# Minimal Cellular Resolutions of Powers of Graphs

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**Abstract**

Let $G$ be a connected graph and let $I(G)$ denote its edge ideal. We classify when $I(G)^n$, for $n \geqslant 1$, admits a minimal Lyubeznik resolution. We also give a characterization for when $I(G)^n$ is bridge-friendly, which, in turn, implies that $I(G)^n$ has a minimal Barile-Macchia cellular resolution.

**Mathematics Subject Classifications:** 13D02, 13F55, 05C65, 05C75, 05E40

## 1 Introduction

It has been a central problem in the study of minimal free resolutions to understand when a monomial ideal admits cellular resolutions and how to construct these resolutions (cf. [1, 2, 3, 4, 5, 8, 10, 11, 12, 17, 20, 24]). Despite much effort, only a few explicit constructions of simplicial complexes that support the free resolution of a monomial ideal in general are known. The resolutions resulting from these explicit constructions are the *Taylor resolution*, *Lyubeznik resolution*, and, in special cases, the *Scarf complex* (see [4, 17, 21]). From discrete Morse theory, cellular resolutions for monomial ideals were constructed in [2, 3, 8, 11, 12]; particularly, a subclass considered in [2, 8, 9] is called the *Barile-Macchia resolution*.

Broadly speaking, discrete Morse theory allows one to "trim down" the Taylor resolution of a monomial ideal in search for the minimal one. This process becomes harder when the number of generators gets large, as the number of "trim down" options increases. Thus, a systematic algorithm is desirable. The Lyubeznik algorithm [17] requires a total order on the generators as its sole input. Batzies and Welker [3], in addition to proving that Lyubeznik resolutions can be derived from discrete Morse theory, noticed that instead of applying the Lyubeznik algorithm to $2^{\mathrm{Gens}(I)}$, for a monomial ideal $I$, one can do so to $\{\sigma \in 2^{\mathrm{Gens}(I)} \colon \operatorname{lcm}(\sigma) = p\}$ for each monomial $p$, using a total order ($\succ_p$) on the

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set of minimal generators $\mathrm{Gens}(I)$ of $I$. This process gives rise to the so-called *generalized* Lyubeznik resolution. Chau and Kara [8] constructed the Barile-Macchia algorithm, which also requires only one total order on $2^{\mathrm{Gens}(I)}$ as input, and has a Barile-Macchia resolution as output. Moreover, with a similar process to that described by Batzies and Welker and a given system of total orders on $\mathrm{Gens}(I)$, one can create a *generalized* Barile-Macchia resolution. Table 1 gives a better perspective on how (generalized) Lyubeznik and Barile-Macchia resolutions are created.

| Algorithm \ Input | One total order | Multiple total orders |
|---|---|---|
| Lyubeznik [17] | Lyubeznik resolution | generalized Lyubeznik resolution |
| Barile-Macchia [2] | Barile-Macchia resolution | generalized Barile-Macchia resolution |

Table 1: Lyubeznik and Barile-Macchia algorithms - Input and Output

Out of these resolutions coming from discrete Morse theory, Lyubeznik resolutions are the only ones that are always simplicial. This is one of the reasons why Lyubeznik resolutions are often non-minimal. On the other hand, Barile-Macchia resolutions have been proven to be minimal in many cases (see, e.g., [2, 6, 8, 9]). The Lyubeznik and Barile-Macchia algorithms, therefore, are vastly different, even though both require only a single total order. Moreover, [8, Theorem 5.14] showed that under mild assumptions, Barile-Macchia resolutions are closer to minimal than their Lyubeznik counterparts. Not much is known about their generalized versions, as generalized Lyubeznik resolutions have only been used in the work of Batzies and Welker [3], and thus have only been shown to be minimal for generic or shellable ideals. In a different article [7], we showed that generic or shellable ideals also have minimal generalized Barile-Macchia resolutions. In other words, the existing literature suggests that the Barile-Macchia algorithm produces resolutions closer to minimal than the Lyubeznik algorithm. On the other hand, classifying monomial ideals whose Lyubeznik or Barile-Macchia resolutions are minimal, or whose Scarf complexes are resolutions, seems out of reach at this time.

In a recent work [14] of Faridi, Hà, Hibi, and Morey, graphs whose edge ideals and their powers admit Scarf resolutions are identified. More precisely, let $G = (V, E)$ be a simple undirected graph (i.e., $G$ contains neither loops nor multiple edges) over the vertex set $V = \{x_1, \ldots, x_r\}$. We will always consider *connected* simple graphs with at least one edge. Let $\Bbbk$ be a field and, by identifying the vertices of $G$ with variables, let $S = \Bbbk[x_1, \ldots, x_r] = \Bbbk[V]$ be a polynomial ring. The *edge ideal* of $G$ is defined to be

$$I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E \rangle \subseteq S.$$

It was proved in [14, Theorem 8.3] that the Scarf complex of $I(G)^n$, for a connected graph $G$, is a resolution (which is necessarily minimal) if and only if either

(S1) $n = 1$ and $G$ is a *gap-free* tree; or

(S2) $n > 1$ and $G$ is either an isolated vertex, an edge, or a path on three vertices.

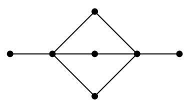
We say that a monomial ideal is *Lyubeznik* (resp. *Barile-Macchia*) if it admits a minimal Lyubeznik (resp. Barile-Macchia) resolution, i.e., there exists a total order on the minimal generating set such that the output of the corresponding algorithm is a minimal resolution. We remark that there are monomial ideals that are Barile-Macchia but not Lyubeznik. In fact, as shall be seen from our results below, $I(P_5)$, the edge ideal of a 5-path, is one such example. On the other hand, whether Lyubeznik monomial ideals are Barile-Macchia is still open. Our work in this paper addresses the following problem:

**Problem 1.** Characterize graphs whose edge ideals and their powers are Lyubeznik and/or Barile-Macchia.

Our results give a complete classification of graphs whose edge ideals and their powers have minimal Lyubeznik resolutions. Our method is based on the observation that having a minimal Lyubeznik resolution descends to HHZ-subideals (see Section 2.5 for the definition of HHZ-subideals) with respect to any given monomial $m$, i.e., ideals generated by subcollections of the generators which divide the given monomial $m$ (see Section 2.5 and Lemma 18). HHZ-subideals were introduced in [16], and appeared briefly prior in [4]. As a consequence, if $H$ is an induced subgraph of $G$ and $I(H)^n$ does not have a minimal Lyubeznik resolution, then neither does $I(G)^n$. This allows us to reduce the problem to finding "forbidden structures".

The problem is considerably more difficult with Barile-Macchia resolutions. Even though having a minimal Barile-Macchia resolution also descends to HHZ-subideals (see Lemma 18), we have not been able to find "forbidden strictures" for this property. It is too computationally expensive to verify this property even for all graphs with nine vertices or fewer.

*Remark* 2. One known example of a graph whose edge ideal does not have a minimal Barile-Macchia resolution is the 9-cycle [8, Remark 4.24]. The smallest example (in terms of both number of vertices and number of edges) was found using an exhaustive search on `SageMath` and is drawn here.



Chau and Kara [8] introduced the notion of *bridge-friendly* monomial ideals (see Definition 10), which implies that the given monomial ideal has a minimal Barile-Macchia resolution, and this resolution can be nicely described. We will classify *chordal* graphs whose edge ideals are bridge-friendly and, thus, admit minimal cellular resolutions.

Let us now describe our main results in more details. For nonnegative integers $a, b, c$, let $L(a, b, c)$ denote the graph consisting of exactly $c$ triangles sharing one edge $\{x, y\}$ such that $a$ and $b$ distinct leaves are attached to the vertices $x$ and $y$, respectively (see Definition 26). Next, associated to a tree $T$ and a $\mathbb{Z}_{\geqslant 0}$-valued function w on the edges of $T$, we let $\mathrm{BF}(T, \mathrm{w})$ be the graph obtained by attaching $\mathrm{w}(e)$ triangles to each edge $e$ in $T$ (see Definition 41). For the edge ideal $I(G)$ itself, we establish in Theorems 31 and 45 the following results.

(**31**) $I(G)$ is Lyubeznik if and only if $G = L(a, b, c)$ for some nonnegative integers $a, b, c$; and

(**45**) if $G$ is a chordal graph, then $I(G)$ is bridge-friendly if and only if $G = \mathrm{BF}(T, \mathrm{w})$, for some tree $T$ and function $\mathrm{w}\colon E(T) \to \mathbb{Z}_{\geqslant 0}$.

To prove Theorem 31, we establish that:

(L1) (forbidden structures) if $G$ is a 5-path $P_5$, 4-cycle $C_4$, 5-cycle $C_5$, 4-complete graph $K_4$, kite graph, gem graph, tadpole graph, butterfly graph, or net graph, then $I(G)$ is not Lyubeznik (Proposition 25);

(L2) (graph-theoretic classification) $G$ does not contain an induced subgraph of the forms listed in (L1) if and only if $G = L(a, b, c)$ for $a, b, c \in \mathbb{Z}_{\geqslant 0}$ (Proposition 28); and

(L3) (Lyubeznik graphs) if $G = L(a, b, c)$, for $a, b, c \in \mathbb{Z}_{\geqslant 0}$, then $I(G)$ is Lyubeznik (Proposition 30).

The proof of Theorem 45 follows in similar steps; particularly, we show that:

(BF1) (forbidden structures) if $G$ is a 4-complete graph $K_4$, gem graph, kite graph, or net graph, then $I(G)$ is not bridge-friendly (Proposition 40);

(BF2) (graph-theoretic classification) a chordal graph $G$ is does not contain an induced subgraph of the forms listed in (BF1) if and only if $G = \mathrm{BF}(T, \mathrm{w})$ for a tree $T$ and a function $\mathrm{w}\colon E(T) \to \mathbb{Z}_{\geqslant 0}$ (Proposition 43); and

(BF3) (bridge-friendly graphs) if $G = \mathrm{BF}(T, \mathrm{w})$ for some tree $T$ and a function $\mathrm{w}\colon E(T) \to \mathbb{Z}_{\geqslant 0}$, then $G$ is chordal (Proposition 42) and $I(G)$ is bridge-friendly (Proposition 44).

The study of higher powers $I(G)^n$, with $n \geqslant 2$, proceeds in a similar fashion, though considerably simpler. We prove in Theorems 37 and 52 that:

(**37**) $I(G)^n$ has a minimal Lyubeznik resolution if and only if $G$ is an edge, or $n = 2$ and $G$ is a path on three vertices; and

(**52**) $I(G)^n$ is bridge-friendly if and only if $G$ is an edge, or $G$ is a path on three vertices, or $n = 2, 3$ and $G$ is the triangle $C_3$.

Theorems 37 and 52 are proven in a manner similar to that of Theorems 31 and 45, although simpler. In particular, we show that:

(Lp) (forbidden structures) if $G$ is $K_{1,3}$, a 4-path $P_4$, a triangle $C_3$, or a 4-cycle $C_4$, then $I(G)^2$ is not Lyubeznik; and for the last three graphs in the list, neither is $I(G)^n$ for all $n \geqslant 2$ (see Proposition 36); and

(BFp) (forbidden structures) if $G$ is a 4-star $K_{1,3}$, a 4-path graph $P_4$, a 4-cycle graph $C_4$, paw graph, diamond graph, or a 4-complete graph $K_4$, then $I(G)^2$ and $I(G)^3$ are not bridge-friendly; and for the first three graphs in the list, $I(G)^n$ is not bridge-friendly for all $n \geqslant 2$ (see Proposition 48).

The article is structured as follows. Section 2 provides the background on Lyubeznik and Barile-Macchia resolutions. Section 3 is devoted to the characterization of graphs whose edge ideals and their powers admit minimal Lyubeznik resolutions. Section 4 focuses on the bridge-friendly property. We remark that throughout the article, we will often use computer computation to verify whether a given explicit monomial ideal is bridge-friendly or admits a minimal Lyubeznik resolution. To that end, we provide examples and code for these two tasks in the Appendix.

## 2 Preliminaries

In this section, we collect basic terminology and notations about graphs and edge ideals of graphs. We also give auxiliary results on Lyubeznik and Barile-Macchia resolutions, bridge-friendly property, and HHZ-subideals.

### 2.1 Graphs and edge ideals of graphs

Throughout the paper, $G = (V, E)$ denotes a connected simple graph with vertex set $V$ and edge set $E$, where $|E| \geqslant 1$. Let $\Bbbk$ be a field and let $S = \Bbbk[V]$ be the polynomial ring whose variables are identified with the vertices of $G$. The *edge ideal* of $G$ is defined as

$$I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E \rangle \subseteq S.$$

A graph $H$ is an *induced subgraph* of $G$ if $V(H) \subset V(G)$ and the edges of $H$ are precisely the edges of $G$ that connect two vertices in $H$. A *tree* is a (connected) graph with no cycles. A *chord* in a cycle is an edge that connects two non-adjacent vertices in the cycle. The graph $G$ is called *chordal* if every cycle of length $\geqslant 4$ has a chord.

For a vertex $x \in V$, its set of *neighbors* is $\{y \in V \mid \{x, y\} \in E\}$. The *distance* between two vertices $x$ and $y$ of $G$, denoted by $\operatorname{dist}_G(x, y)$, is defined as the smallest value $n$ such that there exists a path of length $n$ connecting $x$ and $y$ in $G$. In particular, the neighbors of a vertex are exactly those of distance one from the given vertex.

We shall denote by $K_{1,n-1}$, $P_n$, $C_n$, and $K_n$ the complete bipartite graph of size $(1, n-1)$, the path with $n$ vertices, the cycle on $n$ vertices, and the complete graph on $n$ vertices, respectively. We sometimes refer to those graphs as the *$n$-star*, *$n$-path*, *$n$-cycle*, and *$n$-complete* graphs. Note that the $n$-path $P_n$ is said to have length $n - 1$.

We call the following small graphs by particular names that their shapes represent: *net* ( ), *kite* ( ), *diamond* ( ), *paw* ( ), *gem* ( ), *butterfly* ( ), and *tadpole* ( ).

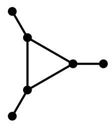


Figure 1: Net

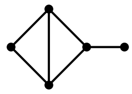


Figure 2: Kite

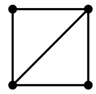


Figure 3: Diamond

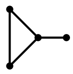


Figure 4: Paw

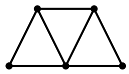


Figure 5: Gem

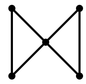


Figure 6: Butterfly

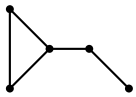


Figure 7: Tadpole

### 2.2 Taylor resolutions

Let $I \subseteq S$ be a homogeneous ideal. A *free resolution* of $S/I$ is a complex of free $S$-modules of the form

$$\mathcal{F} : 0 \to F_p \xrightarrow{\partial_p} F_{p-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \to 0$$

where $H_0(\mathcal{F}) \cong S/I$ and $H_i(\mathcal{F}) \cong 0$ if $i \neq 0$. Moreover, $\mathcal{F}$ is $\mathbb{N}^r$*-graded* if $\partial_i$ is $\mathbb{N}^r$-homogeneous for all $i$, and *minimal* if $\partial_i(F_i) \subseteq (x_1, \dots, x_r)F_{i-1}$ for all $i$.

For a monomial ideal $I \subseteq S$, let $\operatorname{Gens}(I)$ denote its unique set of minimal monomial generators. We consider the full $|\operatorname{Gens}(I)|$-simplex whose vertices are labelled by the monomial generators of $I$. It is well-known (cf. [21]) that the chain complex of this simplex gives a free resolution of $S/I$, which is referred to as the *Taylor resolution.* Set $|\operatorname{Gens}(I)| = c$. Then the Taylor resolution of $S/I$ is of the form

$$0 \to S^{\binom{c}{c}} \to S^{\binom{c}{c-1}} \to \cdots \to S^{\binom{c}{1}} \to S^{\binom{c}{0}} \to 0.$$

We remark that for each integer $i$, one can identify a basis of $S^{\binom{c}{i}}$ with the collection of subsets of $\operatorname{Gens}(I)$ with exactly $i$ elements. We refer interested readers to [19] for a great introduction to simplicial resolutions.

### 2.3 Lyubeznik and Barile-Macchia resolutions

The Lyubeznik and Barile-Macchia resolutions, introduced in [17, 2, 3, 8], are subcomplexes of the Taylor resolution. While the Taylor resolution accounts for all subsets of $\operatorname{Gens}(I)$, the Lyubeznik and Barile-Macchia resolutions are constructed based on the *Lyubeznik-critical* and *Barile-Macchia-critical* subsets of $\operatorname{Gens}(I)$; these are the subsets of $\operatorname{Gens}(I)$ that survive in the Lyubeznik and Barile-Macchia resolutions. The terms

*Lyubeznik-critical* and *Barile-Macchia-critical* depend on the choice of a given total order $(\succ)$ on $\mathrm{Gens}(I)$. In particular, the Lyubeznik critical and Barile-Macchia critical subsets with respect to $(\succ)$ are characterized in Propositions 3 and 7 below.

Given a set of monomials $\sigma$, a monomial $m$, and a total order $(\succ)$ on $\sigma \cup \{m\}$, set

$$\sigma_{\succ m} \coloneqq \{m' \in \sigma \colon m' \succ m\}.$$

**Proposition 3** ([3, Theorem 3.2])**.** *Let $\sigma$ be a subset of* $\mathrm{Gens}(I)$ *and $(\succ)$ a total order on* $\mathrm{Gens}(I)$*. Then, $\sigma$ is Lyubeznik-critical (with respect to $(\succ)$) if and only if*

$$\{m \in \mathrm{Gens}(I) \colon m \mid \mathrm{lcm}(\sigma_{\succ m})\} = \emptyset.$$

**Example 4.** Consider the ideal $I = (xw, xy, yz, zw) \subseteq \Bbbk[x, y, z, w]$ with the total order $wx \succ xy \succ yz \succ zw$.

(a) Let $\sigma_1 = \{xw, xy, yz\}$. Since $zw \mid \mathrm{lcm}(\sigma_1)$ and $yz \succ zw$, the set $\sigma_1$ is not Lyubeznik-critical.

(b) Let $\sigma_2 = \{xw, xy, zw\}$. It is routine to verify that $\sigma_2$ is Lyubeznik-critical. Indeed, set $m_1 = xw, m_2 = xy$, and $m_3 = yz$. Then by brute force, one can see that there exists no $m \in \mathrm{Gens}(I)$ such that either

$$m \mid \mathrm{lcm}(m_1, m_2) \text{ and } m_2 \succ m$$

or

$$m \mid \mathrm{lcm}(m_1, m_2, m_3) \text{ and } m_3 \succ m.$$

**Definition 5.** Let $I$ be a monomial ideal and $(\succ)$ a total order on $\mathrm{Gens}(I)$.

1. Given $\sigma \subseteq \mathrm{Gens}(I)$ and $m \in \mathrm{Gens}(I)$ such that $\mathrm{lcm}(\sigma \cup \{m\}) = \mathrm{lcm}(\sigma \setminus \{m\})$, we say that $m$ is a *bridge* (respectively, *gap*) of $\sigma$ if $m \in \sigma$ (respectively, $m \notin \sigma$).

2. If $m \succ m'$ where $m, m' \in \mathrm{Gens}(I)$, we say that $m$ *dominates* $m'$.

3. The *smallest bridge function* is defined to be

$$\mathrm{sb} : 2^{\mathrm{Gens}(I)} \to \mathrm{Gens}(I) \sqcup \{\emptyset\}$$

where $2^{\mathrm{Gens}(I)}$ denotes the power set of $\mathrm{Gens}(I)$, and $\mathrm{sb}(\sigma)$ is the smallest bridge of $\sigma$ (with respect to $(\succ)$) if $\sigma$ has a bridge and $\emptyset$ otherwise.

4. A monomial $m \in \mathrm{Gens}(I)$ is called a *true gap* of $\sigma \subseteq \mathrm{Gens}(I)$ if

   (a) it is a gap of $\sigma$, and

   (b) the set $\sigma \cup \{m\}$ has no new bridges dominated by $m$. In other words, if $m'$ is a bridge of $\sigma \cup \{m\}$ and $m \succ m'$, then $m'$ is a bridge of $\sigma$.

Equivalently, $m$ is not a true gap of $\sigma$ either if $m$ is not a gap of $\sigma$ or if there exists $m' \prec m$ such that $m'$ is a bridge of $\sigma \cup \{m\}$ but not one of $\sigma$. In the latter case, we call $m'$ a *non-true-gap witness* of $m$ in $\sigma$.

5. A subset $\sigma \subseteq \operatorname{Gens}(I)$ is called *potentially-type-2* if it has a bridge not dominating any of its true gaps, and *type-1* if it has a true gap not dominating any of its bridges. Moreover, $\sigma$ is called *type-2* if it is potentially-type-2 and whenever there exists another potentially-type-2 $\sigma'$ such that

$$\sigma' \setminus \{\operatorname{sb}(\sigma')\} = \sigma \setminus \{\operatorname{sb}(\sigma)\},$$

we have $\operatorname{sb}(\sigma') \succ \operatorname{sb}(\sigma)$.

We provide an explicit example of these concepts.

**Example 6.** Consider the ideal $I = (xw, xy, yz, zw)$ with the total order $wx \succ xy \succ yz \succ zw$.

(a) Let $\sigma_1 = \{xw, xy, yz\}$. It is clear that $zw$ is the only true gap and $xy$ is the only bridge of $\sigma_1$, and by definition, $\sigma_1$ is type-1.

(b) Let $\sigma_2 = \{xw, xy, zw\}$. It is clear that $yz$ is the only gap and $xw$ is the only bridge of $\sigma_2$. However, $yz$ is not a true gap of $\sigma_2$, and by definition, $\sigma_2$ is potentially-type-2. Moreover, $\sigma_2$ is not type-2 since for $\sigma_2' = \{xy, yz, zw\}$ (which one can check to be potentially-type-2), we have

$$\sigma_2' \setminus \{\operatorname{sb}(\sigma_2')\} = \sigma_2 \setminus \{\operatorname{sb}(\sigma_2)\},$$

and $\operatorname{sb}(\sigma_2) = xy \succ yz = \operatorname{sb}(\sigma_2')$.

(c) Let $\sigma_3 = \{xw, xy, yz, zw\}$. All the elements of $\sigma_3$ are its bridges. Hence $\sigma_3$ is potentially-type-2, and by definition, it is easy to see that it is indeed type-2.

We recall that Definition 5 (5) is an equivalent characterization of type-1, type-2, and potentially-type-2 sets, by [8, Theorem 2.24]. Originally, type-1 and type-2 subsets were defined to be the sources and targets, respectively, of the corresponding Morse matching. The following result then follows.

**Proposition 7** ([8, Theorem 2.24])**.** *Let $\sigma$ be a subset of* $\operatorname{Gens}(I)$. *Then, $\sigma$ is Barile-Macchia-critical if and only if it is neither type-1 nor type-2.*

The following theorem describes Lyubeznik and Barile-Macchia resolutions with respect to a given total order $(\succ)$ on $\operatorname{Gens}(I)$.

**Theorem 8** ([3, Proposition 2.2 and Theorem 3.2])**.** *Let $I$ be a monomial ideal with a fixed total order $(\succ)$ on* $\operatorname{Gens}(I)$. *Let $\mathcal{F}$ be the Lyubeznik (respectively, Barile-Macchia) resolution of $S/I$ with respect to $(\succ)$. Then, for any integer $i$, a basis of $\mathcal{F}_i$ can be identified with the collection of Lyubeznik-critical (respectively, Barile-Macchia-critical) subsets of* $\operatorname{Gens}(I)$ *with exactly $i$ elements.*

**Example 9.** Consider our running example in Examples 4 and 6 of $I = (xw, xy, yz, zw)$ in the polynomial ring $\Bbbk[w, x, y, z]$ with the total ordering $xw \succ xy \succ yz \succ zw$. The corresponding Lyubeznik resolution of $S/I$ is

$$\begin{array}{ccccccc} & & S\{xw,xy\} & & & & \\ & & \oplus & & S\{xw\} & & \\ & & S\{xw,zw\} & & \oplus & & \\ S\{xw,xy,zw\} & & \oplus & & S\{xy\} & & \\ \oplus & \to & S\{xy,zw\} & \to & \oplus & \to & S\emptyset \\ S\{xy,yz,zw\} & & \oplus & & S\{yz\} & & \\ & & S\{xy,xz\} & & \oplus & & \\ & & \oplus & & S\{zw\} & & \\ & & S\{zy,zw\} & & & & \end{array}$$

and the corresponding Barile-Macchia resolution of $S/I$ is

$$\begin{array}{ccccccc} & & S\{xw,xy\} & & S\{xw\} & & \\ & & \oplus & & \oplus & & \\ & & S\{xw,zw\} & & S\{xy\} & & \\ S\{xy,yz,zw\} & \to & \oplus & \to & \oplus & \to & S\emptyset \\ & & S\{xy,xz\} & & S\{yz\} & & \\ & & \oplus & & \oplus & & \\ & & S\{zy,zw\} & & S\{zw\} & & \end{array}$$

where the subsets of $\mathrm{Gens}(I)$ here represents the respective *-critical ones.

### 2.4 Bridge-friendly monomial ideals

The terminology "bridge-friendly monomial ideals" was introduced in [8] to ease the process of identifying Barile-Macchia-critical subsets of the generators. The motivation for this concept is partly that potentially-type-2 subsets are easier to check than type-2 subsets.

**Definition 10.** A monomial ideal $I$ is called *bridge-friendly* (with respect to $(\succ)$) if all potentially-type-2 subsets of $\mathrm{Gens}(I)$ are type-2. Equivalently, $I$ is bridge-friendly if and only if Barile-Macchia-critical subsets of $\mathrm{Gens}(I)$ are precisely the ones that have neither bridges nor true gaps (see [8, Corollary 2.28]).

The following is known to experts (cf. [13, Lemma 2.1]): a set $\sigma \in 2^{\mathrm{Gens}(I)}$ is a face of the Scarf complex of $I$ if and only if $\sigma$ has neither bridges nor gaps. This characterization is similar to the definition of Barile-Machia-critical sets of a bridge-friendly ideal. No direct relation is known between bridge-friendly ideals and those that admit the minimal Scarf resolution. One can verify that $I(C_3)$ is bridge-friendly using the code in Appendix A, but its Scarf complex is not a resolution ([14, Theorem 8.3]). As far as we know, it is unknown if the reverse implication holds, which we leave here as a question.

**Question 11.** Is it true that if a monomial ideal $I$ admits the minimal Scarf resolution, then it is bridge-friendly?

As will be seen later, Theorem 45—together with [14, Theorem 8.3]—implies that the answer to the above question is "yes" if $I$ is a power of an edge ideal.

It turns out that generic monomial ideals (in the sense of [4]), and thus most monomial ideals, are bridge-friendly (see [8, Theorem 5.4]), and the Barile-Macchia resolutions of bridge-friendly ideals are minimal (see [8, Theorem 2.29]). We shall identify the failure of being bridge-friendly.

For the remainder of the section, let $\sigma$ denote a non-empty subset of $\operatorname{Gens}(I)$.

**Proposition 12** ([8, Proposition 2.21])**.** *A monomial $m$ is a gap of $\sigma$ such that $\operatorname{sb}(\sigma \cup \{m\}) = m$ if and only if $m$ is a true gap of $\sigma$ that does not dominate any bridge of $\sigma$.*

**Lemma 13.** *A monomial ideal $I$ is not bridge-friendly with respect to $(\succ)$ if and only if there exist a type-1 set $\tau \subseteq \operatorname{Gens}(I)$ and monomials $m_1 \succ m_2$ in $\operatorname{Gens}(I)$ such that:*

1. *The monomials $m_1$ and $m_2$ are true gaps of $\tau$ that do not dominate any bridges (of $\tau$). In particular, $m_1, m_2 \notin \tau$.*

2. *The sets $\tau \cup \{m_1\}$ and $\tau \cup \{m_2\}$ are potentially-type-2.*

*In this case, $m_2$ can be chosen to be the smallest true gap of $\tau$. Moreover, under these conditions, there exists a monomial $m_3 \prec m_2$ such that $m_3$ is a bridge of $\tau \cup \{m_1, m_2\}$. In particular, $m_3 \in \tau$.*

*Proof.* The ideal $I$ is not bridge-friendly if and only if there exists a set $\sigma \in \operatorname{Gens}(I)$ that is potentially-type-2, but not type-2. By definition, this is equivalent to saying that there exists a different potentially-type-2 set $\sigma' \subseteq \operatorname{Gens}(I)$ such that

$$\sigma' \setminus \{\operatorname{sb}(\sigma')\} = \sigma \setminus \{\operatorname{sb}(\sigma)\},$$

and $\operatorname{sb}(\sigma) \succ \operatorname{sb}(\sigma')$. Set $m_1 = \operatorname{sb}(\sigma)$, $m_2 = \operatorname{sb}(\sigma')$, and $\tau = \sigma \setminus \{m_1\} = \sigma' \setminus \{m_2\}$. Then, using Proposition 12, one can see that this condition is equivalent to (1) and (2), as claimed.

We remark that $m_2$ is a true gap of $\tau$ by definition. We can choose $m_2$ to be the smallest true gap of $\tau$ since then $m_2 = \operatorname{sb}(\tau \cup \{m_2\})$ and $\tau \cup \{m_2\}$ is potentially-type-2 by [8, Remark 2.26]. Thus, the previous part can still proceed in this setting. Moreover, since $\sigma$ is potentially-type-2, $m_2$ is not a true gap of $\sigma$ by definition, and so $\tau \cup \{m_1, m_2\} = \sigma \cup \{m_2\}$ has a bridge $m_3 \prec m_2$, as claimed. $\square$

The equivalent condition to non-bridge-friendliness, in fact, says a lot more about the three monomials $m_1 \succ m_2 \succ m_3$. We will first introduce some new terminology.

**Definition 14.** Let $I$ be a monomial ideal, $x$ a variable, $\sigma \subset \operatorname{Gens}(I)$, and $m, m' \in \sigma$. We say that two monomials $m$, $m'$ *share a factor $x^n$ unique within $\sigma$* if $n \geqslant 1$ and we have the following:

(i) $x^n$ divides $m$ and $m'$, and $x^{n+1}$ does not divide either.

(ii) $x^n \nmid m''$ for any $m'' \in \sigma \setminus \{m, m'\}$.

This concept appears many times when we work with gaps that are not true gaps, as shown in the next result.

**Proposition 15.** *If a monomial $m \in \mathrm{Gens}(I)$ is a gap, but not a true gap of $\sigma$, then $m$ and any of its non-true-gap witnesses $m'$ share a factor unique within $\sigma \cup \{m\}$.*

*Proof.* Since $m'$ is not a bridge of $\sigma$, there exists a factor $x^n$ such that $x^n \mid m'$, $x^{n+1} \nmid m'$, and $x^n \nmid m''$ for each $m'' \in \sigma \setminus \{m'\}$. Moreover, since $m'$ is a bridge of $\sigma \cup \{m\}$, we have

$$x^n \mid m' \mid \mathrm{lcm}(\sigma \cup \{m\}),$$

and thus $x^n \mid m$. On the other hand, we have $x^{n+1} \nmid m$ since $\mathrm{lcm}(\sigma \cup \{m\}) = \mathrm{lcm}(\sigma)$. $\square$

*Remark* 16. In the proof of Lemma 13, we showed that $m_2$ is a gap of $\tau \cup \{m_1\}$ and set $m_3$ to be a non-true-gap witness of $m_2$ in $\tau \cup \{m_1\}$. By Proposition 15, $m_2$ and $m_3$ share a factor unique within $\tau \cup \{m_1\}$. It can be checked that $m_3$ is also a non-true-gap witness of $m_1$ in $\tau \cup \{m_2\}$. Therefore, $m_1$ and $m_3$ share a factor unique within $\tau \cup \{m_2\}$.

We immediately obtain a condition to determine true gaps:

**Corollary 17.** *If a monomial $m$ is a gap of $\sigma$ such that*

$$m \mid \mathrm{lcm}(\{m' \in \sigma \colon m' \succ m \textit{ or } m \textit{ and } m' \textit{ do not share a factor unique within } \sigma\}),$$

*then $m$ is a true gap of $\sigma$.*

*Proof.* Suppose that $m$ is not a true gap of $\sigma$. We will derive a contradiction. Indeed, there exists a non-true-gap witness $m_0 \prec m$ of $m$ in $\sigma$. By Proposition 15, $m_0$ and $m$ share a unique factor $x^n$ within $\sigma \cup \{m\}$. In particular, we have $x^n \nmid \mathrm{lcm}(\sigma \setminus \{m_0\})$, and

$$m_0 \notin \{m' \in \sigma \colon m' \succ m \text{ or } m \text{ and } m' \text{ do not share a factor unique within } \sigma\}.$$

Therefore, $x^n$ divides $m$, and does not divide

$$\mathrm{lcm}(\{m' \in \sigma \colon m' \succ m \text{ or } m \text{ and } m' \text{ do not share a factor unique within } \sigma\}),$$

a contradiction, as desired. $\square$

### 2.5 Restriction lemmas and HHZ-subideals

There is a one-to-one correspondence between $\mathbb{N}^r$ and the set of monomials in $S$. Thus, at times we will abuse notations and use monomials (in $S$) and vectors (in $\mathbb{N}^r$) interchangeably.

Fix a monomial ideal $I \subseteq S$ and a monomial $m \in S$. Let $I^{\leqslant m}$ be the monomial ideal generated by elements of $\mathrm{Gens}(I)$ that divide $m$. It is clear that $I^{\leqslant m}$ is always a subideal of $I$. This notation was introduced in [16], although the idea briefly appeared prior in [4]. We will call $I^{\leqslant m}$ a *Herzog-Hibi-Zheng-subideal* (with respect to $m$) of $I$, or *HHZ-subideal* for short.

Let $\mathcal{F} : 0 \to F_p \to F_{p-1} \to \cdots \to F_1 \to F_0 \to 0$ be a (minimal) $\mathbb{N}^r$-graded free resolutions of $S/I$, with $F_i = \bigoplus_{j\in\mathbb{Z}} R(-q_{ij})$. Let $\mathcal{F}^{\leqslant m}$ be the subcomplex of $\mathcal{F}$, whose $i$-th module is $\bigoplus_{q_{ij}\leqslant m} R(-q_{ij})$. Here, for monomials $a$ and $b$, we write $a \leqslant b$ to mean $a \mid b$.

By [16, Lemma 4.4], $\mathcal{F}^{\leqslant m}$ is a (minimal) $\mathbb{N}^r$-graded free resolution of $S/I^{\leqslant m}$. This result is referred to as the *Restriction Lemma.* We will also state analogs of the Restriction Lemma for Lyubeznik, Barile-Macchia resolutions, and the bridge-friendly property. These come from the observation that, for a fixed a monomial $m$, faces $\sigma$ in the simplex over the vertices $\operatorname{Gens}(I)$, whose least common multiple of the vertices divides $m$, are exactly the faces of the simplex over the vertices $\operatorname{Gens}(I^{\leqslant m})$.

**Lemma 18.** *(Restriction Lemma for Barile-Macchia/Lyubeznik resolutions) Let $I$ be a monomial ideal and let $m$ be a monomial in $S$. If $\mathcal{F}$ is a (minimal) Barile-Macchia (respectively, Lyubeznik) resolution of $S/I$, then $\mathcal{F}^{\leqslant m}$ is a (minimal) Barile-Macchia (respectively, Lyubeznik) resolution of $S/I^{\leqslant m}$.*

*Proof.* We will prove the result when $\mathcal{F}$ is a Barile-Macchia resolution of $S/I$, remarking that the proof for when it is a Lyubeznik resolution follows from similar arguments. Let $(\succ)$ be a total order on $\operatorname{Gens}(I)$. As we have $\operatorname{Gens}(I^{\leqslant m}) \subseteq \operatorname{Gens}(I)$, the order $(\succ)$ also defines a total order on $\operatorname{Gens}(I^{\leqslant m})$, which we denote $(\succ_m)$. We have the following claim.

**Claim 19.** *Given $\sigma \in 2^{\operatorname{Gens}(I^{\leqslant m})}$. If $m_i$ is a bridge of $\sigma$ for some $m_i \in \operatorname{Gens}(I)$, then $m_i \in \operatorname{Gens}(I^{\leqslant m})$. In particular, we have $\operatorname{sb}_{\succ}(\sigma) = \operatorname{sb}_{\succ_m}(\sigma)$.*

*Proof.* Indeed, if $m_i$ is a bridge of $\sigma$, then

$$m_i \mid \operatorname{lcm}(\sigma) \mid m$$

where the second divisibility comes from the assumption that $\sigma \in 2^{\operatorname{Gens}(I^{\leqslant m})}$. Since $m_i \in \operatorname{Gens}(I)$ and $m_i \mid m$, it follows from definition that $m_i \in \operatorname{Gens}(I^{\leqslant m})$, as desired. □

Recall that by definition, whether a given set $\sigma \in 2^{\operatorname{Gens}(I^{\leqslant m})}$ is potentially-type-2, type-2, or type-1 (as a set in $2^{\operatorname{Gens}(I^{\leqslant m})}$) depends on the set

$$\{\tau \in 2^{\operatorname{Gens}(I^{\leqslant m})} :\ \operatorname{lcm}(\tau) = \operatorname{lcm}(\sigma)\}$$

and what $\operatorname{sb}_{\succ_m}(\tau)$ is, for each such $\tau$. By Claim 19, we have

$$\{\tau \in 2^{\operatorname{Gens}(I^{\leqslant m})} :\ \operatorname{lcm}(\tau) = \operatorname{lcm}(\sigma)\} = \{\tau \in 2^{\operatorname{Gens}(I)} :\ \operatorname{lcm}(\tau) = \operatorname{lcm}(\sigma)\}$$

and $\operatorname{sb}_{\succ}(\tau) = \operatorname{sb}_{\succ_m}(\tau)$, for each such $\tau$. Therefore, a given set $\sigma \in 2^{\operatorname{Gens}(I^{\leqslant m})}$ is potentially-type-2, type-2, or type-1 as a set in $2^{\operatorname{Gens}(I^{\leqslant m})}$ if and only if it is of the same type as a set in $2^{\operatorname{Gens}(I)}$, and its least common multiple divides $m$. Therefore, the Barile-Macchia resolution of $S/I^{\leqslant m}$ (w.r.t. $(\succ_m)$) is exactly the Barile-Macchia resolution of $S/I$ (w.r.t. $(\succ)$), restricted to only faces of least common multiples that divide $m$. The result now follows. □

The following result follows from similar arguments coupled with Claim 19. We thus state it without proof.

**Lemma 20.** *(Restriction Lemma for bridge-friendly ideals) Let $I$ be a monomial ideal and let $m$ be a monomial in $S$. If $I$ is bridge-friendly with respect to a total order $(\succ)$, then so is $I^{\leqslant m}$ with respect to the total order induced from $(\succ)$.*

As an application of Lemmas 18 and 20, we arrive at the following corollary.

**Corollary 21.** *Let $G$ be a graph and let $H$ be an induced subgraph of $G$. For any integer $n$, if $I(G)^n$ is Lyubeznik, Barile-Macchia, or bridge-friendly, then so is $I(H)^n$.*

*Proof.* It suffices to show that $I(H)^n$ is an HHZ-subideal of $I(G)^n$, as the results would then follow from Lemmas 18 and 20. Let $m$ be the product of all the vertices in $H$. Then we claim that

$$I(H)^n = (I(G)^n)^{\leqslant m^n}.$$

Consider $m' \in \operatorname{Gens}(I(G)^n)$ where $m' \mid m^n$. Then $m' = (x_{11}x_{21}) \cdots (x_{1n}x_{2n}) \mid m^n = \left(\prod_{x\in H} x\right)^n$. Thus all the vertices $x_{11}, x_{21}, \dots, x_{1n}, x_{2n}$ are all in $V(H)$. Therefore $m' \in I(H)^n$, and thus $m' \in \operatorname{Gens}(I(H)^n)$ as all minimal generators of $I(H)^n$ are of degree $2n$. Now consider a generator $m'' \in \operatorname{Gens}(I(H)^n)$. Then $m'' \in I(G)^n$ and therefore $m'' \in \operatorname{Gens}(I(G)^n)$ as all minimal generators of $I(G)^n$ are of degree $2n$. Moreover we can write $m'' = (x'_{11}x'_{21}) \cdots (x'_{1n}x'_{2n})$ for some $x'_{11}, x'_{21}, \dots, x'_{1n}, x'_{2n} \in V(H)$ where $x'_{1i} \neq x'_{2i}$ for any $i \in [n]$. Thus $m'' \mid \left(\prod_{x\in H} x\right)^n = m^n$. To conclude, we have

$$\operatorname{Gens}(I(H)^n) = \operatorname{Gens}((I(G)^n)^{\leqslant m^n}),$$

as claimed. The result now follows. ☐

The following result is obvious from the definition, which allows us to discuss the property of being Lyubeznik, Barile-Macchia, and bridge-friendly for a monomial ideal $I$ and $mI$ interchangeably, where $m$ is any monomial.

**Proposition 22.** *Let $I$ be a monomial ideal and let $m$ be a monomial. Then $I$ is Lyubeznik, Barile-Macchia, or bridge-friendly, if and only if so is $mI$.*

*Proof.* We will prove the result when $I$ is Barile-Macchia, remarking that the proof in the other cases follows from similar arguments. Let $m_1 \succ \cdots \succ m_q$ be a total order on $\operatorname{Gens}(I) = \{m_1, \dots, m_q\}$ such that $I$ is Barile-Macchia with respect to $(\succ)$. Then $\operatorname{Gens}(mI) = \{mm_1, \dots, mm_q\}$. Let $mm_1 > \cdots > mm_q$ be a total order on $\operatorname{Gens}(mI)$. For a $\sigma \in 2^{\operatorname{Gens}(I)}$, let $m\sigma$ denote the set $\{mm' \colon m' \in \sigma\}$. We have the following claim.

**Claim 23.** *Given $\sigma \in 2^{\operatorname{Gens}(I)}$ and $m' \in \operatorname{Gens}(I)$. Then $m'$ is a bridge of $\sigma$ if and only if $mm'$ is a bridge of $m\sigma$. In particular, we have $\operatorname{sb}_{>}(m\sigma) = m \operatorname{sb}_{\succ}(\sigma)$.*

*Proof.* Indeed, we have

$$\begin{aligned} m' \text{ is a bridge of } \sigma &\Leftrightarrow m' \in \sigma \text{ and } m' \mid \operatorname{lcm}(\sigma \setminus \{m'\}) \\ &\Leftrightarrow mm' \in m\sigma \text{ and } mm' \mid \operatorname{lcm}(m\sigma \setminus \{mm'\}) \\ &\Leftrightarrow mm' \text{ is a bridge of } m\sigma, \end{aligned}$$

as desired. □

Recall that by definition, the set of potentially-type-2, type-2, or type-1 subsets of $2^{\mathrm{Gens}(J)}$, for a monomial $J$, depends entirely on the value of the smallest bridge function. Claim 23 implies that there is a bijection between the divisibility relations in $\mathrm{Gens}(I)$ and $\mathrm{Gens}(mI)$. Therefore, a given set $\sigma \in 2^{\mathrm{Gens}(I)}$ is potentially-type-2, type-2, or type-1 if and only if $m\sigma$ is of the same type as a set in $2^{\mathrm{Gens}(mI)}$. Therefore, the Barile-Macchia resolution of $S/I$ (w.r.t. $(\succ)$) is exactly the Barile-Macchia resolution of $S/mI$ (w.r.t. $(>)$). The result now follows. □

The following result serves as an example of HHZ-subideals, which we will use in later parts of the article.

**Proposition 24.** *Let $G$ be the $P_4$, $C_3$, or $C_4$ graph. Then for each positive integer $n$, there exists a generator $f \in \mathrm{Gens}(I(G))$ such that the ideal $fI(G)^n$ is an HHZ-subideal of $I(G)^{n+1}$.*

*Proof.* Fix an integer $n$. We will prove the statement in each case.

1. $G = P_4$, i.e., $I \coloneqq I(G) = (x_1x_2, x_2x_3, x_3x_4)$.
   Set $J \coloneqq (x_1x_2, x_2x_3)$, $m \coloneqq x_2^n(x_1x_3x_4)^{n+1}$, and $f \coloneqq x_3x_4$. We have

   $$I^{n+1} = J^{n+1} + fI^n.$$

   In fact, in this case, it is clear that

   $$\mathrm{Gens}(I^{n+1}) = \mathrm{Gens}(J^{n+1}) \sqcup \mathrm{Gens}(fI^n).$$

   It is straightforward that all elements of $\mathrm{Gens}(I^{n+1})$ that divide $m$ come from $\mathrm{Gens}(fI^n)$. Therefore, the ideal $fI^n$ is the HHZ-subideal of $I^{n+1}$ with respect to $m$, as desired.

2. Similar arguments apply to the case where $G = C_3$, i.e., $I(G) = (x_1x_2, x_2x_3, x_1x_3)$.

3. $G = C_4$, i.e., $I \coloneqq I(G) = (x_1x_2, x_2x_3, x_3x_4, x_1x_4)$.
   Set $J \coloneqq (x_1x_2, x_2x_3, x_3x_4)$, $m \coloneqq (x_2x_3)^n(x_1x_4)^{n+1}$, and $f \coloneqq x_1x_4$. We have

   $$I^{n+1} = J^{n+1} + fI^n.$$

   We claim that the HHZ-subideal of $I^{n+1}$ with respect to $m$ is exactly $fI^n$. Indeed, consider any element $g$ in $\mathrm{Gens}(I^{n+1})$. If $g$ belongs to $fI^n$, then it clearly divides

$m$. On the other hand, if it does not, i.e., $g$ is not divisible by $f = x_1x_4$, then $g$ must be of the form

$$(x_1x_2)^a(x_2x_3)^{n+1-a} \text{ or } (x_2x_3)^a(x_3x_4)^{n+1-a}$$

for some integer $a$. In either case, $g$ does not divide $m$. Thus the claim holds, and therefore $fI^n$ is an HHZ-subideal of $I^{n+1}$. $\square$

We remark that Proposition 24 does not necessarily hold for monomial ideals or edge ideals in general. Consider $I = (xx_1, xx_2, xx_3)$. One can check that for any monomial $m$, the ideal $(I^3)^{\leqslant m}$ does not have the same total Betti numbers as $fI^2$ for any generator $f \in \operatorname{Gens}(I)$ (we note that it suffices to consider only the monomials $m$ that divide $\operatorname{lcm}(\operatorname{Gens}(I^3))$). This implies that $fI^2 \neq (I^3)^{\leqslant m}$ for any monomial $m$ and any generator $f \in \operatorname{Gens}(I)$. In other words, $fI^2$ is not an HHZ-subideal of $I^3$. As a consequence, $fI(K_{1,3})^2$ is not an HHZ-subideal of $I(K_{1,3})^3$ for any generator $f \in \operatorname{Gens}(I(K_{1,3}))$.

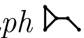
## 3 Powers of edge ideals with minimal Lyubeznik resolutions

In this section, we classify all graphs $G$ and integers $n$ such that $I(G)^n$ is Lyubeznik.

### 3.1 Lyubeznik edge ideals

We will begin with the case $n = 1$, which is considerably more difficult, and follow with the case $n \geqslant 2$. We shall first identify the forbidden structures for Lyubeznik graphs.

**Proposition 25.** *Let $G$ be one of the following graphs:*

1. *The 5-path graph $P_5$.*
2. *The 4-cycle graph $C_4$.*
3. *The 5-cycle graph $C_5$.*
4. *The complete graph $K_4$* .
5. *The kite graph* .
6. *The gem graph* .
7. *The tadpole graph* .
8. *The butterfly graph* .
9. *The net graph* .

*Then $I(G)$ is not Lyubeznik.*

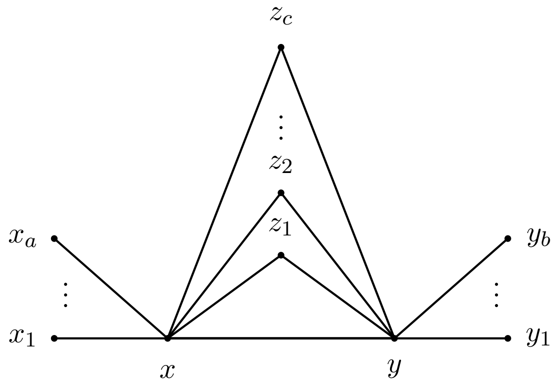
*Proof.* Verified using the code in Appendix A $\square$

Proposition 25, coupled with Corollary 21, shows that any graph whose edge ideal is Lyubeznik cannot contain the graphs listed in Proposition 25 as induced subgraphs. The following construction plays a key role in the classification of Lyubeznik graphs.

**Definition 26.** Let $a, b, c$ be nonnegative integers. We define $L(a, b, c)$ to be the graph whose vertex and edge sets are:

$$V(L(a,b,c)) = \{x, y\} \sqcup \{x_i\}_{i=1}^a \sqcup \{y_j\}_{j=1}^b \sqcup \{z_k\}_{k=1}^c,$$

$$E(L(a,b,c)) = \{xy\} \sqcup \{xx_i\}_{i=1}^a \sqcup \{yy_j\}_{j=1}^b \sqcup \{xz_k\}_{k=1}^c \sqcup \{yz_k\}_{k=1}^c.$$




Roughly speaking, $L(a, b, c)$ consists of exactly $c$ triangles $xyz_k$, for $k = 1, \dots, c$, that share a common edge $\{x, y\}$, together with $a$ leaves $\{x_i\}_{i=1}^{a}$ attached to $x$, and $b$ leaves $\{y_j\}_{j=1}^{b}$ attached to $y$. In particular, $L(a, b, c)$ has $a + b + c + 2$ vertices.

**Example 27.**

0. By convention, $L(0, 0, 0)$ is the graph consisting of exactly one edge.
1. The graph $L(0, 0, c)$ is exactly $c$ triangles sharing one edge. In particular, $L(0, 0, 1)$ is the triangle $C_3$, and $L(0, 0, 2)$ is a diamond $\boxslash$.
2. The graph $L(a, 0, 0)$ is the star graph $K_{1,a+1}$.
3. The graph $L(a, b, 0)$ is a gap-free tree. In fact, $\{L(a, b, 0) \mid a, b \in \mathbb{Z}_{\geqslant 0}\}$ are exactly the connected graphs that admit minimal Scarf resolutions [14, Theorem 8.3].

The forbidden structures in Proposition 25 can now be characterized using graphs of the form $L(a, b, c)$.

**Proposition 28.** *A connected graph is isomorphic to $L(a, b, c)$ for some integers $a, b, c$ if and only if it does not contain, as an induced subgraph, any of the graphs listed in Proposition 25.*

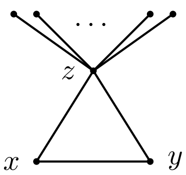
*Proof.* It is clear that for any $a, b$, and $c$, the graph $L(a, b, c)$ does not contain any of the graphs listed in Proposition 25 as an induced subgraph. Hence, we will only show the other direction. Let $G$ be a connected graph that does not contain any of the forbidden graphs. We remark that since $G$ does not contain $P_5$, $C_4$, or $C_5$, it does not contain $P_n$ for any $n \geqslant 5$ or $C_n$ for any $n \geqslant 4$. In particular, $G$ is chordal. Moreover, since $G$ does not contain the $K_4$ graph, it does not contain the $K_n$ graph for any $n \geqslant 4$.

Suppose $G$ contains a cycle with a unique chord as an induced subgraph. By Proposition 25 (2), that cycle must be a diamond. So we can assume that $G$ contains the diamond graph $\boxslash$ formed by $w, x, y, z$ with a chord $wy$. We will show that $G$ is exactly $L(a, b, c)$ for some integers $a, b, c$. Indeed, consider any vertex $u$ of $G$ that does not belong to this diamond. It suffices to show that $N(u)$, the set of neighbors of $u$ among $w, x, y, z$, is $\{w\}$, $\{y\}$ or $\{w, y\}$. Suppose otherwise. Then by symmetry we can assume that $x \in N(u)$. We will derive a contradiction after considering all possibilities.

- $N(u) = \{x\}$: Then $G$ contains the kite graph, a contradiction.
- $N(u) = \{x, w\}$ or $N(u) = \{x, y\}$: Then $G$ contains the gem graph formed by $w, x, y, z, u$, a contradiction.
- $N(u) = \{x, z\}$: Then $G$ contains the $C_4$ graph formed by $w, x, u, z$, a contradiction.
- $N(u) = \{x, w, y\}$ or $N(u) = \{x, w, y, z\}$: Then $G$ contains the $K_4$ graph formed by $w, x, y, u$, a contradiction.
- $N(u) = \{x, w, z\}$ (or $N(u) = \{x, y, z\}$, resp): Then $G$ contains the $C_4$ graph formed by $x, y, z, u$ (or $w, x, y, u$, resp), a contradiction.

We shall consider the case that $G$ does not contain a cycle with a unique chord. If $G$ is a tree, it is straightforward from Proposition 25 (1) that $G$ is $L(a, b, 0)$ for some integers $a, b$. If $G$ is not a tree, then by Proposition 25 (2), it contains a triangle, whose vertices we shall denote by $x, y, z$. By [23, Theorem 2.2], either $G$ is a complete graph, or one vertex of the maximal complete induced subgraph that contains this triangle is a 1-cutset of $G$. Since $G$ cannot contain any $K_n$ graph for any $n \geqslant 4$, the triangle formed by $x, y, z$ must be the maximal complete induced subgraph of $G$ that contains itself. Thus, we may assume that $z$ is a 1-cutset of $G$. By definition, this means that the graph $G \setminus \{z\}$ has at least two connected components. Let $V$ denote the set of vertices in the connected component that contains $x$ and $y$, and $W$ the set of the remaining vertices.

Since $G$ does not contain the tadpole graph, no vertex $w \in W$ satisfies $\mathrm{dist}_G(w, z) \geqslant 2$. In other words, all vertices in $W$ are connected to $z$ in $G$. On the other hand, since $G$ does not contain the butterfly graph, no two vertices in $W$ are connected. Therefore, the subgraph induced by $x, y, z$ and vertices in $W$ is:




Now, we consider vertices in $V$. Since $G$ does not contain any $n$-path graph $P_n$ where $n \geqslant 5$, no vertex $v \in V$ satisfies $\mathrm{dist}_G(v, x) \geqslant 2$ or $\mathrm{dist}_G(x, y) \geqslant 2$. In other words, all vertices in $V$ are connected to either $x$ or $y$ in $G$. Since $G$ does not contain the kite graph, no vertex in $V$ is connected to both $x$ and $y$ in $G$. Suppose there exist two vertices $x', y' \in X$ such that $x'$ is connected to $x$, not $y$, and $y'$ is connected to $y$, not $x$. If $x'$ and $y'$ form an edge, then $G$ contains a $C_4$ graph formed by $x', y', x, y$, a contradiction. Otherwise, $G$ contains the net graph formed by $x, y, z, x', y', z'$ where $z'$ is a fixed vertex in $W$, another contradiction. Thus, all vertices in $V$ are either connected to $x$ and not $y$, or vice versa. Thus $G = L(a, b, 1)$ where $a = |V| - 2$ and $b = |W|$. $\square$

Before proceeding to show that the edge ideals of $L(a, b, c)$ graphs are Lyubeznik, we make the following observation.

*Remark* 29. Let $\mathcal{F}$ be a Lyubeznik resolution of $S/I$, where $I \subseteq S$ is a monomial ideal. Recall that $\mathcal{F}$ is a simplicial resolution (see [19, Section 6]). Then, by Theorem 8, for any integer $i$, a basis of $\mathcal{F}_i$ can be identified with the collection of Lyubeznik-critical subsets of $\operatorname{Gens}(I)$ with exactly $i$ elements. Moreover, the differentials are as follows:

$$\partial(\sigma) = \sum_{\tau \subseteq \sigma, |\tau| = |\sigma|-1} \pm 1 \frac{\operatorname{lcm}(\sigma)}{\operatorname{lcm}(\tau)} \tau,$$

where $\sigma$ is a Lyubeznik-critical subset. We refer to [19, Construction 4.1] for the fact that the coefficients are either 0 or $\pm 1$. From this description, it is easy to see that a Lyubeznik resolution is minimal if and only if each Lyubeznik-critical subset does not have any bridge. We shall make use of this observation often in this section.

**Proposition 30.** *If $G = L(a,b,c)$, for some nonnegative integers $a, b$, and $c$, then the edge ideal $I(G)$ is Lyubeznik.*

*Proof.* If $G = L(0,b,0) = K_{1,b+1}$, then it is clear that $I(G)$ is Taylor, and thus any Lyubeznik resolution of $I(G)$ is minimal. For the rest of the proof, we assume that $G = L(a,b,c)$ where $a \geqslant 1$ or $c \geqslant 1$. Consider the following total order on $\operatorname{Gens}(I(G))$:

$$yy_b \succ \cdots \succ yy_1 \succ xx_a \succ \cdots \succ xx_1 \succ yz_c \succ \cdots \succ yz_1 \succ xz_c \succ \cdots \succ xz_1 \succ xy.$$

We will show that the Lyubeznik resolution of $I(G)$ induced from this total order is minimal. By definition, it suffices to show that any Lyubeznik-critical subset $\sigma$ of $\operatorname{Gens}(I(G))$ has no bridge.

To this end, let $\sigma$ be Lyubeznik-critical; we have

$$xy \nmid \operatorname{lcm}\left(\sigma \cap \{e \in E(G) \colon e \succ xy\}\right).$$

In particular, if an edge incident to $x$ is in $\sigma$, then $\sigma$ does not contain any edge incident to $y$, and vice versa. Suppose that $\sigma$ contains a leaf edge which, without loss of generality, we can assume to be $xx_1$. Then $\sigma$ is a subset of

$$\{xy\} \sqcup \{xx_i\}_{i=1}^{a} \sqcup \{xz_k\}_{k=1}^{c},$$

and thus has no bridge. On the other hand, suppose that $\sigma$ contains no leaf edges. If $\sigma = \{xy\}$, then it clearly has no bridge. Otherwise, without loss of generality, we can assume that $\sigma$ contains $xz_1$. By the same observation, $\sigma$ is then a subset of

$$\{xy\} \sqcup \{xz_k\}_{k=1}^{c},$$

and thus has no bridge. This concludes the proof. $\square$

Combining the preceding three propositions, we obtain the first main result of the paper.

**Theorem 31.** *The edge ideal $I(G)$ is Lyubeznik if and only if $G$ is $L(a,b,c)$ for some nonnegative integers $a, b, c$.*

*Remark* 32. Theorem 31 can be naturally extended to work over not-necessarily-connected graphs. This is because the resolution of the edge ideal of an arbitrary graph can be obtained by tensoring the corresponding resolutions of its connected components, and the next lemma shows how the property of being Lyubeznik behaves with this operation.

**Lemma 33.** *Let $I$ and $J$ be monomial ideals of $R = \Bbbk[x_1, \dots, x_r]$ and $S = \Bbbk[y_1, \dots, y_s]$, respectively. Then $I + J$, the ideal of $R \otimes_\Bbbk S$ generated by $\operatorname{Gens}(I) \sqcup \operatorname{Gens}(J)$, is Lyubeznik if and only if so are $I$ and $J$.*

*Proof.* We first note that by construction, the ideal $I$ of $R$ is Lyubeznik if and only if so is $I(R \otimes_\Bbbk S)$. Thus, we will replace $I$ and $J$ by their respective extensions in $R \otimes_\Bbbk S$.

Assume that $I + J$ is Lyubeznik. Then since

$$I = (I+J)^{\leqslant \prod_{m \in \operatorname{Gens}(I)} m} \quad \text{and} \quad J = (I+J)^{\leqslant \prod_{m \in \operatorname{Gens}(J)} m},$$

the ideals $I$ and $J$ are Lyubeznik too by Lemma 18.

Conversely, assume that $I$ (resp, $J$) is Lyubeznik with respect to a total order $(\succ_I)$ on $\operatorname{Gens}(I)$ (resp, $(\succ_J)$ on $\operatorname{Gens}(J)$). Consider the total order $(\succ)$ on $\operatorname{Gens}(I+J)$ such that for $m, m' \in \operatorname{Gens}(I+J)$, we have $m \succ m'$ if and only if one of the following holds:

- $m \in \operatorname{Gens}(I)$ and $m' \in \operatorname{Gens}(J)$;
- $m, m' \in \operatorname{Gens}(I)$ and $m \succ_I m'$;
- $m, m' \in \operatorname{Gens}(J)$ and $m \succ_J m'$.

We have the following claim.

**Claim 34.** *For any subset $\sigma$ of $\operatorname{Gens}(I+J)$, we have*

$$\begin{aligned}\{m \in \operatorname{Gens}(I+J) \colon m \mid \operatorname{lcm}(\sigma_{\succ m})\} = {} & \{m' \in \operatorname{Gens}(I) \colon m' \mid \operatorname{lcm}((\sigma \cap \operatorname{Gens}(I))_{\succ_I m'})\} \cup \\ & \{m'' \in \operatorname{Gens}(J) \colon m'' \mid \operatorname{lcm}((\sigma \cap \operatorname{Gens}(J))_{\succ_J m''})\}\end{aligned}$$

*Proof.* Since monomials $\operatorname{Gens}(I)$ and $\operatorname{Gens}(J)$ are in different variables, a monomial $m$ in $\operatorname{Gens}(I)$ divides $\operatorname{lcm}(\sigma_{\succ m})$ if and only $m$ divides $\operatorname{lcm}(\sigma_{\succ m} \cap \operatorname{Gens}(I))$, and an analog holds for a monomial in $\operatorname{Gens}(J)$ as well. Moreover, remark that if $m \in \operatorname{Gens}(I)$, we have

$$\sigma_{\succ m} \cap \operatorname{Gens}(I) = (\sigma \cap \operatorname{Gens}(I))_{\succ_I m},$$

and again, an analog holds for a monomial in $\operatorname{Gens}(J)$ as well. The result then follows. ☐

Consider a Lyubeznik-critical subset $\sigma$ of $\operatorname{Gens}(I+J)$ (with respect to $(\succ)$). By definition, Claim 34 implies that $\sigma \cap \operatorname{Gens}(I)$ and $\sigma \cap \operatorname{Gens}(J)$ are Lyubeznik-critical with respect to $(\succ_I)$ and $(\succ_J)$, respectively. Since $I$ and $J$ are Lyubeznik (with respect to their respective order), $\sigma \cap \operatorname{Gens}(I)$ and $\sigma \cap \operatorname{Gens}(J)$ have no bridge. Since the monomials in $\operatorname{Gens}(I)$ and $\operatorname{Gens}(J)$ are in different variables, this implies that

$$\sigma = (\sigma \cap \operatorname{Gens}(I)) \cup (\sigma \cap \operatorname{Gens}(J))$$

has no bridge as well. Therefore, $I + J$ is Lyubeznik, as desired. ☐

We summarize the above discussion in the following theorem.

**Theorem 35.** *Let $G$ be a graph with $G_1, \ldots, G_t$ as its connected components. Then the edge ideal $I(G)$ is Lyubeznik if and only if for each $i \in [t]$, the graph $G_i$ is $L(a_i, b_i, c_i)$ for some nonnegative integers $a_i, b_i, c_i$.*

### 3.2 Higher powers of edge ideals which are Lyubeznik

For the remainder of this section, we will investigate Lyubeznik resolutions of higher powers of edge ideals of graphs.

**Proposition 36.** *Let $G$ be $K_{1,3}$, $P_4$, $C_3$, or $C_4$. Then, the ideal $I(G)^2$ is not Lyubeznik. Moreover, if $G$ is among the last three graphs of the given list, then $I(G)^n$ is not Lyubeznik for any $n \geqslant 2$.*

*Proof.* For the first three graphs in the list, $I(G)^2$ is generated by 6 monomials, so there are 6! possibilities for a total order on the minimal generating set. The first assertion for these graphs can be verified using the code in Appendix A. For $C_4$, one can verify using the code in Appendix A that the ideal

$$(x_1^2x_2^2,\ x_2^2x_3^2,\ x_1^2x_2x_4,\ x_1x_2^2x_3,\ x_2x_3^2x_4,\ x_1x_2x_3x_4),$$

an HHZ-subideal of $(x_1x_2,\ x_2x_3,\ x_3x_4,\ x_1x_4)^2$ with respect to $(x_1x_2x_3)^2x_4$, does not admit a minimal Lyubeznik resolution. Thus, the first assertion for $C_4$ follows from Lemma 18.

The second statement is a direct consequence of the first assertion, Proposition 24, and Lemma 18. $\square$

We are now ready to state the next main result of this section.

**Theorem 37.** *Let $G$ be a graph and let $n \geqslant 2$ be a positive integer. Then the ideal $I(G)^n$ is Lyubeznik if and only if one of the following holds:*

1. *$G$ is an edge; or*
2. *$n = 2$ and $G$ is the path $P_3$ on three vertices.*

*Proof.* If $G$ is an edge, then $I(G)^n$ is principal and hence, is Lyubeznik. If $n = 2$ and $G = P_3$, then we can assume that $I(G)^n = (a^2b^2, b^2c^2, ab^2c)$. One can verify with the code in Appendix A that this ideal is Lyubeznik. The “if” implication then follows.

We now prove the “only if” direction. Assume that $G$ is a graph such that $I(G)^n$ is Lyubeznik. By Proposition 36, $G$ does not contain $C_3, C_4, P_4$ as an induced subgraph. Since any $C_k$ or $P_k$, where $k \geqslant 5$, contains $P_4$ as an induced subgraph, $G$ does not contain these graphs, either. In other words, $G$ must be a star graph $K_{1,k}$ for some integer $k$.

If $n = 2$, then $G$ does not contain $K_{1,3}$ as an induced subgraph by Proposition 36. Thus $k \leqslant 2$, and $G$ is either an edge or a path of length 2.

Consider the case that $n > 2$. We will show that $I(K_{1,2})^n$ is not Lyubeznik. This would imply that $G$ does not contain $K_{1,2}$ as an induced subgraph and, hence, the desired statement would follow. To this end, we will exhibit that any Lyubeznik resolution of $S/I(K_{1,2})^n$ has length at least 3 and, thus, is not minimal, due to the Auslander-Buchsbaum-Serre theorem.

Since the Lyubeznik resolutions of $I(K_{1,2})$ and its powers are the same as those of the ideal generated by two variables, it suffices to consider the ideal $(x, y)$. Fix any total order $(\succ)$ on $\mathrm{Gens}((x, y)^n)$ and let $m_1 \prec m_2 \prec m_3$ be the three smallest monomials with respect to $(\succ)$. We can set write

$$m_1 = x^a y^{n-a} \text{ and } m_2 = x^b y^{n-b},$$

for some integers $a$ and $b$. By symmetry, we can assume that $a > b$.

It suffices to construct a Lyubeznik-critical subset $\sigma$ of $\mathrm{Gens}((x, y)^n)$ of cardinality 3. Consider the sets

$$\sigma_m \coloneqq \{m_1, m_2, m\} \text{ and } \tau_{m'} \coloneqq \{m_1, m_3, m'\},$$

where $m, m' \in \mathrm{Gens}((x, y)^n)$, $m \succ m_2$, and $m' \succ m_3$. Note that $m'$ always exists since $n > 2$. By definition, $\sigma_m$ is Lyubeznik-critical if and only if $m_1 \nmid \mathrm{lcm}(m_2, m)$, and $\tau_{m'}$ is Lyubeznik-critical if and only if $m_1 \nmid \mathrm{lcm}(m_3, m')$ and $m_2 \nmid \mathrm{lcm}(m_3, m')$.

The proof is completed by showing that there always exists $m$ or $m'$ so that either $\sigma_m$ or $\tau_{m'}$ is critical. Indeed, if $a > 1$, then $\sigma_m$ is Lyubeznik-critical for $m = x^{a-1}y^{n-a+1}$. On the other hand, if $a = 1$, i.e., $m_1 = xy^{n-1}$, then it follows that $m_2 = y^n$. In this case, $\tau_{m'}$ is Lyubeznik-critical for any $m'$. □

# 4 Powers of edge ideals that are bridge-friendly

In this section, we will study graphs $G$ and integers $n$ such that $I(G)^n$ is bridge-friendly.

## 4.1 Bridge-friendly graphs and forbidden structures

As in the previous section, we begin with the case when $n = 1$. As we saw in Lemma 13 and Remark 16, if $I$ is not bridge-friendly, then we have three monomials $m_1, m_2, m_3$ such that $m_1$ and $m_2$ (resp. $m_1$ and $m_3$) share a unique factor. In the case of edge ideals, a generator has exactly two factors to share. Thus, we have the following result.

**Proposition 38.** *An edge ideal $I(G)$ is not bridge-friendly (with respect to $(\succ)$) if and only if there exist a type-1 set $\tau \subseteq E(G)$ and monomials $m_1 \succ m_2 \succ m_3$ in $E(G)$ satisfying the conditions in Lemma 13. Moreover, if $m_3 = yz$, then in $\tau \cup \{m_1, m_2\}$, $m_1$ and $m_3$ are the only edges containing $y$, and $m_2$ and $m_3$ are the only edges containing $z$, or vice versa.*

*Proof.* The proof is straightforward from Lemma 13 and Remark 16. □

Proposition 38 roughly says that it is quite restrictive for an edge ideal to not be bridge-friendly. It is known in [8, Theorem 3.11] that edge ideals of trees are bridge-friendly. The next result addresses edge ideals of cycles.

**Proposition 39.** *The edge ideal of an $n$-cycle is bridge-friendly if and only if $n \in \{3, 5, 6\}$.*

*Proof.* The statement can be verified for $n \leqslant 6$ using the code in Appendix A. Suppose that $n \geqslant 7$, and set

$$I(C_n) = (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1).$$

Consider any total order $(\succ)$ on $\mathrm{Gens}(I(C_n))$. Without loss of generality, assume $x_1x_2$ is the smallest monomial with respect to $(\succ)$. Set $\tau = \{x_1x_2, x_3x_4, x_{n-1}x_n\}$, $m_1 = x_2x_3$, $m_2 = x_nx_1$, and $m_3 = x_1x_2$. One can verify that these elements satisfy the condition in Lemma 13. Thus, $I(C_n)$ is not bridge-friendly. $\square$

Proposition 39 suggests that, in examining bridge-friendly edge ideals, one should investigate graphs whose induced cycles are only $C_3, C_5$ or $C_6$. Unfortunately, understanding the class of such graphs poses a challenging problem. Our approach is to focus on special classes of graphs for which the cycle structures are better understood. Particularly, we shall consider *chordal* graphs, i.e., graphs whose only induced cycles are triangles.

The following result gives a few additional "forbidden structures" for being bridge-friendly. We remark here that the property of being bridge-friendly is purely combinatorial without any reference to the base field, so this can be verified by any computer algebra system.

**Proposition 40.** *Let $G$ be one of the following graphs:*

1. *The complete graph $K_4$* .
2. *The gem graph* .
3. *The kite graph* .
4. *The net graph* .

*Then $I(G)$ is not bridge-friendly.*

*Proof.* Verified using the code in Appendix A. $\square$

### 4.2 Bridge-friendly chordal graphs

Propositions 39 and 40, coupled with Corollary 21, state that graphs that contain any forbidden structures are not bridge-friendly. The following definition is crucial in classifying chordal graphs that avoid the forbidden structures described in Proposition 40.

**Definition 41.** Let $T = (V(T), E(T))$ be a tree graph and let $\mathrm{w}\colon E(T) \to \mathbb{Z}_{\geqslant 0}$ be a function. Let $\mathrm{BF}(T, \mathrm{w})$ denote the graph with vertex set

$$V(T) \sqcup \bigsqcup_{e \in E(T)} \{v_{e,1}, v_{e,2}, \ldots, v_{e,\mathrm{w}(e)}\}$$

and edge set

$$E(T) \sqcup \bigsqcup_{yz=e \in E(T)} \{yv_{e,i}\}_{i=1}^{\mathrm{w}(e)} \cup \{zv_{e,i}\}_{i=1}^{\mathrm{w}(e)}.$$

Roughly speaking, $\mathrm{BF}(T, \mathrm{w})$ is obtained from $T$ by, for each edge $e \in E(T)$, attaching $\mathrm{w}(e)$ new triangles along the edge $e$. We use the notation BF as we shall show that these characterize bridge-friendliness amongst chordal graphs. We first characterize them using their combinatorial structure. Going forth, whenever we write $\mathrm{BF}(T, \mathrm{w})$, it is understood that $T$ is a tree and w is a function as above.

**Proposition 42.** *Let $G$ be a graph. Then, $G$ is isomorphic to some $\mathrm{BF}(T, \mathrm{w})$ if and only if $G$ is chordal and any induced 3-cycle of $G$ has a vertex of degree 2.*

*Proof.* Assume that $G = \mathrm{BF}(T, \mathrm{w})$ for some tree $T$ and function $\mathrm{w}\colon E(T) \to \mathbb{Z}_{\geqslant 0}$. It is clear that any of its induced 3-cycle has a vertex of degree 2. It can also be seen that any induced cycle of $G$ is a 3-cycle attached to an edge of $T$. Thus, $G$ is chordal.

We now proceed with the other implication. Let $G$ be a chordal graph such that any of its induced 3-cycle has a vertex of degree 2. For each induced 3-cycle $C$ of $G$, we fix once and for all a vertex $x_C \in C$ of degree 2. Consider the function

$$\begin{aligned} \{\text{induced 3-cycles of } G\} &\to V(G) \times E(G) \\ C &\mapsto (x_C, y_C z_C), \end{aligned}$$

where $y_C z_C$ the edge of $C$ opposite $x_C$. Set

$$\begin{aligned} V &\coloneqq V(G) \setminus \{x_C \colon C \text{ is an induced 3-cycle of } G\}, \\ E &\coloneqq E(G) \setminus \{x_C y_C,\ x_C z_C \colon C \text{ is an induced 3-cycle of } G\}. \end{aligned}$$

Let $T$ be the induced subgraph of $G$ with the vertex set $V$. Since each vertex of $G$ that is not in $T$ is of degree 2, deleting them means deleting the two edges containing it. In other words, $E(T) = E$.

We claim that for each induced 3-cycle $C$ of $G$, the edge $y_C z_C$ is an edge of $T$. Indeed, suppose otherwise that $y_C = x_{C'}$ is a vertex of degree 2 of a different induced 3-cycle $C'$. Then, $C'$ must contain the only two edges containing $y_C$, namely $x_C y_C$ and $y_C z_C$, and thus $C' = C$, a contradiction. Therefore, the claim holds, and hence $T$ is connected.

Since any 3-cycle in $G$ becomes an edge in $T$, the latter has no cycle. Thus, $T$ is a tree. We can define the following edge-weight function

$$\begin{aligned} \mathrm{w} : E = E(T) &\to \mathbb{Z}_{\geqslant 0} \\ e &\mapsto |\{C \colon C \text{ is an induced 3-cycle of } G \text{ such that } y_C z_C = e\}|. \end{aligned}$$

Since the vertices that we delete from $G$ to obtain $T$ are all of degree 2, $T$ is obtained from $G$ by deleting all of these vertices and all pairs of edges containing each of them. This is a bijective process, i.e., we can obtain $G$ from $T$, provided that we know how many vertices are needed to add for each edge of $T$, which is recorded in the function w. Thus, $G = \mathrm{BF}(T, \mathrm{w})$, as desired. $\square$

We are now ready to classify chordal graphs that avoid forbidden structures described in Proposition 40.

**Proposition 43.** *A chordal graph is of the form* $\mathrm{BF}(T, \mathrm{w})$ *if and only if it does not contain any of 4-complete graph , gem graph , kite graph , or net graph  as an induced subgraph.*

*Proof.* It is clear that the graph $\mathrm{BF}(T, \mathrm{w})$ does not contain any of the forbidden graphs as an induced subgraph. We will only show the other implication.

Let $G$ be a chordal graph that does not contain any of the listed forbidden induced subgraphs. If $G$ is a tree or $C_3$, then we are done. By Proposition 42, it suffices to show that any induced triangle $C_3$ in $G$ has a vertex of degree 2. Suppose otherwise that $G$ contains a $C_3$, with vertices $x, y$, and $z$, such that these vertices are of degrees at least 3. Equivalently, $x, y$, and $z$ are each connected to at least a vertex other than the remaining two vertices. We consider the following possibilities on the additional neighbors of $x, y$ and $z$.

- The vertices $x, y$ and $z$ have a common neighbor outside of the triangle $xyz$. In this case, $G$ contains a 4-complete graph , a contradiction.

- The vertices $x, y$ and $z$ have no common neighbor outside of $xyz$, but two of the vertices do. Without loss of generality, assume that for some vertex $w \in V(G)$, $wx, wy \in E(G)$, but $wz \notin E(G)$. Since $z$ is of degree at least 3, there exists a neighbor $z'$ of $z$ other than $x$ and $y$. Clearly, $z'$ cannot be connected to both $x$ and $y$. Assume that $z'x \notin E(G)$.

  If $z'w \in E(G)$, then $G$ contains an induced 4-cycle on the vertices $z'zxw$, and so $G$ is not chordal. If $z'w \notin E(G)$ and $z'y \in E(G)$, then $G$ contains an induced gem graph . If $z'x, z'y, z'w \notin E(G)$, then $G$ contains an induced kite graph . The last two cases both lead to a contradiction.

- No two vertices among $\{x, y, z\}$ have a common neighbor outside of the triangle $xyz$. Since the degrees of these vertices are at least 3, we may assume that there are additional edges $xx', yy', zz'$, with distinct vertices $x', y', z'$. If there is an edge between $x', y'$ and $z'$, say $x'y' \in E(G)$, then $G$ contains an induced $C_4$ over the vertices $xyy'x'$, and so $G$ is not chordal. If there is no edge between $x', y'$ and $z'$, then $G$ contains an induced net graph , a contradiction. $\square$

Before continuing to prove that $\mathrm{BF}(T, \mathrm{w})$ is bridge-friendly, we shall define a particular total order $(\succ)$ on $E(T)$. For a fixed vertex $x_0$ in $T$, we shall view $T$ as a *rooted* tree with root $x_0$. Each vertex $v \in V(T)$ determines a unique path from $v$ to $x_0$. For $i \in \mathbb{N}$, let

$$V_i \coloneqq \{v \in V(T) \mid \mathrm{dist}_T(v, x_0) = i\}$$

be the set of vertices in $T$ whose distance to $x_0$ is $i$. Obviously, $V(T) = \bigcup_{i \in \mathbb{Z}_{\geqslant 0}} V_i$. Let $c_i = |V_i|$, for $i \in \mathbb{Z}_{\geqslant 0}$. We shall consider a specific labeling for the vertices in $T$ given by writing

$$V_i = \{x_{i,j} \mid 1 \leqslant j \leqslant c_i\}$$

(with the convention that $x_{0,1} = x_0$.) With respect to this particular labeling of the vertices in $T$, define the following total order $(\succ)$ on $E(T)$:

$$x_{i,j}x_{i+1,k} \succ x_{i',j'}x_{i'+1,k'} \text{ if } \begin{cases} i < i'; \text{ or} \\ i = i' \text{ and } j < j'; \text{ or} \\ i = i', j = j' \text{ and } k < k'. \end{cases}$$

**Proposition 44.** *If $G = \mathrm{BF}(T, \mathrm{w})$ for a tree $T$ and a function $\mathrm{w}\colon E(T) \to \mathbb{Z}_{\geqslant 0}$, then the edge ideal $I(G)$ is bridge-friendly.*

*Proof.* As before, fix a vertex $x_0$ of $T$ and view $T$ as a tree rooted at $x_0$. Let $V_i$, the labeling $x_{i,j}$'s for the vertices of $T$, and the total order $(\succ)$ be defined as above.

We shall first extend the total order $(\succ)$ to $E(G)$. By induction, it suffices to extend the total order $(\succ)$ to $E(H)$, where $H$ is obtained by attaching $l$ triangles along an edge $e \in E(T)$. Suppose that $e = \{x_{i,j}, x_{i+1,k}\}$ and the $l$ new vertices in $H$ are $v_{e,1}, \dots, v_{e,l}$. Let $e' \succ e$ be the edge immediately before $e$ in the total order $(\succ)$ of $E(T)$. The total order on $E(H)$ is now given by setting:

$$e' \succ x_{i,j}v_{e,1} \succ \cdots \succ x_{i,j}v_{e,l} \succ x_{i+1,k}v_{e,1} \succ \cdots \succ x_{i+1,k}v_{e,l} \succ e.$$

We will show that $I(G)$ is bridge-friendly with respect to $(\succ)$. Suppose otherwise that $I(G)$ is not bridge-friendly. By Proposition 38, there exists a collection of edges $\tau \subseteq E(G)$ and edges $m_1 \succ m_2 \succ m_3$ in $E(G)$ such that if we set $m_3 = yz$, then no other edge in $\tau$ is incident to $y$ or $z$, and $m_1, m_2$ are gaps of $\tau$. We shall arrive at a contradiction. Consider the following possibilities:

- $m_3$ is an edge of $T$, i.e., $m_3 = x_{i,j}x_{i+1,k}$ for some integers $i, j$, and $k$. The only edges that are incident to $x_{i+1,k}$, and are larger than $m_3$, are of the form $x_{i+1,k}v_{m_3,l}$ for some $1 \leqslant l \leqslant \mathrm{w}(m_3)$. So, either $m_1$ or $m_2$ must be of this form. Since they are both gaps of $\tau$, and there are exactly two edges that are incident to $v_{m_3,l}$, we must have $x_{i,j}v_{m_3,l} \in \tau$, a contradiction.

- $m_3 = x_{i,j}v_{e,l}$ for some edge $e = x_{i,j}x_{i+1,k} \in E(T)$ and integer $l$. Then, the only edge that is incident to $v_{e,l}$, other than $m_3$ itself, is $x_{i+1,k}v_{e,l}$, which is smaller than $m_3$. Thus, such $m_1$ and $m_2$ do not exist, a contradiction.

- $m_3 = x_{i+1,k}v_{e,l}$ for some edge $e = x_{i,j}x_{i+1,k} \in E(T)$ and integer $l$. Then, the only edge that is incident to $v_{e,l}$, other than $m_3$ itself, is $x_{i,j}v_{e,l}$, and the only edges that are incident to $x_{i+1,k}$, and are bigger than $m_3$ itself, must be of the form $x_{i+1,k}v_{e,l'}$ for some integer $l'$. We can thus assume that $m_1 = x_{i,j}v_{e,l}$ and $m_2 = x_{i+1,k}v_{e,l'}$. Since $m_2$ is a gap of $\tau$, it follows that $x_{i,j}v_{e,l'} \in \tau$. Observe that $x_{i,j}v_{e,l'}$ and $m_3$ are both in $\tau$ and bigger than $e$, and

$$e = x_{i,j}x_{i+1,k} \mid \operatorname{lcm}(x_{i,j}v_{e,l'}, m_3).$$

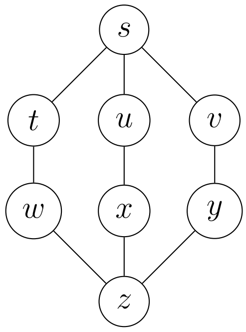
  Hence, by Corollary 17, $e$ is a true gap of $\tau$. This contradicts the fact that $m_2$ can be chosen to be the smallest true gap of $\tau$. $\square$

Combining Propositions 40, 43, and 44, we obtain our next main result.

**Theorem 45.** *Let $G$ be a chordal graph. The edge ideal $I(G)$ is bridge-friendly if and only if $G$ is isomorphic to* $\mathrm{BF}(T, \mathrm{w})$ *for some tree $T$ and function* $\mathrm{w} : E(T) \to \mathbb{Z}_{\geqslant 0}$.

We end the discussion of bridge-friendly edge ideals with an example of a graph $G$ that does not contain any of the forbidden graphs in Proposition 40, and yet $I(G)$ is bridge-friendly.

**Example 46.** Let $G$ be the join of two 6-cycles at three consecutive edges. Note that $G$ is not chordal.




One can verify using the code in Appendix A that $I(G)$ is bridge-friendly with respect to the total order

$$vy \succ ux \succ xz \succ tw \succ yz \succ wz \succ st \succ sv \succ su.$$

*Remark* 47. As with Lyubeznik ideals, it is worth noting that Theorem 45 can be extended to work over disconnected graphs; see Theorem 35 and the discussion preceding it.

### 4.3 Higher powers of edge ideals

For the remainder of this section, we focus on higher powers of edge ideals of graphs. Similar to what happened to the Lyubeznik resolutions, edge ideals whose some higher power is bridge-friendly form a much smaller class.

**Proposition 48.** *Let $G$ be one of the following graphs:*

1. *The 4-star graph $K_{1,3}$.*
2. *The 4-path graph $P_4$.*
3. *The 4-cycle graph $C_4$.*
4. *The paw graph.*
5. *The diamond graph.*
6. *The complete graph $K_4$.*

*Then, the ideals $I(G)^2$ and $I(G)^3$ are not bridge-friendly. Moreover, if $G$ is among the first three graphs, then $I(G)^n$ is not bridge-friendly for any $n \geqslant 2$.*

*Proof.* (1) If $G$ is a 4-star graph $K_{1,3}$, then we can assume that $I(G) = x(x_1, x_2, x_3)$. By Proposition 22, we can instead work on the ideal $J = (x_1, x_2, x_3)$ and derive the same conclusions.

One can verify that $J^2$ and $J^3$ are not bridge-friendly, using the code in Appendix A. Note that for $J^3$, there are only 7! total orders that need checking, since the positions of $x_1^3, x_2^3$, and $x_3^3$ do not matter. Consider $n \geqslant 4$. Compute the HHZ-subideal of $(J^n)^{\leqslant x_1^n x_2^2 x_3^2}$, we get:

$$(x_1^n,\ x_1^{n-1}x_2,\ x_1^{n-1}x_3,\ x_1^{n-2}x_2^2,\ x_1^{n-2}x_2x_3,\ x_1^{n-2}x_3^2,\ x_1^{n-3}x_2^2x_3,\ x_1^{n-3}x_2x_3^2,\ x_1^{n-4}x_2^2x_3^2).$$

By factoring out necessary powers of $x_1$, using Proposition 22, we can instead consider the ideal

$$(x_1^4,\ x_1^3x_2,\ x_1^3x_3,\ x_1^2x_2^2,\ x_1^2x_2x_3,\ x_1^2x_3^2,\ x_1x_2^2x_3,\ x_1x_2x_3^2,\ x_2^2x_3^2).$$

One can verify using the code in Appendix A that this last ideal is not bridge-friendly, and thus neither is $J^n$ nor $I(G)^n$, by Lemma 20.

(2) If $G$ is the 4-path graph $P_4$, then one can check that $I(G)^2$ is not bridge-friendly by exhausting the 6! possible total orders, using the code in Appendix A. Hence $I(G)^n$ is not bridge-friendly for any $n \geqslant 2$, by Proposition 24 and Lemma 20.

(3) If $G$ is the 4-cycle graph $C_4$, i.e., $I(G) = (x_1x_2, x_2x_3, x_3x_4, x_1x_4)$, then we can compute the following HHZ-subideal of $I(G)^2$:

$$(I(G)^2)^{\leqslant (x_1x_2x_3)^2x_4} = (x_1^2x_2^2,\ x_2^2x_3^2,\ x_1^2x_2x_4,\ x_1x_2^2x_3,\ x_2x_3^2x_4,\ x_1x_2x_3x_4).$$

Once again, this ideal can be verified using the code in Appendix A to be not bridge-friendly, by checking all 6! total orders. Therefore, $I(G)^n$ is not bridge-friendly, for all $n \geqslant 2$, by Proposition 24 and Lemma 20.

(4) If $G$ is the paw graph, i.e., $I(G) = (x_1x_2, x_2x_3, x_1x_3, x_3x_4)$, then we can compute the following HHZ-subideal of $I(G)^2$:

$$(I(G)^2)^{\leqslant (x_1x_3x_4)^2x_2} = (x_1^2x_3^2,\ x_3^2x_4^2,\ x_1^2x_2x_3,\ x_1x_2x_3^2,\ x_1x_3^2x_4,\ x_2x_3^2x_4,\ x_1x_2x_3x_4).$$

One can verify using the code in Appendix A that this ideal is not bridge-friendly. Thus $I(G)^2$ is not bridge-friendly by Lemma 20. Moreover, one can check that

$$(I(G)^3)^{\leqslant (x_1x_2x_4)^3x_3^2} = x_1x_2I(G)^2,$$

which is the same as $I(G)^2$. Thus $I(G)^3$ is not bridge-friendly.

(5) If $G$ is the diamond graph, i.e., $I(G) = (x_1x_2, x_2x_3, x_3x_4, x_1x_4, x_2x_4)$, then we can compute the following HHZ-subideals:

$$\begin{aligned}(I(G)^2)^{\leqslant (x_2x_4)^2x_1x_3} &= (x_2^2x_4^2, x_1x_2^2x_3, x_1x_2^2x_4, x_1x_2x_4^2, x_1x_3x_4^2, x_2^2x_3x_4, x_2x_3x_4^2, x_1x_2x_3x_4),\\ (I(G)^3)^{\leqslant (x_2x_4)^3x_1x_3} &= x_2x_4(I(G)^2)^{\leqslant (x_2x_4)^2x_1x_3}.\end{aligned}$$

One can verify using the code in Appendix A that $(I(G)^2)^{\leqslant (x_2x_4)^2x_1x_3}$ is not bridge-friendly. Thus, both $I(G)^2$ and $I(G)^3$ are not bridge-friendly by Lemma 20.

(6) We consider the complete graph $K_4$ on the four vertices $x_1, x_2, x_3, x_4$. One can check that

$$\begin{aligned}(I(K_4)^2)^{\leqslant (x_2x_4)^2x_1x_3} &= (I(H)^2)^{\leqslant (x_2x_4)^2x_1x_3}\\ (I(K_4)^3)^{\leqslant (x_2x_4)^3x_1x_3} &= x_2x_4(I(K_4)^2)^{\leqslant (x_2x_4)^2x_1x_3},\end{aligned}$$

where $H$ is the diamond graph ⊠ in part (5). It is then known that $(I(H)^2)^{\leqslant (x_2x_4)^2x_1x_3}$ is not bridge-friendly and, thus, neither are $I(K_4)^2$ and $I(K_4)^3$ by Lemma 20. □

The result and technique for when $G = C_3$ is a bit different, so we will treat it separately.

**Proposition 49.** *The monomial ideal $I(C_3)^n$ is bridge-friendly if and only if $n \leqslant 3$.*

*Proof.* Set $I(C_3) = (x_1x_2, x_2x_3, x_1x_3)$. One can check that $I(C_3)^2$ and $I(C_3)^3$ are bridge-friendly with respect to the total orders

$$x_2^2x_3^2 \succ x_1x_2^2x_3 \succ x_1^2x_2^2 \succ x_1x_2x_3^2 \succ x_1^2x_3^2x_1^2x_2x_3$$

and

$$\begin{aligned}x_1^3x_3^3 \succ x_2^3x_3^3 \succ x_1^3x_2^3 \succ x_1^2x_2x_3^3 \succ x_1x_2^3x_3^2 \succ x_1^2x_2^3x_3 \succ x_1^3x_2x_3^2 \succ x_1^3x_2^2x_3 \succ x_1^2x_2^2x_3^2\\ \succ x_1x_2^2x_3^3,\end{aligned}$$

respectively. In fact, we have the following claims:

**Claim 50.** *If $(\succ)$ is a total order with respect to which $I(C_3)^2$ is bridge-friendly, then*

$$\min_{\succ} \operatorname{Gens}(I(G)^2) \in \{x_1^2x_2x_3,\ x_1x_2^2x_3,\ x_1x_2x_3^2\}.$$

**Claim 51.** *If $(\succ)$ is a total order with respect to which $I(C_3)^3$ is bridge-friendly, then*

$$\min_{\succ} \operatorname{Gens}(I(C_3)^3) = x_1^2x_2^2x_3^2.$$

The first claim can be verified using the code in Appendix A by checking all 6! total orders. For the second claim, suppose that $(\succ)$ is a total order with respect to which $I(C_3)^3$ is bridge-friendly. By the arguments in the proof of Proposition 24, we have

$$\begin{aligned}(I(C_3)^3)^{\leqslant (x_2x_3)^3x_1^2} &= x_2x_3I(C_3)^2,\\ (I(C_3)^3)^{\leqslant (x_1x_3)^3x_2^2} &= x_1x_3I(C_3)^2,\\ (I(C_3)^3)^{\leqslant (x_1x_2)^3x_3^2} &= x_1x_2I(C_3)^2.\end{aligned}$$

Combining this observation and Claim 50, if $\min_{\succ} \operatorname{Gens}(I(C_3)^3)$ divides $(x_2x_3)^3x_1^2$, then we must have

$$\min_{\succ} \operatorname{Gens}(I(C_3)^3) \in \{x_1^2x_2^2x_3^2,\ x_1x_2^3x_3^2,\ x_1x_2^2x_3^3\}.$$

By symmetry, we have similar statements when permuting $1, 2$, and 3. By checking all 10 possibilities for $\min_{\succ} \operatorname{Gens}(I(C_3)^3)$, the only possibility is $\min_{\succ} \operatorname{Gens}(I(C_3)^3) = x_1^2x_2^2x_3^2$, as claimed.

Back to the proof of Proposition 49. By Proposition 24 and Lemma 20, it suffices to show that $I(C_3)^4$ is not bridge-friendly. Suppose otherwise that there exists a total order $(\prec)$ with respect to which $I(C_3)^4$ is bridge-friendly. By similar arguments as above, we have

$$\begin{aligned}(I(C_3)^4)^{\leqslant (x_2x_3)^4x_1^3} &= x_2x_3I(C_3)^3,\\ (I(C_3)^4)^{\leqslant (x_1x_3)^4x_2^3} &= x_1x_3I(C_3)^3,\\ (I(C_3)^4)^{\leqslant (x_1x_2)^4x_3^3} &= x_1x_2I(C_3)^3.\end{aligned}$$

By Claim 50, if $\min_{\prec} \operatorname{Gens}(I(C_3)^4)$ divides $(x_2x_3)^4x_1^3$, then we must have

$$\min_{\prec} \operatorname{Gens}(I(C_3)^4) = x_1^2x_2^3x_3^3.$$

By symmetry, we have similar statements when permuting $1, 2$, and 3. By checking all generators of $I(C_3)^4$, we conclude that no such element exists, a contradiction. This concludes the proof. $\square$

We are now ready to state the last result of this section.

**Theorem 52.** *Let $G$ be a graph and let $n \geqslant 2$ be a positive integer. Then, the ideal $I(G)^n$ is bridge-friendly if and only if one of the following holds:*

1. *$G$ is $P_2$ or $P_3$; or*

2. *$n = 2, 3$ and $G$ is the triangle $C_3$.*

*Proof.* The "if" implication follows from [8, Corollary 5.5] and Proposition 49. We now show the "only if" direction.

Assume that $I(G)^n$ is bridge-friendly. If $n \geqslant 4$, then by Propositions 48 and 49, $G$ does not contain, as an induced subgraph, any $C_k$, where $k \geqslant 3$, or $P_k$, where $k \geqslant 4$. In other words, $G$ must be a star graph $K_{1,k}$ for some integer $k$. By Proposition 48 (1), $G$ does not contain $K_{1,3}$ as an induced subgraph. Thus $k \leqslant 2$, as desired.

Now, suppose that $n$ is equal to 2 or 3. By Proposition 48 (2) and (3), $G$ is chordal. If $G$ is a tree, then by Proposition 48 (1) and (2), $G$ must be $K_{1,1}$ or $K_{1,2}$, as desired.

It remains to consider the case when $G$ contains, as an induced subgraph, a $C_3$ graph formed by $x, y, z$. If $G = C_3$, then we are done by Proposition 49. Suppose that $G \neq C_3$, i.e., there exists a vertex $w \neq x, y, z$ in $G$. Let $N(w)$ denote the set of neighbors of $w$ among $x, y$, and $z$. We have the following cases:

- $|N(w)| = 1$: Without loss of generality, assume that $N(w) = \{x\}$. In this case, the induced subgraph of $G$ formed by $x, y, z, w$ is a paw graph.

- $|N(w)| = 2$: Without loss of generality, assume that $N(w) = \{x, y\}$. The induced subgraph of $G$ formed by $x, y, z, w$ is a diamond graph ⧄.
- $|N(w)| = 3$, i.e., $N(w) = \{x, y, z\}$: The induced subgraph of $G$ formed by $x, y, z, w$ is a complete graph $K_4$ ⊠.

We arrive at a contradiction to Proposition 48 (4), (5), and (6) in all these cases. This concludes the proof. $\square$

We end the paper with a few questions that we would like to see answered.

**Question 53.** For which connected graph $G$ is the ideal $I(G)$ bridge-friendly, Barile-Macchia and/or generalized Barile-Macchia (cf. [7, 8])?

**Question 54.** For which hypergraph $\mathcal{H}$ is the edge ideal $I(\mathcal{H})$ Scarf, Lyubeznik, Barile-Macchia, and bridge-friendly?

## A Checking Lyubeznik or bridge-friendly monomial ideals

In this appendix, we provide the `SageMath` [22] code for checking whether a given monomial ideal is Lyubeznik or bridge-friendly. We remark that this invokes functions from `Macaulay2` [15], and thus can only run on systems with both computer algebra systems installed. The following functions compute the edge ideal $I(G)$ in the polynomial ring $\mathbb{Z}/2\mathbb{Z}[V(G)]$, given a graph $G$, and the total Betti numbers of a given ideal.

```
from itertools import combinations, permutations
def edgeIdeal(G):
  V, E = G.vertices(), G.edges(labels=False)
  list_of_vars = ",".join([f"x_{v}" for v in V])
  R = macaulay2(f"ZZ/2[{list_of_vars}]")
  x = {v:R.gens()[i] for i, v in zip(range(len(V)), V)}
  return macaulay2.ideal([x[v] * x[w] for v, w in E])
def bettiNumbers(gens):
  I = macaulay2.ideal(gens)
  B = I.res().betti()
  B = dict(B)
  total = {}
  for key in B:
    i = key[0].sage()
    if i not in total:
      total[i] = 0
    total[i] += B[key]
  return [total[i] for i in range(len(total))]
```

We discuss the method behind checking whether a monomial ideal $I$ is Lyubeznik with respect to a given total order $(\succ)$. It is known that Lyubeznik resolutions are constructed independently of the base field. This can be seen from the original construction by Lyubeznik [17], or the fact that these are resolutions coming from discrete Morse theory [3]. Moreover, the entries in the differentials are either 0, or $\pm m$ for some monomial $m$ (Remark 29). Therefore, it suffices to assume that the base field is $\mathbb{Z}/2\mathbb{Z}$. Thus, all the `Macaulay2` verifications are done over the base field $\mathbb{Z}/2\mathbb{Z}$. Lyubeznik-critical subsets of $\text{Gens}(I)$ can be found in finite time just by definition. The following code allows us to check if a monomial ideal $I$ is Lyubeznik with respect to $(\succ)$, by comparing the number of Lyubeznik-critical sets (with a fixed cardinality) to the corresponding Betti number of $I$. If we attain equality for every Betti number, then the Lyubeznik resolution under consideration is minimal. In other words, the coefficients in the differentials of this Lyubeznik resolution are now either 0 or $\pm m$ for some monomial $m \neq 1$. Thus, the Lyubeznik resolution in this case is minimal regardless of $\Bbbk$. We remark that whether $I$'s Betti numbers change when the base field is no longer $\mathbb{Z}/2\mathbb{Z}$ is irrelevant in our analysis.

```
def isOrderLyubeznikMinimal(gens, betti):
  powerset = []
  N = len(gens)
  for size in range(1, N + 1):
    powerset += list(combinations(gens, size))
  tocheck = {frozenset(subset): True for subset in powerset}
  ranks = {}
  mingen = set(gens)
  for sigma in powerset:
    if not tocheck[frozenset(sigma)]: continue
    critical = True
    for i in range(N):
      m = gens[i]
      lcm = LCM([mu for mu in gens[i+1:] if mu in sigma])
      if lcm % m == 0:
        critical = False
        break
    if critical:
      k = len(sigma)
      betti[k] -= 1
      if betti[k] < 0: return False
    else:
      sig = set(sigma)
      complement = mingen.difference(sig)
      for subset in subsets(complement):
        superset = frozenset(sig.union(subset))
        tocheck[superset] = False
  return True
```

To check whether there exists a total order $(\succ)$ on $\text{Gens}(I)$ with respect to which $I$ is

Lyubeznik, we exhaust all of $|\mathrm{Gens}(I)|!$ possible options of $(\succ)$.

```
def isLyubeznikMinimal(I):
  gens = list(I.mingens().entries().flatten())
  betti = bettiNumbers(gens)
  betti = betti + [0] * (len(gens) - len(betti) + 1)
  P = permutations(gens)
  return any(isOrderLyubeznikMinimal(pi, betti) for pi in P)
```

Bridge-friendliness, on the other hand, is purely a combinatorial property that can be verified directly regardless of the base field $\Bbbk$. We first provide the code that helps determine the smallest bridge of a given set of monomials.

```
def isbridge(sigma, m):
  removed = list(sigma)
  removed.remove(m)
  l = LCM(removed)
  if l == LCM([l, m]):
    return tuple(removed)
  return False
def smallestbridge(sigma):
  for m in sigma:
    test = isbridge(sigma, m)
    if test: return m, test
  return False, ()
```

Similar to verifying Lyubeznik-ness, below is the code to verify whether a monomial ideal $I$ is bridge-friendly with respect to a given total order, and to verify whether $I$ is bridge-friendly with respect to *some* order.

```
def isOrderBridgeFriendly(perm):
  N = len(perm)
  marked = {}
  for size in range(N, 2, -1):
    bigSets = combinations(perm, size)
    for sigma in bigSets:
      if sigma in marked:
        del marked[sigma]
        continue
      m, complement = smallestbridge(sigma)
      if not m: continue
      if complement in marked: return False
      marked[complement] = sigma
  return True
def isBridgeFriendly(I):
  gens = list(I.mingens().entries().flatten())
```

```
  P = permutations(gens)
  return any(isOrderBridgeFriendly(perm) for perm in P)
```

Finally, we give a quick tutorial. Below is some code that checks whether $I(C_3)$ and $I(C_3)^2$ are bridge-friendly or Lyubeznik.

```
G = graphs.CycleGraph(3)
I = edgeIdeal(G)
print(isBridgeFriendly(I))
print(isBridgeFriendly(I^2))
print(isLyubeznikMinimal(I))
print(isLyubeznikMinimal(I^2))
```

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