b}_1\cdots\mathbf{b}_{n-t}) = k-t\}\ .$$

**Lemma 10.** *If $n \geqslant k > t \geqslant 0$ then $\mathcal{F}_q(t,k,n)$ is a $(t,k)_q$-intersecting family of length $n$ and $\binom{n-t}{k-t}(q-1)^{k-t}$ words.*

*Proof.* It is readily verified that $\mathcal{F}_q(t,k,n)$ is a $(t,k)_q$-intersecting family of length $n$. The size of $\mathcal{F}_q(t,k,n)$ follows immediately from choosing the $k-t$ nonzero coordinates in the last $n-t$ coordinates and each one of these coordinates can be assigned with $q-1$ possible values. □

The ***support***, $\mathsf{supp}(\mathbf{x})$, of a word $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is the set of coordinates with values different from zero, i.e.,

$$\mathsf{supp}(\mathbf{x}) \triangleq \{i \;:\; \mathbf{x}_i \neq 0,\ 1 \leqslant i \leqslant n\}\ .$$

**Theorem 11.** *If $n \geqslant k > t \geqslant 0$, $n \geqslant (t+1)(k-t+1)$, $q > 2$, and $q \geqslant t+1$, then the $(t,k)_q$-intersecting family $\mathcal{F}_q(t,k,n)$ is a maximum size $(t,k)_q$-intersecting family.*

*Proof.* Let $\mathcal{G}$ be a $(t,k)_q$-intersecting family with words of length $n$ and let $\mathcal{H}$ be the set of supports of the words in $\mathcal{G}$, i.e.,

$$\mathcal{H} \triangleq \{\mathsf{supp}(\mathbf{x}) \ : \ \mathbf{x} \in \mathcal{G}\} \ .$$

The set $\mathcal{H}$ is a $t$-intersecting family since each two words of $\mathcal{G}$ are from a $(t,k)_q$-intersecting family, i.e., they have the same nonzero values in at least $t$ coordinates. Hence, by Theorem 1 we have $|\mathcal{H}| \leqslant \binom{n-t}{k-t}$. Each $k$-subset $\mathbf{c} \in \mathcal{H}$ is a support of words in $\mathcal{G}$ which form a $(t,k)_q$-intersecting family in which $k = n$. Therefore, by Theorem 8 the number of codewords whose support is $\mathbf{c}$ is at most $(q-1)^{k-t}$. Hence, the number of codewords in $\mathcal{G}$ is at most $\binom{n-t}{k-t}(q-1)^{k-t}$, which is the size of $\mathcal{F}_q(t,k,n)$ by Lemma 10. □

**Definition 12.** Two codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of length $n$ with constant-weight $k$ over $\mathbb{Z}_q$ are said to be ***equivalent*** if $\mathcal{C}_1$ can be obtained from $\mathcal{C}_2$ by permuting coordinate positions (columns) and nonzero symbols independently per each coordinate. We say that a maximum size intersecting family or anticode, with fixed parameters, over $\mathbb{Z}_q$ is ***unique*** if any two intersecting families or anticodes, respectively, of maximum size with the fixed parameters are equivalent.

**Theorem 13.** *If $n \geqslant k > t > 1$, $n > (t+1)(k-t+1)$ (if $t = 1$ then $n > 3k-2$), and $q \geqslant t+1$, then $\mathcal{F}_q(t,k,n)$ is a unique maximum size $(t,k)_q$-intersecting family.*

*Proof.* By the proof of Theorem 11, the $(t,k)_q$-intersecting family $\mathcal{G}$ has maximum size $\binom{n-t}{k-t}(q-1)^{k-t}$ only if the $t$-intersecting family $\mathcal{H}$ has maximum size $\binom{n-t}{k-t}$. By Theorem 1 we have that $\mathcal{H}$ is equivalent to a code in which the codewords are all the words of weight $k$ that have *ones* in the first $t$ coordinates. Two codewords of $\mathcal{G}$ whose supports do not intersect on the last $n-t$ coordinates must have the same values in the first $t$ positions.

Assume there exist two codewords $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ that have different values in the first $t$ coordinates and hence their supports have some intersection in the other $n-t$ coordinates. There exists a codeword $\mathbf{z} \in \mathcal{G}$ whose support does not intersect the support of $\mathbf{u}$ in the last $n-t$ coordinates and does not intersect the support of $\mathbf{v}$ in the last $n-t$ coordinates. This codeword $\mathbf{z}$ must have the same values as $\mathbf{u}$ in the first $t$ coordinates and the same values as $\mathbf{v}$ in the first $t$ coordinates, a contradiction since $\mathbf{u}$ and $\mathbf{v}$ have different values in the first $t$ coordinates. Therefore, all the codewords of $\mathcal{G}$ have the same values on the first $t$ coordinates, w.l.o.g. *ones*.

Thus, the unique maximum size $(t,k)_q$-intersecting family containing $\binom{n-t}{k-t}(q-1)^{k-t}$ words, contains all the $\binom{n-t}{k-t}(q-1)^{k-t}$ words of weight $k$ over $\mathbb{Z}_q$ with *ones* in the first $t$ coordinates. □

The requirement that $n > (t+1)(k-t+1)$ in Theorem 13 is due to the fact that for $n = (t+1)(k-t+1)$ there exist two types of optimal $t$-intersecting families (see Theorem 1). The requirement $n > 3k-2$ for $t = 1$ is due to the fact that to have the three codewords $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{z}$, it is required that $n - t \geqslant 3(k-t) - 1$ (implied by the intersection of their supports).

When $n \geqslant 2k - t$, the family $\mathcal{F}_q(t,k,n)$ can be viewed also as an anticode of length $n$, constant weight $k$, and diameter $D = 2(k-t)$ (since two codewords of maximum distance have distinct nonzero entries in the last $n-t$ coordinates, where the weight of each codeword is $k-t$). A constant-weight anticode is defined via its diameter $D$ instead of the minimum intersection $t$, and with this definition we have that $t=\frac{2k-D}{2}$. It follows that the anticode will be the $t$-intersecting family $\mathcal{F}_q(t,k,n)$. Therefore, we continue with the parameter $t$ for the anticodes. Surprisingly, the same proof as in Theorem 11 cannot be used to prove that this anticode is of maximum size for large enough $n$. The reason is that an $(n,D,k)_q$ anticode is not necessarily a $(t,k)_q$ intersecting family. However, we will prove that this intersecting family is also of maximum size as an anticode. Two families of $(n,D,k)_q$ anticodes will be defined now, one for even diameter $D$ and a second for odd diameter $D$.

We start with anticodes having odd diameter $D$. For these anticodes, we will have that $D=2(k-t)-1$ which is an odd integer $D$, where $t=\frac{2k-1-D}{2}$, and $k>t\geqslant 0$. Let

$$\mathcal{A}_q(t,k,n)\triangleq\{(\overbrace{1\cdots 1}^{t\text{ times}}\mathbf{a}\mathbf{b}_1\cdots\mathbf{b}_{n-t-1}):\ \mathbf{a}\in\mathbb{Z}_q\setminus\{0\},\ \mathbf{b}_i\in\mathbb{Z}_q,\ \mathsf{wt}(\mathbf{b}_1\cdots\mathbf{b}_{n-t-1})=k-t-1\}.$$

**Lemma 14.** *If $n\geqslant 2k-t-1$ then $\mathcal{A}_q(t,k,n)$ is an $(n,2(k-t)-1,k)_q$ anticode with $\binom{n-t-1}{k-t-1}(q-1)^{k-t}$ codewords.*

*Proof.* Since $n\geqslant 2k-t-1$ and the weight in the last $n-t-1$ coordinates is exactly $k-t-1$, it follows that there exist two codewords in $\mathcal{A}_q(t,k,n)$ whose nonzero entries in the last $n-t-1$ coordinates are in disjoint coordinates. If two such codewords have different values in $\mathbf{a}$, then the distance between them is $2(k-t)-1$ which is the diameter of the anticode. The size of $\mathcal{A}_q(t,k,n)$ follows immediately by choosing the $k-t-1$ nonzero coordinates, each nonzero $\mathbf{b}_i$ and also $\mathbf{a}$ can be assigned with $q-1$ possible values. □

If $n<2k-t-1$ then $\mathcal{A}_q(t,k,n)$ is still an anticode of size $\binom{n-t-1}{k-t-1}(q-1)^{k-t}$, but its diameter is smaller than $2(k-t)-1$.

**Theorem 15.** *If $n$ is large enough, then $\mathcal{A}_q(t,k,n)$ is a maximum size anticode for given $q>2$, $k>t\geqslant 0$, and diameter $2(k-t)-1$.*

*Proof.* If $k=t+1$, then the diameter of the anticode is 1 and a maximum size anticode is of size $q-1$ and hence we assume that $k>t+1$. Let $\mathcal{B}$ be an $(n,2(k-t)-1,k)_q$ anticode and let $\mathcal{C}$ be the set of supports of the codewords in $\mathcal{B}$, i.e.,

$$\mathcal{C}\triangleq\{\mathsf{supp}(\mathbf{x})\ :\ \mathbf{x}\in\mathcal{B}\}\ .$$

The code $\mathcal{C}$ forms a $(t+1)$-intersecting family since any two words whose supports intersect in less than $t+1$ coordinates have between them distance at least $2(k-t)$ and cannot belong to $\mathcal{B}$.

If $\mathcal{C}$ is a non-trivial $(t+1)$-intersecting family, then $|\mathcal{B}|\leqslant(q-1)^k\cdot|\mathcal{C}|$ and

$$|\mathcal{C}|\leqslant\max\{|\mathcal{V}_1(n,k,t+1)|\,,|\mathcal{V}_2(n,k,t+1)|\}$$

(see Theorem 2 for details). Since $|\mathcal{V}_1(n,k,t+1)|$ and $|\mathcal{V}_2(n,k,t+1)|$ are polynomials of degree $k-t-2$ in $n$, then

$$|\mathcal{B}| \leqslant (q-1)^k \cdot |\mathcal{C}| \leqslant \binom{n-t-1}{k-t-1}(q-1)^{k-t} = \mathcal{A}_q(t,k,n),$$

when $n$ is large enough. Therefore, we assume now that $\mathcal{C}$ is a trivial $(t+1)$-intersecting family.

We partition $\mathcal{C}$ into two subsets

$$\mathcal{C}_1 = \{\mathbf{v} \in \mathcal{C} \ : \ |\mathbf{v} \cap \mathbf{u}| = t+1 \ \text{ for some } \ \mathbf{u} \in \mathcal{C}\},$$
$$\mathcal{C}_2 = \{\mathbf{v} \in \mathcal{C} \ : \ |\mathbf{v} \cap \mathbf{u}| \geqslant t+2 \ \text{ for all } \ \mathbf{u} \in \mathcal{C}\}.$$

We distinguish now between two cases depending whether $\mathcal{C}_2$ is an empty subset or not.

**Case 1:** If $\mathcal{C}_2 \neq \varnothing$ then consider any $\mathbf{u} \in \mathcal{C}_2$. Each codeword of $\mathcal{C}$ intersects $\mathbf{u}$ in at least $t+2$ coordinates. Hence, we have that

$$|\mathcal{C}| \leqslant \sum_{i=t+2}^{k} \binom{k}{i}\binom{n-k}{k-i}.$$

A support of $\mathcal{C}$ can support at most $(q-1)^k$ codewords of $\mathcal{B}$. Hence, a codeword in $\mathcal{C}$ can support at most $(q-1)^k$ codewords of $\mathcal{B}$. This implies that

$$|\mathcal{B}| \leqslant (q-1)^k \sum_{i=t+2}^{k} \binom{k}{i}\binom{n-k}{k-i}. \tag{2}$$

The sum $\sum_{i=t+2}^{k} \binom{k}{i}\binom{n-k}{k-i}$ is a polynomial of degree $k-t-2$ in $n$, while $|\mathcal{A}_q(t,k,n)| = \binom{n-t-1}{k-t-1}(q-1)^{k-t}$ is a polynomial in $n$ whose degree is $k-t-1$. Hence (2) implies that for large enough $n$ we have that $\mathcal{B}$ is smaller than $\mathcal{A}_q(t,k,n)$.

**Case 2:** If $\mathcal{C}_2 = \varnothing$ then $\mathcal{C} = \mathcal{C}_1$. We further claim that for each support $\mathbf{u} \in \mathcal{C}$ there are at most $(q-1)^{k-t}$ codewords in $\mathcal{B}$ whose support is $\mathbf{u}$. Since $\mathcal{C}_2 = \varnothing$, it follows that there exists a support $\mathbf{v} \in \mathcal{C}$ that intersects $\mathbf{u}$ in exactly $t+1$ coordinates. The number of codewords of $\mathcal{B}$ with the support $\mathbf{u}$ such that the values of all these codewords in $\mathbf{u} \cap \mathbf{v}$ are equal is at most $(q-1)^{k-t-1}$, which is the number of distinct assignments to the other $k-t-1$ coordinates of $\mathbf{u}$. Let $\mathbf{x}, \mathbf{y} \in \mathcal{B}$ be two codewords such that $\mathbf{u} = \mathsf{supp}(\mathbf{x})$ and $\mathbf{v} = \mathsf{supp}(\mathbf{y})$. Since the diameter of $\mathcal{B}$ is $2(k-t)-1$, and $|\mathbf{u} \cap \mathbf{v}| = t+1$, it follows that $\mathbf{x}$ and $\mathbf{y}$ have different values in at most one coordinate of $\mathbf{u} \cap \mathbf{v}$.

We distinguish now between four cases to prove that the number of codewords in $\mathcal{B}$ whose support is $\mathbf{u}$ is at most $(q-1)^{k-t}$.

I. If two codewords of $\mathcal{B}$ whose support is $\mathbf{u}$ differ in more than two coordinates of $\mathbf{u} \cap \mathbf{v}$, then there are no codewords in $\mathcal{B}$ whose support is $\mathbf{v}$ since the distance of this codeword from one of these two codewords of $\mathcal{B}$ is at least $2(k-t)$. This contradicts the definition of $\mathbf{v}$.

II. If all the codewords in $\mathcal{B}$ whose support is $\mathbf{u}$ have the same values in $\mathbf{u} \cap \mathbf{v}$, then the number of codewords in $\mathcal{B}$ whose support is $\mathbf{u}$ is at most $(q-1)^{k-t-1}$.

III. If two codewords of $\mathcal{B}$ whose support is $\mathbf{u}$ differ in exactly one coordinate of $\mathbf{u} \cap \mathbf{v}$, then there are $q-1$ different assignments for this coordinate in $\mathbf{u}$. Therefore, the number of distinct values assigned in $\mathbf{u} \cap \mathbf{v}$, for these codewords of $\mathcal{B}$ whose support is $\mathbf{u}$, is at most $q-1$ which implies that the number of codewords in $\mathcal{B}$ whose support is $\mathbf{u}$ is at most $(q-1)^{k-t-1} \cdot (q-1) = (q-1)^{k-t}$.

IV. Assume two codewords of $\mathcal{B}$ whose support is $\mathbf{u}$ differ in exactly two coordinates of $\mathbf{u} \cap \mathbf{v}$. W.l.o.g. let $\mathbf{x}_1$, $\mathbf{x}_2$ be two such codewords that intersect $\mathbf{y}$ exactly at the first $t+1$ coordinates and

$$\mathbf{x}_1 = (\underbrace{111111}_{t+1}\cdots), \quad \mathbf{x}_2 = (\underbrace{221111}_{t+1}\cdots).$$

In this case $\mathbf{y}$ has at most two possible assignments, i.e.,

$$\mathbf{y}_1 = (\underbrace{121111}_{t+1}\cdots), \quad \mathbf{y}_2 = (\underbrace{211111}_{t+1}\cdots)$$

(the case where there exists only one such assignment will be discussed separately.) and if both exist $\mathbf{u}$ has no more possible assignments in $\mathbf{u} \cap \mathbf{v}$. This implies that the number of codewords in $\mathcal{B}$ whose support is $\mathbf{u}$ is at most $2(q-1)^{k-t-1} \leqslant (q-1)^{k-t}$.

Since by Theorem 1 we have that $|\mathcal{C}| \leqslant \binom{n-t-1}{k-t-1}$ for $n \geqslant (t+1)(k-t+1)$, it follows that $|\mathcal{B}| \leqslant \binom{n-t-1}{k-t-1}(q-1)^{k-t} = |\mathcal{A}_q(t,k,n)|$ for large enough $n$ in all these four cases (excluding the subcase considered below).

We continue with the same assumptions as in IV., when there is exactly one assignment to the codeword $\mathbf{y}$ in $\mathbf{u} \cap \mathbf{v}$ and this is the same for each support $\mathbf{v} \in \mathcal{C}$, such that $|\mathbf{u} \cap \mathbf{v}| = t+1$. Define the following set

$$\Gamma(\mathbf{u}) = \{\mathbf{w} \in \mathcal{C} : |\mathbf{u} \cap \mathbf{w}| = t+1\}.$$

The elements of $\Gamma(\mathbf{u})$ form a trivial $(t+1)$-intersecting family on all the $n$ coordinates except for the $k-t+1$ coordinates of $\mathbf{u} \setminus (\mathbf{u} \cap \mathbf{v})$. Hence,

$$|\Gamma(\mathbf{u})| \leqslant \binom{n-k}{k-t-1}.$$

For all $\mathbf{w} \in \mathcal{C} \setminus \Gamma(\mathbf{u})$, we have $|\mathbf{u} \cap \mathbf{w}| \geqslant t+2$. Hence, the set $\mathcal{C} \setminus \Gamma(\mathbf{u})$ can support at most

$$(q-1)^k \sum_{i=t+2}^{k} \binom{k}{i}\binom{n-k}{k-i}$$

codewords of $\mathcal{B}$, which include all the codewords supported by $\mathbf{u}$.

Therefore, we have that

$$|\mathcal{B}| \leqslant (q-1)^{k-t-1}\binom{n-k}{k-t-1} + (q-1)^k \sum_{i=t+2}^{k} \binom{k}{i}\binom{n-k}{k-i}.$$

Similarly to Case 1 it is proved that

$$|\mathcal{B}| \leqslant (q-1)^{k-t-1}\binom{n-k}{k-t-1} + (q-1)^k \sum_{i=t+2}^{k} \binom{k}{i}\binom{n-k}{k-i} \leqslant \binom{n-t-1}{k-t-1}(q-1)^{k-t}$$

for large enough $n$. This is because

$$\binom{n-t-1}{k-t-1}(q-1)^{k-t} - (q-1)^{k-t-1}\binom{n-k}{k-t-1} > (q-1)^{k-t-1}\binom{n-t-1}{k-t-1},$$

the sum $\sum_{i=t+2}^{k}\binom{k}{i}\binom{n-k}{k-i}$ is a polynomial of degree $k-t-2$ in $n$, while $\binom{n-t-1}{k-t-1}$ is a polynomial of degree $k-t-1$.

In all the analyzed cases, we have proved that $|\mathcal{B}| \leqslant |\mathcal{A}_q(t,k,n)|$ for large enough $n$, i.e., $\mathcal{A}_q(t,k,n)$ is a maximum size anticode with diameter $2(k-t)-1$ for large enough $n$. □

The anticode $\mathcal{A}_q(t,k,n)$ is not a $t$-intersecting family of maximum size. The proof of Theorem 15 fails in this case since $\mathcal{C}$ is a $(t+1)$-intersecting family and a $t$-intersecting family can be larger than $\mathcal{A}_q(t,k,n)$. Indeed, there exists such a $t$-intersecting family, which is $\mathcal{F}_q(t,k,n)$.

We can present the value of $n$ in Theorem 15 such that the theorem is true for all values above this $n$. The value should imply that $n$ is large enough when

$$(q-1)^{k-t-1}\binom{n-k}{k-t-1} + (q-1)^k \sum_{i=t+2}^{k} \binom{k}{i}\binom{n-k}{k-i} < \binom{n-t-1}{k-t-1}(q-1)^{k-t}\ .$$

This implies that it is enough to require $n > (q-1)^{t+1}(k-t-1)^2\binom{k}{k/2}+t+1$ in Theorem 15 (see the details of the proof in the Appendix).

Now, we define $\mathcal{A}'_q(t,k,n) \triangleq \mathcal{F}_q(t,k,n)$, i.e., $\mathcal{F}_q(t,k,n)$ is treated as an anticode.

**Lemma 16.** *If $n \geqslant 2k-t$ then $\mathcal{A}'_q(t,k,n)$ is an $(n, 2(k-t), k)_q$ anticode with $\binom{n-t}{k-t}(q-1)^{k-t}$ codewords.*

*Proof.* Since $n \geqslant 2k-t$ and the weight in the last $n-t$ coordinates is exactly $k-t$, it follows that there exist two codewords in $\mathcal{A}'_q(t,k,n)$ whose nonzero entries in the last $n-t$ coordinates are in disjoint coordinates. For two such codewords the distance between them is $2(k-t)$ which is the diameter of the anticode. The size of $\mathcal{A}'_q(t,k,n)$ follows immediately by choosing the $k-t$ nonzero coordinates, each one can be assigned with $q-1$ possible values (see also Lemma 10). □

It will be proved now that similarly to $\mathcal{A}_q(t,k,n)$ also $\mathcal{A}'_q(t,k,n)$ is a maximum size anticode if $n$ is large enough. The proof is simpler than the one of Theorem 15, but the approach is similar.

**Theorem 17.** *If $n$ is large enough, then $\mathcal{A}'_q(t,k,n)$ is a maximum size anticode for given $q > 2$, $k > t \geqslant 0$, and diameter $2(k-t)$.*

*Proof.* Let $\mathcal{B}$ be an $(n, 2(k-t), k)_q$ anticode and let $\mathcal{C}$ be the set of supports of the codewords in $\mathcal{B}$, i.e.,

$$\mathcal{C} \triangleq \{\mathsf{supp}(\mathbf{x}) \ : \ \mathbf{x} \in \mathcal{B}\} \ .$$

The code $\mathcal{C}$ forms a $t$-intersecting family since the diameter $2(k-t)$ implies that each two codewords of $\mathcal{B}$ must have at most $2(k-t)$ positions in which one codeword has a nonzero element of $\mathbb{Z}_q$ and the second codeword has a *zero*.

We partition $\mathcal{C}$ into two subsets

$$\mathcal{C}_1 = \{\mathbf{v} \in \mathcal{C} \ : \ |\mathbf{v} \cap \mathbf{u}| = t \ \text{ for some } \ \mathbf{u} \in \mathcal{C}\},$$
$$\mathcal{C}_2 = \{\mathbf{v} \in \mathcal{C} \ : \ |\mathbf{v} \cap \mathbf{u}| \geqslant t+1 \ \text{ for all } \ \mathbf{u} \in \mathcal{C}\}.$$

We distinguish now between two cases depending whether $\mathcal{C}_2$ is an empty subset or not.

**Case 1:** If $\mathcal{C}_2 = \varnothing$ then $\mathcal{C} = \mathcal{C}_1$ and by Theorem 1 we have that $|\mathcal{C}| \leqslant \binom{n-k}{k-t}$. Consider now two distinct codewords $\mathbf{u}, \mathbf{v} \in \mathcal{C}$ whose intersection is of size $t$. This already implies that their associated codewords in $\mathcal{B}$ have distance $2(k-t)$ and hence their values on the coordinates of $\mathbf{u} \cap \mathbf{v}$ are the same. This implies that the number of codewords in $\mathcal{B}$ whose support is $\mathbf{u}$ is at most $(q-1)^{k-t}$. Hence, the number of codewords in $\mathcal{B}$ is at most $\binom{n-t}{k-t}(q-1)^{k-t}$, i.e., for large enough $n$ we have that $\mathcal{B}$ is not bigger than $\mathcal{A}'_q(t,k,n)$.

**Case 2:** If $\mathcal{C}_2 \neq \varnothing$ then consider any $\mathbf{u} \in \mathcal{C}_2$. Each codeword of $\mathcal{C}$ intersect $\mathbf{u}$ in at least $t+1$ coordinates. Hence, we have that the size of $\mathcal{C}$ is at most

$$\sum_{i=t+1}^{k} \binom{k}{i}\binom{n-k}{k-i}.$$

Each codeword of $\mathcal{C}$ is a support for at most $(q-1)^k$ codewords of $\mathcal{B}$. This implies that

$$|\mathcal{B}| \leqslant (q-1)^k \sum_{i=t+1}^{k} \binom{k}{i}\binom{n-k}{k-i}. \tag{3}$$

$\sum_{i=t+1}^{k} \binom{k}{i}\binom{n-k}{k-i}$ is polynomial of degree $k-t-1$ in $n$, while $\left|\mathcal{A}'_q(t,k,n)\right| = \binom{n-t}{k-t}(q-1)^{k-t}$ is a polynomial in $n$ whose degree is $k-t$. Hence (3) implies that for large enough $n$ we have that $\mathcal{B}$ is smaller than $\mathcal{A}'_q(t,k,n)$.

In both cases, we have proved that $|\mathcal{B}| \leqslant \left|\mathcal{A}'_q(t,k,n)\right|$, i.e., $\mathcal{A}'_q(t,k,n)$ is a maximum size anticode with diameter $2(k-t)$ for large enough $n$. ∎

Similarly to Theorem 15 we can show that for Theorem 17 it is enough to require that $n > (q-1)^t(k-t)^2\binom{k}{k/2} + t + 1$.

We continue to prove that the anticodes $\mathcal{A}_q(t,k,n)$ and $\mathcal{A}'_q(t,k,n)$ are not just optimal. If $n$ is large enough then they are unique for the fixed $q$, $t$, and $k$. We start with $\mathcal{A}'_q(t,k,n)$ with a proof similar to the one of Theorem 13.

**Theorem 18.** *If $n$ is large enough, then $\mathcal{A}'_q(t,k,n)$ is a unique maximum size anticode of length $n$, constant-weight $k$ and with diameter $2(k-t)$ over $\mathbb{Z}_q$.*

*Proof.* Assume that $|\mathcal{B}| = \left|\mathcal{A}'_q(t,k,n)\right|$ and $n$ is large enough in Theorem 17. This implies that $\mathcal{C}_2 = \varnothing$, $\mathcal{C} = \mathcal{C}_1$, and $|\mathcal{C}_1| = \binom{n-t}{k-t}$. By the Erdős-Ko-Rado theorem (Theorem 1), we know that $\mathcal{C}$ consists of all $k$-subsets that include a fixed $t$-subset of coordinates. W.l.o.g., this $t$-subset is $\{1,\ldots,t\}$, and w.l.o.g., $\mathcal{B}$ contains the word $\mathbf{e} = (\overbrace{1\cdots\cdots1}^{k \text{ times}}\overbrace{0\cdots\cdots0}^{n-k \text{ times}})$. Now, any other codeword in $\mathcal{B}$ that has nonzero symbols in the first $t$ coordinates and zeros in the next $k-t$ positions must start with $t$ *ones* (otherwise, the distance from $\mathbf{e}$ will be larger than $2(k-t)$).

Assume to the contrary that there exists a codeword $\mathbf{u}$ in $\mathcal{B}$ that does not start with $t$ *ones*. To have distance at most $2(k-t)$ between $\mathbf{u}$ and $\mathbf{e}$, $\mathbf{u}$ must have at least one nonzero symbol in the next $k-t$ positions. Now, consider a third codeword $\mathbf{v} \in \mathcal{B}$ whose support intersects $\mathbf{e}$ and $\mathbf{u}$ only in the first $t$ positions. The codeword $\mathbf{v}$ must have the same values in the first $t$ positions as $\mathbf{e}$ and the same values in the first $t$ positions as $\mathbf{u}$, which is a contradiction. Therefore, all the codewords in $\mathcal{B}$ start with $t$ *ones* which implies that $\mathcal{B}$ is equivalent to $\mathcal{A}'_q(t,k,n)$. □

The uniqueness of the anticode $\mathcal{A}_q(t,k,n)$ is proved similarly to the uniqueness of the anticode $\mathcal{A}'_q(t,k,n)$.

**Theorem 19.** *If $n$ is large enough and $q > 3$, then $\mathcal{A}_q(t,k,n)$ is unique maximum size anticode of length $n$, constant-weight $k$ and with diameter $2(k-t)-1$ over $\mathbb{Z}_q$.*

*Proof.* Assume that $|\mathcal{B}| = |\mathcal{A}_q(t,k,n)|$ and $n$ is large enough in Theorem 15. This implies that $\mathcal{C}_2 = \varnothing$, $\mathcal{C} = \mathcal{C}_1$, and $|\mathcal{C}_1| = \binom{n-t-1}{k-t-1}$. Since $\mathcal{C}_1$ is a $(t+1)$-intersecting family, it follows by its size that it is a maximum size family and by the Erdős-Ko-Rado theorem (Theorem 1), we know that $\mathcal{C}$ consists of all $k$-subsets that include a fixed $(t+1)$-subset of coordinates. W.l.o.g., this $(t+1)$-subset is $\{1,\ldots,t,t+1\}$, and w.l.o.g., $\mathcal{B}$ contains the word $\mathbf{e} = (\overbrace{1\cdots\cdots1}^{k \text{ times}}\overbrace{0\cdots\cdots0}^{n-k \text{ times}})$.

Now, any other codeword in $\mathcal{B}$ that has nonzero symbols in the first $t+1$ coordinates and zeros in the next $k-t-1$ coordinates must have at least $t$ *ones* in the first $t+1$ coordinates (otherwise, the distance from $\mathbf{e}$ will be larger than $2(k-t)-1$). Our next step is to prove that since $q > 3$, it follows from the structure of $\mathcal{C}$ as explained in the proof of Theorem 15 that all these codewords have *ones* in the first $t$ positions and a nonzero alphabet letter in the next position, where there are codewords with each nonzero letter.

Assume there exists a codeword $\mathbf{u} \in \mathcal{B}$ whose first $t$ coordinates have some values different from *one* and a nonzero symbol $\alpha$ in position $t+1$. Let $\mathbf{v}$ be another codeword in $\mathcal{B}$ with *ones* in the first $t$ positions and a nonzero symbol $\beta \neq \alpha$ in position $t+1$. To avoid distance larger than $2(k-t)-1$ between $\mathbf{u}$ and $\mathbf{v}$, their supports must intersect in one more coordinate outside the first $t+1$ coordinates. There exists a codeword $\mathbf{z} \in \mathcal{C}$ whose support does not intersect the support of $\mathbf{u}$ in the last $n-t-1$ coordinates and does not intersect the support of $\mathbf{v}$ in the last $n-t$ coordinates. Moreover, $\mathbf{z}$ has a nonzero symbol $\gamma \notin \{\alpha, \beta\}$ in position $t+1$. To avoid distance larger than $2(k-t)-1$ between codewords, this codeword $\mathbf{z}$ must have the same values as $\mathbf{u}$ in the first $t$ coordinates and the same values as $\mathbf{v}$ in the first $t$ coordinates, a contradiction since $\mathbf{u}$ and $\mathbf{v}$ have different values in the first $t$ coordinates. Therefore, all the codewords of $\mathcal{C}$ have the same values on the first $t$ coordinates, w.l.o.g. *ones*.

Thus, the unique maximum size $\mathcal{A}_q(t,k,n)$ contains all the $\binom{n-t-1}{k-t-1}(q-1)^{k-t}$ words of weight $k$ over $\mathbb{Z}_q$ with *ones* in the first $t$ coordinates and a nonzero symbol in the $(t+1)$-th coordinate. □

If $q=3$, then the same proof of Theorem 19 works as well. But, we need to consider the case where all codewords of $\mathcal{C}$ whose support is $\mathbf{e}$ either begin with $t+1$ ones or w.l.o.g. with 22 and then $t-1$ ones (there are $2 \cdot (q-1)^{k-t-1}$ such words, which coincides with $(q-1)^{k-t}$ for $q=3$). For any support $\mathbf{v}$ such that $|\mathbf{e} \cap \mathbf{v}| = t+1$, only the words beginning with $12\overbrace{1\cdots\cdots1}^{t-1\ times}$ or $21\overbrace{1\cdots\cdots1}^{t-1\ times}$ can belong to $\mathcal{B}$. Since there are $2^{k-t}$ such words with support $\mathbf{v}$, $\mathcal{B}$ contains all of them. Now, implying $n-t-1 \geqslant 3(k-t-1)$, we consider a third support $\mathbf{z} \in \mathcal{C}$ at maximum distance from both $\mathbf{e}$ and $\mathbf{v}$, i.e., $|\mathbf{e} \cap \mathbf{z}| = |\mathbf{v} \cap \mathbf{z}| = t+1$. It is not difficult to verify that such codewords with support $\mathbf{z}$ cannot exist and the proof is completed.

## 3 A Hierarchy of Anticodes

In this section we define two sequences of anticodes, where each sequence has one anticode which was proved to be optimal asymptotically. It will be proved that any two anticodes in the sequence are incomparable, i.e., one is larger for one range of alphabet sizes and lengths of the codewords and the second is larger for the other alphabet sizes and lengths of the codewords. For comparing two anticodes we should have that the alphabet size $q$, the length of the words $n$, their weight $k$, and their diameter $D$ are the same for the two anticodes.

We start by defining the anticodes in these sequences. For $0 \leqslant \epsilon \leqslant k-t$, let

$$\mathcal{A}_q(t,\epsilon,k,n) \triangleq \{(\overbrace{1\cdots1}^{t\text{ times}}a_1\cdots a_\epsilon b_1\cdots b_{n-t-\epsilon}) \ : \ a_i \in \mathbb{Z}_q \setminus \{0\}, \ b_i \in \mathbb{Z}_q\},$$

where $\mathsf{wt}(b_1\cdots b_{n-t-\epsilon}) = k-t-\epsilon$, be and anticode in this sequence.

The codewords of the anticode $\mathcal{A}_q(t,\epsilon,k,n)$ have three parts: part $\mathbb{A}$ which consists of the first $t$ *ones* in the codeword, part $\mathbb{C}$ which consists of the last $n-t-\epsilon$ entries in the codewords, and part $\mathbb{B}$ which consists of the middle $\epsilon$ entries in the codeword.

We note that $\mathcal{A}_q(t,1,k,n) = \mathcal{A}_q(t,k,n)$ and $\mathcal{A}_q(t,0,k,n) = \mathcal{A}'_q(t,k,n)$. We have already proved in Theorem 15 and Theorem 17, respectively that these anticodes are maximum size anticodes for odd diameter $D = 2(k-t)-1$ and even diameter $D = 2(k-t)$, respectively.

**Lemma 20.** *If $n \geqslant 2k-t-\epsilon$, then the anticode $\mathcal{A}_q(t,\epsilon,k,n)$ has diameter $2k-2t-\epsilon$.*

*Proof.* Consider the following two codewords of $\mathcal{A}_q(t,\epsilon,k,n)$,

$$\mathbf{c} = (\overbrace{1\cdots\cdots 1}^{t \text{ times}}\overbrace{1\cdots\cdots 1}^{\epsilon \text{ times}}\overbrace{1\cdots\cdots 1}^{k-t-\epsilon \text{ times}}\overbrace{0\cdots\cdots 0}^{n-k \text{ times}})$$

and

$$\mathbf{c}' = (\overbrace{1\cdots\cdots 1}^{t \text{ times}}\overbrace{2\cdots\cdots 2}^{\epsilon \text{ times}}\overbrace{0\cdots\cdots 0}^{n-k \text{ times}}\overbrace{1\cdots\cdots 1}^{k-t-\epsilon \text{ times}}).$$

Since $n \geqslant 2k-t-\epsilon$, i.e., $n-t-\epsilon \geqslant 2(k-t-\epsilon)$, it follows that

$$\mathsf{d}(\mathbf{c},\mathbf{c}') = \epsilon + 2(k-t-\epsilon) = 2k-2t-\epsilon.$$

These two codewords are at maximum distance in the anticode $\mathcal{A}_q(t,\epsilon,k,n)$. Hence, the anticode $\mathcal{A}_q(t,\epsilon,k,n)$ has diameter $2k-2t-\epsilon$. □

**Lemma 21.** *The size of the anticode $\mathcal{A}_q(t,\epsilon,k,n)$ is $\binom{n-t-\epsilon}{k-t-\epsilon}(q-1)^{k-t}$.*

*Proof.* The number of possible combinations in part $\mathbb{B}$ is $(q-1)^{\epsilon}$ and the number of the combinations in part $\mathbb{C}$ is $\binom{n-t-\epsilon}{k-t-\epsilon}(q-1)^{k-t-\epsilon}$, which implies the claim in the lemma. □

Since the diameter $D$ of the anticode $\mathcal{A}_q(t,\epsilon,k,n)$ is $2k-2t-\epsilon$, it follows that this diameter can be even or odd, depending whether $\epsilon$ is even or odd, respectively. For $\mathcal{A}_q(t,\epsilon,k,n)$ we have that the diameter is $D = 2k-2t-\epsilon$ and this anticode will be compared with the anticode $\mathcal{A}_q(t',\epsilon',k,n)$, where $\epsilon' = \epsilon+2$, $t' = t-1$, and hence its diameter is $D' = D$.

**Lemma 22.** *When $n \geqslant 2k-t-\epsilon$, the anticode $\mathcal{A}_q(t',\epsilon',k,n) = \mathcal{A}_q(t-1,\epsilon+2,k,n)$ is larger than $\mathcal{A}_q(t,\epsilon,k,n)$ if and only if $q > \frac{n+k-2t-2\epsilon}{k-t-\epsilon}$ or equivalently $n < (q-2)(k-t-\epsilon)+k$ (both anticodes have diameter $2k-2t-\epsilon$).*

*Proof.* By Lemma 20 the diameter of the two anticodes is $2k-2t-\epsilon$. By Lemma 21 the size of $\mathcal{A}_q(t,\epsilon,k,n)$ is $\binom{n-t-\epsilon}{k-t-\epsilon}(q-1)^{k-t}$ and the size of $\mathcal{A}_q(t',\epsilon',k,n)$ is $\binom{n-t-\epsilon-1}{k-t-\epsilon-1}(q-1)^{k-t+1}$. Therefore, $\mathcal{A}_q(t',\epsilon',k,n)$ is larger than $\mathcal{A}_q(t,\epsilon,k,n)$ if and only if

$$\binom{n-t-\epsilon-1}{k-t-\epsilon-1}(q-1)^{k-t+1} > \binom{n-t-\epsilon}{k-t-\epsilon}(q-1)^{k-t},$$

which is equivalent to

$$q-1 > \frac{n-t-\epsilon}{k-t-\epsilon}$$

or

$$n < (q-2)(k-t-\epsilon)+k \ .$$ □

Lemma 22 implies that we can have two sequences of incomparable anticodes for given $t$, $k$, and $D = 2(k-t)$. For any given $q$ if $n$ is large enough then $\mathcal{A}_q(t,0,k,n)$ is the maximum size anticode. However, if $q > \frac{n+k-2t}{k-t}$, then the anticode $\mathcal{A}_q(t-1,2,k,n)$ is larger than $\mathcal{A}_q(t,0,k,n)$. If $q > \frac{n+k-2(t-1)-2}{k-(t-1)-1}$, then the anticode $\mathcal{A}_q(t-2,4,k,n)$ is larger than $\mathcal{A}_q(t-1,2,k,n)$ and as a result also larger than the anticode $\mathcal{A}_q(t,0,k,n)$. We can continue and obtain a sequence of minimum$\{t+1, k-t+1\}$ anticodes (the minimum is due to the fact that $k-t-\epsilon$ cannot be negative), where the first anticode is of maximum size if $n$ is large enough. If $n$ is fixed, then from a certain alphabet size the second anticode is larger. Similar hierarchy can be defined for odd diameter and the anticodes $\mathcal{A}_q(t,\epsilon,k,n)$, where $\epsilon$ is odd. The anticode $\mathcal{A}_q(t,1,k,n)$ is asymptotically of maximum size and each two classes of anticodes for each two different values of $\epsilon$ in this hierarchy are not comparable.

## 4 Conclusion, Discussion, and Future Research

This paper considers maximum size anticodes and maximum size $t$-intersecting families over a non-binary alphabet. Maximality and uniqueness of such anticodes was proved and hierarchy between anticodes was given. Such different anticodes were discussed also in other papers as follows.

The intersecting family $\mathcal{F}_q(t,k,n)$ which was proved to be a maximum size $(t,k)_q$-intersecting family if $n \geqslant 2k-t$ is large enough (and it was referred to as $\mathcal{A}'_q(t,k,n)$ or $\mathcal{A}_q(t,0,k,n)$) was already defined in [10], where it was proved to be a maximum size anticode if a certain structure called a generalized Steiner system with appropriate parameters exists.

There are other anticodes which were proven to be of maximum size in [10]. For $1 \leqslant \epsilon \leqslant k$, the anticode

$$\{(\mathbf{b}_1 \cdots \mathbf{b}_\epsilon \overbrace{1 \cdots\cdots 1}^{t=k-\epsilon \text{ times}} \overbrace{0 \cdots\cdots 0}^{n-k \text{ times}}) \ : \ \mathbf{b}_i \in \mathbb{Z}_q \setminus \{0\}, \ 1 \leqslant i \leqslant \epsilon\}.$$

This anticode is exactly $\mathcal{A}_q(t,\epsilon,k,n)$, where $t = k-\epsilon$, in our hierarchy. It is a maximum size anticode when certain structures exist (see [10] for more details). These structures exist for many parameters. This anticode is an $(n,\epsilon,k)$ anticode with $(q-1)^\epsilon$ codewords.

Two more anticodes, which are of maximum size for certain parameters, were defined in [16]. The first one

$$\{(\mathbf{a}_1 \cdots \mathbf{a}_\epsilon \mathbf{b}_1 \cdots \mathbf{b}_{n-\epsilon}) \ : \ \mathbf{a}_i \in \mathbb{Z}_q \setminus \{0\}, \ \mathrm{wt}(\mathbf{b}_1 \cdots \mathbf{b}_{n-\epsilon}) = k-\epsilon\} \ .$$

This anticode is exactly $\mathcal{A}_q(0,\epsilon,k,n)$ in our hierarchy. It is an $(n, 2k-t, k)$ anticode with $\binom{n-t}{k-t}(q-1)^k$ codewords. This anticode is of maximum size if $n \geqslant (k-t+1)(t+1)$ and $q$ is large enough.

The second one

$$\{(\overbrace{0 \cdots 0}^{t \text{ times}} \mathbf{b}_1 \cdots \mathbf{b}_{n-t}) \ : \ \mathrm{wt}(\mathbf{b}_1 \cdots \mathbf{b}_{n-t}) = k\} \ .$$

is an $(n, n-t, k)$ anticode with $\binom{n-t}{k}(q-1)^k$ codewords. This anticode is of maximum size if $n \leqslant (k+t-1)(t+1)/t$ and $q$ is large enough.

Other maximum size anticodes mentioned in [10] are not relevant for our discussion since either $n = k$ or $n = k+1$. As we see, there are many maximum size anticodes with similar parameters. Each maximum size anticode is of maximum size in different parameters and there are other large anticodes in different lengths and alphabet size which cannot be compared (each one is larger in different parameters). These anticodes were defined for the hierarchy, but it is not proved that they are of maximum size (given their parameters). Some comparisons and hierarchy between the anticodes were given in this paper and there are more comparisons in [10], where also the uniqueness of other anticodes for some given parameters is discussed. As for future research, we would like to prove the maximality of all the anticodes in the hierarchies. We also would like to know whether such hierarchies exist also for intersecting families.

## Appendix

We provide the details why it is enough to require $n > (q-1)^{t+1}(k-t-1)^2\binom{k}{k/2}+t+1$ in Theorem 15. We have to prove for Case 2 that $n$ should satisfy the inequality

$$(q-1)^{k-t}\binom{n-t-1}{k-t-1} > (q-1)^k \sum_{i=t+2}^{k} \binom{k}{i}\binom{n-k}{k-i} + (q-1)^{k-t-1}\binom{n-k}{k-t-1}$$

which is satisfied if

$$(q-1)^{k-t-1}\binom{n-t-1}{k-t-1} > (q-1)^k \sum_{i=t+2}^{k} \binom{k}{i}\binom{n-k}{k-i} \tag{4}$$

is satisfied. Such an $n$ will satisfy also the requirement of Case 1.

Inequality (4) is satisfied if the inequality

$$\binom{n-t-1}{k-t-1} > (q-1)^{t+1} \sum_{i=t+2}^{k} \binom{k}{k/2}\binom{n-k}{k-i} = (q-1)^{t+1}\binom{k}{k/2} \sum_{i=t+2}^{k} \binom{n-k}{k-i} \tag{5}$$

is satisfied.

It is easy to verify that for $n$ large enough (and also for $n > 3k$) we have

$$(k-t-1)\binom{n-k}{k-(t+2)} > \sum_{i=t+2}^{k} \binom{n-k}{k-i}. \tag{6}$$

The inequality (6) is equivalent to

$$(q-1)^{t+1}\binom{k}{k/2}(k-t-1)\binom{n-k}{k-(t+2)} > (q-1)^{t+1}\binom{k}{k/2} \sum_{i=t+2}^{k} \binom{n-k}{k-i}. \tag{7}$$

Inequalities (4) and (7) imply that $n$ large enough which satisfies

$$\binom{n-t-1}{k-t-1} > (q-1)^{t+1}\binom{k}{k/2}(k-t-1)\binom{n-k}{k-(t+2)} > (q-1)^{t+1}\binom{k}{k/2}\sum_{i=t+2}^{k}\binom{n-k}{k-i} \tag{8}$$

also satisfies Inequality (4). This implies that it is enough to require that $n$ satisfies the inequality

$$\binom{n-t-1}{k-t-1} > (q-1)^{t+1}\binom{k}{k/2}(k-t-1)\binom{n-k}{k-(t+2)}. \tag{9}$$

We further note that

$$\binom{n-t-1}{k-t-1}\Big/\binom{n-k}{k-(t+2)} = \frac{1}{k-t-1}\cdot\frac{(n-k+1)(n-k+2)\ldots(n-t-1)}{(n-k)(n-k-1)\ldots(n-2k+t+3)} \tag{10}$$

i.e.,

$$\binom{n-t-1}{k-t-1}\Big/\binom{n-k}{k-(t+2)} > \frac{1}{k-t-1}\cdot(n-t-1). \tag{11}$$

From Inequalities (9) and (11) it is enough to require that

$$\frac{1}{k-t-1}\cdot(n-t-1) > (q-1)^{t+1}(k-t-1)\binom{k}{k/2}. \tag{12}$$

Finally, Inequality (12) implies that it is enough to require

$$n > (q-1)^{t+1}(k-t-1)^2\binom{k}{k/2} + t + 1$$

in Theorem 15 as claimed.

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