On Cube-Free Problems

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Abstract

Eberhard and Pohoata conjectured that every 3-cube-free subset of $[N]$ has size less than $2N/3 + o(N)$. In this paper we show that if we replace $[N]$ with $\mathbb{Z}_N$ the upper bound of $2N/3$ holds, and the bound is tight when N is divisible by 3 since we have $A = \{a \in \mathbb{Z}_N : a \equiv 1, 2 \pmod 3\}$. Inspired by this observation we conjecture that every d-cube-free subset of $\mathbb{Z}_N$ has size less than $(d-1)N/d$ where N is divisible by d, and we show the tightness of this bound by providing an example $B = \{b \in \mathbb{Z}_N : b \equiv 1, 2, \dots, d-1 \pmod d\}$. We prove the conjecture for several interesting cases, including when d is the smallest prime factor of N, or when N is a prime power.

We also discuss some related issues regarding $\{x, dx\}$-free and $\{x, 2x, \dots, dx\}$-free sets. A main ingredient we apply is to arrange all the integers into some square matrix, with $m = d^s \times l$ having the coordinate $(s+1, l - \lfloor l/d \rfloor)$. Here d is a given integer and l is not divisible by d.

Mathematics Subject Classifications: 11B30

1 Introduction

A set is called sum-free if there are no solutions to the equation $x + y = z$. For example, any subset of integers consisting of odd numbers is sum-free, as the sum of any two odd numbers results in an even number. The study on sum-free set traces its roots back to the early 20th century when Schur [9] used a combinatorial argument to show that Fermat's Last Theorem does not hold in the finite field $\mathbb{F}_p$.

In addition to exploring the sum-free problem, the research community has shown considerable interest in generalizations. For instance, one of them is to study the so-called (k, l)-sum-free sets, which is a set with no solutions to $x_1 + \cdots + x_k = y_1 + \cdots + y_l$. In particular, the avoidance density for such sets was recently determined by by Jing and Wu [6, 7], generalizing the line of research for sum-free sets by Bourgain [1], by Eberhard, Green, and Manners [5], and by Eberhard [4].

In this note our primary focus lies in yet another branch of generalization for sum-free problems — the study of cube-free subsets within the cyclic group $\mathbb{Z}_N$. To establish

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the foundation for our exploration, we present the definition of cubes, or more precisely, projective cubes:

Definition 1. Given a multiset $S = \{a_1, \ldots, a_d\}$ of size d, we define the projective d-cube generated by S as

$$\Sigma^* S = \left\{ \sum_{i \in I} a_i : \varnothing \neq I \subset [d] \right\}.$$

Definition 2. We say A is d-cube-free if there does not exist a multiset S of size d with $(\Sigma^* S) \subset A$.

For example, a set is 3-cube-free if it contains no $\{x, y, z, x+y, y+z, x+z, x+y+z\}$ as a subset.

The motivation behind this research is derived from a similar problem concerning cube-free subsets of the set $[N]$, which was conjectured by Eberhard and Pohoata[1]:

Conjecture 3 (Eberhard–Pohoata)**.** Suppose $A \subset [N]$ is 3-cube-free, then

$$|A| \leqslant (2/3 + o(1))N.$$

The equality holds when $A = \{x \equiv 1, 2 \pmod 3\}$ or $A = (N/3, N]$.

It is easy to verify that the two examples are 3-cube-free. However, it is important to note that when discussing the problem within cyclic groups whose order is divisible by 3, the latter condition is no longer 3-cube-free, while the former still holds. This observation suggests the following conjecture:

Conjecture 4. Let A be a d-cube-free subset of $\mathbb{Z}_N$ where $d \mid N$, then

$$|A| \leqslant \frac{d-1}{d} N.$$

The main result of this note verifies Conjecture 4 for many interesting cases:

Theorem 5. *Let A be a d-cube-free subset of $\mathbb{Z}_N$ where $d \mid N$. We have*

$$|A| \leqslant \frac{d-1}{d} N$$

when one of following is true:

(i) $d = 3$.

(ii) *d is the smallest prime factor of N.*

(iii) *N is the power of some prime p.*

Notably, the method we employed in proving Theorem 5 (i) holds considerable promise for addressing similar problems. To establish this theorem, we concentrate on subsets that are free of the diagonal solutions, namely $\{x, 2x, \ldots, (d-1)x\}$-free, and the proofs for Theorem 5 (ii) and Theorem 5 (iii) are subsequently derived from this fundamental idea.

[1] This question was mentioned in a blog of Eberhard: https://seaneberhard.com/2020/01/17/the-avoidance-density-of-kl-sum-free-sets/

2 The tightness of the upper bound

The bound in Conjecture 4 is tight, since we have the following result.

Theorem 6. *Let d, N be positive integers with $d \mid N$, then*

$$A = \{a \in \mathbb{Z}_N : a \equiv 1, 2, \ldots, d-1 \pmod{d}\}$$

is a d-cube-free subset of $\mathbb{Z}_N$.

We are going to prove a lemma to show the correctness of the example, which is based on the Cauchy–Davenport Theorem [2, 3]. Throughout this section we use standard definitions and notations in Additive Combinatorics as given in [10]. Given $A, B \subset \mathbb{Z}$, we write

$$A + B := \{a + b : a \in A, b \in B\}, \quad \text{and} \quad AB := \{ab : a \in A, b \in B\}.$$

When $A = \{x\}$, we simply write $x + B := \{x\} + B$ and $x \cdot B := \{x\}B$.

Theorem 7 (Cauchy–Davenport)**.** *Let $A, B \subset \mathbb{F}_p$, then*

$$|A + B| \geqslant min\{|A| + |B| - 1, p\}.$$

Lemma 8. *Let $t \leqslant d$ be a positive integer, let $a_i \in \mathbb{Z}_d \backslash \{0\}$ be not necessarily distinct elements and let $\lambda_i \in \{0, 1\}$. We define*

$$S_t := \left\{ \sum_{i=1}^{t} \lambda_i a_i : (\lambda_1, \lambda_2, \ldots, \lambda_t) \neq (0, 0, \ldots, 0) \right\}.$$

If $0 \notin S_t$, then $|S_t| \geqslant t$.

Proof. When $d = p$ is a prime, the lemma is indicated by the Cauchy–Davenport Theorem. Indeed, note that

$$S_t \supseteq \{a_1\} + \sum_{i=2}^{t} \{0, a_i\}.$$

We only need to consider the case that the size of the right hand side is strictly less than d, or else $|S_t| = d$ so that $0 \in S_t$.

Now we have for all $k \leqslant t$, $|\{a_1\} + \sum_{i=2}^{k} \{0, a_i\}| < d$. Then the Cauchy–Davenport Theorem implies

$$|\{a_1\} + \sum_{i=2}^{k} \{0, a_i\}| \geqslant |\{a_1\} + \sum_{i=2}^{k-1} \{0, a_i\}| + |\{0, a_k\}| - 1.$$

By using the Cauchy–Davenport Theorem for $(t-1)$ times, we have

$$|\{a_1\} + \sum_{i=2}^{t} \{0, a_i\}| \geqslant |\{a_1\}| + \sum_{i=2}^{t} |\{0, a_i\}| - (t-1) = t.$$

Thus

$$|S_t| \geqslant |\{a_1\} + \sum_{i=2}^{k} \{0, a_i\}| \geqslant t.$$

When d is not a prime, the proof goes by induction on d and then induction on t. According to the discussion above, we have already proved the lemma for prime factors, as a foundation of the induction on d. Now we suppose that the lemma holds in $\mathbb{Z}_k$ with k being all the factors of d and start our induction on t. To begin with, $|S_t| \geqslant t$ for $t = 1, 2$. Indeed, when $t = 2$, $S_t = \{a_1, a_2, a_1 + a_2\}$. It is impossible to have $a_1 = a_2 = a_1 + a_2$, which implies that $a_1 = a_2 = 0$. Now we assume $|S_k| \geqslant k$ for all integers $k \leqslant t$. Note that

$$S_{t+1} \supseteq S_t + \{0, a_{t+1}\} \supseteq S_t.$$

The induction hypothesis gives $|S_{t+1}| \geqslant |S_t| \geqslant t$. Suppose $|S_{t+1}| = t < t + 1$, then

$$S_{t+1} = S_t = S_t + a_{t+1}.$$

Adding up all the elements in three sets respectively, we have

$$\sum_{x \in S_{t+1}} x = \sum_{x \in S_t} x = \sum_{x \in S_t} (x + a_{t+1}).$$

The second equality implies

$$t a_{t+1} \equiv 0 \pmod{d}.$$

Similarly by symmetry we have

$$t a_i \equiv 0 \pmod{d}$$

for $i = 1, 2, \ldots, t + 1$. Let s be the greatest common divisor $\gcd(t, d)$ of t and d, and let $t = st_1, d = sd_1$, then $\gcd(t_1, d_1) = 1$ and $d_1 \mid a_i$. We may assume $s > 1$, or else $d \mid a_i$, i.e. $a_i = 0$. Write $a'_i = a_i / d_1$, then the fact that $d \nmid a_i$ implies

$$a'_i \not\equiv 0 \pmod{s}.$$

Now we consider S'_{t+1} as defined by

$$S'_{t+1} = \left\{ \sum_{i=1}^{t+1} \lambda_i a'_i : (\lambda_1, \lambda_2, \ldots, \lambda_{t+1}) \neq (0, 0, \ldots, 0) \right\}.$$

In other words, S'_{t+1} is defined by dividing every element in S_{t+1} by d_1, thus $|S'_{t+1}| = |S_{t+1}| = t$. So by induction hypothesis there exist $\lambda_1, \lambda_2, \ldots, \lambda_{t+1}$ not all zero such that

$$\sum_{i=1}^{t+1} \lambda_i a'_i \equiv 0 \pmod{s}.$$

Hence

$$\sum_{i=1}^{t+1} \lambda_i a_i \equiv 0 \pmod{d}.$$

Now we finish the proof by showing that $0 \in S_{t+1}$ if $|S_{t+1}| \leqslant t$. □

Proof of Theorem 5 assuming Lemma 8. We may reduce to the case in which $N = d$, by considering the quotient group $\mathbb{Z}_N/(d \cdot \mathbb{Z}_N) \cong \mathbb{Z}_d$. Now, by setting $t = d$, Lemma 8 implies that every d-cube generated by elements in $A = \mathbb{Z}_d \backslash \{0\}$ which is still a subset of A has size at least d. This conflicts with the fact that $|A| = d - 1$, hence such d-cube doesn't exist and thus A is d-cube-free. □

3 Related problems

To start further discussion, we prove the case $d = 3$ in advance. Recall Theorem 5 (i).

Theorem 9. *Let A be a 3-cube-free subset of $\mathbb{Z}_N$ where $3 \mid N$, then*

$$|A| \leqslant \frac{2}{3}N.$$

Proof. The proof consists of two parts discussing whether A contains $\{x, 2x\}$ as a subset for some x, namely $\{x, 2x\}$-free or not. When A is $\{x, 2x\}$-free, it is equivalent to

$$A \cap 2 \cdot A = \varnothing.$$

Here $2 \cdot A$ is defined by $2 \cdot A := \{2a : a \in A\}$, as mentioned in Section 2. Note that for all $a \in \mathbb{Z}_N$, there is at most one pair (b, c) with $b \neq c$ such that $2b = 2c = a$, which implies

$$|2 \cdot A| \geqslant \frac{1}{2}|A|.$$

Thus

$$N \geqslant |A| + |2 \cdot A| \geqslant \frac{3}{2}|A|,$$

and then $A \leqslant 2N/3$.

Now let A be not $\{x, 2x\}$-free, then there exists x such that $x, 2x \in A$. Considering the cube generated by $\{x, x, y\}$ where y is selected among all the elements in A, we have

$$A \cap (A - x) \cap (A - 2x) = \varnothing.$$

By taking the complementary set

$$A^c \cup (A - x)^c \cup (A - 2x)^c = \mathbb{Z}_N.$$

Note that both $(A - x)$ and $(A - 2x)$ are copies of A, thus

$$3|A^c| \geqslant N,$$

and then $|A| \leqslant 2N/3$. □

Actually the proof above can be generalized to all cyclic group, not necessarily $3 \mid N$. It gives a quite trivial upper bound of $2N/3$, but in some cases there might exist a better one, for instance $(5/8 + o(1))N$ conjectured by Long and Wagner [8] where $N = 2^k$.

Inspired by the proof on 3-cube discussing whether $\{x, 2x\}$ is forbidden, one can naturally expect to generalize the proof to larger cubes, which leads to the conjecture as follows.

Conjecture 10. Let A be a $\{x, 2x, \ldots, (d-1)x\}$-free subset of $\mathbb{Z}_N$ where $d \mid N$, then

$$|A| \leqslant \frac{d-1}{d}N.$$

Proof of Conjecture 4 assuming Conjecture 10. It suffices to prove it when A is not $\{x, 2x, \ldots, (d-1)x\}$-free. Now there must be an x such that $x, 2x, \ldots, (d-1)x \in A$. Consider the d-cube generated by $\{x, x, \ldots, x, y\}$ where y is selected among all the elements in A, we have

$$A \cap (A - x) \cap (A - 2x) \cap \cdots \cap (A - (d-1)x) = \varnothing.$$

This implies

$$|A| \leqslant \frac{d-1}{d}N. \qquad \square$$

It must be pointed out that we have a similar bound for $\{x, dx\}$-free subsets as follows.

Theorem 11. *Let A be a $\{x, dx\}$-free subset of $[N]$, then*

$$|A| \leqslant \frac{d}{d+1}N + O(\log N).$$

Proof. Given d, note that every positive integer m can be uniquely written as $m = d^s \times l$ with s being a non-negative integer and l being a positive integer not divisible by d. Thus we can divide all the integers into different chains, each starting with some integer l not divisible by d: $l, dl, d^2l, \ldots$. We denote the chain starting with l by C_l.

It is clear that A is $\{x, dx\}$-free if and only if there are no two elements of A adjacent in one chain. To acquire the upper bound, we just need to consider the extreme cases on different chains independently. Given l and C_l, since only one of $\{d^k l, d^{k+1} l\}$ can be contained in A for all $k \geqslant 0$ such that $d^{k+1} l \leqslant N$, the extreme case appears when the elements are selected alternately. More precisely, when $|[N] \cap C_l|$ is odd, the elements of $A \cap C_l$ take up all the odd positions in C_l; when $|[N] \cap C_l|$ is even, the elements of $A \cap C_l$ take up either all the odd positions or all the even positions in C_l.

Since different chains have different lengths, it is difficult to count $|A \cap C_l|$ respectively and then add them together. Instead, we count them by layers which are defined by

$$L_i := \{x \in \mathbb{Z}_+ : d^{i-1} \mid x,\ d^i \nmid x\}.$$

It is clear that all the integers can be divided into different layers, i.e.

$$\mathbb{Z}_+ = \bigcup_{i=1}^{\infty} L_i.$$

For convenience, we may assume the elements of A take up all the odd positions in C_l no matter whether $|[N] \cap C_l|$ is even or odd, as it does not change the size. Based on this assumption, the maximal A can be write as

$$A = \left(\bigcup_{i=0}^{\infty} L_{2i+1}\right) \cap [N].$$

Also all the layers are pairwise disjoint, thus

$$|A| = \sum_{i=0}^{\infty} |L_{2i+1} \cap [N]|.$$

Suppose $d^s \leqslant N < d^{s+1}$. When s is odd, we have

$$\begin{aligned}
|A| &= \sum_{i=0}^{(s-1)/2} |L_{2i+1} \cap [N]| \\
&= \sum_{i=0}^{(s-1)/2} \left\lfloor \frac{d-1}{d^{2i+1}} N + \frac{d-1}{d} \right\rfloor \\
&= \sum_{i=0}^{(s-1)/2} \left(\frac{d-1}{d^{2i+1}} N + \frac{d-1}{d} \right) + O(s) \\
&= \frac{d}{d+1}(1 - \frac{1}{d^{s+1}})N + \frac{(s+1)(d-1)}{2d} + O(s) \\
&= \frac{d}{d+1} N + O(s) = \frac{d}{d+1} N + O(\log N).
\end{aligned}$$

And when s is even, we have

$$\begin{aligned}
|A| &= \sum_{i=0}^{s/2} |L_{2i+1} \cap [N]| \\
&= \sum_{i=0}^{s/2} \left\lfloor \frac{d-1}{d^{2i+1}} N + \frac{d-1}{d} \right\rfloor \\
&= \sum_{i=0}^{s/2} \left(\frac{d-1}{d^{2i+1}} N + \frac{d-1}{d} \right) + O(s) \\
&= \frac{d}{d+1}(1 - \frac{1}{d^{s+2}})N + \frac{(s+2)(d-1)}{2d} + O(s) \\
&= \frac{d}{d+1} N + O(s) = \frac{d}{d+1} N + O(\log N). \qquad \square
\end{aligned}$$

This bound may be helpful when we consider cube-free subsets of $[N]$. It is worth noticing that the remainder term $O(\log N)$ cannot be removed. Indeed, take the case $d = 2$ as an example and we have the following result.

Theorem 12. [2] *Let A be an $\{x, 2x\}$-free subset of $[N]$, then*

$$|A| \leqslant \frac{2}{3} N + O(\log N).$$

[2] The author appreciates Sean Eberhard for providing the idea of comparing $[N]$ with $[4N]$, which helped give rise to the construction in the following proof.

Moreover, there exists $\varepsilon > 0$, such that for all $n > 0$, there exists $N > n$ such that $[N]$ contains a $\{x, 2x\}$-free subset with size at least $2N/3 + \varepsilon \log N$.

Proof. Take $d = 2$ in Theorem 11 then we have $|A| \leqslant 2N/3 + O(\log N)$. Now consider the sequence $a_n = (4^n - 1)/3$ which converges to infinity, we are going to show that $D(a_n) = 2a_n/3 + n/3$, so that $\{a_n\}$ is the sequence we want. Here $D(N)$ is the size of the largest $\{x, 2x\}$-free subsets of $[N]$.

The proof goes by induction. Recall that

$$D(N) = \sum_{k=0}^{\infty} \left\lfloor \frac{N + 4^k}{2 \cdot 4^k} \right\rfloor .$$

Since $a_{n+1} = 4a_n + 1$, we have

$$\begin{aligned} D(a_{n+1}) &= \sum_{k=0}^{\infty} \left\lfloor \frac{a_{n+1} + 4^k}{2 \cdot 4^k} \right\rfloor \\ &= \sum_{k=0}^{\infty} \left\lfloor \frac{4a_n + 1 + 4^k}{2 \cdot 4^k} \right\rfloor \\ &= \sum_{k=1}^{\infty} \left\lfloor \frac{4a_n + 4^k}{2 \cdot 4^k} \right\rfloor + 2a_n + 1 \\ &= \sum_{k=0}^{\infty} \left\lfloor \frac{a_n + 4^k}{2 \cdot 4^k} \right\rfloor + 2a_n + 1 \\ &= D(a_n) + 2a_n + 1 \end{aligned}$$

It remains to show that $D(1) = 1$, which is clear. □

As for cyclic group case, we can get rid of the remainder term.

Theorem 13. *Let A be a $\{x, dx\}$-free subset of $\mathbb{Z}_N$ and $k = (d, N)$, then*

$$|A| \leqslant \frac{k}{k+1} N.$$

Proof. We are going to count the number of solutions to the equation $x_0 = da$ where x_0 is fixed. Suppose $da \equiv db \pmod N$, then

$$a \equiv b \pmod{N/k}.$$

This implies there will be at most k solutions to the equation. Thus

$$|d \cdot A| \geqslant \frac{1}{k} |A|.$$

Since $A \cap d \cdot A = \varnothing$, we have

$$|A| \leqslant \frac{k}{k+1} N. \qquad \square$$

Corollary 14. *Let A be a $\{x,(d-1)x\}$-free subset of $[N]$ where $d \mid N$, then $k=(d-1,N)\leqslant d-1$ and*

$$|A|\leqslant \frac{k}{k+1}N\leqslant \frac{d-1}{d}N.$$

Since $\{x,(d-1)x\}$ is a subset of $\{x,2x,\dots,(d-1)x\}$, a larger density is implied when the latter is forbidden. But comparing Conjecture 10 with Corollary 14, we find that these two sets give rise to a same density when forbidden(or at least we expect them to).

4 Specific cases

In this section we are going to prove Theorem 5 (ii) and Theorem 5 (iii) by showing Conjecture 10 holds respectively. It must be pointed out that the idea of counting the family of sets partly comes from Long and Wagner [8].

4.1 When d is the smallest prime factor

We define the set $\mathcal{F}$ as

$$\mathcal{F}:=\{\{x,2x,3x,\dots,(d-1)x\}:x\in\mathbb{Z}_N\backslash\{0\}\}.$$

Note that every element B in $\mathcal{F}$ has size exactly $d-1$. Otherwise, there exist $i_1,i_2\in[d-1]$ and $x\in\mathbb{Z}_N\backslash\{0\}$ such that

$$N\mid (i_1-i_2)x.$$

Since $|i_1-i_2|\leqslant d-2$ and N has no prime factors smaller than d, we have

$$(i_1-i_2,N)=1$$

and then $N\mid x$ which is contradictory.

Moreover, observe that every element in $\mathbb{Z}_N\backslash\{0\}$ appears in precisely $d-1$ many different choices of B. Indeed, for all $x_0\in\mathbb{Z}_N\backslash\{0\}$ and $t\in[d-1]$, the congruence equation with respect to x

$$x_0\equiv tx \pmod N$$

has one and only solution. This is because $(t,N)=1$ and thus $0,t,2t,\dots,(N-1)t$ form a complete system of residues modulo N.

Now we are able to figure out the size of $\mathcal{F}$ by double counting all the elements covered.

$$(d-1)|\mathcal{F}|=(d-1)(N-1).$$

Let A be a $\{x,2x,\dots,(d-1)x\}$-free subset, it is clear that for every $B\in\mathcal{F}$ there exist at least one element $a_B\in A^c$. Moreover, since each fixed $x\in\mathbb{Z}_N$ can appear as a_B, for some choice of B, at most $d-1$ times, one has

$$|A^c|\geqslant \frac{|\mathcal{F}|}{d-1}=\frac{N-1}{d-1}\geqslant \frac{N}{d}.$$

Then

$$|A|\leqslant N-\frac{N}{d}=\frac{d-1}{d}N.$$

4.2 When $N = p^l$

We are to prove a better result for an $\{x, 2x, \ldots, (p^d - 1)x\}$-free subset A:

$$|A| \leqslant (1 - \frac{1}{p^d - 1})N.$$

First we define the layers in $\mathbb{Z}_{p^l}$ by

$$L_i := \{x \in \mathbb{Z}_{p^l} : p^{i-1} \mid x, p^i \nmid x\}.$$

For convenience we write

$$L_{[a,b]} := L_a \cup L_{a+1} \cup \cdots \cup L_{b-1} \cup L_b.$$

The proof goes by dividing $\mathbb{Z}_{p^l}$ into $\lceil (l+1)/d \rceil$ blocks, each block consisting of several continuous layers. Indeed, with q being the largest integer such that $qd \leqslant l+1$, the division is

$$\mathbb{Z}_{q^l} = L_{[1,d]} \cup L_{[d+1,2d]} \cup \cdots \cup L_{[(q-1)d+1,qd]} \cup L_{[qd+1,l+1]}.$$

For an integer $a \leqslant l - d + 1$, we define the set $\mathcal{F}_a$ as

$$\mathcal{F}_a := \{\{x, 2x, 3x, \ldots, (p^d - 1)x\} : x \in L_a\}.$$

Note that for any $B \in \mathcal{F}_a$, B has size exactly $p^d - 1$. Otherwise there exists $i_1, i_2 \in [p^d - 1]$ and $x \in L_a$ such that

$$p^l \mid (i_1 - i_2)x.$$

Since $|i_1 - i_2| \leqslant p^d - 2$, $(i_1 - i_2)$ is not divisible by p^d. Recall that $x \in L_a$ with $a \leqslant l - d + 1$, thus we can find a contradiction.

Moreover, observe that for every $B \in \mathcal{F}_a$, $B \subset L_{[a,a+d-1]}$ and every element of $L_{[a,a+d-1]}$ appears in precisely $(p-1)p^{d-1}$(that is, the size of $[p^d - 1] \cap L_1$) different sets in $\mathcal{F}_a$.

Now we are able to figure out the size of $\mathcal{F}_a$ by double counting all the elements covered.

$$(p^d - 1)|\mathcal{F}_a| = (p-1)p^{d-1}|L_{[a,a+d-1]}|.$$

Let A be an $\{x, 2x, \ldots, (p^d - 1)x\}$-free subset. It is clear that for every set $B \in \mathcal{F}_a$ there exists an element $x_B \in B$ with $x_B \notin A$, and every x_B recurs at most $(p-1)p^{d-1}$ times, therefore

$$|A^c \cap L_{[a,a+d-1]}| \geqslant \frac{|\mathcal{F}_a|}{(p-1)p^{d-1}} = \frac{1}{p^d - 1}|L_{[a,a+d-1]}|.$$

$$\frac{|A \cap L_{[a,a+d-1]}|}{|L_{[a,a+d-1]}|} \leqslant 1 - \frac{1}{p^d - 1}.$$

Especially, we set $a = 1 + td, t = 0, 1, \ldots, q - 1$ to obtain

$$\frac{|A \cap L_{[1+td,(t+1)d]}|}{|L_{[1+td,(t+1)d]}|} \leqslant 1 - \frac{1}{p^d - 1}. \quad (1)$$

Since $0 = p^l \notin A$ and $qd + 1 \geqslant l - d + 2$ we have

$$\frac{|A \cap L_{[qd+1,l+1]}|}{|L_{[qd+1,l+1]}|} \leqslant 1 - \frac{1}{|L_{[qd+1,l+1]}|} \leqslant 1 - \frac{1}{p^{d-1}} \leqslant 1 - \frac{1}{p^d - 1}. \tag{2}$$

Finally we combine (1) and (2) to draw the conclusion that in each block A has a density less than $1 - 1/(p^d - 1)$, therefore $|A| \leqslant (1 - 1/(p^d - 1))N$.

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