Factorizations of regular graphs of infinite degree

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Abstract

Let $\mathcal{H} = (H_i : i < \alpha)$ for some ordinal number α be an indexed family of graphs. A family $\mathcal{G} = (G_i : i < \alpha)$ of edge-disjoint subgraphs of a graph G such that for every $i < \alpha$: G_i is isomorphic to H_i, each G_i is a spanning subgraph of G, and $E(G) = \bigcup\{E(G_i) : i < \alpha\}$ is a $\mathcal{H}$*-factorization* of G. Let κ be an infinite cardinal. Kőnig proved in 1936 that every κ-regular graph has a factorization into perfect matchings. We extend this result to the most general factorizations possible. We study indexed families $\mathcal{T} = (T_i : i < \kappa)$ of graphs without isolated vertices such that every connected κ-regular graph has a $\mathcal{T}$-factorization. We prove that if $\mathcal{T}$ is a family of forests each of order at most κ, then every connected κ-regular graph G has a $\mathcal{T}$-factorization. These are the most general assumptions for such a family $\mathcal{T}$ for this statement to hold.

Keywords: infinite graphs, trees, decompositions, factorizations, packings, regular graphs, graph factors

Mathematics Subject Classifications: 05C63, 05C51, 05C70, 05C05, 03E05

1 Introduction

The study of matchings and related notions is arguably one of the most popular topics in graph theory. This includes matchability, factorizations, packings, and decompositions. One of the most studied problems in this area is a 1-factorization, which is a decomposition into perfect matchings. The most natural necessary condition for the existence of a 1-factorization of a given graph is its regularity. In the case of finite graphs, this condition is far from being sufficient, even for very simple classes of graphs. The same applies for infinite locally finite graphs (graphs without vertices of infinite degree). However, it turns out that this condition is indeed sufficient in the case of non-locally-finite graphs, as was shown by Kőnig [5] in 1936. Similarly, Andersen and Thomassen [1] proved in 1980 the following theorem.

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Theorem 1 (Andersen, Thomassen [1]). If κ is an infinite cardinal, then a connected graph G has a spanning κ-regular tree if and only if G is κ-regular.

In this paper, we focus on the most general decomposition properties possible for which a necessary and a sufficient condition is the regularity of a given connected non-locally-finite graph. This covers the mentioned problems such as matchability, factorizations, packings, and decompositions, and generalizes the mentioned results of Kőnig, Andersen and Thomassen.

Let $\mathcal{H} = (H_i : i < \alpha)$ be an indexed family of graphs for some ordinal number α. We say that $\mathcal{H}$ *packs* into a graph G if there exists a family $(G_i : i < \alpha)$ of edge-disjoint subgraphs of G such that G_i is isomorphic to H_i for every $i < \alpha$. If $\mathcal{H}$ packs into G and furthermore $\bigcup\{E(G_i) : i < \alpha\} = E(G)$, then $\mathcal{G}$ is called a $\mathcal{H}$-*decomposition* of G. A *factor* of a graph G is a spanning subgraph of G. A $\mathcal{H}$-decomposition of G such that every element of $\mathcal{H}$ is a factor of G is called a $\mathcal{H}$-*factorization* of G. If λ is a cardinal number, then a factorization into λ-regular subgraphs is simply called a λ-*factorization*.

To extend Kőnig's result we can ask for which families $\mathcal{T}$ of κ graphs each κ-regular connected graph has a $\mathcal{T}$-factorization. To simplify we can assume that no element of $\mathcal{T}$ has an isolated vertex. Since a κ-regular graph G itself can be a tree, no graph in $\mathcal{T}$ contains a cycle. Furthermore, each graph in $\mathcal{T}$ is a factor of G, and therefore it has order κ. We also show that each element of $\mathcal{T}$ necessarily has κ components. We prove that these are the only assumption on $\mathcal{T}$ that we need. The main result of this paper is Theorem 2, which provides a general answer to the problem of factorizations for non-locally finite regular graphs. Furthermore, it provides a complete classification of indexed families $\mathcal{T} = (T_i : i < \kappa)$ of graphs such that each element of $\mathcal{T}$ has no isolated vertex and every κ-regular connected graph has a $\mathcal{T}$-factorization.

Theorem 2. Let κ be an infinite cardinal and let $\mathcal{T} = (T_i : i < \kappa)$ be an indexed family of graphs without isolated vertices. The following statements are equivalent:

1. Every connected κ-regular graph G has a $\mathcal{T}$-factorization.

2. Each element of $\mathcal{T}$ is a forest with κ components, each of order at most κ.

The theorem above can be strengthened if we replace the condition of no isolated vertices with a requirement that at least κ many graphs in $\mathcal{T}$ have at least one edge, at the expense of the clarity of the proof. We can replace the connectivity of G with the assumption that G has at most κ components. The extension of the theorem above for graphs with $\lambda > \kappa$ many components is straightforward. However, we restrict ourselves to connected graphs mainly for the clarity of the result.

We can apply Theorem 2 to various problems by setting a suitable family $\mathcal{T}$. To show that every family $\mathcal{T}'$ of κ forests of order at most κ packs into every κ-regular graph it is enough to partition $\mathcal{T}'$ into κ sets of cardinality κ (possibly omitting isolated vertices). Thus, we obtain a family $\mathcal{T}$ of κ forests with κ components of order at most κ, for which we can apply Theorem 2. The same method may be applied to obtain an arbitrary decomposition into κ non-trivial forests of size at most κ. A λ-factorization for non-zero $\lambda \leqslant \kappa$ may be obtained by setting $\mathcal{T}$ to be a family of κ forests with κ

components, each isomorphic to the λ-regular tree. For $\lambda = 1$ we obtain the mentioned result of Kőnig. One can easily deduce the existence of a κ-regular spanning tree from the existence of a κ-regular spanning forest by connecting its component by transfinite induction. Therefore, for $\lambda = \kappa$ Theorem 2 gives a strengthening of the Theorem 1.

The problem of graph decomposition is closely related to the colouring number. The *colouring number* $\mathrm{Col}(G)$ of a graph G is the least cardinal number μ for which there exists an enumeration $(v_i \colon 0 < |V(G)|)$ of vertices of G such that each vertex $v_i \in V(G)$ has less than μ neighbours of smaller indices. Erdős and Hajnal [3] proved in 1967 that if λ is an infinite cardinal, then there exists a decomposition of a graph G into a union of λ forests if and only if $\mathrm{Col}(G) \leqslant \lambda^+$. Note that the colouring number of the complete graph on κ vertices is equal to κ. Therefore, if κ is an infinite cardinal, then each connected infinite κ-regular graph has a decomposition into λ forests if and only if $\kappa = \lambda$ or $\kappa = \lambda^+$. It follows that if κ is a limit cardinal, then we cannot replace $\mathcal{T} = (T_i \colon i < \kappa)$ with an indexed family of smaller cardinality. However, if $\lambda^+ = \kappa$, then for every connected κ-regular graph G there exists an indexed family $\mathcal{T} = (T_i \colon i < \lambda)$ of forests with κ components, each of order at most κ such that G has a $\mathcal{T}$-factorization. It is unknown to the author if there exists an indexed family $\mathcal{T} = (T_i \colon i < \lambda)$ of forests such that every λ^+-regular graph has a $\mathcal{T}$-factorization.

Problem 3. Let λ be an infinite cardinal. Is there any indexed family $\mathcal{T} = (T_i \colon i < \lambda)$ of forests such that every connected λ^+-regular graph G has a $\mathcal{T}$-factorization?

2 Factorizations and decompositions

By the order of a graph we mean the number of elements of the set of its vertices. When considering a $\mathcal{H}$-decomposition, we always implicitly assume that elements of $\mathcal{H}$ are vertex-disjoint. Lowercase Greek letters always refer to ordinal numbers. We consider each ordinal $\alpha = \{\beta \colon \beta < \alpha\}$ as a well-ordered set with the standard well-ordering of the ordinals. If α, β, γ are ordinals, then we treat the Cartesian products $\alpha \times \beta$ and $\alpha \times \beta \times \gamma$ as well-ordered sets with the lexicographic order induced by the well-ordering of the ordinals. This means that we first make an induction on γ with β and α fixed, then on β with α fixed, and finally on α. Each ordering in this paper is either the lexicographic order on a product of ordinals or the standard well-ordering of ordinal. The κ-regular tree or a tree of degree κ is the unique tree such that each of its vertices has degree κ. For notions of graph theory and set theory which are not defined in this paper see [2, 4].

Our main goal is to prove Theorem 2. We divide its proof into three parts. The first part is the theorem below, which shows the necessity of the conditions in Theorem 2. Theorem 4, also applies to Problem 3.

Theorem 4. Let κ be an infinite cardinal and let $\mathcal{T}$ be an indexed family of at least two graphs without isolated vertices. If the κ-regular tree G has a $\mathcal{T}$-factorization, then each element of $\mathcal{T}$ is a forest with κ components, each of order at most κ.

Proof. Let T be a component of some element of $\mathcal{T}$. If the order of T is greater than κ, then T cannot be a subgraph of G, because the order of G is equal to κ. Hence, each element of $\mathcal{T}$ has order at most κ, and so have its components. The κ-regular tree G does not contain a cycle. Hence, no element of $\mathcal{T}$ contains a cycle. Suppose to the contrary that there exists an element T of $\mathcal{T}$ which has less than κ components.

Consider T as a subgraph of G. We shall show that there exists a vertex u such that each edge incident to u in G is an edge of T. Suppose that this is not the case. Pick an arbitrary vertex r. Suppose that r has less than κ neighbours in T. Then each neighbour of r in $G - E(T)$ lies in a different component of T, and therefore there are κ components of T. It follows that r has κ neighbours in T. Note first that the neighbours in G of any vertex v lie in at least two components of T. Otherwise, we could pick u equal to v. Therefore, there exists a component of T which contains a neighbour of v but not v. We say that each such component is *friendly* with respect to v. Suppose that some neighbours u and v of r in T have the same friendly component C. Let u' be a neighbour of u in C, and v' a neighbour of v in C. The only path between u' and v' in G contains r but this path is not in T. Therefore, u' and v' lie in different components, and u and v have distinct friendly components. Since there are κ neighbours of r in T, there are κ friendly components of T. This gives a contradiction.

Since each edge incident to u is contained in T, then u is an isolated vertex in $G-E(T)$. There exists an element of $\mathcal{T}$ different than T without isolated vertex, therefore there is no $\mathcal{T}$-factorization of G. □

The next part of the proof of Theorem 2 is Theorem 5, which covers the case of factorizations of a κ-regular connected graph into κ many κ-regular forests.

Theorem 5. Let κ be an infinite cardinal. A connected graph G has a factorization into κ many κ-regular forests if and only if G is κ-regular.

Proof. Assume first that G contains a κ-regular spanning forest with at most κ components. It follows that each vertex of G has degree at least κ. Moreover, each vertex of G has degree at most κ because G has κ vertices. This proves the necessity of κ-regularity of G. Therefore, it remains to prove the sufficiency of κ-regularity of G.

Let F be a spanning κ-regular tree of G which exists by Theorem 1. Let v_0 be an arbitrary vertex of G. Consider an enumeration $(v_i\colon i < \kappa)$ of vertices of G such that in the rooted tree (F, v_0) if v_i is a child of v_j, then $i > j$.

For each vertex v_j, in order to ensure that in a step $(m,t) \in \kappa \times \kappa$ in the upcoming induction we do not run out of children of v_j, for fixed $j < \kappa$ we define a partition $\{X_j^m(t) : (m,t) \in \kappa \times \kappa\}$ of the set of children of v_j into $|\kappa \times \kappa|$ many sets each of size κ. Note that every vertex of G except v_0 belongs to exactly one set of the family $\{X_j^m(t)\colon j, m, t < \kappa\}$, because the family $\{X_j^m(t)\colon m, t < \kappa\}$ forms a partition of the set of children of v_j, and each vertex of G except v_0 is a child of some vertex v_j. In the proof, we construct a family $\{C_j^m\colon j, m < \kappa\}$ satisfying the conditions:

(C1) $C_j^m \subset N(v_j) \cap \{v_i\colon i > j\}$,

(C2) $C_j^m \cap C_j^n = \emptyset$, for $m \neq n$,

(C3) $|C_j^m| = \kappa$,

(C4) if $v_jv_i \in E(G)$ and $i > j$, then there exists $m < \kappa$ such that $v_i \in C_j^m$.

The Condition (C1) means that C_j^m is a subset of the children of v_j. It follows that $C_i^m \cap C_j^m = \emptyset$ for $i \neq j$. Combining the Condition (C1) with the Condition (C2), we obtain that $C_{j'}^{m'} \cap C_{j''}^{m''} = \emptyset$ if $j' \neq j''$ or $m' \neq m''$.

First, we describe how to obtain a desired κ-factorization of G into κ many κ-regular forests using the family $\{C_j^m : j, m < \kappa\}$ satisfying the conditions above. We construct a κ-factorization $\mathcal{F} = (F^m : m < \kappa)$ by setting $E(F^m) = \{v_jv_i : v_i \in C_j^m, j < i < \kappa\}$. Throughout the proof we use the index $m < \kappa$ for constructing the m-th element of factorization. If v_j is a vertex in a component F of F^m, then C_j^m is the set of children of v_j in (F, v) where v denotes the vertex $v_i \in F$ with the least index i.

By the Condition (C3), every graph in $\mathcal{F}$ is a κ-regular spanning subgraph of G. Let F^m be an element of $\mathcal{F}$, and assume that there exists a cycle in F^m. The vertex of the greatest index in this cycle has two neighbours in F^m. Hence, it belongs to C_j^m and C_i^m for some distinct $i, j < \kappa$. This contradicts (C1). It follows that F^m is a forest for every $m < \kappa$. By the Conditions (C2) and (C4) every edge of G appears in exactly one element of $\mathcal{F}$. Therefore, $\mathcal{F}$ is a factorization of graph G into κ many regular forests of degree κ.

It remains to construct the family $\{C_j^m : j, m < \kappa\}$. We construct sets $\{A_j^m : j, m < \kappa\}$ and $\{B_j^m : j, m < \kappa\}$, and then we obtain C_j^m by setting $C_j^m = A_j^m \cup B_j^m$ for every $j, m < \kappa$. We shall construct the sets $\{A_j^m : j, m < \kappa\}$ and $\{B_j^m : j, m < \kappa\}$ by transfinite induction on $(m, \tau, i) \in \kappa \times \kappa \times \kappa$ with respect to the lexicographic order on $\kappa \times \kappa \times \kappa$. During step (m, τ, i) we either assign the vertex v_i to $a_j^m(y)$ for some $j, y < \kappa$, we put v_i in B_j^m for some $j < \kappa$, or we proceed to the next step without doing anything. Assigning v_i to $a_j^m(y)$ is equivalent to defining $a_j^m(y)$ as v_i. Without loss of generality, we can assume that $V(G) \cap \kappa = \emptyset$. Initially, we temporarily assign a different ordinal number less than κ to each element of $\{a_j^m(y) : m, j, y < \kappa\}$ but still refer to $a_j^m(y)$ as not defined until some vertex $v_i \in V(G)$ is assigned to it. This is done for technical reasons so that sets containing $a_j^m(y)$ are well-defined throughout the entire proof.

Recall that the index m is related to the m-th factor. After executing steps (m, τ, i) for every $\tau, i < \kappa$ the set $\{a_j^m(y) : y < \kappa\}$ has been defined for every $j < \kappa$, and we define $A_j^m = \{a_j^m(y) : y < \kappa\}$. During steps (m, τ, i) for $\tau, i < \kappa$ we define the set B_j^m by putting vertices in it. At the start of the induction no vertex lies in B_j^m for every $m, j < \kappa$.

For a fixed triple (m, τ, i) let $\sigma_\tau^m(i) = 0$ if no $a_j^m(y)$ has been defined, let $\sigma_\tau^m(i)$ be the least ordinal for which there exist $j, y \leqslant \sigma_\tau^m(i)$ such that $a_j^m(y)$ has not been defined, or let $\sigma_\tau^m(i) = \kappa$ if every element of $\{a_j^m(y) : j, y < \kappa\}$ has already been defined. The parameter above and the family $X_j^m(t)$ shall ensure that every element in the set $\{a_j^m(y) : j, y < \kappa\}$ is defined after executing steps (m, τ', i') for $\tau', i' < \kappa$. In particular, it shall ensure that $|A_j^m| \leqslant |C_j^m| = \kappa$. The index τ in a triple (m, τ, i) is an auxiliary index which serves the purpose of considering every vertex v_i multiple times.

Throughout the induction every vertex can be assigned to more than one element of $\{a_i^{m'}(y) : i, m', y < \kappa\}$ but it can be assigned to at most one of them for the fixed m'. We consider the vertices of G one by one, in the step (m, τ, i) we assign v_i to the unique $a_j^m(y)$, if such exists, such that all the Conditions below are satisfied:

(D1) the vertex v_i has not been assigned to $a^m_{j'}(y')$ nor put to $B^m_{j'}$ for every $j', y' < \kappa$,

(D2) v_i is a neighbour of v_j and $i > j$,

(D3) for every $y' < y$, the vertex $a^m_j(y')$ has already been defined but $a^m_j(y)$ has not been,

(D4) $v_i \notin A^{m'}_j$ for every $m' < m$,

(D5) $v_i \notin B^{m'}_j$ for every $m' < m$,

(D6) $v_i \notin X^{m''}_j(t)$ for every $(m'', t) > (m, \tau)$ and $v_i \notin X^m_{j'}(\tau)$ for every $j' \neq j$,

(D7) $j \leqslant \sigma^m_\tau(i)$ and $y \leqslant \sigma^m_\tau(i)$,

(D8) j is the least index for which the conditions (D2)–(D7) are satisfied for some $y < \kappa$.

If v_i has not been assigned to any $a^m_j(y)$ by the Conditions above (for the fixed m), then we consider putting v_i in B^m_j for some $j < \kappa$. If the Condition (D1) is satisfied and j is the least index for which the Conditions (D2), (D4), (D5) and (D6) are satisfied, then we put v_i in B^m_j. Otherwise, we do nothing and proceed to the next index.

After the induction on $(m, \tau, i) \in \kappa \times \kappa \times \kappa$ we define $C^m_j = A^m_j \cup B^m_j$ for every $j, m < \kappa$. Note that for every $m, j < \kappa$ the sets A^m_j and B^m_j are disjoint. We prove that the family $\{C^m_j : j, m < \kappa\}$, obtained by the recursive construction, satisfies the Conditions (C1)–(C4). The first two Conditions are easy to check and follow directly from the construction. The Condition (C1) are satisfied by the the Conditions (D2) and (D1) in the construction of the sets A^m_j and B^m_j. The Condition (C2) follows from the Conditions (D4) and (D5). We need to show that the Conditions (C3) and (C4) are satisfied.

Now, we prove that $|A^m_j| = \kappa$ for every $m, j < \kappa$. This is equivalent to $\{\sigma^m_\tau(0) : \tau < \kappa\}$ not being bounded by any ordinal less than κ for every $m < \kappa$. We show the following claim.

Claim 1. For every $m < \kappa$ consider $\sigma^m_\tau(0)$ as a function of τ. Then $\sigma^m_\tau(0)$ is strictly increasing on the set $\{\tau < \kappa : \sigma^m_\tau(0) < \kappa\}$

Proof. Fix $m < \kappa$. For any non-zero $\alpha < \kappa$ we have $\sup\{\sigma^m_\tau(0) : \tau < \alpha\} \leqslant \sigma^m_\alpha(0)$. Therefore, it is enough to prove that for every $\alpha < \kappa$ we have $\sigma^m_\alpha(0) < \sigma^m_{\alpha+1}(0)$ or $\sigma^m_\alpha(0) = \kappa$. Denote $s = \sigma^m_\alpha(0)$. Before the execution of step $(m, \alpha, 0)$ the vertex $a^m_j(y)$ has already been defined for every $j < s$ and $y < s$. Let $C = \{a^m_s(y) : y < s\} \cup \{a^m_j(s) : j < s\}$. At least one element of C has not been defined. Fix m and α. In every step on induction on i we consider assigning v_i to $a^m_j(y) \in C$ for every $j, y < \kappa$. Notice that there are κ many vertices v_i (considered in induction on i) such that the Conditions (D1), (D2), (D4), (D5) and (D6) are satisfied for this choice of $j, y < \kappa$. If $a^m_j(y) \in C$ has not been assigned before step i and $v_i \in X^m_j(\alpha)$, then v_i satisfy the Conditions (D1)–(D7) when considered as a candidate for $a^m_j(y')$ for some $y' \leqslant s$. It follows from the second part of (D6) that the Condition (D8) is also satisfied. Thus, $v_i = a^m_j(y')$. Note that there are less than κ

elements in C. Recall that there are κ indices i' such that vertex $v_{i'} \in V(G)$ satisfies the Conditions (D1), (D2), (D4), (D5) and (D6) when it is considered at step i'. Therefore, we assign a vertex for every element of C during the induction on i with fixed m and α. Thus, $\sigma^m_\alpha(0) < \sigma^m_{\alpha+1}(0)$. □

It follows from the claim above that for every $j, m < \kappa$ we have $|A^m_j| = \kappa$. Since $A^m_j \subseteq C^m_j$, we obtained that the family $\{C^m_j : j, m < \kappa\}$ satisfies (C3). It remains to prove that the Condition (C4) holds for $\{C^m_j : j, m < \kappa\}$. Assume that $\{C^m_j : j, m < \kappa\}$ does not satisfy the Condition (C4). Let (i, j) be the least element in $\kappa \times \kappa$ such that $i > j$ and $v_j v_i \in E$ but $v_i \notin C^m_j$ for every $m < \kappa$. Notice that there exists an index $m'' < \kappa$ such that $v_i \notin X^m_{j'}(\tau)$ for every (m, τ, j') such that $m \geqslant m''$, $\tau < \kappa$, and $j' < \kappa$. From Claim 1 and the fact that $\sigma^m_\tau(i)$ is non-decreasing we obtain that $\sigma^m_t(i)$ is unbounded in κ for every m and i. It follows that for every $m < \kappa$ there exists an index $t(m)$ such that $\sigma^m_{t(m)}(i) \geqslant \max\{i, j\} = i$. For $m > m''$ and $\tau \geqslant t(m)$ we consider step (m, τ, i), and we check which of the Conditions (D1)-(D8) would be satisfied for assigning v_i to $a^m_j(y)$ for some $y \leqslant \sigma^m_{t(m)}(i)$.

The Conditions (D4)–(D7) and the Condition (D2) are satisfied for such a choice of y and (i, j) because of the minimality of the pair (i, j). However, it may happen that the Condition (D1), (D3) or (D8) fails. If the Condition (D1) fails, then it also fails for every successive step (m, t', i) within m. Furthermore, the Condition (D3) may be satisfied for at most one y and by the assumption that we did not put v_i to B^m_j.

When assigning v_i to $a^m_j(y)$, the Condition (D2) is satisfied only for $j < i$, hence for at most $|i| < \kappa$ indices j. Therefore, the satisfaction of the Condition (D1) depends on only these indices j' for which $j' < i$. Therefore, for all but at most $|i|$ indices m the Condition (D1) is satisfied at (m, t, i) for every $t < \kappa$. Take m such that $m > m''$ and the condition (D1) is satisfied at (m, t, i) for every $t < \kappa$. It means that v_i is not assigned to any element of $A^m_{j'}$ or put in $B^m_{j'}$ for every $j' < \kappa$ during the induction on (m, t, i) for the fixed m and i. Consider the assigning of v_i to $a^m_j(y)$. By the assumption the Conditions (D4) and (D5) are satisfied. By the choice of m the Conditions (D1) and (D6) are satisfied. Furthermore, by the minimality of (i, j) the index j is the least index for which all the Conditions (D2), (D4), (D5), (D6) are satisfied. Therefore, v_i in step (m, t, i) is assigned to an element of A^m_j or v_i is put in B^m_j, which contradicts the assumption. □

The next theorem allows us to further factorize κ-regular forests from Theorem 5. For an arbitrary graph H denote by $S_H(v, d)$ the *sphere* of radius d and centre v in graph H. Similarly, denote by $B_H(v, d)$ the *ball* of radius d and centre v in graph H.

Theorem 6. Let κ be an infinite cardinal. If $\mathcal{T} = (T^m : m < \kappa)$ is an indexed family of forests without isolated vertices and with κ components each of order at most κ, then there exists a $\mathcal{T}$-factorization of the κ-regular tree.

Proof. Denote the κ-regular tree by G. For $m < \kappa$ let $(t^m_i : i < \kappa)$ be an enumeration of vertices of T^m. We shall define the set $\{y^m_i : m, i < \kappa\}$ and the graph Y^m for every $m < \kappa$ such that $V(Y^m) = \{y^m_i : i < \kappa\}$, $E(Y^m) = \{y^m_i y^m_j : i, j < \kappa, t^m_i t^m_j \in E(T^m)\}$, and the following conditions shall be satisfied:

(E1) $f^m\colon t_i^m \mapsto y_i^m$ is an isomorphism of T^m into Y^m for every $m < \kappa$,

(E2) if $xy \in E(G)$, then there exists the unique triple (m, i, j) such that $i < j$ and $xy = y_i^m y_j^m$,

(E3) $V(Y^m) = V(G)$ for every $m < \kappa$.

Now we show that if the Conditions (E1)–(E3) hold, then the family $(Y^m\colon m < \kappa)$ is a $\mathcal{T}$-factorization of G. The Condition (E3) means that each Y^m is a factor of G. It follows from the conditions (E1) and (E2) that $(Y^m\colon m < \kappa)$ is a $\mathcal{T}$-factorization of G.

Pick any $v_0 \in V(G)$ as the root of G. For every $m < \kappa$ we define $y_0^m = v_0$. First, for every $m < \kappa$ we partition the family of components of T^m into sets T_d^m for every $d < \omega$ in such a way that T_0^m is the singleton of the component containing t_0^m and $|T_d^m| = \kappa$ for every non-zero $d < \omega$. Furthermore, for a component T of T^m denote by x_T the vertex t_i^m with the least index i in T. For induction on $d < \omega$, assume that we already assigned all elements in $B_G(v_0, d)$ to some elements of $\{y_i^m\colon i < \alpha\}$ for every $m < \kappa$ in such a way that the following Conditions are satisfied:

(F1) every vertex in $B_G(v_0, d) - v_0$ has been assigned to exactly one vertex in $\{y_j^m\colon j < \kappa\}$ for every $m < \kappa$, and only vertices in $B_G(v_0, d)$ have been assigned,

(F2) if y_i^m and y_j^m have been defined, then $y_i^m y_j^m \in E(G)$ if and only if $t_i^m t_j^m \in E(T^m)$,

(F3) if xy is an edge in G between two vertices in $B_G(v_0, d)$, then there exists the unique triple (m, i, j) such that $i < j$ and $xy = y_i^m y_j^m$,

(F4) y_i^m has been defined if and only if $t_i^m \in B_T(x_T, d - d')$ for some natural number d' such that $d' \leqslant d$, and for some $T \in T_{d'}^m$.

For $y \in S_G(v_0, d)$ we define $W_d(y)$ as the set of the vertices $t_i^m \in T^m$ such that $y = y_i^m$ for some $m, i < \kappa$. For every $y \in S_G(v_0, d)$ we assign each child of y in G to the unique y_j^m such that $t_i^m \in W_d(y)$ is a neighbour of t_j^m in T^m and y_j^m has not been defined. Moreover, for each m the vertex y_j^m has to be assigned to some child of y. Such an assignment is possible because y has κ children and if we put $d' = d$ in the Condition (F4), we obtain that there are κ vertices of the form y_j^m which are yet unassigned, because they lie outside of the ball $B_T(x_t, d)$, and we can assign κ of them to each child of y. Let $X_{d+1}^m = \{x_T\colon T \in T_{d+1}^m\}$. For each $t_i^m \in X_{d+1}^m$ we assign the vertex y_i^m to some vertex v in $S_G(v_0, d+1)$ such that v has not been defined yet as a y_j^m for every $j < \kappa$. For a fixed m each vertex y_i^m has to be assigned with a different vertex of $S_G(v_0, d+1)$.

Now we show that before executing step d (after executing step $d-1$) the Conditions (F1)–(F4) are satisfied. Each of these conditions are trivially satisfied for $d = 0$. Assume then that $d > 0$. It follows directly from the construction that the conditions (F1), (F2) and (F4) are satisfied. Let $y_i^m y_j^m$ be an edge between two vertices in $E(G)$ and assume that $i < j$. Furthermore, assume that $y_i^m \in S_G(v_0, d)$ and $y_j^m \in S_G(v_0, d+1)$. We can assume that because each vertex in $B_G(v_0, d)$ has vertices assigned to it before executing step d, and only vertices in $S_G(v_0, d+1)$ are assigned to in step d. Notice that if $y_j^m = y_{j'}^{m'}$

for some $j' < \kappa$, $m' \neq m$, then $t_{j'}^{m'} = x_T$ for some $T \in \mathcal{T}^{m'}$. Therefore, no neighbour in $Y^{m'}$ of $y_{j'}^{m'}$ lies in $S_G(v_0, d-1)$. It follows that the condition (F3) is satisfied.

It remains to prove that $\{y_i^m \colon m, i < \kappa\}$ satisfies the Conditions (E1)–(E3). The Condition (E1) is satisfied by (F2) and (F4). The Condition (E2) is satisfied by (F3). It follows directly from the Condition (F1) that the Condition (E3) is satisfied. □

The proof of Theorem 2 follows easily from Theorems 4, 5, and 6. Theorem 4 shows the necessity of the conditions in Theorem 2. By Theorem 5 we obtain a factorization $(Y^m \colon m < \kappa)$ of G into κ many regular forests of degree κ. Then, we partition $\mathcal{T}$ into an indexed family $(\mathcal{U}^m \colon m < \kappa)$ of sets each of cardinality κ. For every $m < \kappa$ the set $\mathcal{U}^m$ is an indexed family of κ forests without isolated vertices and with κ components, each of order at most κ. By Theorem 6, there exists a $\mathcal{U}^m$-factorization $\mathcal{W}^m$ of Y^m for every $m < \kappa$. It follows that $\{W \colon W \in \mathcal{W}^m, m < \kappa\}$ forms a $\mathcal{T}$-factorization of G.

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