Sharp Bounds on Lengths
of Linear Recolouring Sequences

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Abstract

A recolouring sequence, between k-colourings α and β of a graph G, transforms α into β by recolouring one vertex at a time, such that after each recolouring step we again have a proper k-colouring of G. The diameter of the k-recolouring graph, $\operatorname{diam} \mathcal{C}_k(G)$, is the maximum over all pairs α and β of the minimum length of a recolouring sequence from α to β. Much previous work has focused on determining the asymptotics of $\operatorname{diam} \mathcal{C}_k(G)$: Is it $\Theta(|G|)$? Is it $\Theta(|G|^2)$? Or even larger? Here we focus on graphs for which $\operatorname{diam} \mathcal{C}_k(G) = \Theta(|G|)$, and seek to determine more precisely the multiplicative constant implicit in the $\Theta()$. In particular, for each $k \geqslant 3$, for all positive integers p and q we exactly determine $\operatorname{diam} \mathcal{C}_k(K_{p,q})$, up to a small additive constant. We also sharpen a recolouring lemma that has been used in multiple papers, proving an optimal version. This improves the multiplicative constant in various prior results. Finally, we investigate plausible relationships between similar reconfiguration graphs.

Mathematics Subject Classifications: 05C85, 05C15, 05C12

1 Introduction

The *k-recolouring graph*, $\mathcal{C}_k(G)$, has as its vertices the k-colourings of G; two vertices are adjacent precisely when those k-colourings differ in the colour of a single vertex of G. A walk in $\mathcal{C}_k(G)$ between two k-colourings α and β is called a *recolouring sequence* from α to β. It is well-known [15], and easy to prove by induction, that $\mathcal{C}_k(G)$ is connected when $k \geqslant \Delta(G) + 2$. In fact, in this case the diameter of $\mathcal{C}_k(G)$ is linear in $|G|$; specifically [9, Thm. 1], we have $\operatorname{diam} \mathcal{C}_k(G) \leqslant 2|G|$. Much previous research [2–5, 7, 8, 11–13, 16] has focused on determining sufficient conditions on integers k and graph classes $\mathcal{G}$ to imply

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that $\operatorname{diam} \mathcal{C}_k(G) = \Theta(|G|)$ for all $G \in \mathcal{G}$. In this paper we revisit some of these graphs, and prove sharper upper bounds on $\operatorname{diam} \mathcal{C}_k(G)$.

In Section 2, we prove the Optimal Renaming Lemma, which is a sharper version of a lemma used in numerous papers. As a consequence, we improve the upper bounds in these previous works. In Section 3, we consider the complete bipartite graph $K_{p,q}$; for each number k of colours we determine, up to a small additive constant, $\operatorname{diam} \mathcal{C}_k(K_{p,q})$. We show, perhaps surprisingly, that $\operatorname{diam} \mathcal{C}_k(K_{p,q})$ moves through 3 distinct phases, as the ratio q/p decreases. Namely, we prove our main result, stated below.

Theorem 1. *Fix positive integers p, q, k. If $p \leqslant q$ and $k \geqslant 3$, then*

$$
\operatorname{diam} \mathcal{C}_k(K_{p,q}) = \begin{cases} 2p + q + \left\lfloor \frac{q-p}{k-1} \right\rfloor & \text{if } q \geqslant kp \\[2mm] 3p + q + \left\lfloor \frac{q-kp}{\left\lfloor \frac{k^2}{4} \right\rfloor} \right\rfloor - g_1(k,p,q) & \text{if } \left\lceil \frac{k}{2} \right\rceil p \leqslant q \leqslant kp, \\[2mm] 2p + q + \left\lfloor \frac{q-p}{\left\lceil \frac{k}{2} \right\rceil} \right\rfloor - g_2(k,p,q) & \text{if } q \leqslant \left\lceil \frac{k}{2} \right\rceil p, \end{cases}
$$

where $0 \leqslant g_1(k,p,q) \leqslant \left\lfloor \frac{k^2}{4} \right\rfloor$ and $0 \leqslant g_2(k,p,q) \leqslant \left\lfloor \frac{k}{2} \right\rfloor$.

Finally, in Section 4 we consider list-colouring. For each list-assignment L we define $\mathcal{C}_L(G)$, analogous to $\mathcal{C}_k(G)$. Naturally, we look for relationships between $\operatorname{diam} \mathcal{C}_L(G)$ and $\operatorname{diam} \mathcal{C}_k(G)$, particularly when $|L(v)| = k$ for all vertices v. We mainly construct examples disproving many statements we might hope were true. We also pose as open problems some statements that we have neither been able to prove or disprove, regarding relationships between $\operatorname{diam} \mathcal{C}_k(G)$ and $\operatorname{diam} \mathcal{C}_{k+1}(G)$, or between $\operatorname{rad} \mathcal{C}_k(G)$ and $\operatorname{rad} \mathcal{C}_{k+1}(G)$.

Most of our notation is standard. We note that we write $[i]$ for $\{1, \ldots, i\}$ and we write $[i,j]$ for $\{i, \ldots, j\}$.

1.1 Intuition behind Theorem 1

To better understand Theorem 1, it is instructive to compare to each other the upper bounds for its 3 regimes. When $k = 3$, these upper bounds for all regimes are identical; furthermore, in that case the functions g_1 and g_2 are identically zero. We state this formally below and prove it just before Subsection 3.2.

Proposition 2. *For every complete bipartite graph $K_{p,q}$, we have* $\operatorname{diam} \mathcal{C}_3(K_{p,q}) = \left\lfloor \frac{3(p+q)}{2} \right\rfloor$.

In Fig. 1 below, we plot the 3 upper bounds in Theorem 1 as functions in q, for $k = 4$ and p fixed. But where do these bounds come from? A natural guess for colourings α and β that are far apart is to have α use a of the colours, equally, on part U and the remaining $k - a$ of the colours, equally, on part V; and β swaps the parts of all colours, relative to α, again using the colours on each part as equally as possible.

Formalising this intuition does indeed give a pair α, β that is far apart. But we still must specify a, the number of colours used by α on U. When q is large relative to p, namely $q \geqslant kp$, the worst case (largest distance) arises with $a = 1$; and when $q < kp$, the

worst case arises with $a = \left\lfloor \frac{k}{2} \right\rfloor$. For the pair α, β of colourings arising with $a = 1$, the number of recolouring steps needed is shown in Fig. 1 in blue (we call this function $\mathcal{B}$). For the pair α, β of colourings arising with $a = \left\lfloor \frac{k}{2} \right\rfloor$, we have two candidate recolouring strategies, shown in Fig. 1 in orange and green (functions $\mathcal{O}$ and $\mathcal{G}$). The first of these ($\mathcal{O}$) is optimal when $q \leqslant \left\lceil \frac{k}{2} \right\rceil p$, and the second ($\mathcal{G}$) is optimal when $\left\lceil \frac{k}{2} \right\rceil p \leqslant q \leqslant kp$. So our upper bound for the diameter is $\max\{\mathcal{B}, \min\{\mathcal{O}, \mathcal{G}\}\}$.
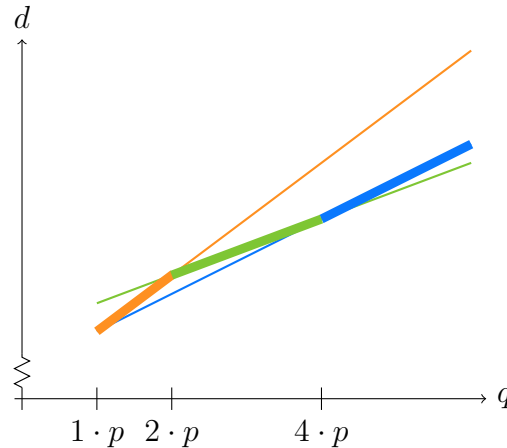

Figure 1: The three upper bounds in Theorem 1 for $k = 4$ and $p \leqslant q \leqslant 6p$. The bold line segments indicate the value of $\operatorname{diam} \mathcal{C}_k(K_{p,q})$, up to the error terms $g_1(k, p, q)$ and $g_2(k, p, q)$.

To prove the upper bounds in Theorem 1, we consider a few candidate recolouring sequences (constructed inductively). We use an averaging argument to show, for each pair of colourings α, β, not just those defined above, that one of these candidates recolours α to β in a number of steps at most the bound in Theorem 1; examples of these candidate recolouring sequences, when $k = 4$, (applied on a pair of extremal colourings α, β) are shown in Fig. 2.

To prove the lower bound in Theorem 1, we show further that indeed no recolouring sequence can do better. We remark briefly about the error terms $g_1(k, p, q)$ and $g_2(k, p, q)$. When p and q are both multiples of $\left\lfloor \frac{k}{2} \right\rfloor \left\lceil \frac{k}{2} \right\rceil = \left\lfloor \frac{k^2}{4} \right\rfloor$, these error terms vanish. In the general case, they arise because we cannot quite use all colours "equally" on a part when the number of colours does not divide the order of that part. (We suspect that $g_1(k, p, q) = O(k)$, but our efforts to prove this encountered various technical complications.) Moreover, the error term $g_2(k, p, q)$ becomes 1 or 0 when q is sufficiently small compared to k. Similarly, as proved in [9, Thm. 20], the diameter of $\mathcal{C}_k(K_{p,q})$ equals $2p + q$ whenever $p \leqslant q \leqslant k - 2$.

As Theorem 1 illustrates, finding the diameter of $\mathcal{C}_k(G)$ for graphs G in even a simple class like complete bipartite graphs is already quite involved. This example hints at the complexity of the problem in general. In this light, proving that $\operatorname{diam} \mathcal{C}_k(G) = n(G) + \mu(G)$ for all $k \geqslant \Delta(G) + 2$, and all graphs G, (see [9, Conj. 1]) would be very interesting. (Here $\mu(G)$ denotes the maximum size of a matching in G.) When we restrict to regular graphs, this problem becomes Regular Cereceda's Conjecture [9, Conj. 3]. In a sense (expressed in the maximum degree), once k is big enough to guarantee connectedness of $\mathcal{C}_k(G)$, the diameter of $\mathcal{C}_k(G)$ would be known as well. For smaller k, one can expect
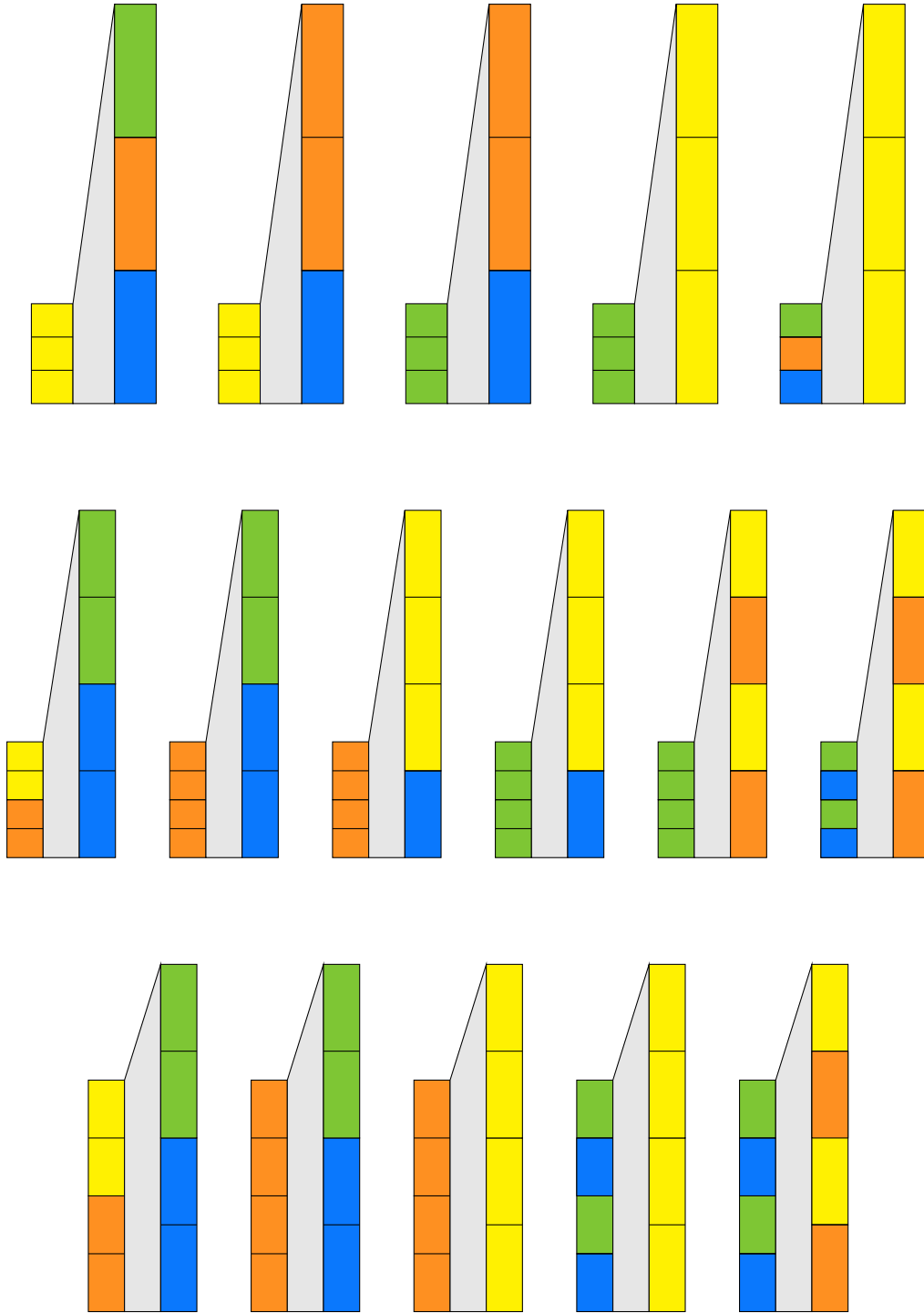
Figure 2: Examples, for $k = 4$, of recolouring sequences between extremal colourings of $K_{p,q}$ in the 3 regimes. Each coloured rectangle shows the proportion of vertices coloured in a particular colour. Each grey area indicates the edges of $K_{p,q}$. Top: An optimal recolouring sequence from α to β when $a = 1$. The number of steps needed is shown in Fig. 1 as $\mathcal{B}$. Middle: One candidate for an optimal recolouring sequence from α to β when $a = \left\lfloor \frac{k}{2} \right\rfloor$. The number of steps needed is shown in Fig. 1 as $\mathcal{G}$. Bottom: The other candidate for an optimal recolouring sequence from α to β when $a = \left\lfloor \frac{k}{2} \right\rfloor$. The number of steps needed is shown in Fig. 1 as $\mathcal{O}$.

many different regimes, and it seems impossible to state a precise answer in general.

2 Optimal Renaming Lemma

Bonamy and Bousquet [5, Lemma 5] proved a variant of the Optimal Renaming Lemma below; for clarity, we call their version the Rough Renaming Lemma. It has been reused in several papers as a subroutine. In a typical application, we are given two colourings α and β that can be efficiently recoloured to some carefully chosen colourings α', β' that induce the same partition into colour classes. The Rough Renaming Lemma is then used as a 'finishing blow' to recolour α' to β', thus establishing a short recolouring sequence between α and β. But the upper bound in the Rough Renaming Lemma, of $2n$, is suboptimal. In fact, this recolouring can be done with only $\lfloor 3n/2 \rfloor$ steps, which is sharp.

So reusing the Rough Renaming Lemma is wasteful, especially in papers that prove a linear bound on the diameter of the reconfiguration graph. In a sense, the ideas needed to prove the Optimal Renaming Lemma were already present in the argument of Bonamy and Bousquet, but authors of subsequent papers seem to be unaware. By making this improvement explicit, we indirectly strengthen the upper bounds in the papers that have used this Rough Renaming Lemma.

Optimal Renaming Lemma. *Let G be a graph on n vertices, fix $k \geqslant \chi(G)$, and let α and β be two proper k-colourings of G that induce the same partition of vertices into colour classes $C_1, \ldots, C_k$. If $\ell \geqslant k+1$, then α can be recoloured to β in $\mathcal{C}_\ell(G)$ in at most $\lfloor 3n/2 \rfloor$ steps, with each vertex recoloured at most 2 times.*

Proof. We define an auxiliary loopless digraph D with vertex set $V(D) := \{C_1, \ldots, C_k\}$, and with (C_i, C_j) a directed edge of D if and only if $\alpha(C_i) = \beta(C_j)$ and $i \neq j$. By construction, the vertices of D have pairwise distinct colours, so each vertex has in-degree and out-degree both at most 1. So the vertices of D can be partitioned into directed paths and directed cycles. Now we sequentially recolour the vertices of each directed path and cycle. For a directed path $C_{i_1}, \ldots, C_{i_t}$ (possibly with $t = 1$), we simply recolour C_{i_1} to $\beta(C_{i_1})$, then recolour C_{i_2} to $\beta(C_{i_2})$, etcetera, finally recolouring C_{i_t} to $\beta(C_{i_t})$. This takes $\sum_{j=1}^{t} |C_{i_j}|$ recolouring steps. Note that every recolouring step induces an updated directed graph D. But this is of no consequence for us since every colour class C_i that is currently coloured $\beta(C_i)$ has in-degree and out-degree in D both 0.

For a directed cycle $C_{i_1}, \ldots, C_{i_t}$ with $t \geqslant 2$, we assume without loss of generality that C_{i_1} has minimum size among $C_{i_1}, \ldots, C_{i_t}$. We first recolour C_{i_1} to some free colour in $[\ell] \setminus \gamma(V(G))$, where γ denotes the current colouring. Note that this cannot create a new arc incident to the current directed cycle. Next, we proceed along the directed cycle, starting by recolouring C_{i_2} to $\beta(C_{i_2})$ and ending by recolouring C_{i_t} to $\beta(C_{i_t})$. To finish, we recolour C_{i_1} again, this time to $\beta(C_{i_1})$. In total, the number of recolouring steps we use is $|C_{i_1}| + \sum_{j=1}^{t} |C_{i_j}|$. This is at most $\lfloor 3/2 \cdot \sum_{j=1}^{t} |C_{i_j}| \rfloor$ because $t \geqslant 2$ and we chose C_{i_1} to be of minimum size. In this way we can recolour every directed path and cycle, one after the other. So the total number of recolouring steps we use is at most $\lfloor 3/2 \cdot \sum_{i=1}^{k} |C_i| \rfloor = \lfloor 3n/2 \rfloor$. Moreover, every vertex is recoloured only once or, if it belonged to the smallest colour class of its directed cycle, twice. $\square$

Corollary 3. *The upper bound in [4, Thm. 6], for $(P_3 + P_1)$-free graphs, improves from $6n$*

to $\lfloor(4+1/2)n\rfloor$; and the upper bound in [4, Thm. 7], for $(2K_2, C_4)$-free graphs, improves from $4n$ to $\lfloor 7n/2\rfloor$.

Proof. In these proofs, the authors use the Rough Renaming Lemma, which promises a recolouring sequence of length at most $2n$, rather than of length at most $\lfloor 3n/2\rfloor$. So we gain $\lceil n/2\rceil$ steps each time the lemma is applied. To prove Theorem 6 it is applied 3 times, while to prove Theorem 7 it is applied 1 time. $\square$

Corollary 4. *In [3] its authors prove that for every $2K_2$-free bipartite n-vertex graph G, and for every $\ell \geqslant \chi(G)+1$, we have $\mathrm{diam}\,\mathcal{C}_\ell(G) \leqslant 4n$. That proof uses the Rough Renaming Lemma once. By instead using The Optimal Renaming Lemma, we improve the bound to $\lfloor 7n/2\rfloor$. In the same paper, the authors also prove that for every $2K_2$-free 3-chromatic n-vertex graph G, and for every $\ell \geqslant \chi(G)+1$, we have $\mathrm{diam}\,\mathcal{C}_\ell(G) \leqslant 14n$. They use the Rough Renaming Lemma three times (once explicitly, and twice implicitly via the result for $2K_2$-free bipartite graphs), so the bound now improves to $\lfloor(12+1/2)n\rfloor$.*

Corollary 5. *In [2], the Rough Renaming Lemma is used once to show that for every {triangle, co-diamond}-free graph G and for every $\ell \geqslant \chi(G)+1$, we have $\mathrm{diam}\,\mathcal{C}_\ell(G) \leqslant 6n$, unless G is isomorphic to $K_{\ell,\ell}$ minus a perfect matching. The Optimal Renaming Lemma improves this bound to $\lfloor(5+1/2)n\rfloor$.*

Corollary 6. *In [16] it is shown that every $3K_1$-free graph G satisfies $\mathrm{diam}\,\mathcal{C}_{\chi(G)+1}(G) \leqslant 4n$. The Rough Renaming Lemma is used once, so the Optimal Renaming Lemma improves this bound to $\lfloor(3+1/2)n\rfloor$.*

3 Reconfiguration graphs of complete bipartite graphs

In this section, we consider k-colourings of $K_{p,q}$ and prove Theorem 1. To begin, we prove the lower bound.

3.1 Lower Bounds for $K_{p,q}$

It is trivial to get a lower bound of the order of the graph, $p+q$, by constructing two colourings that differ on each vertex. To improve this bound, our basic idea is to construct two k-colourings that require the colours on many pairs of vertices (in opposite parts) to "swap"; that is, the old colour of vertex 1 is the new colour of vertex 2 and vice versa. For every such pair, a recolouring step must be "wasted", since the first vertex in the pair to be recoloured cannot immediately receive its desired new colour.

Proof of the lower bound in Theorem 1. Fix integers a, b, k with $1 \leqslant a \leqslant b$ and $k = a+b$. For convenience, we assume through most of the proof that ab divides both p and q; however, near the end we comment about how to remove this assumption. In fact we will mainly care about the cases that $(a,b) \in \{(1, k-1), (\lfloor\frac{k}{2}\rfloor, \lceil\frac{k}{2}\rceil)\}$, but for the time being we consider the more general case above. We define the colourings α and β using ab pairs of sets $U_{i,j}, V_{i,j}$ of sizes (respectively) $\frac{p}{ab}$ and $\frac{q}{ab}$, with $i \in [a]$ and $j \in [a+1, k]$, where U and V are the unions of these sets. The colourings α and β are defined by $\alpha(U_{i,j}) = i = \beta(V_{i,j})$ and $\beta(U_{i,j}) = j = \alpha(V_{i,j})$. For convenience, we also let $U_i := \cup_{j\in[a+1,k]}U_{i,j}$ and $V_j := \cup_{i\in[a]}V_{i,j}$.

Since the colours on U and V swap from α to β, there is always a first moment that U resp. V, and in particular U_i (or V_j), is completely coloured with colours different from i (resp. j). We call this the time when colour i or colour j is *erased*. (We do not care about a colour disappearing and reappearing on a part, since we are only proving a lower bound.) We order the sets U_i (renaming colours if needed) so that colour 1 is erased from U_1 before colour 2 is erased from U_2, which happens before colour 3 is erased from U_3, etc. Similarly, if $j < j'$, then we assume that colour j is erased from V_j before colour j' is erased from $V_{j'}$.

When colour i gets erased from U_i, let $f(i)$ denote the number of the b colours on V that are not yet erased. So there are at least $f(i)$ sets $U_{i,j}$ which have been recoloured entirely with colours different from their corresponding target colour j. Analogously, we define $f(j)$, for each $j \in [a+1, k]$, as the number of colours that are not yet erased on U when j is erased from V (for the first time).

For each pair $(i,j) \in [a] \times [a+1, k]$, one of the two colours i and j must be erased first, when the other is not yet erased. Since we do not consider reoccurence of a colour after being erased, this implies $\sum_{h=1}^{k} f(h) = ab$. More specifically, for all distinct $i \in [a]$ and $j \in [a+1, k]$, we define $X_{i<j}$ to be 1 if colour i is erased from U before colour j is erased from V and define it to be 0 otherwise; and we define $X_{j<i}$ as $1 - X_{i<j}$. Now we get $\sum_{h=1}^{k} f(h) = \sum_{h=1}^{a} f(h) + \sum_{h=a+1}^{b} f(h) = \sum_{h=1}^{a} \sum_{\ell=a+1}^{k} X_{h<\ell} + \sum_{\ell=a+1}^{k} \sum_{h=1}^{a} X_{\ell<h} = \sum_{h=1}^{a} \sum_{\ell=a+1}^{k} (X_{h<\ell} + X_{\ell<h}) = \sum_{h=1}^{a} \sum_{\ell=a+1}^{k} 1 = a(k - (a+1) + 1) = ab$.

When colour i is erased from U, for each of the $f(i)$ colours j that are not yet erased from V, the vertex subset $U_{i,j}$ has been completely recoloured with colours other than j, so all its vertices will need to be recoloured again to reach β. An analogous argument works when each colour j is erased from V. Thus, a lower bound for the number of recolourings needed to transform α into β is

$$p + q + \sum_{i=1}^{a} f(i)\frac{p}{ab} + \sum_{j=a+1}^{k} f(j)\frac{q}{ab} = 2p + q + \sum_{j=a+1}^{k} f(j)\frac{q-p}{ab}. \tag{1}$$

Before continuing, we mention that if $q \geqslant kp$, then we let $(a,b) := (1, k-1)$; and if $q < kp$, then we let $(a,b) := (\lfloor k/2 \rfloor, \lceil k/2 \rceil)$. Depending on the order in which the colours were erased, we get different lower bounds. Recall from above that the first colour erased is either 1 or $a+1$, and the last colour erased is either a or k. Thus, we are in one of the following two cases.

Case 1: $a+1$ is the first colour erased or a is the last colour erased. In the first case $f(a+1) = a$, and in the second case $f(j) \geqslant 1$ for every $a+1 \leqslant j \leqslant k$. In both cases $\sum_{j=a+1}^{k} f(j) \geqslant a$. Hence the number of recolourings needed is at least $2p + q + a\frac{q-p}{ab} = 2p + q + \frac{q-p}{b}$. If $q \geqslant kp$, then $b = k - 1$, so we have the desired bound. And if $q \leqslant \lceil \frac{k}{2} \rceil p$, then $b = \lceil \frac{k}{2} \rceil$, so we again have the desired bound. Finally, suppose that $\lceil \frac{k}{2} \rceil p < q < kp$. In this case, substituting $q > \lceil \frac{k}{2} \rceil p$ shows that the above bound $2p + q + \frac{q-p}{\lceil \frac{k}{2} \rceil}$ exceeds the desired bound: $\frac{q-p}{\lceil \frac{k}{2} \rceil} - \left(p + \frac{q-kp}{\frac{k^2}{4}}\right) \geqslant \frac{\lceil \frac{k}{2} \rceil p - p}{\lceil \frac{k}{2} \rceil} - \left(p + \frac{\lceil \frac{k}{2} \rceil p - kp}{\frac{k^2}{4}}\right) = \left(p - \frac{p}{\lceil \frac{k}{2} \rceil}\right) - \left(p - \frac{p\lfloor \frac{k}{2} \rfloor}{\lceil \frac{k}{2} \rceil \lfloor \frac{k}{2} \rfloor}\right) = 0$. In this case, we are done. So we now assume that we are in Case 2, below.

Case 2: 1 is the first colour erased and k is the last one. We define x and y as follows. Let $a+1$ be the $(x+1)^{th}$ colour to be erased; that is, we first erase x colours

on U. Let a be the $(k-y)^{th}$ colour to be erased; that is, y is the number of colours on V that we erase after erasing colour a from U.

Note that when we erase from U colours 1 up to x, we must recolour all vertices that used these colours with colours from $[a]$. All of these vertices must be recoloured again, before the colours $k-y+1$ up to k are erased from V (since all colours in $[a]$ are erased before any colours in $[k-y+1,k]$). This implies that in addition to (1), we have at least $xy\frac{p}{ab}$ more recolourings. Also, $f(a+1) = a-x$ and $f(j) \geqslant 1$ for all $j \in [a+2, k-y]$. Thus $\sum_{j=a+1}^{k} f(j) \geqslant (a-x) + (k-y-(a+1)) = k-x-y-1$. As a lower bound on the number of recolourings, we thus get

$$2p + q + (k-x-y-1)\frac{q-p}{ab} + xy\frac{p}{ab}. \tag{2}$$

Because we are in Case 2, we know that $a \geqslant 2$, since we first erase colour 1, and all vertices that were coloured 1 must be recoloured with other colours in $[a]$. This implies that $q < kp$ and thus that $a = \left\lfloor \frac{k}{2} \right\rfloor$ and $b = \left\lceil \frac{k}{2} \right\rceil$. By definition, $x \in [a]$ and $y \in [b]$. But in fact we may assume $x \in [a-1]$. (If instead we have $x = a$, then erasing colour a from U will require reintroducing to U some colour in $[a-1]$, before erasing any colour from V. Such a sequence can always be "shortcut" to a sequence that is no longer but that has $x \in [a-1]$.) Similarly, we assume $y \in [b-1]$. So we must show that the minimum value of (2), when $x \in [a-1]$ and $y \in [b-1]$ is at least the value claimed in the theorem. We have 2 subcases, depending on whether or not $q \geqslant \left\lceil \frac{k}{2} \right\rceil p$.

First suppose that $q \geqslant \left\lceil \frac{k}{2} \right\rceil p$. Since $\max\{x,y\} \leqslant b-1 \leqslant \left\lceil \frac{k}{2} \right\rceil - 1 \leqslant \frac{q-p}{p}$, we see that $(k-x-y-1)\frac{q-p}{ab} + xy\frac{p}{ab}$ is non-increasing in x and y; specifically, when we increase x or y, the increase in the second term is at most the decrease in the first term. So the minimum is attained when both x and y are maximal, i.e. $x = a-1$ and $y = b-1$. In that case $(k-x-y-1)\frac{q-p}{ab} + xy\frac{p}{ab} = \frac{q-p}{ab} + (ab-k+1)\frac{p}{ab} = p + \frac{q-kp}{ab} = p + \frac{q-kp}{\left\lceil \frac{k^2}{4} \right\rceil}$.

Now suppose instead that $q < \left\lceil \frac{k}{2} \right\rceil p$. Consider an arbitrary pair (x,y). If $x \notin \{1, a-1\}$, then we can either increase or decrease x without increasing the value of (2); by repeating this process, we can assume that $x \in \{1, a-1\}$ and, similarly, that $y \in \{1, b-1\}$.

Due to the symmetry of (2) in x and y, evaluating (2) with $(x,y) = (a-1,1)$ gives the same value as evaluating with $(x,y) = (1, a-1)$. So the case $(x,y) = (a-1,1)$ is settled once we have settled the case $(x,y) \in \{(1, b-1), (1,1)\}$. Thus, we assume that $(x,y) \in \{(1,1), (1, b-1), (a-1, b-1)\}$. When $(x,y) = (a-1, b-1)$, as in the previous paragraph, we get the lower bound $p + \frac{q-kp}{ab}$. Since $q < \left\lceil \frac{k}{2} \right\rceil p = bp$, we have $aq - q = (a-1)q < (a-1)bp = abp - bp$. Thus, we get $p + \frac{q-kp}{ab} = \frac{pab+q-kp}{ab} > \frac{(aq-q+bp)+q-kp}{ab} = \frac{aq-ap}{ab} = \frac{q-p}{b} = \frac{q-p}{\left\lceil \frac{k}{2} \right\rceil}$, as desired. When $(x,y) = (1, b-1)$, we get $(k-x-y-1)\frac{q-p}{ab} + xy\frac{p}{ab} = (a-1)\frac{q-p}{ab} + \frac{(b-1)p}{ab} = \frac{q-p}{b} + \frac{bp-q}{ab} > \frac{q-p}{b}$. When $(x,y) = (1,1)$, the conclusion is immediate if $b \geqslant 3$, since then $\frac{(a+b-3)(q-p)}{ab} \geqslant \frac{q-p}{b}$. Instead suppose $a = b = 2$. Now (2) gives the lower bound $2p + q + \frac{q-p}{4} + \frac{p}{4} = 2p + q + \frac{q}{4}$. By assumption $2p = \left\lceil \frac{k}{2} \right\rceil p > q$. Thus, $\frac{q}{4} > \frac{q}{4} + \frac{q-2p}{4} = \frac{q-p}{2} = \frac{q-p}{\left\lceil \frac{k}{2} \right\rceil}$.

The paragraph above finishes the proof when ab divides both p and q. Now we handle the general case.

- We begin with the first regime; that is, when $q \geqslant kp$. We show that in the lower bound we do not lose anything due to rounding except for what comes from the floor

(in the statement of the theorem). In the construction of α and β, we make each set $U_{1,j}$ of size $\left\lceil \frac{p}{ab} \right\rceil$ or size $\left\lfloor \frac{p}{ab} \right\rfloor$, and each set $V_{1,j}$ of size $\left\lceil \frac{q}{ab} \right\rceil$ or size $\left\lfloor \frac{q}{ab} \right\rfloor$. Furthermore, we take the larger sets (those of size $\left\lceil \frac{p}{ab} \right\rceil$ or size $\left\lceil \frac{q}{ab} \right\rceil$) to be the ones with smaller indices. Now it is straightforward to check that $|V_{1,j}| - |U_{1,j}| - \left\lfloor \frac{q-p}{ab} \right\rfloor \geqslant 0$ for all $j \in [2,k]$. (For example, it is enough to consider 2 cases based on whether or not q (mod b) $\geqslant p$ (mod b).)

Recall that when $q \geqslant kp$, we let $a := 1$ and $b := k - 1$. Thus, we must be in Case 1; that is, colour 1 is not the first colour erased. In other words, $X_{j<1} = 1$ for some $j \in [2,k]$. Now the argument proving (1) can be refined to give the following lower bound:

$$d(\alpha, \beta) = \sum_{j=2}^{k} \left(|U_{1,j}| + |V_{1,j}| + (|U_{1,j}|X_{1<j} + |V_{1,j}|X_{j<1}) \right)$$

$$= |U| + |V| + \sum_{j=2}^{k} \left((|U_{1,j}|X_{1<j} + |U_{1,j}|X_{j<1}) + (|V_{1,j}| - |U_{1,j}|)X_{j<1} \right)$$

$$= 2|U| + |V| + \sum_{j=2}^{k} (|V_{1,j}| - |U_{1,j}|)X_{j<1}$$

$$\geqslant 2p + q + \min_{j \in [2,k]} (|V_{1,j}| - |U_{1,j}|)$$

$$\geqslant 2p + q + \left\lfloor \frac{q-p}{b} \right\rfloor.$$

- Now we consider the second regime; in particular, $q \geqslant \left\lceil \frac{k}{2} \right\rceil p$. If $q \equiv p$ (mod $\left\lfloor \frac{k^2}{4} \right\rfloor$), then instead of a term $xy\frac{p}{ab}$ (in the calculation above), we have xy terms that are each equal to $\left\lfloor \frac{p}{ab} \right\rfloor$ or $\left\lceil \frac{p}{ab} \right\rceil$. So the number of terms that are rounded is $xy \leqslant (a-1)(b-1) = ab - k + 1$; thus, $g_1(k,p,q) < ab - k + 1$. Next, we consider the case when $q \not\equiv p$ (mod $\left\lfloor \frac{k^2}{4} \right\rfloor$). Let q' be the largest integer which is at most q and satisfies $q' \equiv p$ (mod $\left\lfloor \frac{k^2}{4} \right\rfloor$). In that case, we obtain a lower bound on the diameter observing that $\operatorname{diam} \mathcal{C}_k(K_{p,q}) \geqslant \operatorname{diam} \mathcal{C}_k(K_{p,q'}) + (q - q')$; we let the $q - q'$ additional vertices satisfy $\alpha(v) \neq \beta(v)$. By comparing the main terms in the estimate of the diameter, we can conclude an upper bound on $g_1(k,p,q)$. For this, we check two cases.

  If $\left\lceil \frac{k}{2} \right\rceil p \leqslant q'$, then we get $g_1(k,p,q) \leqslant g_1(k,p,q') + 1$, since $\left\lfloor \frac{q-kp}{ab} \right\rfloor - \left\lfloor \frac{q'-kp}{ab} \right\rfloor \leqslant 1$. So assume instead that $q' < \left\lceil \frac{k}{2} \right\rceil p \leqslant q$. We have analogously $g_1(k,p,q) \leqslant g_1(k,p,\left\lceil \frac{k}{2} \right\rceil p) + 1 = g_2(k,p,\left\lceil \frac{k}{2} \right\rceil p) + 1 \leqslant \left\lfloor \frac{k}{2} \right\rfloor + 1$. For this equality, note that in Theorem 1 the bounds for the second and third regimes are equal when $q = \left\lceil \frac{k}{2} \right\rceil p$. For this second inequality, we applied the upper bound for g_2, which we prove in the following paragraph.

- Finally, we consider the third regime. If $q \equiv p$ (mod $\left\lfloor \frac{k^2}{4} \right\rfloor$), then we can ensure that $|V_{i,j}| - |U_{i,j}| = \frac{q-p}{ab}$ for every $(i,j) \in [a] \times [a+1,k]$. Now the analysis is nearly

identical to that above when p and q are both divisible by $\left\lfloor \frac{k^2}{4} \right\rfloor$; consequently, $g_2(k, p, q) = 0$. In the more general case, let $q := q' + r$, where $q' \equiv p \pmod{\left\lfloor \frac{k^2}{4} \right\rfloor}$ and $0 \leqslant r < \left\lfloor \frac{k^2}{4} \right\rfloor$. Now $\operatorname{diam} \mathcal{C}_k(K_{p,q}) \geqslant \operatorname{diam} \mathcal{C}_k(K_{p,q'}) + r$. This holds because any pair of colourings (α, β) of $K_{p,q'}$ can be extended to colourings of $K_{p,q}$ such that each of the additional r vertices has different colours in α and in β. In the third regime, $\operatorname{diam} \mathcal{C}_k(K_{p,q}) \geqslant \operatorname{diam} \mathcal{C}_k(K_{p,q'}) + r$ implies $g_2(k, p, q) \leqslant \left\lfloor \frac{q-p}{b} \right\rfloor - \left\lfloor \frac{q'-p}{b} \right\rfloor \leqslant \left\lceil \frac{q-q'}{\left\lceil \frac{k}{2} \right\rceil} \right\rceil \leqslant \left\lfloor \frac{k}{2} \right\rfloor$. $\qquad \square$

Now we can sketch the proof of Proposition 2. For convenience, we restate it.

Proposition 2. *For every complete bipartite graph $K_{p,q}$, we have $\operatorname{diam} \mathcal{C}_3(K_{p,q}) = \left\lfloor \frac{3(p+q)}{2} \right\rfloor$.*

Proof. Let $k = 3$. Note that each of the three upper bounds in Theorem 1 (for easier reference, see Theorem 11 below) is equal to $3(p+q)/2$; these bounds are $2p+q+(q-p)/2$ and $3p + q + (q - 3p)/2$ and $2p + q + (q - p)/2$. Since the diameter is always an integer, in all 3 regimes Theorem 11 gives the desired upper bound. So we just need a matching lower bound. For this, we note that the lower bound in the previous proof, stated for the case $q \geqslant kp$, actually works for all values of p and q. (Although when $k \geqslant 4$ and $q < kp$, it is not sharp.) But here it matches our upper bound for all p and q. $\qquad \square$

3.2   Upper Bounds for $K_{p,q}$

In the remainder of this section, we prove the upper bounds in Theorem 1. First, we prove a proposition and a lemma that will be used in the proofs of the three statements of the theorem.

Definition 7. Fix a complete bipartite graph $K_{p,q}$, where p and q are positive integers. By symmetry, we assume $p \leqslant q$. Fix an integer $k \geqslant 3$ and $[k]$-colourings α and β of $K_{p,q}$. Denote the parts of $K_{p,q}$ by U and V, with $|U| = p$ and $|V| = q$. Let $C_1 := \alpha(V) \cap \beta(V)$, let $C_2 := \alpha(V) \setminus C_1$, and let $C_3 := \beta(V) \setminus C_1$. By symmetry, we assume that $|C_3| \geqslant |C_2|$; if not, then we swap α and β.

Lemma 8. *Fix positive integers p, q, k and $[k]$-colourings α and β as in Definition 7. If there exists $c \in [k]$ used on V by neither α nor β, then $d(\alpha, \beta) \leqslant 2p + q$.*

Proof. Recolour each vertex of U with c, recolour each $v \in V$ with $\beta(v)$, and recolour each $u \in U$ with $\beta(u)$. The number of recolouring steps is at most $p + q + p = 2p + q$. $\quad \square$

Remark 9. *Throughout this section, we assume Definition 7. When proving the upper bounds for Theorem 1, we assume that $\alpha(V) \cup \beta(V) = [k]$, since otherwise we are done by Lemma 8. That is, C_1, C_2, C_3 partition $[k]$. Note also that $C_2 \neq \emptyset$ and $C_3 \neq \emptyset$, since U is coloured under both β and α.*

Proposition 10. *Fix integers p, q, k, the graph $K_{p,q}$, and $[k]$-colourings α and β, as in Theorem 7. Assume that $\alpha(U) = \beta(V) = C_3$ and $\alpha(V) = \beta(U) = C_2$, where $C_2 \cup C_3 = [k]$. If $|C_3| \geqslant |C_2|$, then*

$$d(\alpha, \beta) \leqslant q + 2p + \min\left\{ \frac{q-p}{|C_3|}, \frac{q - kp}{|C_2||C_3|} + p \right\}.$$

Proof. Fix a non-empty subset $C_3' \subset C_3$ (not equal to the whole set C_3; this is possible, since $|C_3| \geqslant \lceil k/2 \rceil \geqslant 2$) and a non-empty subset $C_2' \subseteq C_2$ (possibly equal to C_2).

We now sketch a possible series of recolourings from α to β. First recolour $\alpha^{-1}(C_3') \subset U$ (with colours from $C_3 \backslash C_3'$). Then recolour each vertex $v \in \alpha^{-1}(C_2') \subset V$, with its colour $\beta(v)$ whenever $\beta(v) \in C_2'$, and otherwise recolour each such v with an arbitrary colour in $C_3' \cup (C_2 \backslash C_2')$. Now recolour all of U with C_2', such that U does not use any colour of C_3, and such that each $u \in U$ is coloured with $\beta(u)$ if possible. Next, we recolour each remaining vertex $v \in V$ with $\beta(v)$; finally, we recolour each vertex $u \in U$ with $\beta(u)$.

In total, the number of steps we used is $|\alpha^{-1}(C_3')| + |\alpha^{-1}(C_2')| + |\alpha^{-1}(C_2 \backslash C_2')| + |\alpha^{-1}(C_2') \cap \beta^{-1}(C_3 \backslash C_3')| + |\beta^{-1}(C_2 \backslash C_2')| = p + q + |\alpha^{-1}(C_3')| + |\alpha^{-1}(C_2') \cap \beta^{-1}(C_3 \backslash C_3')| + |\beta^{-1}(C_2 \backslash C_2')|$.

Rather than fix sets C_2' and C_3', we just fix their sizes $|C_2'|$ and $|C_3'|$. Let $f_2 := \frac{|C_2'|}{|C_2|}$ and $f_3 := \frac{|C_3 \backslash C_3'|}{|C_3|}$. Averaging over all possible sets C_2' and C_3' of these fixed sizes, the number of steps we use is $p + q + (1 - f_3)p + f_2 f_3 q + (1 - f_2)p = 2p + q + f_2 f_3 q + (1 - f_2 - f_3)p$. Doing this for $|C_2'| = |C_2|$ and $|C_3'| = |C_3| - 1$, as well as for $|C_2'| = 1$ and $|C_3'| = 1$ results in

$$d(\alpha, \beta) \leqslant 2p + q + \min\left\{ \frac{q - p}{|C_3|}, \frac{q}{|C_2||C_3|} + \left(1 - \frac{1}{|C_2|} - \frac{1}{|C_3|}\right)p \right\}.$$

Noting that $\frac{1}{|C_2|} + \frac{1}{|C_3|} = \frac{|C_2| + |C_3|}{|C_2||C_3|}$ and $|C_2| + |C_3| = k$ results in the formulation of the proposition. $\qquad\square$

We are now ready to prove the upper bounds of Theorem 1. For ease of presentation, we restate the upper bounds. Since the diameter is always an integer, the bounds also hold when taking the floor function.

Theorem 11. *Fix positive integers p, q, k. If $p \leqslant q$ and $k \geqslant 3$, then*

$$\operatorname{diam} \mathcal{C}_k(K_{p,q}) = \begin{cases} 2p + q + \left\lfloor \frac{q-p}{k-1} \right\rfloor & \text{if } q \geqslant kp \\[2mm] 3p + q + \left\lfloor \frac{q-kp}{\left\lfloor \frac{k^2}{4} \right\rfloor} \right\rfloor - g_1(k, p, q) & \text{if } \left\lceil \frac{k}{2} \right\rceil p \leqslant q \leqslant kp, \\[2mm] 2p + q + \left\lfloor \frac{q-p}{\left\lceil \frac{k}{2} \right\rceil} \right\rfloor - g_2(k, p, q) & \text{if } q \leqslant \left\lceil \frac{k}{2} \right\rceil p, \end{cases}$$

where $0 \leqslant g_1(k, p, q) \leqslant \left\lfloor \frac{k^2}{4} \right\rfloor$ and $0 \leqslant g_2(k, p, q) \leqslant \left\lfloor \frac{k}{2} \right\rfloor$.

Proof. Fix $K_{p,q}$, $[k]$, α, β, C_1, C_2, C_3 as in Definition 7.

If $|C_1| = 0$, then we are done by Theorem 10; below, we check this for the 3 regimes:

- In the first regime, it is sufficient to note that $q - kp \geqslant 0$ and $|C_3||C_2| \geqslant k - 1$. Together, these imply that $\frac{q-kp}{|C_2||C_3|} + p \leqslant \frac{q-kp}{k-1} + p = \frac{q-p}{k-1}$.

- In the second regime, since $q - kp \leqslant 0$ and $|C_2||C_3| \leqslant \left\lfloor \frac{k^2}{4} \right\rfloor$, we have

$$\frac{q-kp}{|C_2||C_3|} + p \leqslant p + \frac{q-kp}{\left\lfloor \frac{k^2}{4} \right\rfloor}.$$

- In the third regime, it is sufficient to note that $|C_3| \geqslant \lceil k/2 \rceil$ and thus $\frac{q-p}{|C_3|} \leqslant \frac{q-p}{\lceil \frac{k}{2} \rceil}$.

Thus, we assume below that $|C_1| \geqslant 1$.

We proceed by induction on k. The base case is $k = 3$, and thus $|C_1| = |C_2| = |C_3| = 1$. When $k = 3$, we also denote by C_i the single colour of each set C_i. We describe 2 recolouring sequences; by an averaging argument we will show that at least one of them has length at most $p + q + \frac{p+q}{2}$ (when $k = 3$, all 3 upper bounds above are equal).

As a first method to recolour α into β, we: (i) recolour $\alpha^{-1}(C_2)$ with C_1, (ii) recolour U with C_2, and (iii) recolour $\beta^{-1}(C_3)$ with colour C_3. This implies

$$d(\alpha, \beta) \leqslant p + \left| \alpha^{-1}(C_2) \right| + \left| \beta^{-1}(C_3) \right|. \tag{3}$$

As a second method, we: (i) recolour $\alpha^{-1}(C_1)$ with C_2, (ii) recolour each $u \in U$ with C_1, (iii) recolour each $v \in V$ with C_3, (iv) recolour each $u \in U$ with $\beta(u) = C_2$, and (v) recolour each $v \in \beta^{-1}(C_1)$ with $\beta(v) = C_1$. The number of steps this takes is $|\alpha^{-1}(C_1)| + p + q + p + |\beta^{-1}(C_1)|$. That is,

$$d(\alpha, \beta) \leqslant 2p + q + |\alpha^{-1}(C_1)| + |\beta^{-1}(C_1)|. \tag{4}$$

Since $|\alpha^{-1}(C_1)| + |\alpha^{-1}(C_2)| = |V| = |\beta^{-1}(C_1)| + |\beta^{-1}(C_3)|$, adding (3) and (4) gives $2d(\alpha, \beta) \leqslant 3(p+q)$, as desired. This concludes the base case.

For the induction step, we assume $k \geqslant 4$ and the statement is true for $k-1$ colours. For a fixed colour $c \in C_1$, we let $y_0 := |\alpha^{-1}(c) \cap \beta^{-1}(c)|$, $y_1 := |\alpha^{-1}(c) \setminus \beta^{-1}(c)|$, $y_2 := |\beta^{-1}(c) \setminus \alpha^{-1}(c)|$, and $y := |\alpha^{-1}(c) \cup \beta^{-1}(c)| = y_0 + y_1 + y_2$. Again we will use two recolouring sequences, which imply the following two statements.

Claim 1.
$$d(\alpha, \beta) \leqslant \operatorname{diam} \mathcal{C}_{k-1}(K_{p, q-y}) + y - y_0 \tag{5}$$

Proof. We (i) recolour $\beta^{-1}(c) \setminus \alpha^{-1}(c)$ in colour c. Now (ii) every vertex w in $U \cup (V \setminus (\beta^{-1}(c) \cup \alpha^{-1}(c)))$ can be recoloured into $\beta^{-1}(u)$ using at most $\operatorname{diam} \mathcal{C}_{k-1}(K_{p, q-y})$ steps. Finally, we (iii) recolour every $v \in \alpha^{-1}(c) \setminus \beta^{-1}(c)$ with $\beta(v)$. In total, the number of steps we use is at most $y_2 + \operatorname{diam} \mathcal{C}_{k-1}(K_{p, q-y}) + y_1$. $\diamondsuit$

Claim 2.
$$d(\alpha, \beta) \leqslant 2p + q + y + y_0 \tag{6}$$

Proof. We (i) recolour $\alpha^{-1}(c)$ from C_2, (ii) recolour U with c, (iii) recolour every $v \in V \setminus \beta^{-1}(c)$ with $\beta(v)$, (iv) recolour every $v \in \beta^{-1}(c)$ with an arbitrary colour from C_3, (v) recolour every $u \in U$ with $\beta(u)$, and (vi) recolour $\beta^{-1}(c)$ with c. In total, the number of steps we use is $(y_0 + y_1) + p + q + p + (y_0 + y_2) = 2p + q + y + y_0$. $\diamondsuit$

With regards to the 3 cases in the theorem, we have 3 possibilities for q: (a) $q \geqslant kp$, (b) $kp > q > \lceil k/2 \rceil p$, and (c) $\lceil k/2 \rceil p \geqslant q$. We also have 3 possibilities for $q - y$: (i) $q - y \geqslant (k-1)p$, (ii) $(k-1)p > q - y > \lceil (k-1)/2 \rceil p$, and (iii) $\lceil (k-1)/2 \rceil p \geqslant q - y$. These possibilities for q and $q - y$ combine below to give us 9 cases: (a.i) through (c.iii), although we handle (a.i) and (b.i) together, and also handle (c.i) and (c.ii) together.

- **(a.i) and (b.i): $q > \lceil \frac{k}{2} \rceil p$ and $q - y \geqslant (k-1)p$.** Adding $(k-2)$ times (5) to (6), then dividing by $k-1$ gives

$$d(\alpha, \beta) - (2p + q) \leqslant \frac{1}{k-1} \left[(k-2) \left(-y_0 + \frac{q - y - p}{k-2} \right) + y + y_0 \right] \leqslant \frac{q - p}{k - 1}.$$

  If $q \geqslant kp$, then this gives the desired bound on $d(\alpha, \beta)$. But if $kp > q > \lceil k/2 \rceil p$, then

$$\frac{q - p}{k - 1} = p + \frac{q - kp}{k - 1} \leqslant p + \frac{q - kp}{\left\lfloor \frac{k^2}{4} \right\rfloor},$$

  since $k - 1 = \lfloor (4k - 4)/4 \rfloor \leqslant \lfloor k^2/4 \rfloor$.

- **(a.ii): $q \geqslant kp$ and $(k-1)p > q - y > \lceil \frac{k-1}{2} \rceil p$.** From (5), we have

$$d(\alpha, \beta) - (2p + q) \leqslant p - y_0 + \frac{q - y - (k-1)p}{\left\lfloor \frac{(k-1)^2}{4} \right\rfloor} < p = \frac{kp - p}{k - 1} \leqslant \frac{q - p}{k - 1}.$$

- **(a.iii): $q \geqslant kp$ and $\lceil \frac{k-1}{2} \rceil p \geqslant q - y$.** From (5), we have

$$d(\alpha, \beta) - (2p + q) \leqslant -y + \frac{q - y - p}{\lceil (k-1)/2 \rceil} + y - y_0 \leqslant \frac{\lceil (k-1)/2 \rceil p - p}{\lceil (k-1)/2 \rceil}$$

$$= p - \frac{p}{\lceil (k-1)/2 \rceil} \leqslant p - \frac{p}{k-1} = \frac{pk - p - p}{k - 1} \leqslant \frac{q - p}{k - 1}.$$

- **(b.ii) $kp > q > \lceil \frac{k}{2} \rceil p$ and $(k-1)p > q - y > \lceil \frac{k-1}{2} \rceil p$.**

  Adding $\left\lfloor \frac{(k-1)^2}{4} \right\rfloor$ times (5) to (6), then dividing by $\left\lfloor \frac{(k-1)^2}{4} \right\rfloor + 1 \leqslant \left\lfloor \frac{k^2}{4} \right\rfloor$, gives

$$d(\alpha, \beta) - (2p + q) \leqslant \left(\left\lfloor \frac{(k-1)^2}{4} \right\rfloor \left(p - y + \frac{q - y - (k-1)p}{\lfloor (k-1)^2/4 \rfloor} + y - y_0 \right) + y + y_0 \right) \Bigg/ \left(\left\lfloor \frac{(k-1)^2}{4} \right\rfloor + 1 \right)$$

$$\leqslant \left(p \left(\left\lfloor \frac{(k-1)^2}{4} \right\rfloor + 1 \right) + q - kp \right) \Bigg/ \left(\left\lfloor \frac{(k-1)^2}{4} \right\rfloor + 1 \right) \leqslant p + \frac{q - kp}{\left\lfloor \frac{k^2}{4} \right\rfloor}.$$

- **(b.iii): $kp > q > \lceil \frac{k}{2} \rceil p$ and $\lceil \frac{k-1}{2} \rceil p \geqslant q - y$.** From (5), we have

$$d(\alpha, \beta) - (2p + q) \leqslant -y_0 + \frac{q - y - p}{\left\lceil \frac{k-1}{2} \right\rceil} \leqslant \frac{\left\lceil \frac{k-1}{2} \right\rceil p - p}{\left\lceil \frac{k-1}{2} \right\rceil}$$

$$= p - \frac{p}{\left\lfloor \frac{k}{2} \right\rfloor} = p - \frac{p \left\lceil \frac{k}{2} \right\rceil}{\left\lfloor \frac{k^2}{4} \right\rfloor} \leqslant p + \frac{q - kp}{\left\lfloor \frac{k^2}{4} \right\rfloor}.$$

  Here the last equality holds because $\left\lfloor \frac{k}{2} \right\rfloor \left\lceil \frac{k}{2} \right\rceil = \left\lfloor \frac{k^2}{4} \right\rfloor$, and the last inequality holds because $q - kp \geqslant \left\lceil \frac{k}{2} \right\rceil p - kp = -\left\lfloor \frac{k}{2} \right\rfloor p \geqslant -\left\lceil \frac{k}{2} \right\rceil p$.

- **(c.i) and (c.ii): $\lceil \frac{k}{2} \rceil p \geqslant q$ and $q - y > \lceil \frac{k-1}{2} \rceil p$.** This case can only occur when $\lceil \frac{k}{2} \rceil > \lceil \frac{k-1}{2} \rceil$; that is, when k is odd. So let $r := (k-1)/2$. Since $\lceil \frac{k}{2} \rceil < k - 1$, we must show that $d(\alpha, \beta) \leqslant 3p + q + \frac{q-kp}{\lceil \frac{k^2}{4} \rceil}$. (That is, we must be in (c.ii); in fact, (c.i) is impossible.) If $d(\alpha, \beta) - (2p + q) \leqslant \frac{q-p}{r+1}$, then we have

$$d(\alpha, \beta) - (2p + q) \leqslant \frac{q-p}{r+1} = p + \frac{q - p - p\lceil \frac{k}{2} \rceil}{\lceil \frac{k}{2} \rceil} = p + \frac{q - p\lceil \frac{k}{2} \rceil}{\lceil \frac{k}{2} \rceil} + \frac{-p}{\lceil \frac{k}{2} \rceil}$$

$$\leqslant p + \frac{q - p\lceil \frac{k}{2} \rceil}{\lceil \frac{k^2}{4} \rceil} + \frac{-p\lfloor \frac{k}{2} \rfloor}{\lfloor \frac{k^2}{4} \rfloor} = p + \frac{q - pk}{\lfloor \frac{k^2}{4} \rfloor}.$$

So instead we assume $d(\alpha, \beta) - (2p + q) > \frac{q-p}{r+1}$. By (6) and (5), respectively, this implies the following.

$$y + y_0 > \frac{q-p}{r+1} \tag{7}$$

$$p - y_0 + \frac{q - y - 2rp}{r^2} > \frac{q-p}{r+1} \tag{8}$$

(If we are in (c.ii), then (8) follows directly from the induction hypothesis. But if we are in (c.i), then we note that the left side of (8) is greater than the upper bound guaranteed by the induction hypothesis; this can be verified by a bit of algebra, which we omit.) Since we are in (c.i) or (c.ii), we have $pr = p\lceil \frac{k-1}{2} \rceil < q - y = q - (y + y_0) + y_0 < q - \frac{q-p}{r+1} + y_0 = \frac{rq+p}{r+1} + y_0$; equivalently,

$$(r^2 + r - 1)p < rq + (r+1)y_0. \tag{9}$$

Adding r^2 times (8) to (7) gives

$$r^2 p - (r^2 - 1)y_0 + q - 2rp \geqslant \frac{r^2 + 1}{r+1}(q - p)$$

$$r^3 p + r^2 p - (r^2 - 1)(r+1)y_0 + qr + q - 2r^2 p - 2rp \geqslant (r^2 + 1)q - (r^2 + 1)p$$

$$(r^3 - 2r + 1)p \geqslant (r^2 - r)q + (r^2 - 1)(r+1)y_0$$

$$(r^2 + r - 1)p \geqslant rq + (r+1)^2 y_0.$$

But this final inequality contradicts (9), which finishes the case.

- **(c.iii) $\lceil \frac{k}{2} \rceil p \geqslant q$ and $\lceil \frac{k-1}{2} \rceil p \geqslant q - y$.** If k is even, then $\lceil \frac{k-1}{2} \rceil = \lceil \frac{k}{2} \rceil$, so (5) gives

$$d(\alpha, \beta) - (2p + q) \leqslant -y + \frac{q - y - p}{\lceil \frac{k-1}{2} \rceil} + y - y_0 \leqslant \frac{q-p}{\lceil \frac{k}{2} \rceil}.$$

If k is odd, then adding $\frac{k-1}{2}$ times (5) to (6), and dividing by $\frac{k+1}{2}$, gives

$$d(\alpha, \beta) - (2p + q) \leqslant \frac{1}{\frac{k+1}{2}} \left(\left(\frac{k-1}{2} \right) \frac{q - y - p}{\frac{k-1}{2}} - \frac{k-1}{2} y_0 + (y + y_0) \right)$$

$$\leqslant \frac{q-p}{\lceil \frac{k}{2} \rceil} - \frac{\frac{k-3}{2}}{\frac{k+1}{2}} y_0 \leqslant \frac{q-p}{\lceil \frac{k}{2} \rceil}.$$

This finishes the last of the 9 cases in the induction step, and thus concludes the proof. □

4   Non-relationships between $\mathcal{C}_k(G)$, $\mathcal{C}_{k+1}(G)$, and $\mathcal{C}_L(G)$

In [7], the authors sketched some typical behaviour of configuration graphs. In particular, they mentioned that by increasing the number of colours, the recolouring sequences typically become shorter and shorter. Note that two factors are at play. If the number of colours, k, increases, then this also increases both the total number of colourings and the number of possible local changes, the degrees in the reconfiguration graph. The first factor often leads to a higher diameter, but the second to a lower diameter. Here, we give some exceptions to the typical behaviour. A classical example from [14] and [10, Prop. 2] is the following.

Example 12. If $G = K_{m,m} \backslash M$ ($K_{m,m}$ minus a perfect matching), then $\mathcal{C}_k(G)$ is connected if and only if $k \geqslant 3$ and $k \neq m$.

The key idea in Theorem 12 is that if $k = m$, then G has a frozen colouring, namely, the one that uses each colour once on each part; see Fig. 4. (Otherwise, by the Pigeonhole Principle each part has a colour that is used either 0 times or at least 2 times.) Theorem 12 is the inspiration for more counterexamples. Bard [1, 17] asked whether $\mathcal{C}_k(G)$ being Hamiltonian implies that $\mathcal{C}_{k+1}(G)$ is Hamiltonian. In https://github.com/StijnCambie/reconfiguration (document Check_Ham), it is e.g. verified that if $G = K_{5,5} \backslash M$ then the reconfiguration graph $\mathcal{C}_3(G)$ is Hamiltonian. Since $\mathcal{C}_5(G)$ is disconnected and thus not Hamiltonian, this answers the question of Bard negatively (either for $k = 3$ or for $k = 4$, but it is not clear which one). This counterexample was mentioned in [17, Sec. 9.6], attributed to the first author of the present paper, but no proof was provided; so the code referenced above provides this proof.

Our next example modifies $K_{4,4} \backslash M$, by replacing one vertex with 3 (each of which inherits exactly 2 of its 3 initial neighbours). In each of the 4-colourings shown (see the right of Fig. 3), the colours on many of the vertices are frozen. Not surprisingly, the two 4-colourings are far apart.

Example 13. The graph G presented in Fig. 3 is bipartite with order 10, matching number 4, and degeneracy $\delta^\star(G) = 2$. For $k \geqslant 3$, $\mathcal{C}_k(G)$ is connected. It satisfies $\operatorname{diam} \mathcal{C}_3(G) = 15$ and $\operatorname{diam} \mathcal{C}_4(G) = 17$, thus $\operatorname{diam} \mathcal{C}_k(G) < \operatorname{diam} \mathcal{C}_{k+1}(G)$ for $k = 3$. So G is a graph for which an additional colour can increase the diameter of the recolouring graph. Furthermore, $\operatorname{rad} \mathcal{C}_3(G) = 15$ and $\operatorname{rad} \mathcal{C}_4(G) = 14$.

We verify these numbers in https://github.com/StijnCambie/reconfiguration (document NonMonotoneC3vsC4). As we know from [9, Cor. 14], for k sufficiently large, the diameter of the reconfiguration graph stabilises at the minimum possible bound (this is the graph's order plus its matching number). The graph in Theorem 13 illustrates that for small k, this behaviour can be harder to predict. By simply taking many disjoint copies of it, we obtain infinitely many graphs G with $\frac{\operatorname{diam} \mathcal{C}_{k+1}(G)}{\operatorname{diam} \mathcal{C}_k(G)} = 17/15$. We do not know if this ratio can become arbitrarily large, while staying finite.

Question 14. Fix an integer $k \geqslant 3$. Does there exist a sequence of graphs $(G_i)_{i \in \mathbb{N}}$ such that $\operatorname{diam} \mathcal{C}_{k+1}(G_i) < \infty$ for all i, but $\lim_{i \to \infty} \frac{\operatorname{diam} \mathcal{C}_{k+1}(G_i)}{\operatorname{diam} \mathcal{C}_k(G_i)} = \infty$?

Interestingly, we were unable to find an analogue of Theorem 13 for radius. An affirmative answer to the following question would imply that the ratio $\frac{\operatorname{diam} \mathcal{C}_{k+1}(G)}{\operatorname{diam} \mathcal{C}_k(G)}$ is actually bounded by 2, whenever it is well-defined.
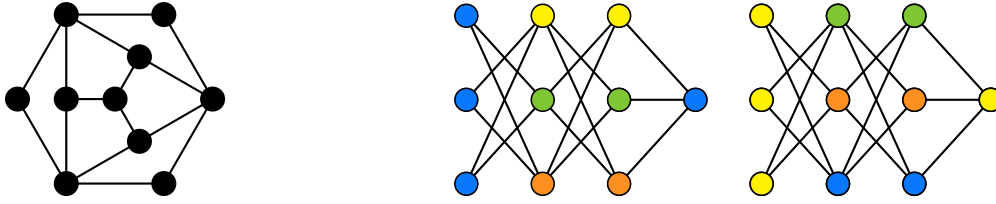
Figure 3: Left: A graph G for which $\operatorname{diam} \mathcal{C}_3(G) < \operatorname{diam} \mathcal{C}_4(G)$. Right: Two [4]-colourings of G (drawn differently from the figure on the left) that in $\mathcal{C}_4(G)$ lie at distance 17.

Question 15. Fix an integer $k \geqslant 3$ and let G be a graph such that $\mathcal{C}_k(G)$ and $\mathcal{C}_{k+1}(G)$ are both connected. Is it necessarily true that $\operatorname{rad} \mathcal{C}_k(G) \geqslant \operatorname{rad} \mathcal{C}_{k+1}(G)$?

Next, we compare the diameter of the k-colour reconfiguration graph and the list reconfiguration graph for a k-list-assignment. Despite identical lists imposing the most restrictions, we prove that the k-colour reconfiguration graph is not extremal; more precisely, for arbitrary k-list-assignments L, no relationships hold in general between the connectedness (or diameter) of $\mathcal{C}_k(G)$ and that of $\mathcal{C}_L(G)$.

Proposition 16. *For each of the four statements below, there are examples of graphs G, integers k, and list-assignments L with $|L(v)| = k$ for every $v \in V(G)$ that satisfy the statement.*

(a) $\mathcal{C}_k(G)$ is disconnected, while $\mathcal{C}_L(G)$ is connected

(b) $\operatorname{diam} \mathcal{C}_L(G) < \operatorname{diam} \mathcal{C}_k(G) < \infty$

(c) $\mathcal{C}_L(G)$ is disconnected, while $\mathcal{C}_k(G)$ is connected

(d) $\operatorname{diam} \mathcal{C}_k(G) < \operatorname{diam} \mathcal{C}_L(G) < \infty$

Proof. We prove the first two statements together, and the third and fourth thereafter.

(a,b) The first two statements are true almost trivially. Take an n-vertex graph G and a value k for which $\operatorname{diam} \mathcal{C}_k(G) > n$. Let the lists $L(v)$, for all $v \in V$, be disjoint; so $\operatorname{diam} \mathcal{C}_L(G) = n$. Here we can choose $\operatorname{diam} \mathcal{C}_k(G)$ to be finite or infinite (equivalently $\mathcal{C}_k(G)$ can be connected or disconnected). For example, with $G = K_{m,m}$, when either $k > 2$ or $k = 2$. By instead letting $G := P_n$ and $k := 3$, by [6, Thm. 3] we know that $\frac{\operatorname{diam} \mathcal{C}_k(G)}{\operatorname{diam} \mathcal{C}_L(G)}$ can be arbitrarily large (while being finite).

(c) Fix $m \geqslant 4$, let G be the graph $K_{m,m} \backslash M$ for a perfect matching M, and let $k := m - 1$. We denote the vertices in the two parts by $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$, with $v_i w_i \notin E(G)$ for all $i \in [m]$. Let $L(v_i) = L(w_i) := [m] \backslash \{i\}$. Then there exist L-colourings that are frozen in $\mathcal{C}_L(G)$, e.g. $\varphi(v_i) = \varphi(w_i) = (i+1) \bmod m$; see Fig. 4. As such, $\mathcal{C}_L(G)$ is disconnected; in contrast, $\mathcal{C}_k(G)$ is connected, as proven in [10] (the intuition behind this final statement is that on each part some colour must appear at least twice, by pigeonhole principle; so we can recolour the whole part with that colour).

(d) Let $k := 4$, let $G := K_{18,18}$, and call its parts U and V. Recall that $54 = n(G) + \mu(G) \leqslant \operatorname{diam} \mathcal{C}_4(G) \leqslant 54$; the upper bound is true by Theorem 1. So it suffices to construct a 4-assignment L for G such that $\operatorname{diam} \mathcal{C}_L(G) \geqslant 55$. This is what we do.

For every triple (i, j, ℓ) with $i \in \{1, 3, 5\}$, with $j \in \{2, 4\}$, and with $\ell \in [5] \backslash \{i, j\}$, for unique vertices $u \in U$ and $v \in V$ we set $L(u) = L(v) := [5] \backslash \{\ell\}$, and define L-colourings α and β by $\alpha(u) = i = \beta(v)$ and $\beta(u) = j = \alpha(v)$. These colourings are depicted in Fig. 5.
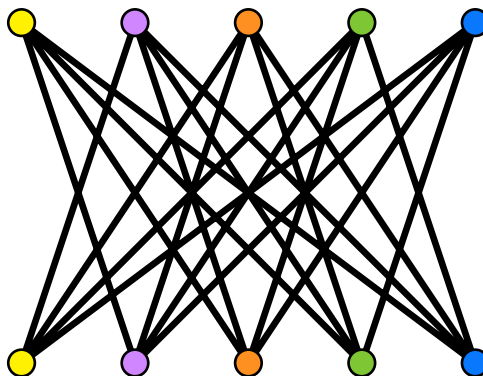
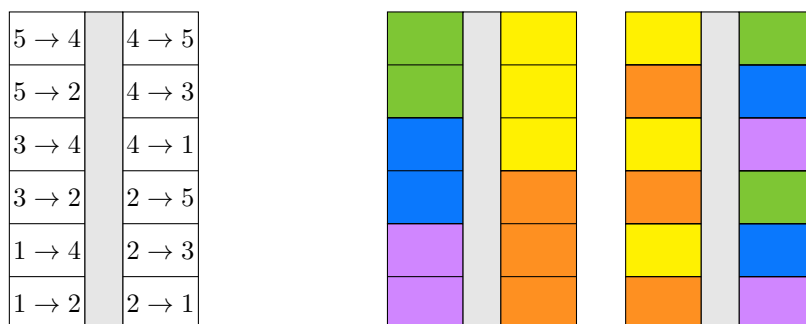Figure 4: A frozen colouring of $K_{5,5} \backslash M$

Figure 5: Initial and final L-colouring of $K_{18,18}$ where every "box" represents 3 vertices (having different lists), which requires at least 55 recolourings. Left: These colourings are shown as $\alpha(v) \to \beta(v)$. Right: These colourings are shown with explicit colours.

Note that due to the choice of lists, we cannot colour either part with only a single colour. So each part must always use at least 2 colours; so 4 colours in total. This implies that we can only move one colour at a time between U and V. When recolouring α into β, we may assume, by possibly renaming colours, that 1 is the first colour completely removed from its part (which must be U), followed next by 2 (removed from V), and followed next by 3 (removed from U). (Note that completely removing a colour from a part and then immediately again using that colour on that part will always be inefficient.) While removing colours 1, 2, 3 from their parts, we need to recolour some vertices without giving each of them its final colour; we call each instance of this a *bad* recolouring. We will prove that, to recolour α to β, we need at least 19 bad recolourings. This will finish the proof, since then the total number of recolourings we use to get from α to β is at least $19 + |U| + |V| = 19 + 36 = 55$, as desired.

We start by recolouring vertices (in U) initially coloured 1 with either 3 or 5; we do this for at least 6 vertices, so begin with at least 6 bad recolourings. Next we remove colour 2 from V. Initially, 2 is used on 9 vertices of V; since 6 of these are coloured with 3 or 5 in β, we perform at least 6 more bad recolourings. Initially, 6 vertices of U are coloured with 3. In β, we use colour 4 on exactly 3 of these; but colour 4 is not yet available for use on U, so we need 3 more bad recolourings. In addition, consider the vertex $u \in U$ that corresponds to triple $(1, 4, 5)$; that is $\alpha(u) = 1$, $\beta(u) = 4$, and $L(u) = [5] \setminus \{5\} = [4]$. When 1 was removed from U, vertex u must have been recoloured

with 3. So now that colour 3 is being removed, we need another bad recolouring.

When we next completely remove a colour from V, it is either 1 or 4. If 4, then we need at least 3 more bad recolourings, since colour 5 is not yet available to use on V. But if we instead remove 1, then we will also incur at least 3 more bad recolourings (the vertices in V with $\alpha(v) = 2, \beta(v) = 1$ all have been recoloured badly as well). Thus, the total number of bad recolourings is at least $6 + 6 + 3 + 1 + 3 = 19$, as desired. $\qquad\square$

The following result, combined with Remark 18, extends Theorem 16(d), exhibiting an even larger (unbounded) gap between $\operatorname{diam} \mathcal{C}_L(G)$ and $\operatorname{diam} \mathcal{C}_k(G)$.

Lemma 17. *For all $n \geqslant 1$ there exists an n-vertex graph G_n with a 4-fold list-assignment L, such that $c_1 \cdot n \leqslant \frac{\operatorname{diam} \mathcal{C}_L(G_n)}{\operatorname{diam} \mathcal{C}_4(G_n)} \leqslant c_2 \cdot n$, where c_1 and c_2 are some uniform positive constants.*

Proof. For each positive integer k, we construct a graph G_k with $|V(G_k)| = 260k$ and a 4-assignment L such that $(k^2 - 1)/4 + 255k \leqslant \operatorname{diam} \mathcal{C}_L(G_k) \leqslant 2k^2 + 255k$ and $260k \leqslant \operatorname{diam} \mathcal{C}_4(G_k) \leqslant 691k$. We use these graphs G_k to prove the result when $n = 260k$ for some positive integer k. It is straightforward (but a bit tedious) to extend the result to all positive integers n; so we just sketch the ideas. In general when $n \geqslant 260$, we pick an integer k such that $260k \leqslant n < 260(k + 1)$, and we take our graph G to be an n-vertex subgraph of G_{k+1} that contains as an induced subgraph G_k. Now $n/2 \leqslant 260k \leqslant n < 260(k + 1) \leqslant 2n$, so by applying the bounds for G_k and G_{k+1}, we lose at most a constant factor in both the upper and lower bounds.

The most interesting part of our construction is the lower bound on $\operatorname{diam} \mathcal{C}_L(G_k)$, so we begin with that. We start with a path of order k and "hang off" of each path vertex a copy of $K_{4,255}$ that uses its lists to enforce that the recolouring of the path simulates the recolouring of a path with only 3 colours (and no lists). Since [6] showed that $(k^2 - 1)/4 \leqslant \operatorname{diam} \mathcal{C}_3(P_k) \leqslant 2k^2$, we are done. Thus, we need only to describe these lists and show how they enforce the desired simulation of $\mathcal{C}_3(P_k)$.

Let H be the complete bipartite graph $K_{4,255}$. To the four vertices v_1, v_2, v_3, v_4 in the part of size 4, assign four disjoint lists: $L(v_1) = \{1, 2, 3, 4\}, L(v_2) = \{5, 6, 7, 8\}, L(v_3) = \{9, 10, 11, 12\}, L(v_4) = \{13, 14, 15, 16\}$. In the part of size $255 = 4^4 - 1$, assign each vertex a unique list that contains precisely one element from each of $L(v_1), L(v_2), L(v_3), L(v_4)$, except that no vertex is assigned $\{1, 5, 9, 13\}$. Note that in every proper L-colouring c of H, we must have $c(v_1) = 1, c(v_2) = 5, c(v_3) = 9$ and $c(v_4) = 13$. From this it follows that $\mathcal{C}_L(H)$ is connected, as any two L-colourings only differ on the part of size 255 (since that is an independent set, in fact the diameter of $\mathcal{C}_L(H)$ is 255). In what follows we call v_1 the *special vertex* of H and we only need that it is coloured 1 in every L-colouring.

Now consider a path $P_k = p_1, \ldots, p_k$ on k vertices. Let G_k denote the graph formed by taking the disjoint union of P_k and k copies $H_1, \ldots, H_k$ of H, and then for every $i \in [k]$ adding an edge between p_i and the special vertex of H_i.

Here we tacitly assumed that each H_i is equipped with the list-assignment L described above. Moreover, we assign each vertex of P the list $\{1, a, b, c\}$, where $\{a, b, c\}$ is disjoint from $\{1, 2, \ldots, 16\}$. Since in a proper L-colouring each special vertex must have colour 1, it follows that each vertex of P_k must use a colour in $\{a, b, c\}$. In other words, restricted to P the L-colourings correspond precisely to the 3-colourings of P. As noted above, by [6, Thm. 3], we know that $(k^2 - 1)/4 \leqslant \operatorname{diam} \mathcal{C}_3(P_k) \leqslant 2k^2$.

It follows that $\operatorname{diam}\mathcal{C}_L(G_k) = \operatorname{diam}\mathcal{C}_L(P_k) + k \cdot \operatorname{diam}\mathcal{C}_L(H) = \operatorname{diam}\mathcal{C}_3(P_k) + 255k = \Theta(k^2)$, because any two L-colourings of G_k only differ on P_k and the size-255 parts of the copies of H, taking note that these parts are separated from each other by the special vertices that have the same colour in every L-colouring.

Now we must prove that $\operatorname{diam}\mathcal{C}_4(G_k) = \Theta(k)$. The lower bound is trivial, since $|V(G_k)| = (255 + 4 + 1)k$ and permuting the colors (with no fixed points) in an arbitrary colouring α gives a colouring β with $d(\alpha, \beta) \geqslant |V(G_k)| = 260k$. For the upper bound, let α, β be two 4-colourings of G_k, and let $\alpha(P_k), \beta(P_k)$ be their restriction to P_k. As $4 = \Delta(P_k) + 2$, it follows (e.g. see [9, Thm. 1]) that $\operatorname{diam}\mathcal{C}_4(P_k) \leqslant 2k$. Thus there is a sequence $\mathcal{S}$ of at most $2k$ vertex recolourings that transforms $\alpha(P_k)$ into $\beta(P_k)$ while maintaining a proper colouring of P_k, but not necessarily of G_k.

Consider an arbitrary step of $\mathcal{S}$ where we need to recolour some vertex p_i of P_k, without loss of generality from colour 1 to colour 2. This recolouring step is also allowed in G_k unless the special vertex $v(H_i)$ of H_i has colour 2 as well. In this case, we first recolour $v(H_i)$ to either 3 or 4. This can be done in one step if either 3 or 4 does not appear in the size-255 part B_i of H_i. Otherwise, we recolour B_i to remove either 3 or 4 from B_i, using at most $\lfloor 255/2 \rfloor = 127$ steps, and then recolour $v(H_i)$ to either 3 or 4. In total we need at most 128 extra recolourings to accomodate the recolouring of p_i. Doing this for every step of $\mathcal{S}$, we need at most $129 \cdot 2k$ steps to recolour α in such a way that it agrees with β on the vertices of P_k.

After recolouring P_k, we must finally recolour each copy of H, which can be done in at most 433 steps per copy. (This needs a small argument; it does not follow directly from Theorem 1 because throughout the colour $\beta(p_i)$ is forbidden at the special vertex of H_i.) Here is one way to prove this upper bound. Let $\widehat{\alpha}$ denote the current colouring. For some colour $c \in L(v_1)$, we first remove c from part B_i (using $\beta(w)$ for each $w \in B_i$ when possible, and using some other colour in $\beta(B_i)$ when needed); then we recolour $\{v_1, v_2, v_3, v_4\}$ with c; then we recolour each $w \in B_i \setminus \beta^{-1}(c)$ with $\beta(w)$; then we recolour each v_i with $\beta(v_i)$; and finally we recolour each $w \in \beta^{-1}(c)$ with c. We choose colour c to minimise $|\widehat{\alpha}^{-1}(c) \cap V| + |\beta^{-1}(c) \cap V|$. Thus, the number of recolouring steps is at most $|\widehat{\alpha}^{-1}(c) \cap V| + 4 + |B_i| + 4 + |\beta^{-1}(c) \cap V| = 255 + 8 + |\widehat{\alpha}^{-1}(c) \cap V| + |\beta^{-1}(c) \cap V| \leqslant 263 + \lfloor 255 \cdot 2/3 \rfloor = 263 + 170 = 433$. Thus, in total the distance between α and β is at most $129 \cdot 2k + 433k = 691k$. $\qquad\square$

Remark 18. *The construction above can be generalized to k-colourings and ℓ-fold list-assignments for any $\ell \geqslant k \geqslant 4$, by replacing the graph H with a larger complete bipartite graph that has $\geqslant 1 + \ell - k$ uniquely L-coloured vertices, and replacing the path with a $(k-2)$-colourable n-vertex graph with $(k-1)$-colour diameter $\Theta(n^2)$ and k-colour diameter $\Theta(n)$, the existence of which is guaranteed by constructions in [6]. In this modification of Theorem 17 the constants c_1 and c_2 depend on k and ℓ.*

References

[1] S. Bard, *Gray code numbers of complete multipartite graphs*, PhD Thesis, 2014.

[2] M. Belavadi and K. Cameron, *Recoloring some hereditary graph classes*, Discrete Applied Mathematics, 361 (2025), 389–401.

[3] M. Belavadi, K. Cameron, and E. Hildred, *Frozen colourings in $2K_2$-free graphs*, Electron. J. Combin. **32** (2025), no. 2, #P2.29.

[4] M. Belavadi, K. Cameron, and O. Merkel, *Reconfiguration of vertex colouring and forbidden induced subgraphs*, European J. Combin., 118 (2024), Paper No. 103908, 10.

[5] M. Bonamy and N. Bousquet, *Recoloring graphs via tree decompositions*, European J. Combin., 69 (2018), 200–213.

[6] M. Bonamy, M. Johnson, I. Lignos, V. Patel, and D. Paulusma, *Reconfiguration graphs for vertex colourings of chordal and chordal bipartite graphs*, J. Comb. Optim., 27(1) (2014), 132–143.

[7] N. Bousquet, L. Feuilloley, M. Heinrich, and M. Rabie, *Short and local transformations between $(\Delta + 1)$-colorings*, Innov. Graph Theory **2** (2025), 119–156.

[8] N. Bousquet and G. Perarnau, *Fast recoloring of sparse graphs*, European J. Combin., 52 (2016), 1–11.

[9] S. Cambie, W. Cames van Batenburg, and D. W. Cranston, *Optimally reconfiguring list and correspondence colourings*, European J. Combin., 115 (2024), 20.

[10] L. Cereceda, J. van den Heuvel, and M. Johnson, *Mixing 3-colourings in bipartite graphs*, European J. Combin., 30(7) (2009), 1593–1606.

[11] D. W. Cranston, *List-recoloring of sparse graphs*, European J. Combin., 105 (2022), Paper No. 103562, 11.

[12] Z. Dvořák and C. Feghali, *An update on reconfiguring 10-colorings of planar graphs*, Electron. J. Combin., 27(4) (2020), #P 4.51.

[13] Z. Dvořák and C. Feghali, *A Thomassen-type method for planar graph recoloring*, European J. Combin., 95 (2021), Paper No. 103319, 12.

[14] J. E. Dunbar, S. M. Hedetniemi, S. T. Hedetniemi, D. P. Jacobs, J. Knisely, R. C. Laskar, and D. F. Rall, *Fall colorings of graphs*, J. Combin. Math. Combin. Comput., 33 (2000), 257–273. Papers in honour of Ernest J. Cockayne.

[15] M. Jerrum, *A very simple algorithm for estimating the number of k-colorings of a low-degree graph*, Random Structures Algorithms, 7(2) (1995), 157–165.

[16] O. Merkel, *Recolouring weakly chordal graphs and the complement of triangle-free graphs*, Discrete Math., 345(3) (2022), Paper No. 112708, 5.

[17] T. Mütze, *Combinatorial Gray codes—an updated survey*, Electron. J. Combin., #DS26 (2023).