Cyclic Ordering of Split Matroids

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Abstract

There is a long list of open questions rooted in the same underlying problem: understanding the structure of bases or common bases of matroids. These conjectures suggest that matroids may possess much stronger structural properties than are currently known. One example is related to cyclic orderings of matroids. A rank-r matroid is called cyclically orderable if its ground set admits a cyclic ordering such that any interval of r consecutive elements forms a basis. In this paper, we show that if the ground set of a split matroid decomposes into pairwise disjoint bases, then it is cyclically orderable. This result answers a conjecture of Kajitani, Ueno, and Miyano in a special case, and also strengthens Gabow's conjecture for this class of matroids. Our proof is algorithmic, hence it provides a procedure for determining a cyclic ordering in question using a polynomial number of independence oracle calls.

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1 Introduction

Throughout the paper, we denote a matroid by $M = (S, \mathcal{B})$, where S is a finite ground set and $\mathcal{B}$ is the *family of bases*, satisfying the so-called *basis axioms*: (B1) $\emptyset \in \mathcal{B}$, and (B2) for any $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 - B_2$, there exists $f \in B_2 - B_1$ such that $B_1 - e + f \in \mathcal{B}$. The latter property, called the *basis exchange axiom*, is one of the most fundamental tools in matroid theory. Nevertheless, it only provides a local characterization of the relationship between bases, which presents a significant stumbling block to further progress.

A rank-r matroid $M = (S, \mathcal{B})$ with $|S| = n$ is cyclically orderable if there exists an ordering $S = \{s_1, \ldots, s_n\}$ such that $\{s_i, s_{i+1}, \ldots, s_{i+r-1}\} \in \mathcal{B}$ for all $i \in [n]$, where indices are understood in a cyclic order. While studying the structure of symmetric exchanges in

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matroids, Gabow [12] formulated a beautiful conjecture, stating that every matroid whose ground set decomposes into two disjoint bases is cyclically orderable. This question was also raised independently by Wiedemann [20] and by Cordovil and Moreira [6]. The conjecture makes a stronger claim: for a fixed partition, the cyclic ordering can be chosen such that the elements of the two bases in the partition form contiguous intervals.

Conjecture 1 (Gabow)**.** Let $M = (S, \mathcal{B})$ be a matroid and $S = B_1 \cup B_2$ be a partition of the ground set into two disjoint bases. Then, M has a cyclic ordering in which the elements of B_1 and B_2 form intervals.

It is not difficult to see that the statement holds for strongly base orderable matroids. The conjecture was settled for graphic matroids [14, 6, 20], sparse paving matroids [5], matroids of rank at most 4 [16] and 5 [15], split matroids [4], and regular matroids [2]. However, the existence of a cyclic ordering remains open in general, even without the constraint of the bases forming intervals.

In [14], Kajitani, Ueno, and Miyano proposed a conjecture that would provide a full characterization of cyclically orderable matroids. A matroid $M = (S, \mathcal{B})$ with *rank function* r_M is called *uniformly dense* if $|S| \cdot r_M(X) \geqslant r_M(S) \cdot |X|$ holds for all $X \subseteq S$. It is not difficult to see that a cyclically orderable matroid is necessarily uniformly dense as well, and the conjecture states that this condition is also sufficient.

Conjecture 2 (Kajitani, Ueno, and Miyano)**.** A matroid is cyclically orderable if and only if it is uniformly dense.

Despite the fact that the conjecture would provide entirely new insights into the structure of matroids, very little progress has been made so far. Van den Heuvel and Thomassé [18] showed that the conjecture is true if $|S|$ and $r(S)$ are coprimes, and Bonin's result [5] for sparse paving matroids remains true also in this more general setting.

It is worth taking a moment to consider the interpretation of the uniformly dense property. By the matroid union theorem of Edmonds and Fulkerson [7], the ground set of a matroid $M = (S, \mathcal{B})$ can be covered by k bases if and only if $k \cdot r_M(X) \geqslant |X|$ holds for all $X \subseteq S$. Using this, a matroid is uniformly dense if and only if its ground set can be covered by $\lceil |S|/r_M(S) \rceil$ bases. In other words, the ground set can be decomposed in 'almost' disjoint bases, where almost means that the total overlapping between distinct bases is bounded by $r_M(S) - 1$. In particular, any matroid whose ground set decomposes into pairwise disjoint bases is uniformly dense. This observation motivates the following strengthening of Gabow's conjecture.

Conjecture 3. Let $M = (S, \mathcal{B})$ be a matroid and $S = B_1 \cup \cdots \cup B_k$ be a partition of the ground set into k pairwise disjoint bases. Then, M has a cyclic ordering in which the elements of B_i form an interval for each $i \in [k]$.

To the best of our knowledge, Conjecture 3 has not been previously considered and remains open even for very restricted classes of matroids, such as strongly base orderable matroids. Our main contribution is proving the conjecture for the class of split matroids. Split matroids were first introduced by Joswig and Schröter [13] while studying matroid polytopes from a geometric point of view. Since then, this class of matroids has gained importance in many contexts, primarily due to the work of Ferroni and Schröter [8, 9, 10, 11].

Theorem 4. *Conjecture 3 is true for split matroids.*

It is worth emphasizing that our proof is algorithmic, hence it provides a procedure for determining a cyclic ordering in question using a polynomial number of independence oracle calls.

Remark 5. In fact, we prove a slightly stronger statement: in the cyclic ordering obtained, the bases $B_1, \dots, B_k$ form intervals that follow each other in this order.

The rest of the paper is organized as follows. Basic definitions and notation are introduced in Section 2. We prove Conjecture 3 for split matroids in Section 3. Finally, in Section 4, we give a list of related open questions and conjectures that are subject of future research.

2 Preliminaries

General notation. We denote the set of *nonnegative integers* by $\mathbb{Z}_+$. For $k \in \mathbb{Z}_+$, we use $[k] = \{1, \dots, k\}$. Given a ground set S, the *difference* of $X, Y \subseteq S$ is denoted by $X - Y$. If Y consists of a single element y, then $X - \{y\}$ and $X \cup \{y\}$ are abbreviated as $X - y$ and $X + y$, respectively. The *symmetric difference* of X and Y is denoted by $X \triangle Y \coloneqq (X - Y) \cup (Y - X)$.

Split matroids. For basic definitions on matroids, we refer the reader to [17]. Let S be a ground set of size at least r, $\mathcal{H} = \{H_1, \dots, H_q\}$ be a (possibly empty) collection of subsets of S, and $r, r_1, \dots, r_q$ be nonnegative integers satisfying

$$|H_i \cap H_j| \leqslant r_i + r_j - r \text{ for distinct } i, j \in [q], \tag{H1}$$

$$|S - H_i| + r_i \geqslant r \text{ for all } i \in [q]. \tag{H2}$$

Then the corresponding *elementary split matroid* $M = (S, \mathcal{B})$ is given by $\mathcal{B} = \{X \subseteq S \mid |X| = r,\ |X \cap H_i| \leqslant r_i \text{ for all } i \in [q]\}$; see [1] for details. It is easy to see that the underlying hypergraph can be chosen in such a way that

$$r_i \leqslant r - 1 \text{ for all } i \in [q], \tag{H3}$$

$$|H_i| \geqslant r_i + 1 \text{ for all } i \in [q]. \tag{H4}$$

The representation is called *non-redundant* if all of (H1)–(H4) hold. A set $F \subseteq S$ is called *H_i-tight* if $|F \cap H_i| = r_i$. Finally, a *split matroid* is the direct sum of a single elementary split matroid and some (maybe zero) uniform matroids. The connection between elementary and connected split matroids is given by the following result [1].

Lemma 6 (Bérczi, Király, Schwarcz, Yamaguchi and Yokoi)**.** *The classes of connected split matroids and connected elementary split matroids coincide.*

A nice feature of split matroids is that they generalize paving and sparse paving matroids: paving matroids correspond to the special case when $r_i = r - 1$ for all $i \in [q]$, while we get back the class of sparse paving matroids if, in addition, $|H_i| = r$ holds for all $i \in [q]$. However, unlike the class of paving matroids, split matroids are closed not only under truncation and taking minors but also under duality [13]. The following result appeared in [1].

Lemma 7 (Bérczi, Király, Schwarcz, Yamaguchi and Yokoi)**.** *Let M be a rank-r elementary split matroid with a non-redundant representation $\mathcal{H} = \{H_1, \ldots, H_q\}$ and $r, r_1, \ldots, r_q$. Let F be a set of size r.*

(a) If F is H_i-tight for some index $i \in [q]$ then F is a basis of M.

(b) If F is both H_i-tight and H_j-tight for distinct $i, j \in [q]$ then $H_i \cap H_j \subseteq F \subseteq H_i \cup H_j$.

By Lemma 7(a), any set of size r that is tight with respect to one of the hyperedges is a basis. We will use this observation throughout without explicitly citing the lemma, to avoid repeatedly referring to part (a).

3 Proof of Theorem 4

Proof of Theorem 4. Throughout the proof, we use the following notational convention: given an ordered sequence $X_1, \ldots, X_k$ of sets or elements $x^1, \ldots, x^k$, indices are meant cyclically, meaning that $X_{k+1} = X_1$, $x^{k+1} = x^1$, $X_0 = X_k$ and $x^0 = x^k$. In addition, we interpret the set $\{x_i, \ldots, x_j\}$ to be empty when $i > j$.

The theorem clearly holds if $k = 1$, while the case when $k = 2$ was proved in [4]. Therefore, we assume that $k \geqslant 3$. Let $M = (S, \mathcal{B})$ be a split matroid and $S = B_1 \cup \cdots \cup B_k$ be a partition of its ground set into k pairwise disjoint bases. First we show that it suffices to consider connected split matroids. To see this, let $M_1 = (S_1, \mathcal{B}_1), \ldots, M_t = (S_t, \mathcal{B}_t)$ be the connected components of M, where $|S_j| = n_j$ and the rank of M_j is r_j for $j \in [t]$. For all $i \in [k]$ and $j \in [t]$, let $B_i^j := B_i \cap S_j$. Then, $S_j = B_1^j \cup \cdots \cup B_k^j$ is a decomposition of S_j into pairwise disjoint bases of M_j. Let $S_j = \{s_1^j, \ldots, s_{n_j}^j\}$ be a cyclic ordering of M_j in which the elements of B_i^j form the interval $I_i^j := \{s^j_{(i-1)\cdot r_j+1}, \ldots, s^j_{i\cdot r_j}\}$ for each $i \in [k]$. Then,

$$S = \{I_1^1, I_1^2, \ldots, I_1^t, I_2^1, I_2^2, \ldots, I_2^t, \ldots, I_k^1, I_k^2, \ldots, I_k^t\}$$

is a cyclic ordering of M in which B_i forms an interval for each $i \in [k]$. Since Conjecture 3 clearly holds for uniform matroids, the combination of the above observation and Lemma 6 allows us to assume that M is a rank-r elementary split matroid, defined by a non-redundant representation $\mathcal{H}$.

The high-level idea of the algorithm is as follows. We build up the orderings of the elements of the bases simultaneously in phases. At the beginning of the j-th phase, the first $(j-1)$ elements in each of the bases are ordered and the goal is to find the j-th element for all of them. We denote the first $(j-1)$ elements that we have already ordered in the i-th basis by $(b_1^i, \dots, b_{j-1}^i)$. The elements that are not yet ordered will be referred to as *remaining elements* in B_i and their set is denoted by C_i, that is, $C_i = B_i - \{b_1^i, \dots, b_{j-1}^i\}$. The goal is to choose b_j^i in such a way that $(C_i - b_j^i) \cup (b_1^{i+1}, \dots, b_j^{i+1})$ forms a basis for all $i \in [k]$; we call such a choice $(b_j^1, \dots, b_j^k)$ *valid*. Note that the condition is satisfied in the beginning as it simply requires $C_i = B_i$ to be a basis for each $i \in [k]$. If valid choices exist up to the r-th phase, then we get a cyclic ordering of the matroid with the desired properties simply by putting the ordered bases after each other. However, if the next elements cannot be chosen while satisfying the above constraints, we will slightly modify the order of the first $(j-1)$ elements to allow further steps.

Now we turn to the detailed description of the proof. For ease of discussion, we present it as an indirect proof; however, it implicitly implies an algorithm as described above. Let $j \in [r+1]$ be maximal with respect to the property that, for all $i \in [k]$, there exist $b_1^i, \dots, b_{j-1}^i \in B_i$ such that

$$(b_\ell^i, \dots, b_{j-1}^i) \cup C_i \cup (b_1^{i+1}, \dots, b_{\ell-1}^{i+1}) \text{ forms a basis for all } i \in [k], \ell \in [j], \qquad (\star)$$

where $C_i = B_i - \{b_1^i, \dots, b_{j-1}^i\}$. If $j = r+1$ then we are done. Therefore, suppose that $j \leqslant r$. In particular, this means that there is no valid choice of j-th elements in the bases. Let $R_i := C_i \cup \{b_1^{i+1}, \dots, b_{j-1}^{i+1}\}$ for all $i \in [k]$. Then, R_i is a basis by applying $(\star)$ for $\ell = j$.

Claim 8. *For all $i \in [k]$, there exist distinct elements $p_i, q_i \in C_i$ and a hyperedge H_i with value r_i satisfying the following:*

(a) $p_k \in H_k - H_1$ and $p_i \in H_i \cap H_{i+1}$ for all $i \in [k-1]$,

(b) $q_k \notin H_k$ and $q_i \notin H_i \cup H_{i+1}$ for all $i \in [k-1]$,

(c) R_{i-1} is H_i-tight for all $i \in [k]$.

Proof. Let $p_1 \in C_1$ be an arbitrary element. By the basis exchange property for R_1 and B_2, there exists an element $p_2 \in C_2$ such that $R_1 - p_1 + p_2$ forms a basis. By the repeated application of this argument we get $p_i \in C_i$ such that $R_i - p_i + p_{i+1}$ forms a basis for all $i \in [k-1]$.

If $R_k - p_k + p_1$ forms a basis, then $(p_1, \dots, p_k)$ is a valid choice, contradicting the maximality of j. Therefore, there exists a hyperedge H_1 with value r_1 such that $|H_1 \cap (R_k - p_k + p_1)| > r_1$. Since R_k is a basis, we conclude that R_k is H_1-tight, $p_k \notin H_1$, $p_1 \in H_1$ and $|H_1 \cap (R_k - p_k + p_1)| = r_1 + 1$. By the basis exchange property, there exists an element $q_1 \in C_1 - p_1$ such that $R_k - p_k + q_1$ forms a basis, implying $q_1 \notin H_1$. As the choice $(q_1, p_2, \dots, p_k)$ cannot be valid, there exists a hyperedge H_2 with value r_2 such that $|H_2 \cap (R_1 - q_1 + p_2)| > r_2$. Since R_1 and $R_1 - p_1 + p_2$ are both bases, we conclude that $|H_2 \cap (R_1 - q_1 + p_2)| = r_2 + 1$, R_1 is H_2-tight, $p_1 \in H_2$, $p_2 \in H_2$, and $q_1 \notin H_2$.

From this point, we proceed for each $2 \leqslant i \leqslant k$ in increasing order. Assume that we already know that there exists a hyperedge H_i and an element $q_{i-1} \in C_{i-1} - p_{i-1}$ such that R_{i-1} is H_i-tight, $p_{i-1} \in H_i$, $p_i \in H_i$, $q_{i-1} \notin H_{i-1}$ and $q_{i-1} \notin H_i$ – this holds for $i = 2$ by the above. By the basis exchange property, there exists an element $q_i \in C_i - p_i$ such that $R_{i-1} - q_{i-1} + q_i$ forms a basis, implying $q_i \notin H_i$. Assume further that $i \leqslant k-1$.

As the choice $(q_1, \dots, q_i, p_{i+1}, \dots, p_k)$ cannot be valid, there exists a hyperedge H_{i+1} with value r_{i+1} such that $|H_{i+1} \cap (R_i - q_i + p_{i+1})| > r_{i+1}$. Since R_i and $R_i - p_i + p_{i+1}$ are both bases, we conclude that $|H_{i+1} \cap (R_i - q_i + p_{i+1})| = r_i + 1$, R_i is H_{i+1}-tight, $p_i \in H_{i+1}$, $p_{i+1} \in H_{i+1}$, and $q_i \notin H_{i+1}$.

Therefore, we get elements $p_1, \dots, p_k, q_1, \dots, q_k$ and hyperedges $H_1, \dots, H_k$ with values $r_1, \dots, r_k$ satisfying conditions (a)–(c) of the claim. □

It is worth noting that the hyperedges $H_1, \dots, H_k$ provided by the claim are not necessarily distinct. We give an analogous claim where the roles of $p_1, \dots, p_k$ and $q_1, \dots, q_k$ are interchanged. The proof follows the same reasoning as in Claim 8; however, we include it here for completeness.

Claim 9. *For all $i \in [k]$, there exist a hyperedge H'_i with value r'_i satisfying the following:*

(a) $p_k \in H'_1 - H'_k$ and $p_i \notin H'_i \cup H'_{i+1}$ for all $i \in [k-1]$,

(b) $q_k \in (H_1 \cap H'_k) - H'_1$ and $q_i \in H'_i \cap H'_{i+1}$ for all $i \in [k-1]$,

(c) R_{i-1} is H'_i-tight for all $i \in [k]$,

(d) $H_i \cap H'_i \subseteq R_{i-1} \subseteq H_i \cup H'_i$ for all $i \in [k]$.

Proof. By Claim 8, $R_i - q_i + q_{i+1}$ is H_{i+1}-tight and hence a basis for $i \in [k-1]$. If $R_k - q_k + q_1$ forms a basis, then $(q_1, \dots, q_k)$ is a valid choice, contradicting the maximality of j. Therefore, there exists a hyperedge H'_1 with value r'_1 such that $q_k \notin H'_1$, $q_1 \in H'_1$ and R_k is H'_1-tight. Since $q_1 \in H'_1$ but $q_1 \notin H_1$, the hyperedges H_1 and H'_1 are distinct. Since R_k is both H_1- and H'_1-tight, Lemma 7(b) implies that $H_1 \cap H'_1 \subseteq R_k \subseteq H_1 \cup H'_1$, thus $q_k \in H_1$, $p_k \in H'_1$ and $p_1 \notin H'_1$.

By Claim 8, R_k is H_1-tight, $p_1 \in H_1$ and $q_k \in H_1$, thus $R_k - q_k + p_1$ is also H_1-tight and so a basis. Moreover, $R_i - p_i + p_{i+1}$ and $R_i - q_i + q_{i+1}$ are H_{i+1}-tight for $i \in [k-1]$. Fix any index $i \in [k-1]$. As the choice $(p_1, \dots, p_i, q_{i+1}, \dots, q_k)$ cannot be valid, there exists a hyperedge H'_{i+1} with value r'_{i+1} such that $p_i \notin H'_{i+1}$, $q_{i+1} \in H'_{i+1}$, and R_i is H'_{i+1}-tight. Since $q_{i+1} \in H'_{i+1}$ but $q_{i+1} \notin H_{i+1}$, the hyperedges H_{i+1} and H'_{i+1} are distinct. As R_i is both H_{i+1}- and H'_{i+1}-tight, Lemma 7(b) gives $H_{i+1} \cap H'_{i+1} \subseteq R_i \subseteq H_{i+1} \cup H'_{i+1}$. In particular, $q_i \in H'_{i+1}$ and $p_{i+1} \notin H'_{i+1}$.

Thus we get hyperedges $H'_1, \dots, H'_k$ with values $r'_1, \dots, r'_k$ satisfying conditions (a)–(d) of the claim. □

Again, let us note that the hyperedges $H'_1, \dots, H'_k$ provided by the claim are not necessarily distinct.

Claim 10. *For all $i \in [k-1]$ and $x \in C_i$, either $x \in (H_i \cap H_{i+1}) - (H'_i \cup H'_{i+1})$ or $x \in (H'_i \cap H'_{i+1}) - (H_i \cup H_{i+1})$. For $x \in C_k$, either $x \in (H_k \cap H'_1) - (H'_k \cup H_1)$ or $x \in (H'_k \cap H_1) - (H_k \cup H'_1)$.*

Proof. Although the proofs are similar, we treat the cases $i \in [k-1]$ and $i = k$ separately, as the reasoning differs slightly.

Case 1. $i \in [k-1]$.

Consider any element $x \in C_i$ for some $i \in [k-1]$. By Claim 9, we know that $x \in H_{i+1} \cup H'_{i+1}$. We distinguish two cases based on which set x belongs to.

Case 1.1. $x \in H_{i+1}$.

If $x \notin H_i$, then $(q_1, \dots, q_{i-1}, x, p_{i+1}, \dots, p_k)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because, by Claim 8, R_ℓ is $H_{\ell+1}$-tight for all $\ell \in [k]$, so if $i \geqslant 2$ then $R_{i-1} - q_{i-1} + x$ and $R_i - x + p_{i+1}$ are H_i- and H_{i+1}-tight, respectively, while if $i = 1$ then $R_k - p_k + x$ and $R_1 - x + p_2$ are H_1- and H_2-tight, respectively, since $p_k \notin H_1$, again by Claim 8. Thus we have $x \in H_i$.

By Claim 9, $H_i \cap H'_i \subseteq R_{i-1}$. As $x \notin R_{i-1}$, we have $x \notin H'_i$. If $x \in H'_{i+1}$, then $(p_1, \dots p_{i-1}, x, q_{i+1}, \dots, q_k)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because, by Claim 9, R_ℓ is $H'_{\ell+1}$-tight for all $\ell \in [k]$, so if $i \geqslant 2$ then $R_{i-1} - p_{i-1} + x$ and $R_i - x + q_{i+1}$ are H'_i- and H'_{i+1}-tight, respectively, while if $i = 1$ then $R_k - q_k + x$ and $R_1 - x + q_2$ are H'_1- and H'_2-tight, respectively, since $q_k \notin H'_1$, again by Claim 9. Thus we have $x \notin H'_{i+1}$.

Case 1.2. $x \in H'_{i+1}$.

If $x \notin H'_i$, then $(p_1, \dots, p_{i-1}, x, q_{i+1}, \dots, q_k)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because, by Claim 9, R_ℓ is $H'_{\ell+1}$-tight for all $\ell \in [k]$, so if $i \geqslant 2$ then $R_{i-1} - p_{i-1} + x$ and $R_i - x + q_{i+1}$ are H'_i- and H'_{i+1}-tight, respectively, while if $i = 1$ then $R_k - q_k + x$ and $R_1 - x + q_2$ are H'_1- and H'_2-tight, respectively, since $q_k \notin H'_1$,again by Claim 9. Thus we have $x \in H'_i$.

By Claim 9, $H_i \cap H'_i \subseteq R_{i-1}$. As $x \notin R_{i-1}$, we have $x \notin H_i$. If $x \in H_{i+1}$, then $(q_1, \dots, q_{i-1}, x, p_{i+1}, \dots, p_k)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because, by Claim 8, R_ℓ is $H_{\ell+1}$-tight for all $\ell \in [k]$, so if $i \geqslant 2$ then $R_{i-1} - q_{i-1} + x$ and $R_i - x + p_{i+1}$ are H_i- and H_{i+1}-tight, respectively, while if $i = 1$ then $R_k - p_k + x$ and $R_1 - x + p_2$ are H_1- and H_2-tight, respectively, since $p_k \notin H_1$, again by Claim 8. Thus we have $x \notin H_{i+1}$.

At a high level, the statement for $x \in C_k$ follows by replacing H_1 with H'_1 and H'_1 with H_1 in the above argument, while using the appropriate notion of tightness throughout.

Case 2. $i = k$.

Consider any element $x \in C_k$. By Claim 9, we know that $x \in H'_1 \cup H_1$. We distinguish two cases based on which set x belongs to.

Case 2.1. $x \in H'_1$.

If $x \notin H_k$, then $(q_1, \dots, q_{k-1}, x)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because, by Claims 8 and 9, R_{k-1} and R_k are H_k- and H'_1-tight, respectively, so $R_{k-1} - q_{k-1} + x$ and $R_k - x + q_1$ are H_k- and H'_1-tight, respectively. Thus we have $x \in H_k$.

By Claim 9, $H_k \cap H'_k \subseteq R_{k-1}$. As $x \notin R_{k-1}$, we have $x \notin H'_k$. If $x \in H_1$, then $(p_1, \dots p_{k-1}, x)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because, by Claims 9 and 8, R_{k-1} and R_k are H'_k- and H_1-tight, so $R_{k-1} - p_{k-1} + x$ and $R_k - x + p_1$ are H'_k- and H_1-tight, respectively. Thus we have $x \notin H_1$.

Case 2.2. $x \in H_1$.

If $x \notin H'_k$, then $(p_1, \dots, p_{k-1}, x)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because, by Claims 9 and 8, R_{k-1} and R_k are H'_k- and H_1-tight, respectively, so $R_{k-1} - p_{k-1} + x$ and $R_k - x + p_1$ are H'_k- and H_1-tight, respectively. Thus we have $x \in H'_k$.

By Claim 9, $H_k \cap H'_k \subseteq R_{k-1}$. As $x \notin R_{k-1}$, we have $x \notin H_k$. If $x \in H'_1$, then $(q_1, \dots, q_{k-1}, x)$ is a valid choice for the set of j-th elements, contradicting the maximality of j. This is because, by Claims 8 and 9, R_{k-1} and R_k are H_k- and H'_1-tight for all $\ell \in [k]$, so $R_{k-1} - q_{k-1} + x$ and $R_k - x + q_1$ are H_k- and H'_1-tight, respectively. Thus we have $x \notin H'_1$. $\square$

For all $i \in [k-1]$, we define $\widehat{C}_i \coloneqq \{x \in C_i \mid x \in (H_i \cap H_{i+1}) - (H'_i \cup H'_{i+1})\}$ and $\widehat{C}'_i \coloneqq \{x \in C_i \mid x \in (H'_i \cap H'_{i+1}) - (H_i \cup H_{i+1})\}$. We further set $\widehat{C}_k \coloneqq \{x \in C_k \mid x \in (H_k \cap H'_1) - (H'_k \cup H_1)\}$ and $\widehat{C}'_k \coloneqq \{x \in C_k \mid x \in (H'_k \cap H_1) - (H_k \cup H'_1)\}$. By Claim 10, $C_i = \widehat{C}_i \cup \widehat{C}'_i$ and $\widehat{C}_i \cap \widehat{C}'_i = \emptyset$ holds for each $i \in [k]$.

Claim 11. *There exists an $s \in \mathbb{Z}_+$ such that $|\widehat{C}_i| = |\widehat{C}'_i| = s$ for each $i \in [k]$.*

Proof. As B_2 is a basis, we have $|B_2 \cap H_2| \leqslant r_2$. Since $|R_1 \cap H_2| = r_2$, we get $|\widehat{C}_2| = |C_2 \cap H_2| \leqslant |C_1 \cap H_2| = |\widehat{C}_1|$. A repeated application of the same argument leads to $|\widehat{C}_1| \geqslant |\widehat{C}_2| \geqslant \dots \geqslant |\widehat{C}_k|$. Similarly, as B_1 is a basis, we have $|B_1 \cap H'_1| \leqslant r'_1$. Since $|R_k \cap H'_1| = r'_1$, we get $|\widehat{C}'_1| = |C_1 \cap H'_1| \leqslant |C_k \cap H'_1| = |\widehat{C}_k|$. A repeated application of the same argument leads to $|\widehat{C}_k| \geqslant |\widehat{C}'_1| \geqslant |\widehat{C}'_2| \dots \geqslant |\widehat{C}'_k|$. Finally, as B_1 is a basis, we have $|B_1 \cap H_1| \leqslant r_1$. Since $|R_k \cap H_1| = r_1$, we get $|\widehat{C}_1| = |C_1 \cap H_1| \leqslant |C_k \cap H_1| = |\widehat{C}'_k|$.

Concluding the above, we get $|\widehat{C}_1| \geqslant |\widehat{C}_2| \geqslant \dots \geqslant |\widehat{C}_k| \geqslant |\widehat{C}'_1| \geqslant |\widehat{C}'_2| \geqslant \dots \geqslant |\widehat{C}'_k| \geqslant |\widehat{C}_1|$, finishing the proof of the claim. $\square$

Claim 12. *For each $i \in [k]$, B_i is both H_i- and H'_i-tight.*

Proof. Recall that R_{i-1} is H_i-tight by Claim 8. Therefore, by Claim 11, we have

$$\begin{aligned}
|B_i \cap H_i| &= |(B_i \cap R_{i-1}) \cap H_i| + |(B_i - R_{i-1}) \cap H_i| \\
&= |(B_i \cap R_{i-1}) \cap H_i| + |\widehat{C}_i| \\
&= |(B_i \cap R_{i-1}) \cap H_i| + |\widehat{C}_{i-1}| \\
&= |R_{i-1} \cap H_i| \\
&= r_i.
\end{aligned}$$

Similarly, recall that R_{i-1} is H'_i-tight by Claim 9. Therefore, by Claim 11, we have

$$\begin{aligned}
|B_i \cap H'_i| &= |(B_i \cap R_{i-1}) \cap H'_i| + |(B_i - R_{i-1}) \cap H'_i| \\
&= |(B_i \cap R_{i-1}) \cap H'_i| + |\widehat{C}'_i| \\
&= |(B_i \cap R_{i-1}) \cap H'_i| + |\widehat{C}'_{i-1}| \\
&= |R_{i-1} \cap H'_i| \\
&= r'_i.
\end{aligned}$$

This concludes the proof of the claim. $\square$

By Claim 8 we know that $p_k \in H_k$ and $p_k \notin H_1$ therefore $H_1 \neq H_k$. This means that there must exist consecutive indices p and $p+1$, where $1 \leqslant p \leqslant k-1$, such that $H_p \neq H_{p+1}$. By definition, we know that $|H_p \cap H_{p+1}| \geqslant |\widehat{C}_p| = s$. Assume first that $p \geqslant 2$. Since R_{p-1} is H_p-tight by Claim 8 and $\widehat{C}'_{p-1} \cap H_p = \emptyset$, we get $|R_{p-1}| = r \geqslant r_p + s$. Since R_p is H_{p+1}-tight by Claim 8 and $\widehat{C}'_p \cap H_{p+1} = \emptyset$, we get $|R_p| = r \geqslant r_{p+1} + s$. Consider now the case $p = 1$. Since R_k is H_1-tight by Claim 8 and $\widehat{C}_k \cap H_1 = \emptyset$, we get $|R_k| = r \geqslant r_1 + s$. Since R_1 is H_2-tight by Claim 8 and $\widehat{C}'_1 \cap H_2 = \emptyset$, we get $|R_1| = r \geqslant r_2 + s$.

These observations give

$$s \leqslant |H_p \cap H_{p+1}| \leqslant r_p + r_{p+1} - r \leqslant (r-s) + (r-s) - r = r - 2s,$$

thus $s \leqslant r/3$. As $r = |B_i| = j - 1 + |C_i| = j - 1 + |\widehat{C}_i| + |\widehat{C}'_i| = j - 1 + 2s \leqslant j - 1 + 2r/3$, we get $j - 1 \geqslant r/3$. In particular, this means that at least one element is already ordered in each of $B_1, \ldots, B_k$.

Now we turn our attention to the elements that have been already ordered. Consider the elements b^i_t for all $i \in [k]$, $t \in [j-1]$. Our goal is to show that the set of hyperedges containing these elements also have a specific structure.

Claim 13. *We have the following.*

(a) For all $t \in [j-1]$, $b^i_t \in (H_i \triangle H'_i) \cap (H_{i+1} \triangle H'_{i+1})$ for all $i \in [k]$.

(b) For all $t \in [j-1]$, either $\{b^i_t, b^{i+1}_t\} \subseteq H_{i+1}$ or $\{b^i_t, b^{i+1}_t\} \subseteq H'_{i+1}$.

(c) For all $t \in [j-1]$, the set $\{b^i_t, \ldots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \ldots, b^{i+1}_{t-1}\}$ is both H_{i+1}- and H'_{i+1}-tight.

Proof. Most of the proof is verifying (a) for all $t \in [j-1]$ in a decreasing order; once this is established, (b) and (c) follow easily. Assume that the statement holds for all indices in $[j-1]$ strictly greater than t. When $t = j-1$, this assumption is vacuous since no such indices exist. We first prove that (a) holds for t. Consider an $i \in [k]$. As $b^i_t \in B_i$ and B_i is H_i- and H'_i-tight by Claim 12, we get that $b^i_t \in H_i \cup H'_i$ by Lemma 7(b).

We first prove that $b^i_t \in (H_i \triangle H'_i)$. Suppose indirectly that $b^i_t \in H_i \cap H'_i$. As $b^i_t \notin R_i$, it is contained in at most one of H_{i+1} and H'_{i+1} by Claim 9 – we consider those scenarios separately.

Case 1. $b^i_t \notin H_{i+1}$.

We distinguish two cases based on the value of i.

Case 1.1. $i \in [k-1]$.

Swap b^i_t with q_i in the ordering of B_i. We claim that $(\star)$ remains true. This is because $q_i \notin H_{i+1}$ by Claim 8 and $\{b^i_m, \ldots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \ldots, b^{i+1}_{m-1}\}$ is H_{i+1}-tight for all $m > t$ by assumption, thus we get that $\{b^i_m, \ldots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \ldots, b^{i+1}_{m-1}\} - q_i + b^i_t$ remains H_{i+1}-tight for all $m > t$. Similarly, $q_i \in H'_i$ by Claim 9 and $\{b^{i-1}_m, \ldots, b^{i-1}_{j-1}\} \cup C_{i-1} \cup$

$\{b^i_1, \dots, b^i_{m-1}\}$ is H'_i-tight for all $m > t$ by assumption, thus we get that $\{b^{i-1}_m, \dots, b^{i-1}_{j-1}\} \cup C_{i-1} \cup \{b^i_1, \dots, b^i_{m-1}\} - b^i_t + q_i$ remains H'_i-tight for all $m > t$.

After the modification, $(q_1, \dots, q_{i-1}, p_i, \dots, p_k)$ becomes a valid choice for the j-th phase. To see this, assume first that $i \geqslant 2$. Then, by Claim 8, we have $q_{i-1} \notin H_i$, $p_i \in H_i$ and $q_i \notin H_i$, so $R_{i-1} - q_{i-1} - b^i_t + p_i + q_i$ remains H_i-tight. If $i = 1$ then, by Claim 8, we have $p_k \notin H_1$, $p_1 \in H_1$ and $q_1 \notin H_1$, so $R_k - p_k - b^1_t + p_1 + q_1$ is H_1-tight. Also by Claim 8, we have $p_{i+1} \in H_{i+1}$, $p_i \in H_{i+1}$ and $q_i \notin H_{i+1}$, so $R_i - p_i - q_i + b^i_t + p_{i+1}$ remains H_2-tight. This contradicts the maximal choice of j.

Case 1.2. $i = k$.

Recall that we are in the case when $b^k_t \notin H_1$. Swap b^k_t with p_k in the ordering of B_k. We claim that ($\star$) remains true. This is because $p_k \notin H_1$ by Claim 8 and $\{b^k_m, \dots, b^k_{j-1}\} \cup C_k \cup \{b^1_1, \dots, b^1_{m-1}\}$ is H_1-tight for all $m > t$ by assumption, thus we get that $\{b^k_m, \dots, b^k_{j-1}\} \cup C_k \cup \{b^1_1, \dots, b^1_{m-1}\} - p_k + b^k_t$ remains H_1-tight for all $m > t$. Similarly, $p_k \in H_k$ by Claim 8 and $\{b^{k-1}_m, \dots, b^{k-1}_{j-1}\} \cup C_{k-1} \cup \{b^k_1, \dots, b^k_{m-1}\}$ is H_k-tight for all $m > t$ by assumption, thus we get that $\{b^{k-1}_m, \dots, b^{k-1}_{j-1}\} \cup C_{k-1} \cup \{b^k_1, \dots, b^k_{m-1}\} - b^k_t + p_k$ remains H_k-tight for all $m > t$.

After the modification, $(p_1, \dots, p_{k-1}, q_k)$ becomes a valid choice for the j-th phase. This is because, by Claim 9, we have $p_{k-1} \notin H'_k$, $p_k \notin H'_k$ and $q_k \in H'_k$, so $R_{k-1} - p_{k-1} - b^k_t + p_k + q_k$ remains H'_k-tight. By Claims 8 and 9, we have $p_1 \in H_1$, $p_k \notin H_1$ and $q_k \in H_1$, so $R_k - p_k - q_k + b^k_t + p_1$ remains H_1-tight. This contradicts the maximal choice of j.

Case 2. $b^i_t \notin H'_{i+1}$.

We distinguish two cases based on the value of i

Case 2.1. $i \in [k-1]$.

Swap b^i_t with p_i in the ordering of B_i. We claim that ($\star$) remains true. This is because $p_i \notin H'_{i+1}$ by Claim 9 and $\{b^i_m, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_{m-1}\}$ is H'_{i+1}-tight for all $m > t$ by assumption, thus we get that $\{b^i_m, \dots, b^i_{j-1}\} \cup C_i \cup \{b^{i+1}_1, \dots, b^{i+1}_{m-1}\} - p_i + b^i_t$ remains H'_{i+1}-tight for all $m > t$. Similarly, $p_i \in H_i$ by Claim 8 and $\{b^{i-1}_m, \dots, b^{i-1}_{j-1}\} \cup C_{i-1} \cup \{b^i_1, \dots, b^i_{m-1}\}$ is H_i-tight for all $m > t$ by assumption, thus we get that $\{b^{i-1}_m, \dots, b^{i-1}_{j-1}\} \cup C_{i-1} \cup \{b^i_1, \dots, b^i_{m-1}\} - b^i_t + p_i$ remains H_i-tight for all $m > t$.

After the modification, $(p_1, \dots, p_{i-1}, q_i, \dots, q_k)$ becomes a valid choice for the j-th phase. To see this, assume first that $i \geqslant 2$. Then, by Claim 9, we have $p_{i-1} \notin H'_i$, $p_i \notin H'_i$ and $q_i \in H'_i$, so $R_{i-1} - p_{i-1} - b^i_t + p_i + q_i$ remains H'_i-tight. If $i = 1$ then, by Claim 9, we have $q_k \notin H'_1$, $p_1 \notin H'_1$ and $q_1 \in H'_1$, so $R_k - q_k - b^1_t + p_1 + q_1$ remains H'_1-tight. Also by Claim 9, we have $q_{i+1} \in H'_{i+1}$, $p_i \notin H'_{i+1}$ and $q_i \in H'_{i+1}$, so $R_i - p_i - q_i + b^i_t + q_{i+1}$ remains H'_{i+1}-tight. This contradicts the maximal choice of j.

Case 2.2. $i = k$.

Recall that we are in the case when $b^k_t \notin H'_1$. Swap b^k_t with q_k in the ordering of B_k. We claim that ($\star$) remains true. This is because $q_k \notin H'_1$ by Claim 9 and $\{b^k_m, \dots, b^k_{j-1}\} \cup C_k \cup \{b^1_1, \dots, b^1_{m-1}\}$ is H'_1-tight for all $m > t$ by assumption, thus we get that $\{b^k_m, \dots, b^k_{j-1}\} \cup C_k \cup \{b^1_1, \dots, b^1_{m-1}\} - q_k + b^k_t$ remains H'_1-tight for all $m > t$. Similarly, $q_k \in H'_k$ by Claim 9 and $\{b^{k-1}_m, \dots, b^{k-1}_{j-1}\} \cup C_{k-1} \cup \{b^k_1, \dots, b^k_{m-1}\}$ is H'_k-tight for all $m > t$ by assumption, thus we get that $\{b_m^{k-1}, \ldots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \ldots, b_{m-1}^k\} - b_t^k + q_k$ remains H_k'-tight for all $m > t$.

After the modification, $(q_1, \ldots, q_{k-1}, p_k)$ becomes a valid choice for the j-th phase. This is because, by Claim 8, we have $q_{k-1} \notin H_k$, $p_k \in H_k$ and $q_k \notin H_k$, so $R_{k-1} - q_{k-1} - b_t^k + p_k + q_k$ remains H_k-tight. Also by Claim 9, we have $q_1 \in H_1'$, $p_k \in H_1'$ and $q_k \notin H_1'$, so $R_k - p_k - q_k + b_t^k + q_1$ remains H_1'-tight. This contradicts the maximal choice of j.

Summarizing the above, we get $b_t^i \in (H_i \triangle H_i')$. We now prove that $b_t^i \in (H_{i+1} \triangle H_{i+1}')$. We know that $b_t^i \notin (H_{i+1} \cap H_{i+1}')$, so it suffices to show that $b_t^i \in (H_{i+1} \cup H_{i+1}')$. Suppose indirectly that $b_t^i \notin (H_{i+1} \cup H_{i+1}')$. We consider two cases based on whether $b_t^i \in H_i - (H_i' \cup H_{i+1} \cup H_{i+1}')$ or $b_t^i \in H_i' - (H_i \cup H_{i+1} \cup H_{i+1}')$.

Case 1. $b_t^i \in H_i - (H_i' \cup H_{i+1} \cup H_{i+1}')$.

We distinguish two cases based on the value of i.

Case 1.1. $i \in [k-1]$.

Swap b_t^i with p_i in the ordering of B_i. We claim that $(\star)$ remains true. This is because $p_i \notin H_{i+1}'$ by Claim 9 and $\{b_m^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_{m-1}^{i+1}\}$ is H_{i+1}'-tight for all $m > t$ by assumption, thus we get that $\{b_m^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_{m-1}^{i+1}\} - p_i + b_t^i$ remains H_{i+1}'-tight for all $m > t$. Similarly, $p_i \in H_i$ by Claim 8 and $\{b_m^{i-1}, \ldots, b_{j-1}^{i-1}\} \cup C_{i-1} \cup \{b_1^i, \ldots, b_{m-1}^i\}$ is H_i-tight for all $m > t$ by assumption, thus we get that $\{b_m^{i-1}, \ldots, b_{j-1}^{i-1}\} \cup C_{i-1} \cup \{b_1^i, \ldots, b_{m-1}^i\} - b_t^i + p_i$ remains H_i-tight for all $m > t$.

After the modification, $(q_1, \ldots, q_i, p_{i+1}, \ldots, p_k)$ becomes a valid choice for the j-th phase. To see this, assume first that $i \geqslant 2$. Then, by Claim 9, we have $q_{i-1} \in H_i'$, $p_i \notin H_i'$ and $q_i \in H_i'$, so $R_{i-1} - q_{i-1} - b_t^i + p_i + q_i$ remains H_i'-tight. If $i = 1$ then, by Claim 9, we have $p_k \in H_1'$, $p_1 \notin H_1'$ and $q_1 \in H_1'$, so $R_k - p_k - b_t^1 + p_1 + q_1$ remains H_1'-tight. Also by Claim 8, we have $p_{i+1} \in H_{i+1}$, $p_i \in H_{i+1}$ and $q_i \notin H_{i+1}$, so $R_i - p_i - q_i + b_t^i + p_{i+1}$ remains H_{i+1}-tight. This contradicts the maximal choice of j.

Case 1.2. $i = k$.

Recall that we are in the case when $b_t^k \in H_k - (H_k' \cup H_1 \cup H_1')$. Swap b_t^k with p_k in the ordering of B_k. We claim that $(\star)$ remains true. This is because $p_k \notin H_1$ by Claim 8 and $\{b_m^k, \ldots, b_{j-1}^k\} \cup C_k \cup \{b_1^1, \ldots, b_{m-1}^1\}$ is H_1-tight for all $m > t$ by assumption, thus we get that $\{b_m^k, \ldots, b_{j-1}^k\} \cup C_k \cup \{b_1^1, \ldots, b_{m-1}^1\} - p_k + b_t^k$ remains H_1-tight for all $m > t$. Similarly, $p_k \in H_k$ by Claim 8 and $\{b_m^{k-1}, \ldots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \ldots, b_{m-1}^k\}$ is H_k-tight for all $m > t$ by assumption, thus we get that $\{b_m^{k-1}, \ldots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \ldots, b_{m-1}^k\} - b_t^k + p_k$ remains H_k-tight for all $m > t$.

After the modification, $(q_1, \ldots, q_k)$ becomes a valid choice for the j-th phase. This is because, by Claim 8, we have $q_{k-1} \notin H_k$, $p_k \in H_k$ and $q_k \notin H_k$, so $R_{k-1} - q_{k-1} - b_t^k + p_k + q_k$ remains H_k-tight. Also by Claim 9, $q_1 \in H_1'$, $q_k \notin H_1'$ and $p_k \in H_1'$, so $R_k - p_k - q_k + b_t^k + q_1$ remains H_1'-tight. This contradicts the maximal choice of j.

Case 2. $b_t^i \in H_i' - (H_i \cup H_{i+1} \cup H_{i+1}')$.

We distinguish two cases based on the value of i.

Case 2.1. $i \in [k-1]$.

Swap b_t^i with q_i in the ordering of B_i. We claim that ($\star$) remains true. This is because $q_i \notin H_{i+1}$ by Claim 8 and $\{b_m^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_{m-1}^{i+1}\}$ is H_{i+1}-tight for all $m > t$ by assumption, thus we get that $\{b_m^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_{m-1}^{i+1}\} - q_i + b_t^i$ remains H_{i+1}-tight for all $m > t$. Similarly, $q_i \in H_i'$ by Claim 9 and $\{b_m^{i-1}, \ldots, b_{j-1}^{i-1}\} \cup C_{i-1} \cup \{b_1^i, \ldots, b_{m-1}^i\}$ is H_i'-tight for all $m > t$ by assumption, thus we get that $\{b_m^{i-1}, \ldots, b_{j-1}^{i-1}\} \cup C_{i-1} \cup \{b_1^i, \ldots, b_{m-1}^i\} - b_t^i + q_i$ remains H_i'-tight for all $m > t$.

After the modification, $(p_1, \ldots, p_i, q_{i+1}, \ldots, q_k)$ becomes a valid choice for the j-th phase. To see this, assume first that $i \geqslant 2$. Then, by Claim 8, we have $p_{i-1} \in H_i$, $p_i \in H_i$ and $q_i \notin H_i$, so $R_{i-1} - p_{i-1} - b_t^i + p_i + q_i$ remains H_i-tight. If $i = 1$ then, by Claim 8 and 9, we have $q_k \in H_1$, $p_1 \in H_1$ and $q_1 \notin H_1$, so $R_k - q_k - b_t^1 + p_1 + q_1$ remains H_1-tight. Also by Claim 9, we have $q_{i+1} \in H_{i+1}'$, $p_i \notin H_{i+1}'$ and $q_i \in H_{i+1}'$, so $R_i - p_i - q_i + b_t^i + q_{i+1}$ remains H_{i+1}'-tight. This contradicts the maximal choice of j.

Case 2.2. $i = k$.

Recall that we are in the case when $b_t^k \in H_k' - (H_k \cup H_1 \cup H_1')$. Swap b_t^k with q_k in the ordering of B_k. We claim that ($\star$) remains true. This is because $q_k \notin H_1'$ by Claim 9 and $\{b_m^k, \ldots, b_{j-1}^k\} \cup C_k \cup \{b_1^1, \ldots, b_{m-1}^1\}$ is H_1'-tight for all $m > t$ by assumption, thus we get that $\{b_m^k, \ldots, b_{j-1}^k\} \cup C_k \cup \{b_1^1, \ldots, b_{m-1}^1\} - q_k + b_t^k$ remains H_1'-tight for all $m > t$. Similarly, $q_k \in H_k'$ by Claim 9 and $\{b_m^{k-1}, \ldots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \ldots, b_{m-1}^k\}$ is H_k'-tight for all $m > t$ by assumption, thus we get that $\{b_m^{k-1}, \ldots, b_{j-1}^{k-1}\} \cup C_{k-1} \cup \{b_1^k, \ldots, b_{m-1}^k\} - b_t^k + q_k$ remains H_k'-tight for all $m > t$.

After the modification, $(p_1, \ldots, p_k)$ becomes a valid choice for the j-th phase. This is because, by Claim 9, we have $p_{k-1} \notin H_k'$, $p_k \notin H_k'$ and $q_k \in H_k'$, so $R_{k-1} - p_{k-1} - b_t^k + p_k + q_k$ remains H_k'-tight. By Claims 8 and 9, $p_1 \in H_1$, $q_k \in H_1$ and $p_k \notin H_1$, so $R_k - p_k - q_k + b_t^k + p_1$ remains H_1-tight. This contradicts the maximal choice of j.

This finishes the proof of (a), that is, $b_t^i \in (H_i \triangle H_i') \cap (H_{i+1} \triangle H_{i+1}')$. To prove the remaining two properties, observe that $\{b_{t+1}^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_t^{i+1}\} - b_t^{i+1} + b_t^i$ is a basis by ($\star$). Note that $\{b_{t+1}^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_t^{i+1}\}$ is both H_{i+1}- and H_{i+1}'-tight, while $b_t^i, b_t^{i+1} \in (H_{i+1} \triangle H_{i+1}')$. Hence $\{b_t^i, b_t^{i+1}\} \subseteq H_{i+1}$ or $\{b_t^i, b_t^{i+1}\} \subseteq H_{i+1}'$, for otherwise the basis would intersect H_{i+1} or H_{i+1}' in too many elements. This implies

$$\begin{aligned} &|(\{b_{t+1}^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_t^{i+1}\}) \cap H_{i+1}| \\ &= |(\{b_{t+1}^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_t^{i+1}\} - b_t^{i+1} + b_t^i) \cap H_{i+1}| \end{aligned}$$

and

$$\begin{aligned} &|(\{b_{t+1}^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_t^{i+1}\}) \cap H_{i+1}'| \\ &= |(\{b_{t+1}^i, \ldots, b_{j-1}^i\} \cup C_i \cup \{b_1^{i+1}, \ldots, b_t^{i+1}\} - b_t^{i+1} + b_t^i) \cap H_{i+1}'|, \end{aligned}$$

which means that properties (b) and (c) hold as well. □

Claims 12 and 13 imply that B_i is tight with respect to H_i, H_i', H_{i+1} and H_{i+1}'. We know that $H_m \neq H_{m+1}$ for some $m < k$. Then, $B_m \subseteq H_m \cup H_{m+1}$ which implies $q_m \in B_m \subseteq H_m \cup H_{m+1}$. However, by Claim 8, $q_m \notin H_m \cup H_{m+1}$, a contradiction. This concludes the proof of the theorem. □

4 Further remarks and open problems

4.1 Comments on Conjecture 2

The most important result toward verifying Conjecture 2 is due to Van den Heuvel and Thomassé [18].

Theorem 14 (Van den Heuvel and Thomassé)**.** *Let $M = (S, \mathcal{B})$ be a loopless matroid with rank function $r\colon 2^S \to \mathbb{Z}_+$ and $|S| = n$, and let g denote the greatest common divisor of $r(S)$ and n. Then, there exists a partition $S = G_1 \cup \cdots \cup G_{n/g}$ into sets of size g such that $\bigcup_{t=0}^{r(S)/g-1} G_{i+t}$ is a basis for all $i \in [n/g]$ if and only if $r(S) \cdot |X| \leqslant n \cdot r(X)$ for $X \subseteq S$.*

In particular, Theorem 14 settles Conjecture 2 in the affirmative if $r(S)$ and n are coprimes. Therefore, to prove Conjecture 2, it would be enough to verify that, when M is uniformly dense, the elements inside each G_i admit an ordering that together induces a cyclic ordering of M. Unfortunately, such an approach cannot work as shown by the following example.

Example 15. Let $S = \{a_1, \ldots, a_{10}\}$ and consider the sparse paving matroid defined by the following hyperedges: $\{a_1, a_2, a_3, a_{10}\}$, $\{a_1, a_2, a_4, a_9\}$, $\{a_1, a_3, a_4, a_5\}$, $\{a_2, a_3, a_4, a_6\}$, $\{a_3, a_5, a_6, a_7\}$, $\{a_4, a_5, a_6, a_8\}$, $\{a_5, a_7, a_8, a_9\}$, $\{a_6, a_7, a_8, a_{10}\}$,$\{a_1, a_7, a_9, a_{10}\}$, $\{a_2, a_8, a_9, a_{10}\}$, with the value of r being 4; see Section 2 for the definition. If $G_i = \{a_{2i-1}, a_{2i}\}$ for all $i \in [5]$, then it is not difficult to check that $G_i \cup G_{i+1}$ is a basis for all $i \in [10]$.

However, we claim that the pairs in the sets G_i cannot be ordered in such a way that we get a cyclic ordering of the matroid M. To see this, observe that each G_i is contained in two of the hyperedges, which excludes two of the four possible orderings of the neighboring groups G_{i-1} and G_{i+1}. Due to the exclusion of these ordering possibilities, it is not difficult to verify that no suitable ordering exists.

4.2 Exchange distance of basis sequences

Note that Gabow's conjecture can be interpreted as follows: for any two disjoint bases B_1 and B_2 of a matroid M of rank r, there is a sequence of r symmetric exchanges that transforms the pair (B_1, B_2) into (B_2, B_1). The closely related problem of transforming a sequence $(B_1, \ldots, B_k)$ of bases into another $(B'_1, \ldots, B'_k)$ was proposed by White [19]. Let $(B_1, \ldots, B_k)$ be a sequence of k bases of a matroid M, and assume that there exist $e \in B_i$, $f \in B_j$ for some $1 \leqslant i < j \leqslant k$ such that both $B_i - e + f$ and $B_j - f + e$ are bases. Then we say that the sequence $(B_1, \ldots, B_{i-1}, B_i - e + f, B_{i+1}, \ldots, B_{j-1}, B_j - f + e, B_{j+1}, \ldots, B_k)$ is obtained from the original one by a *symmetric exchange*. Accordingly, two sequences of bases are called *equivalent* if one can be obtained from the other by a composition of symmetric exchanges. White studied the following question: what is the characterization of two sequences of bases being equivalent?

There is an easy necessary condition. Two sequences $(B_1, \ldots, B_k)$ and $(B'_1, \ldots, B'_k)$ are called *compatible* if the union of the B_is as a multiset coincides with the union of the B'_is as a multiset. Compatibility is obviously a necessary condition for two sequences being equivalent, and White conjectured that it is also sufficient.

Conjecture 16 (White)**.** Two sequences of k bases are equivalent if and only if they are compatible.

In this context, Gabow's conjecture would verify White's conjecture for two pairs of bases of the form (B_1, B_2) and (B_2, B_1). Note that, however, the conjecture says nothing on the minimum number of exchanges needed to transform one of the pairs into the other. As a common generalization of Gabow's conjecture and the special case of White's conjecture when $k = 2$, Hamidoune [6] proposed an optimization variant.

Conjecture 17 (Hamidoune)**.** Let (B_1, B_2) and (B_1', B_2') be compatible basis pairs of a rank-r matroid $M = (S, \mathcal{B})$. Then, (B_1, B_2) can be transformed into (B_1', B_2') by using at most r symmetric exchanges.

In [3], Bérczi, Mátravölgyi and Schwarcz formulated a weighted extension of Hamidoune's conjecture. Let $M = (S, \mathcal{B})$ be a matroid and $w\colon S \to \mathbb{R}_+$ be a weight function on the elements of the ground set S. Given a pair (B_1, B_2) of bases, we define the *weight of a symmetric exchange* $B_1 - e + f$ and $B_2 - f + e$ to be $w(e)/2 + w(f)/2$, that is, the average of the weights of the exchanged elements.

Conjecture 18 (Bérczi, Mátravölgyi, Schwarcz)**.** Let (B_1, B_2) and (B_1', B_2') be compatible basis pairs of a matroid $M = (S, \mathcal{B})$, and let $w\colon S \to \mathbb{R}_+$. Then, (B_1, B_2) can be transformed into (B_1', B_2') by using symmetric exchanges of total weight at most $w(B_1)/2 + w(B_2)/2 = w(B_1')/2 + w(B_2')/2$.

By setting the weights to be identically 1, we get back Hamidoune's conjecture. The question naturally arises: can we formulate extensions of Conjectures 17 and 18 for basis sequences of length greater than two?

Let $(B_1, \dots, B_k)$ be a sequence of k bases of a matroid M, and assume that there exists distinct indices $\{i_1, \dots, i_q\} \subseteq [k]$ and $e_j \in B_{i_j}$ such that $B_{i_j} - e_j + e_{j+1}$ is a basis for each $j \in [q]$. Then, we say that the sequence $(B_1', \dots, B_k')$ where $B_\ell' = B_{i_j} - e_j + e_{j+1}$ if $\ell = i_j$ for some $j \in [q]$ and $B_\ell' = B_\ell$ otherwise, is obtained by a *cyclic exchange*. As a generalization of Conjecture 17, we propose the following.

Conjecture 19. Let $(B_1, \dots, B_k)$ and $(B_1', \dots, B_k')$ be compatible sequences of k bases of a rank-r matroid. Then, $(B_1, \dots, B_k)$ can be transformed into $(B_1', \dots, B_k')$ by using at most r cyclic exchanges.

Given a weight function $w\colon S \to \mathbb{R}_+$ on the elements of the ground set, let us define the *weight of a cyclic exchange* that moves elements $e_j \in B_{i_j}$ for $j \in [q]$ to be $\frac{1}{k}\sum_{j=1}^{q} w(e_j)$. As a generalization of Conjecture 18, the weighted counterpart is as follows.

Conjecture 20. Let $(B_1, \dots, B_k)$ and $(B_1', \dots, B_k')$ be compatible sequences of k bases of a matroid $M = (S, \mathcal{B})$, and let $w\colon S \to \mathbb{R}_+$. Then, $(B_1, \dots, B_k)$ can be transformed into $(B_1', \dots, B_k')$ by using cyclic exchanges of total weight at most $\frac{1}{k}\sum_{i=1}^{k} w(B_i) = \frac{1}{k}\sum_{i=1}^{k} w(B_i')$.

Note that in both cases, the bounds are tight in the sense that r cyclic exchanges are definitely needed to transform the sequence $(B_1, \dots, B_{k-1}, B_k)$ into $(B_2, \dots, B_k, B_1)$.

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References

[1] K. Bérczi, T. Király, T. Schwarcz, Y. Yamaguchi, and Y. Yokoi. Hypergraph characterization of split matroids. *Journal of Combinatorial Theory, Series A*, 194:105697, 2023.

[2] K. Bérczi, B. Mátravölgyi, and T. Schwarcz. Reconfiguration of basis pairs in regular matroids. *Journal of Combinatorial Theory, Series B*, 177:105–142, 2026.

[3] K. Bérczi, B. Mátravölgyi, and T. Schwarcz. Weighted exchange distance of basis pairs. *Discrete Applied Mathematics*, 349:130–143, 2024.

[4] K. Bérczi and T. Schwarcz. Exchange distance of basis pairs in split matroids. *SIAM Journal on Discrete Mathematics*, 38(1):132–147, 2024.

[5] J. E. Bonin. Basis-exchange properties of sparse paving matroids. *Advances in Applied Mathematics*, 50(1):6–15, 2013.

[6] R. Cordovil and M. L. Moreira. Bases-cobases graphs and polytopes of matroids. *Combinatorica*, 13(2):157–165, 1993.

[7] J. Edmonds and D. R. Fulkerson. Transversals and matroid partition. *Journal of Research of the National Bureau of Standards (B)*, 69:147–153, 1965.

[8] L. Ferroni and B. Schröter. Enumerating the faces of split matroid polytopes. arXiv:2310.05487, 2023.

[9] L. Ferroni and B. Schröter. The Merino–Welsh conjecture for split matroids. *Annals of combinatorics*, 27(3):737–748, 2023.

[10] L. Ferroni and B. Schröter. Tutte polynomials of matroids as universal valuative invariants. *arXiv preprint arXiv:2403.17696*, 2024.

[11] L. Ferroni and B. Schröter. Valuative invariants for large classes of matroids. *Journal of the London Mathematical Society*, 110(3):e12984, 2024.

[12] H. Gabow. Decomposing symmetric exchanges in matroid bases. *Mathematical Programming*, 10(1):271–276, 1976.

[13] M. Joswig and B. Schröter. Matroids from hypersimplex splits. *Journal of Combinatorial Theory, Series A*, 151:254–284, 2017.

[14] Y. Kajitani, S. Ueno, and H. Miyano. Ordering of the elements of a matroid such that its consecutive w elements are independent. *Discrete Mathematics*, 72(1-3):187–194, 1988.

[15] D. Kotlar. On circuits and serial symmetric basis-exchange in matroids. *SIAM Journal on Discrete Mathematics*, 27(3):1274–1286, 2013.

[16] D. Kotlar and R. Ziv. On serial symmetric exchanges of matroid bases. *Journal of Graph Theory*, 73(3):296–304, 2013.

[17] J. Oxley. *Matroid Theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.

[18] J. van den Heuvel and S. Thomassé. Cyclic orderings and cyclic arboricity of matroids. *Journal of Combinatorial Theory, Series B*, 102(3):638–646, 2012.

[19] N. L. White. A unique exchange property for bases. *Linear Algebra and its Applications*, 31:81–91, 1980.

[20] D. Wiedemann. Cyclic base orders of matroids. Manuscript, 1984.