The Lie algebra $\mathfrak{sl}_4(\mathbb{C})$ and the hypercubes

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Abstract

We describe a relationship between the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$ and the hypercube graphs. Consider the $\mathbb{C}$-algebra P of polynomials in four commuting variables. We turn P into an $\mathfrak{sl}_4(\mathbb{C})$-module on which each element of $\mathfrak{sl}_4(\mathbb{C})$ acts as a derivation. Then P becomes a direct sum of irreducible $\mathfrak{sl}_4(\mathbb{C})$-modules $P = \sum_{N\in\mathbb{N}} P_N$, where P_N is the Nth homogeneous component of P. For $N \in \mathbb{N}$ we construct some additional $\mathfrak{sl}_4(\mathbb{C})$-modules $\mathrm{Fix}(G)$ and T. For these modules the underlying vector space is described as follows. Let X denote the vertex set of the hypercube $H(N,2)$, and let V denote the $\mathbb{C}$-vector space with basis X. For the automorphism group G of $H(N,2)$, the action of G on X turns V into a G-module. The vector space $V^{\otimes 3} = V\otimes V\otimes V$ becomes a G-module such that $g(u\otimes v\otimes w) = g(u)\otimes g(v)\otimes g(w)$ for $g \in G$ and $u, v, w \in V$. The subspace $\mathrm{Fix}(G)$ of $V^{\otimes 3}$ consists of the vectors in $V^{\otimes 3}$ that are fixed by every element in G. Pick $\varkappa \in X$. The corresponding subconstituent algebra T of $H(N,2)$ is the subalgebra of $\mathrm{End}(V)$ generated by the adjacency map A of $H(N,2)$ and the dual adjacency map A^* of $H(N,2)$ with respect to $\varkappa$. In our main results, we turn $\mathrm{Fix}(G)$ and T into $\mathfrak{sl}_4(\mathbb{C})$-modules, and display $\mathfrak{sl}_4(\mathbb{C})$-module isomorphisms $P_N \to \mathrm{Fix}(G) \to T$. We describe the $\mathfrak{sl}_4(\mathbb{C})$-modules P_N, $\mathrm{Fix}(G)$, T from multiple points of view.

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1 Introduction

The subconstituent algebra of a distance-regular graph was introduced in [51–53]. This algebra is finite-dimensional, semisimple, and noncommutative in general. Its basic prop-

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erties are described in [1, 10, 56]. The subconstituent algebra has been used to study tridiagonal pairs [1,26,27,30,55], spin models [9,16,17,43,44], codes [22,48,50], projective geometries [21, 36–38, 49], quantum groups [3, 13, 28, 29, 58], DAHA of rank one [33–35], and some areas of mathematical physics [4–6, 11]. Further applications can be found in the survey [19].

For someone seeking an introduction to the subconstituent algebra, the hypercube graphs offer an attractive and accessible example. For these graphs the subconstituent algebra is described in [23]. It is apparent from [23] that for the hypercube graphs, the subconstituent algebra is closely related to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

In the present paper, we will investigate the hypercube graphs using an approach that is different from the one in [23]. We will use the S_3-symmetric approach that was suggested in [57]. This approach potentially provides more information and is more aesthetically pleasing, because it removes the need to fix a base vertex. As it turns out, in the S_3-symmetric approach the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$ plays the key role. Consequently, we will begin our investigation with a detailed study of $\mathfrak{sl}_4(\mathbb{C})$. For the rest of this section, we summarize our results.

Our main topic is the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$, although the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ will play a supporting role. Recall that $\mathfrak{sl}_2(\mathbb{C})$ has a basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and Lie bracket

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

We now consider $\mathfrak{sl}_4(\mathbb{C})$. We show that $\mathfrak{sl}_4(\mathbb{C})$ has a generating set with six generators

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_1^* = \operatorname{diag}(1, 1, -1, -1),$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2^* = \operatorname{diag}(1, -1, 1, -1),$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad A_3^* = \operatorname{diag}(1, -1, -1, 1).$$

Using these generators, we give a presentation of $\mathfrak{sl}_4(\mathbb{C})$ by generators and relations. We show that A_1, A_2, A_3 form a basis for a Cartan subalgebra $\mathbb{H}$ of $\mathfrak{sl}_4(\mathbb{C})$. We show that A_1^*, A_2^*, A_3^* form a basis for a Cartan subalgebra $\mathbb{H}^*$ of $\mathfrak{sl}_4(\mathbb{C})$. Let i, j, k denote a permutation of $\{1,2,3\}$. We show that A_i, A_i^* commute. We show that A_j, A_k^* generate a Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. We show that A_j, A_k, A_j^*, A_k^* generate a Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. We call this Lie subalgebra the ith Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. We display an automorphism τ of $\mathfrak{sl}_4(\mathbb{C})$ that swaps $A_1 \leftrightarrow A_1^*$, $A_2 \leftrightarrow A_2^*$, $A_3 \leftrightarrow A_3^*$.

Next, we bring in a polynomial algebra P. Let x, y, z, w denote mutually commuting indeterminates, and consider the polynomial algebra $P = \mathbb{C}[x, y, z, w]$. We turn P into an $\mathfrak{sl}_4(\mathbb{C})$-module on which each element of $\mathfrak{sl}_4(\mathbb{C})$ acts as a derivation. We display two bases for P. The first basis is

$$x^r y^s z^t w^u \qquad r, s, t, u \in \mathbb{N}. \tag{1.1}$$

Define

$$x^* = \frac{x+y+z+w}{2}, \qquad y^* = \frac{x+y-z-w}{2},$$
$$z^* = \frac{x-y+z-w}{2}, \qquad w^* = \frac{x-y-z+w}{2}.$$

The second basis is

$$x^{*r} y^{*s} z^{*t} w^{*u} \qquad r, s, t, u \in \mathbb{N}. \tag{1.2}$$

We show how the $\mathfrak{sl}_4(\mathbb{C})$-generators act on the bases (1.1) and (1.2). As we will see, the basis (1.1) diagonalizes $\mathbb{H}^*$ and the basis (1.2) diagonalizes $\mathbb{H}$. We display an automorphism σ of the algebra P that sends

$$x \leftrightarrow x^*, \qquad y \leftrightarrow y^*, \qquad z \leftrightarrow z^*, \qquad w \leftrightarrow w^*.$$

We show that for $\varphi \in \mathfrak{sl}_4(\mathbb{C})$ the equation $\tau(\varphi) = \sigma\varphi\sigma^{-1}$ holds on P.

Next, we consider the homogeneous components of P. For $N \in \mathbb{N}$ let P_N denote the subspace of P consisting of the homogeneous polynomials that have total degree N. One basis for P_N consists of the polynomials in (1.1) that have total degree N. Another basis for P_N consists of the polynomials in (1.2) that have total degree N. By construction, the sum $P = \sum_{N \in \mathbb{N}} P_N$ is direct. We show that each summand P_N is an irreducible $\mathfrak{sl}_4(\mathbb{C})$-submodule of P. We show that on the $\mathfrak{sl}_4(\mathbb{C})$-module P_N, each $\mathbb{H}$-weight space has dimension one and each $\mathbb{H}^*$-weight space has dimension one. By construction, each $\mathfrak{sl}_4(\mathbb{C})$-generator is diagonalizable on P_N. We show that the eigenvalues are $\{N-2n\}_{n=0}^N$, and for $0 \leqslant n \leqslant N$ the $(N-2n)$-eigenspace has dimension $(n+1)(N-n+1)$.

Next, we display a Hermitian form $\langle\, ,\, \rangle$ on P with respect to which the basis (1.1) is orthogonal and the basis (1.2) is orthogonal. For each basis we compute the square norm of the basis vectors. For $f, g \in P$ we show that $\langle \sigma f, \sigma g \rangle = \langle f, g \rangle$ and

$$\langle A_i f, g \rangle = \langle f, A_i g \rangle, \qquad \langle A_i^* f, g \rangle = \langle f, A_i^* g \rangle \qquad i \in \{1,2,3\}.$$

We describe the inner product between each basis vector in (1.1) and each basis vector in (1.2). We express these inner products in two ways; using a generating function and as hypergeometric sums. We display some orthogonality relations and recurrence relations that involve the hypergeometric sums. For $N \in \mathbb{N}$ we display a polynomial $\mathcal{P}^\vee$ in six variables with the following property: for $r, s, t, u \in \mathbb{N}$ such that $r + s + t + u = N$,

$$\mathcal{P}^\vee(s, t, u; A_1, A_2, A_3)x^N = x^r y^s z^t w^u;$$
$$\mathcal{P}^\vee(s, t, u; A_1^*, A_2^*, A_3^*)x^{*N} = x^{*r} y^{*s} z^{*t} w^{*u}.$$

Using this, we show that P_N has a basis

$$A_1^s A_2^t A_3^u x^N \qquad s, t, u \in \mathbb{N}, \qquad s + t + u \leqslant N$$

and a basis

$$A_1^{*s} A_2^{*t} A_3^{*u} x^{*N} \qquad s, t, u \in \mathbb{N}, \qquad s + t + u \leqslant N.$$

Next, for $i \in \{1, 2, 3\}$ we introduce a "lowering map" $L_i \in \mathrm{End}(P)$ and a "raising map" $R_i \in \mathrm{End}(P)$. We define

$$L_1 = D_x D_y - D_z D_w, \qquad L_2 = D_x D_z - D_w D_y, \qquad L_3 = D_x D_w - D_y D_z,$$

where D_x, D_y, D_z, D_w are the partial derivatives with respect to x, y, z, w respectively. Let $i \in \{1, 2, 3\}$. We show that $L_i(P_N) = P_{N-2}$ for $N \in \mathbb{N}$, where $P_{-1} = 0$ and $P_{-2} = 0$. By construction L_1, L_2, L_3 mutually commute. We define

$$R_1 = M_x M_y - M_z M_w, \qquad R_2 = M_x M_z - M_w M_y, \qquad R_3 = M_x M_w - M_y M_z,$$

where M_x, M_y, M_z, M_w denote multiplication by x, y, z, w respectively. Let $i \in \{1, 2, 3\}$. By construction, $R_i(P_N) \subseteq P_{N+2}$ for $N \in \mathbb{N}$. Also by construction, R_1, R_2, R_3 are injective and mutually commute. Let $i \in \{1, 2, 3\}$. We give the action of L_i, R_i on the basis (1.1) and the basis (1.2). We show that

$$\langle L_i f, g \rangle = \langle f, R_i g \rangle, \qquad \langle R_i f, g \rangle = \langle f, L_i g \rangle \qquad f, g \in P.$$

We show that each of L_i, R_i commutes with σ. We show that each of L_i, R_i commutes with each of A_j, A_k, A_j^*, A_k^* where j, k are the elements in $\{1, 2, 3\} \backslash \{i\}$. We show that for $N \in \mathbb{N}$ the following sum is orthogonal and direct:

$$P_N = R_i(P_{N-2}) + \mathrm{Ker}(L_i) \cap P_N.$$

Expanding on this, we obtain an orthogonal direct sum

$$P_N = \sum_{\ell=0}^{\lfloor N/2 \rfloor} R_i^\ell \Big(\mathrm{Ker}(L_i) \cap P_{N-2\ell} \Big) \tag{1.3}$$

and an orthogonal direct sum

$$P = \sum_{N \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} R_i^\ell \Big(\mathrm{Ker}(L_i) \cap P_N \Big). \tag{1.4}$$

Next, we investigate the summands in (1.4). For each summand in (1.4) we display an orthogonal basis. We show that for $N, \ell \in \mathbb{N}$ the corresponding summand in (1.4) is an irreducible submodule for the ith Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. This irreducible submodule has dimension $(N+1)^2$. It is isomorphic to $\mathbb{V}_N \otimes \mathbb{V}_N$, where $\mathbb{V}_N$ denotes the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module with dimension $N+1$.

Next, for $N \in \mathbb{N}$ we consider the sum

$$\sum_{\ell \in \mathbb{N}} R_i^\ell \Big(\mathrm{Ker}(L_i) \cap P_N \Big). \tag{1.5}$$

As we investigate (1.5), it is convenient to define $\Omega \in \mathrm{End}(P)$ as follows. For $N \in \mathbb{N}$ the subspace P_N is an eigenspace for Ω with eigenvalue N. For $i \in \{1,2,3\}$ we show that

$$[L_i, R_i] = \Omega + 2I, \qquad [\Omega, R_i] = 2R_i, \qquad [\Omega, L_i] = -2L_i.$$

Consequently, P becomes an $\mathfrak{sl}_2(\mathbb{C})$-module on which E, F, H act as follows:

element φ	E	F	H
action of φ on P	$-L_i$	R_i	$-\Omega - 2I$

We show that the subspace (1.5) is an $\mathfrak{sl}_2(\mathbb{C})$-submodule of P. We express (1.5) as an orthogonal direct sum of irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules. The irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules in the sum are mutually isomorphic; they are all highest-weight with highest weight $-N-2$.

Next, we introduce some maps $C_1, C_2, C_3 \in \mathrm{End}(P)$. Let i, j, k denote a permutation of $\{1,2,3\}$. We show that

$$\begin{aligned} \frac{(\Omega + 2I)^2}{2} - L_i R_i - R_i L_i &= \frac{4A_j^2 + 4A_k^{*2} - (A_j A_k^* - A_k^* A_j)^2}{8} \\ &= \frac{4A_j^{*2} + 4A_k^2 - (A_j^* A_k - A_k A_j^*)^2}{8}. \end{aligned}$$

Call this common value C_i. We interpret C_i using the concept of a Casimir operator. We compute the action of C_i on the basis (1.1) and the basis (1.2). We show that C_i commutes with each of $\Omega, L_i, R_i, A_j, A_k, A_j^*, A_k^*, \sigma$. We show that

$$\langle C_i f, g \rangle = \langle f, C_i g \rangle \qquad\qquad f, g \in P.$$

We show that (1.5) is an eigenspace for the action of C_i on P; the eigenvalue is $N(N+2)/2$.

Let $i \in \{1,2,3\}$ and $N \in \mathbb{N}$. From our previous discussion we draw the following conclusions about (1.3). For $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ the ℓ-summand in (1.3) is an irreducible submodule for the ith Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. This ℓ-summand has dimension $(N-2\ell+1)^2$ and is isomorphic to $\mathbb{V}_{N-2\ell} \otimes \mathbb{V}_{N-2\ell}$. This ℓ-summand is an eigenspace for the action of C_i on P_N, with eigenvalue $(N-2\ell)(N-2\ell+2)/2$.

We will return to the decomposition (1.3) later in this section.

Next, we bring in the hypercube graphs. For the rest of this section, fix $N \in \mathbb{N}$. We consider the N-cube $H(N,2)$. Let X denote the vertex set of $H(N,2)$ and let V denote the vector space with basis X. For $x \in X$ the set $\Gamma(x)$ consists of the vertices in X that are adjacent to x. The adjacency map $\mathsf{A} \in \operatorname{End}(V)$ satisfies

$$\mathsf{A}x = \sum_{\xi \in \Gamma(x)} \xi, \qquad\qquad x \in X.$$

The vector space $V^{\otimes 3} = V \otimes V \otimes V$ has a basis

$$X^{\otimes 3} = \{x \otimes y \otimes z | x, y, z \in X\}.$$

Let G denote the automorphism group of $H(N,2)$. The action of G on X turns V into a G-module. The vector space $V^{\otimes 3}$ becomes a G-module such that for $g \in G$ and $u, v, w \in V$,

$$g(u \otimes v \otimes w) = g(u) \otimes g(v) \otimes g(w).$$

Define the subspace

$$\operatorname{Fix}(G) = \{v \in V^{\otimes 3} | g(v) = v \ \forall g \in G\}.$$

We turn $\operatorname{Fix}(G)$ into an $\mathfrak{sl}_4(\mathbb{C})$-module as follows. Define $A^{(1)}, A^{(2)}, A^{(3)} \in \operatorname{End}(V^{\otimes 3})$ such that for $x \otimes y \otimes z \in X^{\otimes 3}$,

$$\begin{aligned}
A^{(1)}(x \otimes y \otimes z) &= \mathsf{A}x \otimes y \otimes z, \\
A^{(2)}(x \otimes y \otimes z) &= x \otimes \mathsf{A}y \otimes z, \\
A^{(3)}(x \otimes y \otimes z) &= x \otimes y \otimes \mathsf{A}z.
\end{aligned}$$

For notational convenience, define $\theta^*_i = N - 2i$ for $0 \leqslant i \leqslant N$. For $x, y \in X$ let $\partial(x,y)$ denote the path-length distance between x, y. Define $A^{*(1)}, A^{*(2)}, A^{*(3)} \in \operatorname{End}(V^{\otimes 3})$ such that for $x \otimes y \otimes z \in X^{\otimes 3}$,

$$\begin{aligned}
A^{*(1)}(x \otimes y \otimes z) &= x \otimes y \otimes z \, \theta^*_{\partial(y,z)}, \\
A^{*(2)}(x \otimes y \otimes z) &= x \otimes y \otimes z \, \theta^*_{\partial(z,x)}, \\
A^{*(3)}(x \otimes y \otimes z) &= x \otimes y \otimes z \, \theta^*_{\partial(x,y)}.
\end{aligned}$$

We show that $\mathrm{Fix}(G)$ is invariant under $A^{(i)}$ and $A^{*(i)}$ for $i \in \{1,2,3\}$. We show that $\mathrm{Fix}(G)$ is an $\mathfrak{sl}_4(\mathbb{C})$-module on which

$$A_i = A^{(i)}, \qquad A_i^* = A^{*(i)} \qquad i \in \{1,2,3\}.$$

We endow $V^{\otimes 3}$ with a Hermitian form $\langle\, , \rangle$ with respect to which the basis $X^{\otimes 3}$ is orthonormal. We display an $\mathfrak{sl}_4(\mathbb{C})$-module isomorphism $\ddagger : P_N \to \mathrm{Fix}(G)$ such that

$$\langle f, g\rangle = \langle f^{\ddagger}, g^{\ddagger}\rangle \qquad f, g \in P_N.$$

Our treatment of $H(N,2)$ follows the S_3-symmetric approach discussed in [57]; see Note 3.7 below.

Next, we consider the subconstituent algebras of $H(N,2)$. Recall the adjacency map $\mathsf{A} \in \mathrm{End}(V)$ for $H(N,2)$. For the rest of this section, fix $\varkappa \in X$. The corresponding dual adjacency map $\mathsf{A}^* = \mathsf{A}^*(\varkappa) \in \mathrm{End}(V)$ satisfies

$$\mathsf{A}^* x = \theta^*_{\partial(x,\varkappa)} x, \qquad x \in X.$$

By construction, the map A^* is diagonalizable with eigenvalues $\{\theta_i^*\}_{i=0}^N$. The subconstituent algebra $T = T(\varkappa)$ is the subalgebra of $\mathrm{End}(V)$ generated by A, A^*; see [23, Definition 2.1]. By [23, Corollary 14.15] we have $\dim T = \binom{N+3}{3}$.

We mention some bases for the vector space T. As we will see, the adjacency map A is diagonalizable with eigenvalues $\theta_i = N - 2i$ $(0 \leqslant i \leqslant N)$. For $0 \leqslant i \leqslant N$ let $\mathsf{E}_i \in \mathrm{End}(V)$ denote the primitive idempotent of A associated with θ_i. For $x \in X$ and $0 \leqslant i \leqslant N$, define the set $\Gamma_i(x) = \{y \in X | \partial(x,y) = i\}$. For $0 \leqslant i \leqslant N$ define $\mathsf{A}_i \in \mathrm{End}(V)$ such that

$$\mathsf{A}_i x = \sum_{\xi \in \Gamma_i(x)} \xi, \qquad x \in X.$$

For $0 \leqslant i \leqslant N$ define $\mathsf{E}_i^* = \mathsf{E}_i^*(\varkappa) \in \mathrm{End}(V)$ such that

$$\mathsf{E}_i^* x = \begin{cases} x, & \text{if } \partial(x,\varkappa) = i; \\ 0, & \text{if } \partial(x,\varkappa) \neq i, \end{cases} \qquad x \in X.$$

By construction, E_i^* is the primitive idempotent of A^* for the eigenvalue θ_i^*. For $0 \leqslant i \leqslant N$ define $\mathsf{A}_i^* = \mathsf{A}_i^*(\varkappa) \in \mathrm{End}(V)$ such that

$$\mathsf{A}_i^* x = 2^N \langle \mathsf{E}_i \varkappa, x\rangle x, \qquad x \in X.$$

For notational convenience, let the set $\mathcal{P}_N''$ consist of the 3-tuples of integers (h,i,j) such that

$$\begin{aligned} &0 \leqslant h,i,j \leqslant N, \qquad && h+i+j \text{ is even}, \qquad && h+i+j \leqslant 2N, \\ &h \leqslant i+j, && i \leqslant j+h, && j \leqslant h+i. \end{aligned}$$

As we will see in Lemma 18.3, the vector space T has a basis

$$\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j \qquad (h,i,j) \in \mathcal{P}''_N$$

and a basis

$$\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j \qquad (h,i,j) \in \mathcal{P}''_N.$$

Define $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \mathcal{A}^{(3)} \in \mathrm{End}(T)$ such that for $(h,i,j) \in \mathcal{P}''_N$,

$$\mathcal{A}^{(1)}\big(\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j\big) = \theta_h \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j,$$
$$\mathcal{A}^{(2)}\big(\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j\big) = \theta_i \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j,$$
$$\mathcal{A}^{(3)}\big(\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j\big) = \theta_j \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j.$$

Define $\mathcal{A}^{*(1)}, \mathcal{A}^{*(2)}, \mathcal{A}^{*(3)} \in \mathrm{End}(T)$ such that for $(h,i,j) \in \mathcal{P}''_N$,

$$\mathcal{A}^{*(1)}\big(\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j\big) = \theta^*_h \mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j,$$
$$\mathcal{A}^{*(2)}\big(\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j\big) = \theta^*_j \mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j,$$
$$\mathcal{A}^{*(3)}\big(\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j\big) = \theta^*_i \mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j.$$

We show that T is an $\mathfrak{sl}_4(\mathbb{C})$-module on which

$$A_i = \mathcal{A}^{(i)}, \qquad A^*_i = \mathcal{A}^{*(i)} \qquad i \in \{1,2,3\}.$$

For $x, y \in X$ define a map $e_{x,y} \in \mathrm{End}(V)$ that sends $y \mapsto x$ and all other vertices to 0. Note that $\{e_{x,y}\}_{x,y \in X}$ form a basis for $\mathrm{End}(V)$. We endow $\mathrm{End}(V)$ with a Hermitian form $\langle\, , \rangle$ with respect to which the basis $\{e_{x,y}\}_{x,y\in X}$ is orthonormal. We display an $\mathfrak{sl}_4(\mathbb{C})$-module isomorphism $\vartheta : P_N \to T$ such that

$$\langle f, g \rangle = \langle \vartheta(f), \vartheta(g) \rangle \qquad f, g \in P_N.$$

We show that for $r, s, t, u \in \mathbb{N}$ such that $r + s + t + u = N$, the map ϑ sends

$$x^r y^s z^t w^u \mapsto \frac{r!s!t!u!}{(N!)^{1/2}} \mathsf{E}^*_j \mathsf{A}_h \mathsf{E}^*_i,$$
$$x^{*r} y^{*s} z^{*t} w^{*u} \mapsto \frac{r!s!t!u!}{(N!)^{1/2}} \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j,$$

where

$$h = t + u, \qquad i = u + s, \qquad j = s + t.$$

We return our attention to the decomposition of P_N given in (1.3). Referring to that decomposition, let us take $i = 1$. We show that the isomorphism $\vartheta : P_N \to T$ maps the given decomposition of P_N to the Wedderburn decomposition of T from [23, Theorems 14.10, 14.14].

The main results of this paper are Theorems 17.28, 17.34, 18.16, 18.18, 18.20, 18.21, 18.22, 18.30.

2 Preliminaries

We now begin our formal argument. The following concepts and notation will be used throughout the paper. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\mathbb{C}$ denote the field of complex numbers. Every vector space, algebra, and tensor product that we discuss, is understood to be over $\mathbb{C}$. Every algebra without the Lie prefix that we discuss, is understood to be associative and have a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra. For a nonzero vector space V, the algebra $\mathrm{End}(V)$ consists of the $\mathbb{C}$-linear maps from V to V. Let I denote the multiplicative identity in $\mathrm{End}(V)$. An element $B \in \mathrm{End}(V)$ is called *diagonalizable* whenever V is spanned by the eigenspaces of B. Assume that B is diagonalizable, and let $\{V_i\}_{i=0}^N$ denote an ordering of the eigenspaces of B. The sum $V = \sum_{i=0}^N V_i$ is direct. For $0 \leqslant i \leqslant N$ let θ_i denote the eigenvalue of B for V_i. For $0 \leqslant i \leqslant N$ define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ if $j \neq i$ $(0 \leqslant j \leqslant N)$. We call E_i the *primitive idempotent* of B associated with V_i (or θ_i). We have (i) $E_i E_j = \delta_{i,j} E_i$ $(0 \leqslant i, j \leqslant N)$; (ii) $I = \sum_{i=0}^N E_i$; (iii) $B = \sum_{i=0}^N \theta_i E_i$; (iv) $BE_i = \theta_i E_i = E_i B$ $(0 \leqslant i \leqslant N)$; (v) $V_i = E_i V$ $(0 \leqslant i \leqslant N)$. Moreover

$$E_i = \prod_{\substack{0 \leqslant j \leqslant N \\ j \neq i}} \frac{B - \theta_j I}{\theta_i - \theta_j} \qquad (0 \leqslant i \leqslant N). \tag{2.1}$$

The maps $\{E_i\}_{i=0}^N$ form a basis for the subalgebra of $\mathrm{End}(V)$ generated by B. For $\alpha \in \mathbb{C}$ let $\overline{\alpha}$ denote the complex-conjugate of α. For a positive real number α, let $\alpha^{1/2}$ denote the positive square root of α. Let $\mathcal{B}$ denote an algebra. By an *automorphism* of $\mathcal{B}$ we mean an algebra isomorphism $\mathcal{B} \to \mathcal{B}$. Let the algebra $\mathcal{B}^{\mathrm{opp}}$ consist of the vector space $\mathcal{B}$ and the following multiplication. For $a, b \in \mathcal{B}$ the product ab (in $\mathcal{B}^{\mathrm{opp}}$) is equal to ba (in $\mathcal{B}$). By an *antiautomorphism* of $\mathcal{B}$ we mean an algebra isomorphism $\mathcal{B} \to \mathcal{B}^{\mathrm{opp}}$. We will be discussing Lie algebras. Background information about Lie algebras can be found in [8, 25].

3 The Lie algebras $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_4(\mathbb{C})$

For an integer $n \geqslant 1$, the algebra $\mathrm{Mat}_n(\mathbb{C})$ consists of the $n \times n$ matrices with all entries in $\mathbb{C}$. The Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ consists of the vector space $\mathrm{Mat}_n(\mathbb{C})$ and Lie bracket

$$[\varphi, \phi] = \varphi\phi - \phi\varphi \qquad \varphi, \phi \in \mathrm{Mat}_n(\mathbb{C}).$$

The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, consisting of the matrices in $\mathfrak{gl}_n(\mathbb{C})$ that have trace 0. In this paper we will mainly consider $\mathfrak{sl}_4(\mathbb{C})$, although $\mathfrak{sl}_2(\mathbb{C})$ will play a supporting role.

Example 3.1. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has a basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and Lie bracket

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

We describe a presentation of $\mathfrak{sl}_2(\mathbb{C})$ by generators and relations.

Definition 3.2. Define a Lie algebra $\mathbf{L}$ by generators A, A^* and relations

$$[A, [A, A^*]] = 4A^*, \qquad [A^*, [A^*, A]] = 4A.$$

Lemma 3.3. *There exists a Lie algebra isomorphism $\natural : \mathbf{L} \to \mathfrak{sl}_2(\mathbb{C})$ that sends*

$$A \mapsto E + F, \qquad A^* \mapsto H.$$

Proof. We have

$$[E + F, [E + F, H]] = [E + F, 2F - 2E] = 4[E, F] = 4H,$$
$$[H, [H, E + F]] = [H, 2E - 2F] = 4(E + F).$$

Thus the matrices $E + F, H$ satisfy the relations in Definition 3.2. Consequently, there exists a Lie algebra homomorphism $\natural : \mathbf{L} \to \mathfrak{sl}_2(\mathbb{C})$ that sends $A \mapsto E + F$ and $A^* \mapsto H$. Since E, F, H form a basis for $\mathfrak{sl}_2(\mathbb{C})$, there exists a $\mathbb{C}$-linear map $\mathfrak{sl}_2(\mathbb{C}) \to \mathbf{L}$ that sends

$$E \mapsto \frac{2A - [A, A^*]}{4}, \qquad F \mapsto \frac{[A, A^*] + 2A}{4}, \qquad H \mapsto A^*.$$

One checks that this map is the inverse of $\natural$. The map $\natural$ is a bijection, and hence a Lie algebra isomorphism. □

For the rest of this paper, we identify $\mathbf{L}$ and $\mathfrak{sl}_2(\mathbb{C})$ via the isomorphism $\natural$ in Lemma 3.3.

Lemma 3.4. *The following is a basis for the vector space $\mathfrak{sl}_2(\mathbb{C})$:*

$$A, \qquad A^*, \qquad [A, A^*].$$

Proof. The matrices A, A^* generate $\mathfrak{sl}_2(\mathbb{C})$. □

Next, we describe $\mathfrak{sl}_4(\mathbb{C})$. We will give a basis for $\mathfrak{sl}_4(\mathbb{C})$, and a presentation of $\mathfrak{sl}_4(\mathbb{C})$ by generators and relations. For $1 \leqslant i, j \leqslant 4$ let $E_{i,j} \in \mathrm{Mat}_4(\mathbb{C})$ have (i, j)-entry 1 and all other entries 0. The following is a basis for $\mathfrak{sl}_4(\mathbb{C})$:

$$E_{i,j} \quad (1 \leqslant i, j \leqslant 4, \quad i \neq j), \qquad E_{i,i} - E_{i+1,i+1} \quad (1 \leqslant i \leqslant 3). \tag{3.1}$$

Serre gave a presentation of $\mathfrak{sl}_4(\mathbb{C})$ by generators and relations, see [25, p. 99]. We will use a different presentation, that is better suited to our purpose.

Definition 3.5. We define a Lie algebra $\mathbb{L}$ by generators

$$A_i, \quad A_i^* \qquad i \in \{1, 2, 3\}$$

and the following relations.

(i) For distinct $i, j \in \{1, 2, 3\}$,

$$[A_i, A_j] = 0, \qquad [A_i^*, A_j^*] = 0.$$

(ii) For $i \in \{1, 2, 3\}$,

$$[A_i, A_i^*] = 0.$$

(iii) For distinct $i, j \in \{1, 2, 3\}$,

$$[A_i, [A_i, A_j^*]] = 4A_j^*, \qquad [A_j^*, [A_j^*, A_i]] = 4A_i.$$

(iv) For mutually distinct $h, i, j \in \{1, 2, 3\}$,

$$[A_h, [A_i^*, A_j]] = [A_h^*, [A_i, A_j^*]] = [A_j, [A_i^*, A_h]] = [A_j^*, [A_i, A_h^*]].$$

Lemma 3.6. *There exists a Lie algebra isomorphism* $\sharp : \mathbb{L} \to \mathfrak{sl}_4(\mathbb{C})$ *that sends*

$$A_1 \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_1^* \mapsto \operatorname{diag}(1, 1, -1, -1),$$
$$A_2 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2^* \mapsto \operatorname{diag}(1, -1, 1, -1),$$
$$A_3 \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad A_3^* \mapsto \operatorname{diag}(1, -1, -1, 1).$$

Proof. Consider the six matrices from the lemma statement. One checks that these six matrices satisfy the relations in Definition 3.5. Consequently, there exists a Lie algebra homomorphism $\sharp : \mathbb{L} \to \mathfrak{sl}_4(\mathbb{C})$ that sends each $\mathbb{L}$-generator to the corresponding matrix. In (3.1) we gave a basis for $\mathfrak{sl}_4(\mathbb{C})$. There exists a $\mathbb{C}$-linear map $\mathfrak{sl}_4(\mathbb{C}) \to \mathbb{L}$ that acts on this basis as follows. The map sends

$$E_{1,2} \mapsto \frac{4A_1 + 2[A_2^*, A_1] + 2[A_3^*, A_1] + [A_2^*, [A_3^*, A_1]]}{16},$$
$$E_{2,1} \mapsto \frac{4A_1 - 2[A_2^*, A_1] - 2[A_3^*, A_1] + [A_2^*, [A_3^*, A_1]]}{16},$$
$$E_{3,4} \mapsto \frac{4A_1 + 2[A_2^*, A_1] - 2[A_3^*, A_1] - [A_2^*, [A_3^*, A_1]]}{16},$$
$$E_{4,3} \mapsto \frac{4A_1 - 2[A_2^*, A_1] + 2[A_3^*, A_1] - [A_2^*, [A_3^*, A_1]]}{16}$$

and

$$E_{1,3} \mapsto \frac{4A_2 + 2[A_3^*, A_2] + 2[A_1^*, A_2] + [A_3^*, [A_1^*, A_2]]}{16},$$
$$E_{3,1} \mapsto \frac{4A_2 - 2[A_3^*, A_2] - 2[A_1^*, A_2] + [A_3^*, [A_1^*, A_2]]}{16},$$
$$E_{4,2} \mapsto \frac{4A_2 + 2[A_3^*, A_2] - 2[A_1^*, A_2] - [A_3^*, [A_1^*, A_2]]}{16},$$
$$E_{2,4} \mapsto \frac{4A_2 - 2[A_3^*, A_2] + 2[A_1^*, A_2] - [A_3^*, [A_1^*, A_2]]}{16}$$

and

$$E_{1,4} \mapsto \frac{4A_3 + 2[A_1^*, A_3] + 2[A_2^*, A_3] + [A_1^*, [A_2^*, A_3]]}{16},$$
$$E_{4,1} \mapsto \frac{4A_3 - 2[A_1^*, A_3] - 2[A_2^*, A_3] + [A_1^*, [A_2^*, A_3]]}{16},$$
$$E_{2,3} \mapsto \frac{4A_3 + 2[A_1^*, A_3] - 2[A_2^*, A_3] - [A_1^*, [A_2^*, A_3]]}{16},$$
$$E_{3,2} \mapsto \frac{4A_3 - 2[A_1^*, A_3] + 2[A_2^*, A_3] - [A_1^*, [A_2^*, A_3]]}{16}$$

and

$$E_{1,1} - E_{2,2} \mapsto \frac{A_2^* + A_3^*}{2},$$
$$E_{2,2} - E_{3,3} \mapsto \frac{A_1^* - A_2^*}{2},$$
$$E_{3,3} - E_{4,4} \mapsto \frac{A_2^* - A_3^*}{2}.$$

One checks that the above map $\mathfrak{sl}_4(\mathbb{C}) \to \mathbb{L}$ is the inverse of $\sharp$. The map $\sharp$ is a bijection, and hence a Lie algebra isomorphism. □

Note 3.7. (See [57, Definition 4.1].) The universal enveloping algebra $U(\mathbb{L})$ is a homomorphic image of the S_3-symmetric tridiagonal algebra $\mathbb{T}(2, 0, 0, 4, 4)$.

For the rest of this paper, we identify the Lie algebras $\mathbb{L}$ and $\mathfrak{sl}_4(\mathbb{C})$ via the isomorphism $\sharp$ from Lemma 3.6.

Lemma 3.8. *The following is a basis for the vector space $\mathfrak{sl}_4(\mathbb{C})$:*

$$A_1, \quad A_2, \quad A_3, \quad A_1^*, \quad A_2^*, \quad A_3^*,$$
$$[A_1, A_2^*], \quad [A_2, A_3^*], \quad [A_3, A_1^*], \quad [A_1^*, A_2], \quad [A_2^*, A_3], \quad [A_3^*, A_1],$$
$$[A_1^*, [A_2^*, A_3]], \quad [A_2^*, [A_3^*, A_1]], \quad [A_3^*, [A_1^*, A_2]].$$

Proof. There are 15 matrices in the list, and the dimension of $\mathfrak{sl}_4(\mathbb{C})$ is 15. By linear algebra, it suffices to show that the listed matrices span $\mathfrak{sl}_4(\mathbb{C})$. It is clear from the proof of Lemma 3.6, that the listed matrices span $\mathfrak{sl}_4(\mathbb{C})$. □

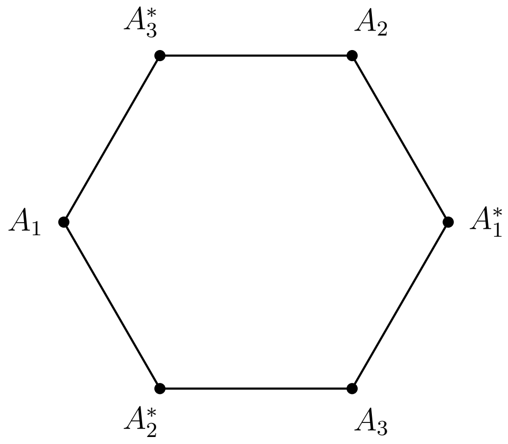

Figure 1: Nonadjacent matrices commute. Adjacent matrices generate a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Lemma 3.9. *For distinct $i, j \in \{1, 2, 3\}$ there exists a Lie algebra homomorphism $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_4(\mathbb{C})$ that sends*

$$A \mapsto A_i, \qquad A^* \mapsto A_j^*.$$

This homomorphism is injective.

Proof. To see that the homomorphism exists, compare the relations in Definition 3.2 and Definition 3.5(iii). The homomorphism is injective by Lemmas 3.4, 3.8. □

Corollary 3.10. *For distinct $i, j \in \{1, 2, 3\}$ the matrices A_i, A_j^* generate a Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ that is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. By Lemma 3.9. □

Our presentation of $\mathfrak{sl}_4(\mathbb{C})$ is described by the diagram in Figure 1.

Lemma 3.11. *For distinct $j, k \in \{1, 2, 3\}$ there exists a Lie algebra homomorphism $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_4(\mathbb{C})$ that sends*

$$(A, 0) \mapsto A_j, \qquad (A^*, 0) \mapsto A_k^*, \qquad (0, A) \mapsto A_k, \qquad (0, A^*) \mapsto A_j^*.$$

This homomorphism is injective.

Proof. The homomorphism exists by Lemma 3.9 and since each of A_j, A_k^* commutes with each of A_j^*, A_k. The homomorphism is injective because the matrices

$$A_j, \qquad A_k, \qquad A_j^*, \qquad A_k^*, \qquad [A_j, A_k^*], \qquad [A_j^*, A_k]$$

are linearly independent by Lemma 3.8. □

Corollary 3.12. *For distinct $j, k \in \{1,2,3\}$ the matrices A_j, A_k, A_j^*, A_k^* generate a Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ that is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.*

Proof. By Lemma 3.11. □

Definition 3.13. Let $i \in \{1,2,3\}$. By the *ith Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to* $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, we mean the Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ generated by A_j, A_k, A_j^*, A_k^* where j, k are the elements in $\{1,2,3\} \backslash \{i\}$.

4 The automorphism τ of $\mathfrak{sl}_4(\mathbb{C})$

We continue to discuss the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$. In this section, we introduce an automorphism τ of $\mathfrak{sl}_4(\mathbb{C})$ that swaps A_i, A_i^* for $i \in \{1,2,3\}$.

Definition 4.1. Define $\Upsilon \in \mathrm{Mat}_4(\mathbb{C})$ by

$$\Upsilon = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Note that $\Upsilon^2 = I$.

Lemma 4.2. *For $i \in \{1,2,3\}$ we have*

$$A_i \Upsilon = \Upsilon A_i^*, \qquad A_i^* \Upsilon = \Upsilon A_i.$$

Proof. By matrix multiplication using Lemma 3.6 and the comment above Lemma 3.8. □

By an *automorphism* of the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$, we mean a Lie algebra isomorphism $\mathfrak{sl}_4(\mathbb{C}) \to \mathfrak{sl}_4(\mathbb{C})$.

Lemma 4.3. *There exists an automorphism τ of $\mathfrak{sl}_4(\mathbb{C})$ that sends $\varphi \mapsto \Upsilon \varphi \Upsilon^{-1}$ for all $\varphi \in \mathfrak{sl}_4(\mathbb{C})$. Moreover, $\tau^2 = \mathrm{id}$.*

Proof. By $\Upsilon^2 = I$ and linear algebra. □

Lemma 4.4. *The automorphism τ from Lemma 4.3 swaps*

$$A_i \leftrightarrow A_i^* \qquad i \in \{1,2,3\}.$$

Proof. By Lemmas 4.2, 4.3. □

We will be discussing Cartan subalgebras of $\mathfrak{sl}_4(\mathbb{C})$. The definition of a Cartan subalgebra can be found in [8, p. 23].

Lemma 4.5. *The following* (i)–(iv) *hold.*

(i) *The elements A_1, A_2, A_3 form a basis for a Cartan subalgebra $\mathbb{H}$ of $\mathfrak{sl}_4(\mathbb{C})$.*

(ii) *The elements* A_1^*, A_2^*, A_3^* *form a basis for a Cartan subalgebra* $\mathbb{H}^*$ *of* $\mathfrak{sl}_4(\mathbb{C})$.

(iii) *The automorphism* τ *swaps* $\mathbb{H} \leftrightarrow \mathbb{H}^*$.

(iv) *The Lie algebra* $\mathfrak{sl}_4(\mathbb{C})$ *is generated by* $\mathbb{H}, \mathbb{H}^*$.

Proof. The matrices A_1, A_2, A_3 are linearly independent, so they form a basis for a subspace $\mathbb{H}$ of $\mathfrak{sl}_4(\mathbb{C})$. Similarly, the matrices A_1^*, A_2^*, A_3^* form a basis for a subspace $\mathbb{H}^*$ of $\mathfrak{sl}_4(\mathbb{C})$. The subspaces $\mathbb{H}$, $\mathbb{H}^*$ satisfy (iii) by Lemma 4.4. The subspaces $\mathbb{H}$, $\mathbb{H}^*$ satisfy (iv) by construction. The subspace $\mathbb{H}^*$ is a Cartan subalgebra of $\mathfrak{sl}_4(\mathbb{C})$, because $\mathbb{H}^*$ consists of the diagonal matrices in $\mathfrak{sl}_4(\mathbb{C})$. The subspace $\mathbb{H}$ is a Cartan subalgebra of $\mathfrak{sl}_4(\mathbb{C})$, because $\mathbb{H}$ is the image of a Cartan subalgebra $\mathbb{H}^*$ under an automorphism τ of $\mathfrak{sl}_4(\mathbb{C})$. □

Definition 4.6. Let W denote an $\mathfrak{sl}_4(\mathbb{C})$-module, and consider the action of $\mathbb{H}$ on W. A common eigenspace for this action is called an $\mathbb{H}$-*weight space* for W. An $\mathbb{H}^*$-weight space for W is similarly defined.

5 An $\mathfrak{sl}_4(\mathbb{C})$-action on the polynomial algebra $\mathbb{C}[x, y, z, w]$

Let x, y, z, w denote mutually commuting indeterminates, and consider the algebra $\mathbb{C}[x, y, z, w]$ of polynomials in x, y, z, w that have all coefficients in $\mathbb{C}$. We abbreviate $P = \mathbb{C}[x, y, z, w]$. The following is a basis for P:

$$x^r y^s z^t w^u \qquad r, s, t, u \in \mathbb{N}. \tag{5.1}$$

For $N \in \mathbb{N}$ let P_N denote the subspace of P consisting of the homogeneous polynomials that have total degree N. We call P_N the *Nth homogeneous component* of P. The sum $P = \sum_{N \in \mathbb{N}} P_N$ is direct. For notational convenience, define $P_{-1} = 0$ and $P_{-2} = 0$.

Definition 5.1. For $N \in \mathbb{N}$ let the set $\mathcal{P}_N$ consist of the 4-tuples of natural numbers (r, s, t, u) such that $r + s + t + u = N$. An element of $\mathcal{P}_N$ is called a *profile of degree N*.

Note that

$$|\mathcal{P}_N| = \binom{N+3}{3}. \tag{5.2}$$

Lemma 5.2. *For* $N \in \mathbb{N}$ *the following is a basis for* P_N*:*

$$x^r y^s z^t w^u \qquad (r, s, t, u) \in \mathcal{P}_N. \tag{5.3}$$

Moreover, P_N *has dimension* $\binom{N+3}{3}$.

Proof. Routine. □

Our next goal is to turn P into an $\mathfrak{sl}_4(\mathbb{C})$-module. To this end, we first consider P_1. Note that x, y, z, w is a basis for P_1.

Lemma 5.3. *The vector space P_1 becomes an $\mathfrak{sl}_4(\mathbb{C})$-module that satisfies* (i)–(vi) *below:*

(i) *A_1 swaps $x \leftrightarrow y$ and $z \leftrightarrow w$;*

(ii) *A_2 swaps $x \leftrightarrow z$ and $y \leftrightarrow w$;*

(iii) *A_3 swaps $x \leftrightarrow w$ and $y \leftrightarrow z$;*

(iv) *A_1^* sends*

$$x \mapsto x, \qquad y \mapsto y, \qquad z \mapsto -z, \qquad w \mapsto -w;$$

(v) *A_2^* sends*

$$x \mapsto x, \qquad y \mapsto -y, \qquad z \mapsto z, \qquad w \mapsto -w;$$

(vi) *A_3^* sends*

$$x \mapsto x, \qquad y \mapsto -y, \qquad z \mapsto -z, \qquad w \mapsto w.$$

Proof. By Lemma 3.6 and the comment above Lemma 3.8. □

We will be discussing derivations of P. The Lie algebra $\mathfrak{gl}(P)$ consists of the vector space $\mathrm{End}(P)$ and Lie bracket

$$[\varphi, \phi] = \varphi\phi - \phi\varphi \qquad\qquad \varphi, \phi \in \mathrm{End}(P).$$

A *derivation of P* is an element $\mathcal{D} \in \mathrm{End}(P)$ such that

$$\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g) \qquad\qquad f, g \in P. \tag{5.4}$$

Let the set $\mathrm{Der}(P)$ consist of the derivations of P. One checks that $\mathrm{Der}(P)$ is a Lie subalgebra of $\mathfrak{gl}(P)$.

Lemma 5.4. *For $\mathcal{D} \in \mathrm{Der}(P)$ we have $\mathcal{D}(1) = 0$.*

Proof. We have

$$\mathcal{D}(1) = \mathcal{D}(1^2) = \mathcal{D}(1)1 + 1\mathcal{D}(1) = 2\mathcal{D}(1).$$

Therefore $\mathcal{D}(1) = 0$. □

Lemma 5.5. *For $\mathcal{D} \in \mathrm{Der}(P)$ and $f \in P$ and $n \in \mathbb{N}$,*

$$\mathcal{D}(f^n) = nf^{n-1}\mathcal{D}(f).$$

Proof. Use (5.4) and Lemma 5.4 and induction on n. □

Lemma 5.6. *For $\mathcal{D} \in \mathrm{End}(P)$ the following are equivalent:*

(i) $\mathcal{D} \in \mathrm{Der}(P)$;

(ii) *for* $r, s, t, u \in \mathbb{N}$,

$$\mathcal{D}(x^r y^s z^t w^u) = r x^{r-1} y^s z^t w^u \mathcal{D}(x) + s x^r y^{s-1} z^t w^u \mathcal{D}(y) \\ + t x^r y^s z^{t-1} w^u \mathcal{D}(z) + u x^r y^s z^t w^{u-1} \mathcal{D}(w);$$

(iii) *for all polynomials* f, g *in the* P*-basis* (5.1),

$$\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g).$$

Proof. (i) ⇒ (ii) By (5.4) and Lemma 5.5,

$$\begin{aligned}\mathcal{D}(x^r y^s z^t w^u) &= \mathcal{D}(x^r) y^s z^t w^u + x^r \mathcal{D}(y^s) z^t w^u + x^r y^s \mathcal{D}(z^t) w^u + x^r y^s z^t \mathcal{D}(w^u) \\ &= r x^{r-1} y^s z^t w^u \mathcal{D}(x) + s x^r y^{s-1} z^t w^u \mathcal{D}(y) + t x^r y^s z^{t-1} w^u \mathcal{D}(z) \\ &\quad + u x^r y^s z^t w^{u-1} \mathcal{D}(w).\end{aligned}$$

(ii) ⇒ (iii) Routine.
(iii) ⇒ (i) The map $\mathcal{D}$ satisfies condition (5.4) since $\mathcal{D}$ is $\mathbb{C}$-linear. □

Lemma 5.7. *For* $\mathcal{D} \in \mathrm{Der}(P)$ *the following are equivalent:*

(i) $\mathcal{D} = 0$;

(ii) $\mathcal{D} = 0$ *on* P_1.

Proof. (i) ⇒ (ii) Clear.
(ii) ⇒ (i) Each of $\mathcal{D}(x), \mathcal{D}(y), \mathcal{D}(z), \mathcal{D}(w)$ is zero, so $\mathcal{D} = 0$ by Lemma 5.6(i),(ii). □

Lemma 5.8. *For a* $\mathbb{C}$*-linear map* $\mathcal{D}_1 : P_1 \to P$ *there exists a unique* $\mathcal{D} \in \mathrm{Der}(P)$ *such that the restriction* $\mathcal{D}|_{P_1} = \mathcal{D}_1$.

Proof. Concerning existence, by linear algebra there exists $\mathcal{D} \in \mathrm{End}(P)$ that acts on the P-basis (5.1) as follows. For $r, s, t, u \in \mathbb{N}$,

$$\mathcal{D}(x^r y^s z^t w^u) = r x^{r-1} y^s z^t w^u \mathcal{D}_1(x) + s x^r y^{s-1} z^t w^u \mathcal{D}_1(y) \\ + t x^r y^s z^{t-1} w^u \mathcal{D}_1(z) + u x^r y^s z^t w^{u-1} \mathcal{D}_1(w).$$

Note that

$$\mathcal{D}(x) = \mathcal{D}_1(x), \qquad \mathcal{D}(y) = \mathcal{D}_1(y), \qquad \mathcal{D}(z) = \mathcal{D}_1(z), \qquad \mathcal{D}(w) = \mathcal{D}_1(w).$$

Therefore $\mathcal{D}|_{P_1} = \mathcal{D}_1$. By these comments and Lemma 5.6(i),(ii) we have $\mathcal{D} \in \mathrm{Der}(P)$. We have shown that there exists $\mathcal{D} \in \mathrm{Der}(P)$ such that $\mathcal{D}|_{P_1} = \mathcal{D}_1$. The uniqueness of $\mathcal{D}$ follows from Lemma 5.7. □

Proposition 5.9. *There exists a unique Lie algebra homomorphism* der : $\mathfrak{sl}_4(\mathbb{C}) \to \mathrm{Der}(P)$ *such that for all* $\varphi \in \mathfrak{sl}_4(\mathbb{C})$ *the following coincide:*

(i) *the action of* $\operatorname{der}(\varphi)$ *on* P_1*;*

(ii) *the action of* φ *on* P_1 *from Lemma 5.3.*

The map der *is injective.*

Proof. For $\varphi \in \mathfrak{sl}_4(\mathbb{C})$ we define $\operatorname{der}(\varphi)$ as follows. Consider the action of φ on P_1 from Lemma 5.3. By Lemma 5.8, there exists a unique $\mathcal{D} \in \operatorname{Der}(P)$ such that $\mathcal{D} = \varphi$ on P_1. Define $\operatorname{der}(\varphi) = \mathcal{D}$. So far, we have a map $\operatorname{der} : \mathfrak{sl}_4(\mathbb{C}) \to \operatorname{Der}(P)$. The map der is $\mathbb{C}$-linear by construction. We show that der is a Lie algebra homomorphism. For $\varphi, \phi \in \mathfrak{sl}_4(\mathbb{C})$ we show that

$$\operatorname{der}[\varphi, \phi] = [\operatorname{der}\varphi, \operatorname{der}\phi]. \tag{5.5}$$

Each side of (5.5) is a derivation of P. For each side of (5.5) the restriction to P_1 coincides with the action of $[\varphi, \phi]$ on P_1. The two sides of (5.5) are equal in view of Lemma 5.8. We have shown that the Lie algebra homomorphism der exists. This map is unique by Lemma 5.8, and injective because $\mathfrak{sl}_4(\mathbb{C})$ acts faithfully on P_1. □

By Proposition 5.9, the vector space P becomes an $\mathfrak{sl}_4(\mathbb{C})$-module on which the elements of $\mathfrak{sl}_4(\mathbb{C})$ act as derivations.

Proposition 5.10. *The* $\mathfrak{sl}_4(\mathbb{C})$*-generators* A_1, A_2, A_3 *and* A_1^*, A_2^*, A_3^* *act on the* P*-basis* (5.1) *as follows. For* $r, s, t, u \in \mathbb{N}$*,*

(i) *the vector*

$$A_1(x^r y^s z^t w^u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{r-1}y^{s+1}z^t w^u$	r
$x^{r+1}y^{s-1}z^t w^u$	s
$x^r y^s z^{t-1} w^{u+1}$	t
$x^r y^s z^{t+1} w^{u-1}$	u

(ii) *the vector*

$$A_2(x^r y^s z^t w^u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{r-1}y^s z^{t+1} w^u$	r
$x^r y^{s-1} z^t w^{u+1}$	s
$x^{r+1}y^s z^{t-1} w^u$	t
$x^r y^{s+1} z^t w^{u-1}$	u

(iii) *the vector*

$$A_3(x^r y^s z^t w^u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{r-1}y^s z^t w^{u+1}$	r
$x^r y^{s-1} z^{t+1} w^u$	s
$x^r y^{s+1} z^{t-1} w^u$	t
$x^{r+1} y^s z^t w^{u-1}$	u

(iv) $A_1^*(x^r y^s z^t w^u) = (r+s-t-u)x^r y^s z^t w^u$;

(v) $A_2^*(x^r y^s z^t w^u) = (r-s+t-u)x^r y^s z^t w^u$;

(vi) $A_3^*(x^r y^s z^t w^u) = (r-s-t+u)x^r y^s z^t w^u$.

Proof. By Lemmas 5.3 and 5.6. □

We have a comment.

Lemma 5.11. *For $N \in \mathbb{N}$ the subspace P_N is a submodule of the $\mathfrak{sl}_4(\mathbb{C})$-module P.*

Proof. By Proposition 5.10, we have

$$A_i(P_N) \subseteq P_N, \qquad A_i^*(P_N) \subseteq P_N$$

for $i \in \{1, 2, 3\}$. □

Referring to Lemma 5.11, the submodule P_N is irreducible by [31, p. 97].

Let $N \in \mathbb{N}$, and consider the $\mathfrak{sl}_4(\mathbb{C})$-module P_N. By Proposition 5.10(iv)–(vi), the P_N-basis (5.3) diagonalizes the Cartan subalgebra $\mathbb{H}^*$. Consequently, P_N is the direct sum of its $\mathbb{H}^*$-weight spaces. Our next goal is to describe these weight spaces.

Lemma 5.12. *For natural numbers r, s, t, u and R, S, T, U we have*

$$r = R, \qquad s = S, \qquad t = T, \qquad u = U$$

if and only if

$$\begin{aligned} r+s+t+u &= R+S+T+U, \\ r+s-t-u &= R+S-T-U, \\ r-s+t-u &= R-S+T-U, \\ r-s-t+u &= R-S-T+U. \end{aligned}$$

Proof. Because the matrix Υ in Definition 4.1 is invertible. □

Lemma 5.13. *Let $N \in \mathbb{N}$ and consider the $\mathfrak{sl}_4(\mathbb{C})$-module P_N.*

(i) *Each $\mathbb{H}^*$-weight space has dimension one.*

(ii) *The $\mathbb{H}^*$-weight spaces are in bijection with $\mathcal{P}_N$.*

(iii) *For $(r, s, t, u) \in \mathcal{P}_N$ the following holds on the corresponding $\mathbb{H}^*$-weight space:*

$$A_1^* = (r + s - t - u)I, \qquad A_2^* = (r - s + t - u)I, \qquad A_3^* = (r - s - t + u)I.$$

Proof. By Proposition 5.10(iv)–(vi) and Lemma 5.12. □

As an aside, we describe the $\mathbb{H}^*$-weight spaces from another point of view. We will make a change of variables.

Lemma 5.14. *Let $N \in \mathbb{N}$. Pick $(r, s, t, u) \in \mathcal{P}_N$ and recall that*

$$N = r + s + t + u.$$

Define

$$\lambda = r + s - t - u, \qquad \mu = r - s + t - u, \qquad \nu = r - s - t + u.$$

Then

$$r = \frac{N + \lambda + \mu + \nu}{4}, \qquad s = \frac{N + \lambda - \mu - \nu}{4},$$
$$t = \frac{N - \lambda + \mu - \nu}{4}, \qquad u = \frac{N - \lambda - \mu + \nu}{4}.$$

Proof. Because the matrix Υ in Definition 4.1 satisfies $\Upsilon^2 = I$. □

Definition 5.15. For $N \in \mathbb{N}$, let the set $\mathcal{P}'_N$ consist of the 3-tuples (λ, μ, ν) such that

$$\lambda, \mu, \nu \in \{N, N-2, N-4, \ldots, -N\}, \qquad N + \lambda + \mu + \nu \text{ is divisible by } 4,$$
$$N + \lambda + \mu + \nu \geqslant 0, \qquad N + \lambda - \mu - \nu \geqslant 0,$$
$$N - \lambda + \mu - \nu \geqslant 0, \qquad N - \lambda - \mu + \nu \geqslant 0.$$

Lemma 5.16. *For $N \in \mathbb{N}$, there exists a bijection $\mathcal{P}_N \to \mathcal{P}'_N$ that sends*

$$(r, s, t, u) \mapsto (r + s - t - u, r - s + t - u, r - s - t + u).$$

The inverse bijection sends

$$(\lambda, \mu, \nu) \mapsto \left(\frac{N + \lambda + \mu + \nu}{4}, \frac{N + \lambda - \mu - \nu}{4}, \frac{N - \lambda + \mu - \nu}{4}, \frac{N - \lambda - \mu + \nu}{4} \right).$$

Proof. This is readily checked using Lemma 5.14. □

Lemma 5.17. *Let $N \in \mathbb{N}$ and consider the $\mathfrak{sl}_4(\mathbb{C})$-module P_N. The $\mathbb{H}^*$-weight spaces are in bijection with $\mathcal{P}'_N$, in such a way that for $(\lambda, \mu, \nu) \in \mathcal{P}'_N$ the following holds on the corresponding $\mathbb{H}^*$-weight space:*

$$A_1^* = \lambda I, \qquad A_2^* = \mu I, \qquad A_3^* = \nu I.$$

Proof. By Definition 5.15 and Lemmas 5.13, 5.16. □

We have a comment.

Lemma 5.18. *For $i \in \{1, 2, 3\}$ and $N \in \mathbb{N}$ the eigenvalues of A_i^* on P_N are $\{N-2n\}_{n=0}^N$. For $0 \leqslant n \leqslant N$ the $(N-2n)$-eigenspace for A_i^* on P_N has dimension $(n+1)(N-n+1)$.*

Proof. We first prove our assertions for A_1^*. By Definition 5.15 and Lemma 5.17, the eigenvalues of A_1^* on P_N are $\{N-2n\}_{n=0}^N$. Pick a natural number n at most N, and let W denote the $(N-2n)$-eigenspace for A_1^* on P_N. The subspace W is a direct sum of $\mathbb{H}^*$-weight spaces. The $\mathbb{H}^*$-weight spaces in question correspond (via Lemma 5.13(ii),(iii)) to the elements $(r, s, t, u) \in \mathcal{P}_N$ such that $r + s = N - n$ and $t + u = n$. There are exactly $N-n+1$ nonnegative integer solutions to $r+s = N-n$. There are exactly $n+1$ nonnegative integer solutions to $t+u = n$. Consequently, there are exactly $(n+1)(N-n+1)$ elements $(r, s, t, u) \in \mathcal{P}_N$ such that $r+s = N-n$ and $t+u = n$. This shows that W has dimension $(n+1)(N-n+1)$. We have proved our assertions for A_1^*. Our assertions for A_2^*, A_3^* are similarly proved. □

We have been discussing $\mathbb{H}^*$. In Section 7, we will have a similar discussion about $\mathbb{H}$.

6 Some derivations and multiplication maps

Recall the polynomial algebra $P = \mathbb{C}[x, y, z, w]$. In the previous section, we turned P into an $\mathfrak{sl}_4(\mathbb{C})$-module. In this section, we describe the $\mathfrak{sl}_4(\mathbb{C})$-module P using four partial derivatives and four multiplication maps.

Consider the partial derivatives

$$D_x = \frac{\partial}{\partial x}, \qquad D_y = \frac{\partial}{\partial y}, \qquad D_z = \frac{\partial}{\partial z}, \qquad D_w = \frac{\partial}{\partial w}.$$

These derivatives act as follows on the P-basis (5.1). For $r, s, t, u \in \mathbb{N}$,

$$D_x\big(x^r y^s z^t w^u\big) = r x^{r-1} y^s z^t w^u, \qquad D_y\big(x^r y^s z^t w^u\big) = s x^r y^{s-1} z^t w^u, \tag{6.1}$$

$$D_z\big(x^r y^s z^t w^u\big) = t x^r y^s z^{t-1} w^u, \qquad D_w\big(x^r y^s z^t w^u\big) = u x^r y^s z^t w^{u-1}. \tag{6.2}$$

For $N \in \mathbb{N}$ we have

$$D_x(P_N) = P_{N-1}, \qquad D_y(P_N) = P_{N-1}, \qquad D_z(P_N) = P_{N-1}, \qquad D_w(P_N) = P_{N-1}.$$

Lemma 6.1. *The following* (i)–(iv) *hold:*

(i) D_x *is the unique element in* $\mathrm{Der}(P)$ *that sends*

$$x \mapsto 1, \qquad y \mapsto 0, \qquad z \mapsto 0, \qquad w \mapsto 0;$$

(ii) D_y *is the unique element in* $\mathrm{Der}(P)$ *that sends*

$$x \mapsto 0, \qquad y \mapsto 1, \qquad z \mapsto 0, \qquad w \mapsto 0;$$

(iii) D_z *is the unique element in* $\mathrm{Der}(P)$ *that sends*

$$x \mapsto 0, \qquad y \mapsto 0, \qquad z \mapsto 1, \qquad w \mapsto 0;$$

(iv) D_w *is the unique element in* $\mathrm{Der}(P)$ *that sends*

$$x \mapsto 0, \qquad y \mapsto 0, \qquad z \mapsto 0, \qquad w \mapsto 1.$$

Proof. By Lemma 5.6 and (6.1), (6.2). □

Definition 6.2. We define $M_x, M_y, M_z, M_w \in \mathrm{End}(P)$ as follows. For $f \in P$,

$$M_x(f) = xf, \qquad M_y(f) = yf, \qquad M_z(f) = zf, \qquad M_w(f) = wf.$$

The maps M_x, M_y, M_z, M_w act as follows on the P-basis (5.1). For $r, s, t, u \in \mathbb{N}$,

$$M_x\big(x^r y^s z^t w^u\big) = x^{r+1} y^s z^t w^u, \qquad M_y\big(x^r y^s z^t w^u\big) = x^r y^{s+1} z^t w^u, \tag{6.3}$$
$$M_z\big(x^r y^s z^t w^u\big) = x^r y^s z^{t+1} w^u, \qquad M_w\big(x^r y^s z^t w^u\big) = x^r y^s z^t w^{u+1}. \tag{6.4}$$

For $N \in \mathbb{N}$ we have

$$M_x(P_N) \subseteq P_{N+1}, \qquad M_y(P_N) \subseteq P_{N+1}, \qquad M_z(P_N) \subseteq P_{N+1}, \qquad M_w(P_N) \subseteq P_{N+1}.$$

Lemma 6.3. (See [46, p. 550].) *The following relations hold.*

(i) *For* $a \in \{x, y, z, w\}$,

$$[D_a, M_a] = I.$$

(ii) *For distinct* $a, b \in \{x, y, z, w\}$,

$$[D_a, D_b] = 0, \qquad [M_a, M_b] = 0, \qquad [D_a, M_b] = 0.$$

Proof. We check that $[D_x, M_x] = I$. For $f \in P$,

$$\begin{aligned}[D_x, M_x](f) &= D_x M_x(f) - M_x D_x(f) = D_x(xf) - x D_x(f) \\ &= D_x(x) f + x D_x(f) - x D_x(f) = f.\end{aligned}$$

Therefore $[D_x, M_x] = I$. The remaining assertions are checked in a similar way. □

Remark 6.4. The subalgebra of End(P) generated by D_x, D_y, D_z, D_w and M_x, M_y, M_z, M_w is called the fourth Weyl algebra, see [46, p. 550].

Proposition 6.5. *On the* $\mathfrak{sl}_4(\mathbb{C})$*-module* P,

$$A_1 = M_y D_x + M_x D_y + M_w D_z + M_z D_w,$$
$$A_2 = M_z D_x + M_w D_y + M_x D_z + M_y D_w,$$
$$A_3 = M_w D_x + M_z D_y + M_y D_z + M_x D_w$$

and also

$$A_1^* = M_x D_x + M_y D_y - M_z D_z - M_w D_w,$$
$$A_2^* = M_x D_x - M_y D_y + M_z D_z - M_w D_w,$$
$$A_3^* = M_x D_x - M_y D_y - M_z D_z + M_w D_w.$$

Proof. To verify these equations, apply each side to a P-basis vector from (5.1), and evaluate the result using Proposition 5.10 along with (6.1)–(6.4). □

7 A basis for P that diagonalizes $\mathbb{H}$

We continue to discuss the $\mathfrak{sl}_4(\mathbb{C})$-module $P = \mathbb{C}[x, y, z, w]$. In Lemma 4.5 we described the Cartan subalgebras $\mathbb{H}$, $\mathbb{H}^*$ of $\mathfrak{sl}_4(\mathbb{C})$. In Section 5, we displayed a basis for P that diagonalizes $\mathbb{H}^*$. In this section, we display a basis for P that diagonalizes $\mathbb{H}$.

Recall that x, y, z, w form a basis for P_1.

Definition 7.1. We define some vectors in P_1:

$$x^* = \frac{x + y + z + w}{2}, \qquad y^* = \frac{x + y - z - w}{2},$$
$$z^* = \frac{x - y + z - w}{2}, \qquad w^* = \frac{x - y - z + w}{2}.$$

Recall the matrix Υ from Definition 4.1.

Lemma 7.2. *The following* (i)–(iii) *hold:*

(i) *the vectors* x^*, y^*, z^*, w^* *form a basis for* P_1*;*

(ii) Υ *is the transition matrix from the basis* x, y, z, w *to the basis* x^*, y^*, z^*, w^**;*

(iii) Υ *is the transition matrix from the basis* x^*, y^*, z^*, w^* *to the basis* x, y, z, w.

Proof. By Definitions 4.1, 7.1 and $\Upsilon^2 = I$. □

Lemma 7.3. *We have*

$$x = \frac{x^* + y^* + z^* + w^*}{2}, \qquad y = \frac{x^* + y^* - z^* - w^*}{2},$$
$$z = \frac{x^* - y^* + z^* - w^*}{2}, \qquad w = \frac{x^* - y^* - z^* + w^*}{2}.$$

Proof. This is a reformulation of Lemma 7.2(iii). □

Lemma 7.4. *Referring to the* $\mathfrak{sl}_4(\mathbb{C})$*-module* P_1*,*

(i) A_1 *sends*

$$x^* \mapsto x^*, \qquad y^* \mapsto y^*, \qquad z^* \mapsto -z^*, \qquad w^* \mapsto -w^*;$$

(ii) A_2 *sends*

$$x^* \mapsto x^*, \qquad y^* \mapsto -y^*, \qquad z^* \mapsto z^*, \qquad w^* \mapsto -w^*;$$

(iii) A_3 *sends*

$$x^* \mapsto x^*, \qquad y^* \mapsto -y^*, \qquad z^* \mapsto -z^*, \qquad w^* \mapsto w^*;$$

(iv) A_1^* *swaps* $x^* \leftrightarrow y^*$ *and* $z^* \leftrightarrow w^*$*;*

(v) A_2^* *swaps* $x^* \leftrightarrow z^*$ *and* $y^* \leftrightarrow w^*$*;*

(vi) A_3^* *swaps* $x^* \leftrightarrow w^*$ *and* $y^* \leftrightarrow z^*$*.*

Proof. By Lemmas 4.2 and 7.2. □

We have been discussing P_1. Next we consider P.

Lemma 7.5. *The following is a basis for* P*:*

$$x^{*r}y^{*s}z^{*t}w^{*u} \qquad\qquad r,s,t,u \in \mathbb{N}. \tag{7.1}$$

Proof. The vectors x^*, y^*, z^*, w^* form a basis for P_1. □

Lemma 7.6. *For* $N \in \mathbb{N}$ *the following is a basis for* P_N*:*

$$x^{*r}y^{*s}z^{*t}w^{*u} \qquad\qquad (r,s,t,u) \in \mathcal{P}_N. \tag{7.2}$$

Proof. By Lemma 7.5 and since each of x^*, y^*, z^*, w^* is homogeneous with total degree one. □

Lemma 7.7. *Each* $\mathcal{D} \in \mathrm{Der}(P)$ *acts on the* P*-basis* (7.1) *as follows. For* $r,s,t,u \in \mathbb{N}$*,*

$$\begin{aligned}\mathcal{D}(x^{*r}y^{*s}z^{*t}w^{*u}) = {} & rx^{*r-1}y^{*s}z^{*t}w^{*u}\mathcal{D}(x^*) + sx^{*r}y^{*s-1}z^{*t}w^{*u}\mathcal{D}(y^*) \\ & + tx^{*r}y^{*s}z^{*t-1}w^{*u}\mathcal{D}(z^*) + ux^{*r}y^{*s}z^{*t}w^{*u-1}\mathcal{D}(w^*).\end{aligned}$$

Proof. Similar to the proof of Lemma 5.6$\big($(i) $\Rightarrow$ (ii)$\big)$. □

Proposition 7.8. *The* $\mathfrak{sl}_4(\mathbb{C})$*-generators* A_1, A_2, A_3 *and* A_1^*, A_2^*, A_3^* *act on the* P*-basis* (7.1) *as follows. For* $r,s,t,u \in \mathbb{N}$*,*

(i) $A_1(x^{*r}y^{*s}z^{*t}w^{*u}) = (r+s-t-u)x^{*r}y^{*s}z^{*t}w^{*u}$;

(ii) $A_2(x^{*r}y^{*s}z^{*t}w^{*u}) = (r-s+t-u)x^{*r}y^{*s}z^{*t}w^{*u}$;

(iii) $A_3(x^{*r}y^{*s}z^{*t}w^{*u}) = (r-s-t+u)x^{*r}y^{*s}z^{*t}w^{*u}$;

(iv) *the vector*

$$A_1^*(x^{*r}y^{*s}z^{*t}w^{*u})$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{*r-1}y^{*s+1}z^{*t}w^{*u}$	r
$x^{*r+1}y^{*s-1}z^{*t}w^{*u}$	s
$x^{*r}y^{*s}z^{*t-1}w^{*u+1}$	t
$x^{*r}y^{*s}z^{*t+1}w^{*u-1}$	u

(v) *the vector*

$$A_2^*(x^{*r}y^{*s}z^{*t}w^{*u})$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{*r-1}y^{*s}z^{*t+1}w^{*u}$	r
$x^{*r}y^{*s-1}z^{*t}w^{*u+1}$	s
$x^{*r+1}y^{*s}z^{*t-1}w^{*u}$	t
$x^{*r}y^{*s+1}z^{*t}w^{*u-1}$	u

(vi) *the vector*

$$A_3^*(x^{*r}y^{*s}z^{*t}w^{*u})$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{*r-1}y^{*s}z^{*t}w^{*u+1}$	r
$x^{*r}y^{*s-1}z^{*t+1}w^{*u}$	s
$x^{*r}y^{*s+1}z^{*t-1}w^{*u}$	t
$x^{*r+1}y^{*s}z^{*t}w^{*u-1}$	u

Proof. By Lemmas 7.4 and 7.7. □

Let $N \in \mathbb{N}$, and consider the $\mathfrak{sl}_4(\mathbb{C})$-module P_N. By Proposition 7.8(i)–(iii), the P_N-basis (7.2) diagonalizes $\mathbb{H}$. Consequently, P_N is the direct sum of its $\mathbb{H}$-weight spaces.

Lemma 7.9. *Let $N \in \mathbb{N}$ and consider the $\mathfrak{sl}_4(\mathbb{C})$-module P_N.*

(i) *Each $\mathbb{H}$-weight space has dimension one.*

(ii) *The $\mathbb{H}$-weight spaces are in bijection with $\mathcal{P}_N$.*

(iii) *For $(r, s, t, u) \in \mathcal{P}_N$ the following holds on the corresponding $\mathbb{H}$-weight space:*

$$A_1 = (r + s - t - u)I, \qquad A_2 = (r - s + t - u)I, \qquad A_3 = (r - s - t + u)I.$$

Proof. By Lemma 5.12 and Proposition 7.8(i)–(iii). □

Lemma 7.10. *Let $N \in \mathbb{N}$ and consider the $\mathfrak{sl}_4(\mathbb{C})$-module P_N. The $\mathbb{H}$-weight spaces are in bijection with $\mathcal{P}'_N$, in such a way that for $(\lambda, \mu, \nu) \in \mathcal{P}'_N$ the following holds on the corresponding $\mathbb{H}$-weight space:*

$$A_1 = \lambda I, \qquad A_2 = \mu I, \qquad A_3 = \nu I.$$

Proof. By Definition 5.15 and Lemmas 5.16, 7.9. □

Lemma 7.11. *For $i \in \{1, 2, 3\}$ and $N \in \mathbb{N}$ the eigenvalues of A_i on P_N are $\{N - 2n\}_{n=0}^{N}$. For $0 \leqslant n \leqslant N$ the $(N - 2n)$-eigenspace for A_i on P_N has dimension $(n + 1)(N - n + 1)$.*

Proof. Similar to the proof of Lemma 5.18. □

Proposition 7.12. *There exists an automorphism σ of P that sends*

$$x \leftrightarrow x^*, \qquad y \leftrightarrow y^*, \qquad z \leftrightarrow z^*, \qquad w \leftrightarrow w^*.$$

We have $\sigma^2 = \mathrm{id}$. For $i \in \{1, 2, 3\}$ the following holds on the $\mathfrak{sl}_4(\mathbb{C})$-module P:

$$A_i^* = \sigma A_i \sigma^{-1}, \qquad A_i = \sigma A_i^* \sigma^{-1}. \tag{7.3}$$

Proof. The vector space P_1 has a basis x, y, z, w and a basis x^*, y^*, z^*, w^*. Therefore, there exists an automorphism σ of P that sends

$$x \mapsto x^*, \qquad y \mapsto y^*, \qquad z \mapsto z^*, \qquad w \mapsto w^*.$$

By Definition 7.1 and Lemma 7.3, the automorphism σ sends

$$x^* \mapsto x, \qquad y^* \mapsto y, \qquad z^* \mapsto z, \qquad w^* \mapsto w.$$

Therefore $\sigma^2 = \mathrm{id}$. Let $i \in \{1, 2, 3\}$. Comparing Propositions 5.10, 7.8 we find that $A_i^* \sigma = \sigma A_i$ holds on P. This yields (7.3). □

Recall the automorphism τ of $\mathfrak{sl}_4(\mathbb{C})$ from Lemma 4.3.

Proposition 7.13. *For $\varphi \in \mathfrak{sl}_4(\mathbb{C})$ the following holds on the $\mathfrak{sl}_4(\mathbb{C})$-module P:*

$$\tau(\varphi) = \sigma \varphi \sigma^{-1}. \tag{7.4}$$

Proof. By (7.3) and Lemma 4.4, the following holds on P:

$$\tau(A_i) = \sigma A_i \sigma^{-1}, \qquad \tau(A_i^*) = \sigma A_i^* \sigma^{-1} \qquad i \in \{1, 2, 3\}.$$

This yields (7.4) because the Lie algebra $\mathfrak{sl}_4(\mathbb{C})$ is generated by

$$A_i, \quad A_i^* \qquad i \in \{1, 2, 3\}.$$

□

We have a comment about the automorphism σ of P from Proposition 7.12. The map σ acts on the P-bases (5.1) and (7.1) as follows. For $r, s, t, u \in \mathbb{N}$ the map σ swaps

$$x^r y^s z^t w^u \leftrightarrow x^{*r} y^{*s} z^{*t} w^{*u}.$$

Moreover, $\sigma(P_N) = P_N$ for $N \in \mathbb{N}$.

8 More derivations and multiplication maps

We continue to discuss the $\mathfrak{sl}_4(\mathbb{C})$-module $P = \mathbb{C}[x, y, z, w]$. In Section 6, we investigated the derivations D_x, D_y, D_z, D_w and the multiplication maps M_x, M_y, M_z, M_w. In this section, we introduce the analogous derivations D_{x^*}, D_{y^*}, D_{z^*}, D_{w^*} and multiplication maps M_{x^*}, M_{y^*}, M_{z^*}, M_{w^*}.

The next result is motivated by Lemma 6.1.

Lemma 8.1. *The following* (i)–(iv) *hold.*

(i) *There exists a unique element* $D_{x^*} \in \mathrm{Der}(P)$ *that sends*

$$x^* \mapsto 1, \qquad y^* \mapsto 0, \qquad z^* \mapsto 0, \qquad w^* \mapsto 0.$$

(ii) *There exists a unique element* $D_{y^*} \in \mathrm{Der}(P)$ *that sends*

$$x^* \mapsto 0, \qquad y^* \mapsto 1, \qquad z^* \mapsto 0, \qquad w^* \mapsto 0.$$

(iii) *There exists a unique element* $D_{z^*} \in \mathrm{Der}(P)$ *that sends*

$$x^* \mapsto 0, \qquad y^* \mapsto 0, \qquad z^* \mapsto 1, \qquad w^* \mapsto 0.$$

(iv) *There exists a unique element* $D_{w^*} \in \mathrm{Der}(P)$ *that sends*

$$x^* \mapsto 0, \qquad y^* \mapsto 0, \qquad z^* \mapsto 0, \qquad w^* \mapsto 1.$$

Proof. By Lemma 5.8. □

Lemma 8.2. *The derivations D_{x^*}, D_{y^*}, D_{z^*}, D_{w^*} act as follows on the P-basis* (7.1)*. For $r, s, t, u \in \mathbb{N}$,*

$$D_{x^*}\left(x^{*r}y^{*s}z^{*t}w^{*u}\right) = rx^{*r-1}y^{*s}z^{*t}w^{*u}, \qquad D_{y^*}\left(x^{*r}y^{*s}z^{*t}w^{*u}\right) = sx^{*r}y^{*s-1}z^{*t}w^{*u},$$
$$D_{z^*}\left(x^{*r}y^{*s}z^{*t}w^{*u}\right) = tx^{*r}y^{*s}z^{*t-1}w^{*u}, \qquad D_{w^*}\left(x^{*r}y^{*s}z^{*t}w^{*u}\right) = ux^{*r}y^{*s}z^{*t}w^{*u-1}.$$

Proof. By Lemmas 7.7, 8.1. □

Lemma 8.3. *For $N \in \mathbb{N}$ we have*

$$D_{x^*}(P_N) = P_{N-1}, \qquad D_{y^*}(P_N) = P_{N-1}, \qquad D_{z^*}(P_N) = P_{N-1}, \qquad D_{w^*}(P_N) = P_{N-1}.$$

Proof. By Lemmas 7.6, 8.2. □

Lemma 8.4. *We have*

$$D_{x^*} = \frac{D_x + D_y + D_z + D_w}{2}, \qquad D_{y^*} = \frac{D_x + D_y - D_z - D_w}{2},$$
$$D_{z^*} = \frac{D_x - D_y + D_z - D_w}{2}, \qquad D_{w^*} = \frac{D_x - D_y - D_z + D_w}{2}.$$

Proof. For these equations, each side is a derivation. So by Lemma 5.7, these equations hold if they hold on P_1. By Definition 7.1 or Lemma 7.3, these equations hold on P_1. □

Lemma 8.5. *We have*

$$D_x = \frac{D_{x^*} + D_{y^*} + D_{z^*} + D_{w^*}}{2}, \qquad D_y = \frac{D_{x^*} + D_{y^*} - D_{z^*} - D_{w^*}}{2},$$
$$D_z = \frac{D_{x^*} - D_{y^*} + D_{z^*} - D_{w^*}}{2}, \qquad D_w = \frac{D_{x^*} - D_{y^*} - D_{z^*} + D_{w^*}}{2}.$$

Proof. By Lemma 8.4 and linear algebra. □

Lemma 8.6. *The automorphism σ of P satisfies*

$$D_{x^*} = \sigma D_x \sigma^{-1}, \qquad D_{y^*} = \sigma D_y \sigma^{-1}, \qquad D_{z^*} = \sigma D_z \sigma^{-1}, \qquad D_{w^*} = \sigma D_w \sigma^{-1}.$$

Proof. Compare (6.1), (6.2) with Lemma 8.2 using the comment at the end of Section 7. □

Definition 8.7. We define $M_{x^*}, M_{y^*}, M_{z^*}, M_{w^*} \in \mathrm{End}(P)$ as follows. For $f \in P$,

$$M_{x^*}(f) = x^* f, \qquad M_{y^*}(f) = y^* f, \qquad M_{z^*}(f) = z^* f, \qquad M_{w^*}(f) = w^* f.$$

The maps $M_{x^*}, M_{y^*}, M_{z^*}, M_{w^*}$ act as follows on the P-basis (7.1). For $r, s, t, u \in \mathbb{N}$,

$$M_{x^*}\left(x^{*r}y^{*s}z^{*t}w^{*u}\right) = x^{*r+1}y^{*s}z^{*t}w^{*u}, \qquad M_{y^*}\left(x^{*r}y^{*s}z^{*t}w^{*u}\right) = x^{*r}y^{*s+1}z^{*t}w^{*u},$$
$$M_{z^*}\left(x^{*r}y^{*s}z^{*t}w^{*u}\right) = x^{*r}y^{*s}z^{*t+1}w^{*u}, \qquad M_{w^*}\left(x^{*r}y^{*s}z^{*t}w^{*u}\right) = x^{*r}y^{*s}z^{*t}w^{*u+1}.$$

For $N \in \mathbb{N}$ we have

$$M_{x^*}(P_N) \subseteq P_{N+1}, \qquad M_{y^*}(P_N) \subseteq P_{N+1}, \qquad M_{z^*}(P_N) \subseteq P_{N+1}, \qquad M_{w^*}(P_N) \subseteq P_{N+1}.$$

Lemma 8.8. *We have*

$$M_{x^*} = \frac{M_x + M_y + M_z + M_w}{2}, \qquad M_{y^*} = \frac{M_x + M_y - M_z - M_w}{2},$$
$$M_{z^*} = \frac{M_x - M_y + M_z - M_w}{2}, \qquad M_{w^*} = \frac{M_x - M_y - M_z + M_w}{2}.$$

Proof. By Definitions 6.2, 7.1, 8.7. □

Lemma 8.9. *We have*

$$M_x = \frac{M_{x^*} + M_{y^*} + M_{z^*} + M_{w^*}}{2}, \qquad M_y = \frac{M_{x^*} + M_{y^*} - M_{z^*} - M_{w^*}}{2},$$
$$M_z = \frac{M_{x^*} - M_{y^*} + M_{z^*} - M_{w^*}}{2}, \qquad M_w = \frac{M_{x^*} - M_{y^*} - M_{z^*} + M_{w^*}}{2}.$$

Proof. By Lemma 7.3 and Definitions 6.2, 8.7. □

Lemma 8.10. *The automorphism σ of P satisfies*

$$M_{x^*} = \sigma M_x \sigma^{-1}, \qquad M_{y^*} = \sigma M_y \sigma^{-1}, \qquad M_{z^*} = \sigma M_z \sigma^{-1}, \qquad M_{w^*} = \sigma M_w \sigma^{-1}.$$

Proof. We prove the first equation. We have $M_{x^*}\sigma = \sigma M_x$ because for all $f \in P$,

$$M_{x^*}\sigma(f) = x^*\sigma(f) = \sigma(x)\sigma(f) = \sigma(xf) = \sigma M_x(f).$$

The first equation is proved. The remaining equations are similarly proved. □

Lemma 8.11. *The following relations hold.*

(i) *For $a \in \{x^*, y^*, z^*, w^*\}$,*

$$[D_a, M_a] = I.$$

(ii) *For distinct $a, b \in \{x^*, y^*, z^*, w^*\}$,*

$$[D_a, D_b] = 0, \qquad [M_a, M_b] = 0, \qquad [D_a, M_b] = 0.$$

Proof. In Lemma 6.3 conjugate each term by σ, and evaluate the result using Lemmas 8.6, 8.10. □

Proposition 8.12. *On the $\mathfrak{sl}_4(\mathbb{C})$-module P,*

$$A_1^* = M_{y^*}D_{x^*} + M_{x^*}D_{y^*} + M_{w^*}D_{z^*} + M_{z^*}D_{w^*},$$
$$A_2^* = M_{z^*}D_{x^*} + M_{w^*}D_{y^*} + M_{x^*}D_{z^*} + M_{y^*}D_{w^*},$$
$$A_3^* = M_{w^*}D_{x^*} + M_{z^*}D_{y^*} + M_{y^*}D_{z^*} + M_{x^*}D_{w^*}$$

and also

$$A_1 = M_{x^*}D_{x^*} + M_{y^*}D_{y^*} - M_{z^*}D_{z^*} - M_{w^*}D_{w^*},$$
$$A_2 = M_{x^*}D_{x^*} - M_{y^*}D_{y^*} + M_{z^*}D_{z^*} - M_{w^*}D_{w^*},$$
$$A_3 = M_{x^*}D_{x^*} - M_{y^*}D_{y^*} - M_{z^*}D_{z^*} + M_{w^*}D_{w^*}.$$

Proof. In Proposition 6.5 conjugate each term by σ, and evaluate the result using (7.3) along with Lemmas 8.6, 8.10. □

9 A Hermitian form on P

We continue to discuss the $\mathfrak{sl}_4(\mathbb{C})$-module $P = \mathbb{C}[x, y, z, w]$. In this section, we endow the vector space P with a Hermitian form $\langle\, ,\, \rangle$ with respect to which the P-bases (5.1) and (7.1) are orthogonal.

Definition 9.1. Let W denote a vector space. A *Hermitian form on* W is a function $\langle\, ,\, \rangle : W \times W \to \mathbb{C}$ such that:

(i) $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$ for all $f, g, h \in W$;

(ii) $\langle \alpha f, g\rangle = \alpha\langle f, g\rangle$ for all $\alpha \in \mathbb{C}$ and $f, g \in W$;

(iii) $\langle f, g\rangle = \overline{\langle g, f\rangle}$ for all $f, g \in W$.

For a Hermitian form $\langle\, ,\, \rangle$ on W, we abbreviate $\|f\|^2 = \langle f, f\rangle$ for all $f \in W$.

Definition 9.2. We endow the vector space P with a Hermitian form $\langle\, ,\, \rangle$ with respect to which the basis vectors

$$x^r y^s z^t w^u \qquad r, s, t, u \in \mathbb{N}$$

are mutually orthogonal and

$$\|x^r y^s z^t w^u\|^2 = r!s!t!u! \qquad r, s, t, u \in \mathbb{N}. \tag{9.1}$$

Lemma 9.3. *The homogeneous components* $\{P_N\}_{N\in\mathbb{N}}$ *are mutually orthogonal with respect to* $\langle\, ,\, \rangle$.

Proof. By Lemma 5.2 and Definition 9.2. □

Lemma 9.4. *For* $N \in \mathbb{N}$ *the* P_N*-basis*

$$x^r y^s z^t w^u \qquad (r, s, t, u) \in \mathcal{P}_N$$

and the P_N*-basis*

$$\frac{x^r y^s z^t w^u}{r!s!t!u!} \qquad (r, s, t, u) \in \mathcal{P}_N$$

are dual with respect to $\langle\, ,\, \rangle$.

Proof. We invoke Definition 9.2. For $(r, s, t, u) \in \mathcal{P}_N$ and $(R, S, T, U) \in \mathcal{P}_N$ we have

$$\left\langle \frac{x^r y^s z^t w^u}{r!s!t!u!}, x^R y^S z^T w^U \right\rangle = \delta_{r,R}\delta_{s,S}\delta_{t,T}\delta_{u,U}.$$

□

Lemma 9.5. *For $N \in \mathbb{N}$,*

$$x^{*N} = \frac{N!}{2^N} \sum_{(r,s,t,u) \in \mathcal{P}_N} \frac{x^r y^s z^t w^u}{r!s!t!u!}. \tag{9.2}$$

Proof. By Definition 7.1,

$$x^{*N} = \Big(\frac{x+y+z+w}{2}\Big)^N.$$

In this equation, expand the right-hand side using the multinomial theorem. □

Lemma 9.6. *For $N \in \mathbb{N}$ and $(r,s,t,u) \in \mathcal{P}_N$,*

$$\Big\langle x^r y^s z^t w^u, x^{*N} \Big\rangle = \frac{N!}{2^N}.$$

Proof. To verify the result, eliminate x^{*N} using (9.2) and evaluate the result using Definition 9.2. □

Lemma 9.7. *For $f, g \in P$ we have*

$$\langle D_x f, g \rangle = \langle f, M_x g \rangle, \qquad \langle D_y f, g \rangle = \langle f, M_y g \rangle, \tag{9.3}$$

$$\langle D_z f, g \rangle = \langle f, M_z g \rangle, \qquad \langle D_w f, g \rangle = \langle f, M_w g \rangle. \tag{9.4}$$

Proof. Without loss of generality, we may assume that f, g are contained in the P-basis (5.1). Under this assumption (9.3), (9.4) are routinely checked using Definition 9.2 and (6.1)–(6.4). □

Lemma 9.8. *For $f, g \in P$ we have*

$$\langle M_x f, g \rangle = \langle f, D_x g \rangle, \qquad \langle M_y f, g \rangle = \langle f, D_y g \rangle, \tag{9.5}$$

$$\langle M_z f, g \rangle = \langle f, D_z g \rangle, \qquad \langle M_w f, g \rangle = \langle f, D_w g \rangle. \tag{9.6}$$

Proof. By Definition 9.1(iii) and Lemma 9.7. □

Lemma 9.9. *For $i \in \{1,2,3\}$ and $f, g \in P$,*

$$\langle A_i f, g \rangle = \langle f, A_i g \rangle, \qquad \langle A_i^* f, g \rangle = \langle f, A_i^* g \rangle.$$

Proof. We first show that $\langle A_1 f, g \rangle = \langle f, A_1 g \rangle$. By Proposition 6.5, the following holds on P:

$$A_1 = M_y D_x + M_x D_y + M_w D_z + M_z D_w.$$

Using this and Lemmas 9.7, 9.8 we obtain

$$\begin{aligned} \langle A_1 f, g \rangle &= \langle M_y D_x f, g \rangle + \langle M_x D_y f, g \rangle + \langle M_w D_z f, g \rangle + \langle M_z D_w f, g \rangle \\ &= \langle D_x f, D_y g \rangle + \langle D_y f, D_x g \rangle + \langle D_z f, D_w g \rangle + \langle D_w f, D_z g \rangle \\ &= \langle f, M_x D_y g \rangle + \langle f, M_y D_x g \rangle + \langle f, M_z D_w g \rangle + \langle f, M_w D_z g \rangle \\ &= \langle f, A_1 g \rangle. \end{aligned}$$

We have shown that $\langle A_1 f, g \rangle = \langle f, A_1 g \rangle$. The other assertions are similarly shown. □

Proposition 9.10. *With respect to* $\langle\, , \rangle$ *the vectors*

$$x^{*r}y^{*s}z^{*t}w^{*u} \qquad\qquad r,s,t,u \in \mathbb{N} \tag{9.7}$$

are mutually orthogonal and

$$\|x^{*r}y^{*s}z^{*t}w^{*u}\|^2 = r!s!t!u! \qquad\qquad r,s,t,u \in \mathbb{N}. \tag{9.8}$$

Proof. We first show that the vectors (9.7) are mutually orthogonal. To this end, let f, g denote vectors from (9.7) such that $\langle f, g\rangle \neq 0$. We show that $f = g$. Write

$$f = x^{*r}y^{*s}z^{*t}w^{*u}, \qquad\qquad g = x^{*R}y^{*S}z^{*T}w^{*U}.$$

By Lemma 9.3 and since $\langle f, g\rangle \neq 0$,

$$r + s + t + u = R + S + T + U. \tag{9.9}$$

By Lemma 9.9 and the construction,

$$\frac{\langle A_1 f, g\rangle}{\langle f, g\rangle} = \frac{\langle f, A_1 g\rangle}{\langle f, g\rangle}, \qquad \frac{\langle A_2 f, g\rangle}{\langle f, g\rangle} = \frac{\langle f, A_2 g\rangle}{\langle f, g\rangle}, \qquad \frac{\langle A_3 f, g\rangle}{\langle f, g\rangle} = \frac{\langle f, A_3 g\rangle}{\langle f, g\rangle}.$$

Evaluating these equations using Proposition 7.8, we obtain

$$r + s - t - u = R + S - T - U, \tag{9.10}$$
$$r - s + t - u = R - S + T - U, \tag{9.11}$$
$$r - s - t + u = R - S - T + U. \tag{9.12}$$

By Lemma 5.12 and (9.9)–(9.12),

$$r = R, \qquad s = S, \qquad t = T, \qquad u = U.$$

Therefore $f = g$. We have shown that the vectors (9.7) are mutually orthogonal. Next we prove (9.8). For the rest of this proof, fix $N \in \mathbb{N}$. We will prove that

$$\|x^{*r}y^{*s}z^{*t}w^{*u}\|^2 = r!s!t!u! \qquad\qquad (r,s,t,u) \in \mathcal{P}_N. \tag{9.13}$$

Our proof of (9.13) is by induction on $s + t + u$. First assume that $s + t + u = 0$. Then $r = N$ and $s = t = u = 0$. We must show that $\|x^{*N}\|^2 = N!$. To this end, in (9.2) take the square norm of each side to obtain

$$\begin{aligned}
\|x^{*N}\|^2 &= \Bigg\| \frac{N!}{2^N} \sum_{(r,s,t,u)\in\mathcal{P}_N} \frac{x^r y^s z^t w^u}{r!s!t!u!} \Bigg\|^2 \\
&= \frac{(N!)^2}{4^N} \sum_{(r,s,t,u)\in\mathcal{P}_N} \frac{\|x^r y^s z^t w^u\|^2}{(r!s!t!u!)^2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{N!}{4^N} \sum_{(r,s,t,u)\in \mathcal{P}_N} \frac{N!}{r!s!t!u!} \\
&= N! 4^{-N} (1+1+1+1)^N \\
&= N!.
\end{aligned}$$

We have proved (9.13) for $s+t+u=0$. Next assume that $s+t+u \geqslant 1$. Then $s \geqslant 1$ or $t \geqslant 1$ or $u \geqslant 1$. We take each case in turn.

Case $s \geqslant 1$. Define

$$R = r+1, \qquad S = s-1, \qquad T = t, \qquad U = u$$

and note that $S+T+U < s+t+u$. By Proposition 7.8(iv) and Lemma 9.9,

$$\begin{aligned}
R\|x^{*r}y^{*s}z^{*t}w^{*u}\|^2 &= \langle x^{*r}y^{*s}z^{*t}w^{*u}, A_1^* x^{*R}y^{*S}z^{*T}w^{*U}\rangle \\
&= \langle A_1^* x^{*r}y^{*s}z^{*t}w^{*u}, x^{*R}y^{*S}z^{*T}w^{*U}\rangle \\
&= s\|x^{*R}y^{*S}z^{*T}w^{*U}\|^2.
\end{aligned}$$

By this and induction,

$$\|x^{*r}y^{*s}z^{*t}w^{*u}\|^2 = \frac{s}{R}\|x^{*R}y^{*S}z^{*T}w^{*U}\|^2 = \frac{s}{R} R!S!T!U! = r!s!t!u!.$$

Case $t \geqslant 1$. Define

$$R = r+1, \qquad S = s, \qquad T = t-1, \qquad U = u$$

and note that $S+T+U < s+t+u$. By Proposition 7.8(v) and Lemma 9.9,

$$\begin{aligned}
R\|x^{*r}y^{*s}z^{*t}w^{*u}\|^2 &= \langle x^{*r}y^{*s}z^{*t}w^{*u}, A_2^* x^{*R}y^{*S}z^{*T}w^{*U}\rangle \\
&= \langle A_2^* x^{*r}y^{*s}z^{*t}w^{*u}, x^{*R}y^{*S}z^{*T}w^{*U}\rangle \\
&= t\|x^{*R}y^{*S}z^{*T}w^{*U}\|^2.
\end{aligned}$$

By this and induction,

$$\|x^{*r}y^{*s}z^{*t}w^{*u}\|^2 = \frac{t}{R}\|x^{*R}y^{*S}z^{*T}w^{*U}\|^2 = \frac{t}{R} R!S!T!U! = r!s!t!u!.$$

Case $u \geqslant 1$. Define

$$R = r+1, \qquad S = s, \qquad T = t, \qquad U = u-1$$

and note that $S+T+U < s+t+u$. By Proposition 7.8(vi) and Lemma 9.9,

$$\begin{aligned}
R\|x^{*r}y^{*s}z^{*t}w^{*u}\|^2 &= \langle x^{*r}y^{*s}z^{*t}w^{*u}, A_3^* x^{*R}y^{*S}z^{*T}w^{*U}\rangle \\
&= \langle A_3^* x^{*r}y^{*s}z^{*t}w^{*u}, x^{*R}y^{*S}z^{*T}w^{*U}\rangle \\
&= u\|x^{*R}y^{*S}z^{*T}w^{*U}\|^2.
\end{aligned}$$

By this and induction,

$$\|x^{*r}y^{*s}z^{*t}w^{*u}\|^2 = \frac{u}{R}\|x^{*R}y^{*S}z^{*T}w^{*U}\|^2 = \frac{u}{R} R!S!T!U! = r!s!t!u!.$$

We have proven (9.13), and (9.8) follows. □

The following result is motivated by Lemma 9.4.

Lemma 9.11. *For $N \in \mathbb{N}$ the P_N-basis*

$$x^{*r} y^{*s} z^{*t} w^{*u} \qquad\qquad (r,s,t,u) \in \mathcal{P}_N$$

and the P_N-basis

$$\frac{x^{*r} y^{*s} z^{*t} w^{*u}}{r!s!t!u!} \qquad\qquad (r,s,t,u) \in \mathcal{P}_N$$

are dual with respect to $\langle\, ,\, \rangle$.

Proof. Use Proposition 9.10. □

The following result is motivated by Lemma 9.5. Recall the automorphism σ of P from Proposition 7.12.

Lemma 9.12. *For $N \in \mathbb{N}$,*

$$x^N = \frac{N!}{2^N} \sum_{(r,s,t,u)\in\mathcal{P}_N} \frac{x^{*r} y^{*s} z^{*t} w^{*u}}{r!s!t!u!}.$$

Proof. Apply the automorphism σ to each side of (9.2). □

Proposition 9.13. *For $f, g \in P$ we have*

$$\langle \sigma f, \sigma g \rangle = \langle f, g \rangle. \tag{9.14}$$

Proof. Without loss of generality, we may assume that f, g are contained in the P-basis (5.1). Comparing Definition 9.2 and Proposition 9.10, we routinely obtain (9.14). □

The following result is motivated by Lemma 9.6.

Lemma 9.14. *For $N \in \mathbb{N}$ and $(r,s,t,u) \in \mathcal{P}_N$,*

$$\left\langle x^N, x^{*r} y^{*s} z^{*t} w^{*u} \right\rangle = \frac{N!}{2^N}.$$

Proof. Apply the automorphism σ to everything in Lemma 9.6, and evaluate the result using Proposition 9.13. □

The next result is motivated by Lemmas 9.7, 9.8.

Lemma 9.15. *For $f, g \in P$ we have*

$$\langle D_{x^*} f, g \rangle = \langle f, M_{x^*} g \rangle, \qquad \langle D_{y^*} f, g \rangle = \langle f, M_{y^*} g \rangle,$$
$$\langle D_{z^*} f, g \rangle = \langle f, M_{z^*} g \rangle, \qquad \langle D_{w^*} f, g \rangle = \langle f, M_{w^*} g \rangle$$

and also

$$\langle M_{x^*} f, g \rangle = \langle f, D_{x^*} g \rangle, \qquad \langle M_{y^*} f, g \rangle = \langle f, D_{y*} g \rangle,$$
$$\langle M_{z^*} f, g \rangle = \langle f, D_{z^*} g \rangle, \qquad \langle M_{w^*} f, g \rangle = \langle f, D_{w^*} g \rangle.$$

Proof. To prove the first equation, observe that

$$\begin{aligned}\langle D_{x^*} f, g\rangle &= \langle \sigma D_x \sigma^{-1} f, g\rangle = \langle D_x \sigma^{-1} f, \sigma^{-1} g\rangle \\ &= \langle \sigma^{-1} f, M_x \sigma^{-1} g\rangle = \langle f, \sigma M_x \sigma^{-1} g\rangle = \langle f, M_{x^*} g\rangle.\end{aligned}$$

The remaining equations are similarly proved. □

Let $N \in \mathbb{N}$. Our next goal is to compute all the inner products between the P_N-basis

$$x^r y^s z^t w^u \qquad (r, s, t, u) \in \mathcal{P}_N$$

and the P_N-basis

$$x^{*r} y^{*s} z^{*t} w^{*u} \qquad (r, s, t, u) \in \mathcal{P}_N.$$

We will express these inner products in two ways: using a generating function, and as hypergeometric sums.

Proposition 9.16. *Let $N \in \mathbb{N}$. For $(r, s, t, u) \in \mathcal{P}_N$ and $(R, S, T, U) \in \mathcal{P}_N$, the inner product*

$$\left\langle x^r y^s z^t w^u, x^{*R} y^{*S} z^{*T} w^{*U} \right\rangle \tag{9.15}$$

is equal to

$$\frac{r!s!t!u!}{2^N}$$

times the coefficient of $x^r y^s z^t w^u$ in

$$(x + y + z + w)^R (x + y - z - w)^S (x - y + z - w)^T (x - y - z + w)^U.$$

Proof. Expand the inner product (9.15) using Definition 7.1, and evaluate the result using Definition 9.2. □

We bring in some notation. For $\alpha \in \mathbb{C}$ define

$$(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1) \qquad n \in \mathbb{N}.$$

The following result is a routine application of [40, Line (6)].

Proposition 9.17. *Let $N \in \mathbb{N}$. For $(r, s, t, u) \in \mathcal{P}_N$ and $(R, S, T, U) \in \mathcal{P}_N$,*

$$\begin{aligned}&\left\langle x^r y^s z^t w^u, x^{*R} y^{*S} z^{*T} w^{*U} \right\rangle \\ &= \frac{N!}{2^N} \sum_{\substack{a,b,c,d,e,f \in \mathbb{N} \\ a+b+c+d+e+f \leqslant N}} \frac{(-s)_{a+b}(-t)_{c+d}(-u)_{e+f}(-S)_{c+e}(-T)_{a+f}(-U)_{b+d}}{(-N)_{a+b+c+d+e+f}} \frac{2^{a+b+c+d+e+f}}{a!b!c!d!e!f!}\end{aligned}$$

Proof. This is [40, Line (6)] applied to the character algebra of the Klein four-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. For this character algebra the first and second eigenmatrix is 2Υ, where Υ is from Definition 4.1. The matrix $\tilde{\Omega}$ mentioned in [40, Line (6)] is given by

$$\tilde{\Omega} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

□

10 Some polynomials

In this section, we consider some polynomials that are motivated by Proposition 9.17. Background information about this general topic can be found in [20, 24, 39, 47].

Throughout this section, fix $N \in \mathbb{N}$. Let $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3$ denote mutually commuting indeterminates.

Definition 10.1. Define the polynomial

$$\begin{aligned}&\mathcal{P}(\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3) \\ &= \sum_{\substack{a,b,c,d,e,f \in \mathbb{N} \\ a+b+c+d+e+f \leqslant N}} \frac{(-\lambda_1)_{a+b}(-\lambda_2)_{c+d}(-\lambda_3)_{e+f}(-\mu_1)_{c+e}(-\mu_2)_{a+f}(-\mu_3)_{b+d}}{(-N)_{a+b+c+d+e+f}} \frac{2^{a+b+c+d+e+f}}{a!b!c!d!e!f!}.\end{aligned}$$

Note 10.2. The polynomial $\mathcal{P}(\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3)$ is symmetric in $\lambda_1, \lambda_2, \lambda_3$ and symmetric in μ_1, μ_2, μ_3. Moreover,

$$\mathcal{P}(\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3) = \mathcal{P}(\mu_1, \mu_2, \mu_3; \lambda_1, \lambda_2, \lambda_3). \tag{10.1}$$

Lemma 10.3. *For $(r, s, t, u) \in \mathcal{P}_N$ and $(R, S, T, U) \in \mathcal{P}_N$,*

$$\left\langle x^r y^s z^t w^u, x^{*R} y^{*S} z^{*T} w^{*U} \right\rangle = \frac{N!}{2^N} \mathcal{P}(s, t, u; S, T, U).$$

Proof. By Proposition 9.17 and Definition 10.1. □

Proposition 10.4. *For $(r, s, t, u) \in \mathcal{P}_N$ we have*

$$x^r y^s z^t w^u = \frac{N!}{2^N} \sum_{(R,S,T,U) \in \mathcal{P}_N} \frac{\mathcal{P}(s, t, u; S, T, U)}{R!S!T!U!} x^{*R} y^{*S} z^{*T} w^{*U}, \tag{10.2}$$

$$x^{*r} y^{*s} z^{*t} w^{*u} = \frac{N!}{2^N} \sum_{(R,S,T,U) \in \mathcal{P}_N} \frac{\mathcal{P}(s, t, u; S, T, U)}{R!S!T!U!} x^R y^S z^T w^U. \tag{10.3}$$

Proof. Use linear algebra, invoking Definition 9.2 and Proposition 9.10 and Lemma 10.3. □

Next, we give an orthogonality relation. The following result is a special case of [40, Theorem 1.1].

Proposition 10.5. *For $(r, s, t, u) \in \mathcal{P}_N$ and $(r', s', t', u') \in \mathcal{P}_N$,*

$$\sum_{(R,S,T,U) \in \mathcal{P}_N} \frac{\mathcal{P}(s, t, u; S, T, U)\mathcal{P}(s', t', u'; S, T, U)}{R!S!T!U!} = \delta_{s,s'} \delta_{t,t'} \delta_{u,u'} \frac{4^N r!s!t!u!}{(N!)^2}.$$

Proof. We consider the inner product

$$\left\langle x^r y^s z^t w^u, x^{r'} y^{s'} z^{t'} w^{u'} \right\rangle \tag{10.4}$$

from two points of view. On one hand, we evaluate (10.4) using Definition 9.2. On the other hand, we eliminate the two arguments in (10.4) using (10.2), and evaluate the result using Proposition 9.10. Comparing the two points of view, we get the orthogonality relation. □

Next, we give some recurrence relations.

Proposition 10.6. *The following* (i)–(iii) *hold for* $(r,s,t,u) \in \mathcal{P}_N$ *and* $(R,S,T,U) \in \mathcal{P}_N$.

(i) $(R+S-T-U)\mathcal{P}(s,t,u;S,T,U)$ *is a linear combination with the following terms and coefficients:*

Term	Coefficient
$\mathcal{P}(s+1,t,u;S,T,U)$	r
$\mathcal{P}(s-1,t,u;S,T,U)$	s
$\mathcal{P}(s,t-1,u+1;S,T,U)$	t
$\mathcal{P}(s,t+1,u-1;S,T,U)$	u

(ii) $(R-S+T-U)\mathcal{P}(s,t,u;S,T,U)$ *is a linear combination with the following terms and coefficients:*

Term	Coefficient
$\mathcal{P}(s,t+1,u;S,T,U)$	r
$\mathcal{P}(s-1,t,u+1;S,T,U)$	s
$\mathcal{P}(s,t-1,u;S,T,U)$	t
$\mathcal{P}(s+1,t,u-1;S,T,U)$	u

(iii) $(R-S-T+U)\mathcal{P}(s,t,u;S,T,U)$ *is a linear combination with the following terms and coefficients:*

Term	Coefficient
$\mathcal{P}(s,t,u+1;S,T,U)$	r
$\mathcal{P}(s-1,t+1,u;S,T,U)$	s
$\mathcal{P}(s+1,t-1,u;S,T,U)$	t
$\mathcal{P}(s,t,u-1;S,T,U)$	u

Proof. (i) By Lemma 9.9 we have

$$\left\langle A_1(x^r y^s z^t w^u), x^{*R} y^{*S} z^{*T} w^{*U} \right\rangle = \left\langle x^r y^s z^t w^u, A_1(x^{*R} y^{*S} z^{*T} w^{*U}) \right\rangle. \tag{10.5}$$

Evaluate the left-hand side of (10.5) using Proposition 5.10(i) and Lemma 10.3. Evaluate the right-hand side of (10.5) using Proposition 7.8(i) and Lemma 10.3. The result follows. (ii), (iii) Similar to the proof of (i) above. □

We just gave some recurrence relations. Next, we simplify these relations using a change of variables.

Definition 10.7. Define the polynomial

$$\begin{aligned}&\mathcal{P}^\vee(\lambda_1,\lambda_2,\lambda_3;\mu_1,\mu_2,\mu_3)\\&\quad=\mathcal{P}\left(\lambda_1,\lambda_2,\lambda_3;\frac{N+\mu_1-\mu_2-\mu_3}{4},\frac{N-\mu_1+\mu_2-\mu_3}{4},\frac{N-\mu_1-\mu_2+\mu_3}{4}\right).\end{aligned}$$

Lemma 10.8. *For* $(r,s,t,u)\in\mathcal{P}_N$ *and* $(R,S,T,U)\in\mathcal{P}_N$,

$$\mathcal{P}(s,t,u;S,T,U)=\mathcal{P}^\vee(s,t,u;R+S-T-U,R-S+T-U,R-S-T+U).$$

Proof. By Lemma 5.14 and Definition 10.7. □

Recall the set $\mathcal{P}'_N$ from Definition 5.15.

Proposition 10.9. *The following* (i)–(iii) *hold for* $(r,s,t,u)\in\mathcal{P}_N$ *and* $(\lambda,\mu,\nu)\in\mathcal{P}'_N$.

(i) $\lambda\mathcal{P}^\vee(s,t,u;\lambda,\mu,\nu)$ *is a linear combination with the following terms and coefficients:*

Term	Coefficient
$\mathcal{P}^\vee(s+1,t,u;\lambda,\mu,\nu)$	r
$\mathcal{P}^\vee(s-1,t,u;\lambda,\mu,\nu)$	s
$\mathcal{P}^\vee(s,t-1,u+1;\lambda,\mu,\nu)$	t
$\mathcal{P}^\vee(s,t+1,u-1;\lambda,\mu,\nu)$	u

(ii) $\mu\mathcal{P}^\vee(s,t,u;\lambda,\mu,\nu)$ *is a linear combination with the following terms and coefficients:*

Term	Coefficient
$\mathcal{P}^\vee(s,t+1,u;\lambda,\mu,\nu)$	r
$\mathcal{P}^\vee(s-1,t,u+1;\lambda,\mu,\nu)$	s
$\mathcal{P}^\vee(s,t-1,u;\lambda,\mu,\nu)$	t
$\mathcal{P}^\vee(s+1,t,u-1;\lambda,\mu,\nu)$	u

(iii) $\nu\mathcal{P}^\vee(s,t,u;\lambda,\mu,\nu)$ *is a linear combination with the following terms and coefficients:*

Term	Coefficient
$\mathcal{P}^\vee(s,t,u+1;\lambda,\mu,\nu)$	r
$\mathcal{P}^\vee(s-1,t+1,u;\lambda,\mu,\nu)$	s
$\mathcal{P}^\vee(s+1,t-1,u;\lambda,\mu,\nu)$	t
$\mathcal{P}^\vee(s,t,u-1;\lambda,\mu,\nu)$	u

Proof. Evaluate Proposition 10.6(i)–(iii) using Lemma 5.16 and Lemma 10.8. □

Proposition 10.10. *The following hold for* $(r,s,t,u)\in\mathcal{P}_N$:

(i) $\mathcal{P}^\vee(s,t,u;A_1,A_2,A_3)x^N = x^r y^s z^t w^u$;

(ii) $\mathcal{P}^\vee(s,t,u;A_1^*,A_2^*,A_3^*)x^{*N} = x^{*r} y^{*s} z^{*t} w^{*u}$.

Proof. (i) It suffices to show that for $(R,S,T,U) \in \mathcal{P}_N$,

$$\left\langle \mathcal{P}^\vee(s,t,u;A_1,A_2,A_3)x^N, x^{*R}y^{*S}z^{*T}w^{*U} \right\rangle = \left\langle x^r y^s z^t w^u, x^{*R}y^{*S}z^{*T}w^{*U} \right\rangle. \tag{10.6}$$

Using in order Lemma 9.9, Proposition 7.8(i)–(iii), and Lemmas 10.8, 9.14, 10.3,

$$\begin{aligned}
&\left\langle \mathcal{P}^\vee(s,t,u;A_1,A_2,A_3)x^N, x^{*R}y^{*S}z^{*T}w^{*U} \right\rangle \\
&\quad = \left\langle x^N, \mathcal{P}^\vee(s,t,u;A_1,A_2,A_3)x^{*R}y^{*S}z^{*T}w^{*U} \right\rangle \\
&\quad = \left\langle x^N, x^{*R}y^{*S}z^{*T}w^{*U} \right\rangle \mathcal{P}^\vee(s,t,u;R+S-T-U,R-S+T-U,R-S-T+U) \\
&\quad = \left\langle x^N, x^{*R}y^{*S}z^{*T}w^{*U} \right\rangle \mathcal{P}(s,t,u;S,T,U) \\
&\quad = \frac{N!}{2^N}\mathcal{P}(s,t,u;S,T,U) \\
&\quad = \left\langle x^r y^s z^t w^u, x^{*R}y^{*S}z^{*T}w^{*U} \right\rangle.
\end{aligned}$$

(ii) Similar to the proof of (i) above. □

Proposition 10.11. *The following hold.*

(i) *P_N has a basis*

$$A_1^s A_2^t A_3^u x^N \qquad s,t,u \in \mathbb{N}, \qquad s+t+u \leqslant N. \tag{10.7}$$

(ii) *P_N has a basis*

$$A_1^{*s} A_2^{*t} A_3^{*u} x^{*N} \qquad s,t,u \in \mathbb{N}, \qquad s+t+u \leqslant N. \tag{10.8}$$

Proof. (i) The number of vectors in (10.7) is equal to $\binom{N+3}{3}$, and this is the dimension of P_N. By linear algebra, it suffices to show that P_N is spanned by the vectors (10.7). By Lemma 5.2 and Proposition 10.10(i), P_N is spanned by the vectors (10.7).
(ii) Similar to the proof of (i) above. □

11 Some $\mathfrak{sl}_2(\mathbb{C})$-actions on P

We continue to discuss the $\mathfrak{sl}_4(\mathbb{C})$-module $P = \mathbb{C}[x,y,z,w]$. Throughout this section we fix distinct $i,j \in \{1,2,3\}$. In Corollary 3.10 we saw that A_i, A_j^* generate a Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ that is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. The $\mathfrak{sl}_4(\mathbb{C})$-module P becomes an $\mathfrak{sl}_2\mathbb{C})$-module, by restricting the $\mathfrak{sl}_4(\mathbb{C})$ action to $\mathfrak{sl}_2\mathbb{C})$. In the present section we investigate the $\mathfrak{sl}_2(\mathbb{C})$-module P.

Let us briefly review the theory of finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules. The following material can be found in [25, Sections 6, 7]. Each finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module is a direct sum of irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules. For $N \in \mathbb{N}$, up to isomorphism there exists a unique irreducible $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{V}_N$ of dimension $N+1$. The $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{V}_N$ has a basis $\{v_n\}_{n=0}^N$ such that

$$\begin{aligned} Hv_n &= (N-2n)v_n && (0 \leqslant n \leqslant N), \\ Fv_n &= (n+1)v_{n+1} && (0 \leqslant n \leqslant N-1), \qquad Fv_N = 0, \\ Ev_n &= (N-n+1)v_{n-1} && (1 \leqslant n \leqslant N), \qquad Ev_0 = 0. \end{aligned}$$

We mention another basis for $\mathbb{V}_N$. Define

$$u_n = v_n \binom{N}{n}^{-1} \qquad (0 \leqslant n \leqslant N).$$

The vectors $\{u_n\}_{n=0}^N$ form a basis for $\mathbb{V}_N$, and

$$\begin{aligned} Hu_n &= (N-2n)u_n && (0 \leqslant n \leqslant N), \\ Fu_n &= (N-n)u_{n+1} && (0 \leqslant n \leqslant N-1), \qquad Fv_N = 0, \\ Eu_n &= nu_{n-1} && (1 \leqslant n \leqslant N), \qquad Ev_0 = 0. \end{aligned}$$

Definition 11.1. Let W denote a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module. Decompose W into a direct sum of irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodules:

$$W = W_1 + W_2 + \cdots + W_m. \tag{11.1}$$

Pick $N \in \mathbb{N}$ and consider the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{V}_N$. By the *multiplicity with which* $\mathbb{V}_N$ *appears in* W we mean the number of summands in (11.1) that are isomorphic to $\mathbb{V}_N$ (this number does not depend on the choice of decomposition, see [18, Chapter IV]). We say that $\mathbb{V}_N$ is an *irreducible component* of W whenever $\mathbb{V}_N$ appears in W with nonzero multiplicity.

Proposition 11.2. *For $N \in \mathbb{N}$ the $\mathfrak{sl}_2(\mathbb{C})$-module P_N has the following irreducible components:*

$$\mathbb{V}_{N-2n} \qquad 0 \leqslant n \leqslant \lfloor N/2 \rfloor. \tag{11.2}$$

For $0 \leqslant n \leqslant \lfloor N/2 \rfloor$ the module $\mathbb{V}_{N-2n}$ appears in P_N with multiplicity $N-2n+1$.

Proof. Pick $M \in \mathbb{N}$ such that $\mathbb{V}_M$ is an irreducible component of P_N. The eigenvalues of A_j^* on P_N are $\{N-2n\}_{n=0}^N$. The integer M is a nonnegative eigenvalue of A_j^* on $\mathbb{V}_M$, so there exists a natural number $n \leqslant \lfloor N/2 \rfloor$ such that $M = N-2n$. By these comments, $\mathbb{V}_M$ is included in the list (11.2). For $0 \leqslant n \leqslant \lfloor N/2 \rfloor$ let $\mathrm{mult}(\mathbb{V}_{N-2n})$ denote the multiplicity with which $\mathbb{V}_{N-2n}$ appears in P_N. Counting A_j^* eigenspace dimensions and using Lemma 5.18, we obtain

$$(n+1)(N-n+1) = \sum_{\ell=0}^{n} \mathrm{mult}(\mathbb{V}_{N-2\ell}), \qquad 0 \leqslant n \leqslant \lfloor N/2 \rfloor.$$

From these equations we find that for $0 \leqslant n \leqslant \lfloor N/2 \rfloor$,

$$\begin{aligned}\mathrm{mult}(\mathbb{V}_{N-2n}) &= (n+1)(N-n+1) - n(N-n+2)\\ &= N-2n+1.\end{aligned}$$

□

12 The maps L_1, L_2, L_3 and R_1, R_2, R_3

We continue to discuss the $\mathfrak{sl}_4(\mathbb{C})$-module $P = \mathbb{C}[x, y, z, w]$. In this section we do the following. For $i \in \{1, 2, 3\}$ we introduce a "lowering map" $L_i \in \mathrm{End}(P)$ and a "raising map" $R_i \in \mathrm{End}(P)$. For these maps we discuss the injectivity and surjectivity. We also discuss how these maps are related to the $\mathfrak{sl}_4(\mathbb{C})$-generators and the Hermitian form $\langle\, ,\, \rangle$.

Definition 12.1. Define $L_1, L_2, L_3 \in \mathrm{End}(P)$ by

$$L_1 = D_x D_y - D_z D_w, \qquad L_2 = D_x D_z - D_w D_y, \qquad L_3 = D_x D_w - D_y D_z.$$

Lemma 12.2. *We have*

$$L_1 = D_{x^*} D_{y^*} - D_{z^*} D_{w^*}, \qquad L_2 = D_{x^*} D_{z^*} - D_{w^*} D_{y^*}, \qquad L_3 = D_{x^*} D_{w^*} - D_{y^*} D_{z^*}.$$

Proof. To verify these equations, eliminate $D_{x^*}, D_{y^*}, D_{z^*}, D_{w^*}$ using Lemma 8.4 and evaluate the results using Definition 12.1. □

Recall the automorphism σ of P from Proposition 7.12.

Lemma 12.3. *For $i \in \{1, 2, 3\}$ the map L_i commutes with σ.*

Proof. By Definition 12.1 and Lemmas 8.6, 12.2 we obtain

$$\sigma L_i \sigma^{-1} = \sigma\Big(D_x D_y - D_z D_w\Big)\sigma^{-1} = D_{x^*} D_{y^*} - D_{z^*} D_{w^*} = L_i.$$

□

Lemma 12.4. *The following hold:*

(i) *for distinct $i, j \in \{1, 2, 3\}$ we have $[L_i, L_j] = 0$;*

(ii) *for $i \in \{1, 2, 3\}$ and $N \in \mathbb{N}$ we have $L_i(P_N) \subseteq P_{N-2}$.*

Proof. (i) By Definition 12.1 and since D_x, D_y, D_z, D_w mutually commute.
(ii) By Definition 12.1 and the comments above Lemma 6.1. □

Next, we describe how L_1, L_2, L_3 act on the P-basis (5.1).

Lemma 12.5. *For $r, s, t, u \in \mathbb{N}$ we have*

$$\begin{aligned} L_1(x^r y^s z^t w^u) &= rsx^{r-1}y^{s-1}z^t w^u - tux^r y^s z^{t-1}w^{u-1}, \\ L_2(x^r y^s z^t w^u) &= rtx^{r-1}y^s z^{t-1}w^u - sux^r y^{s-1}z^t w^{u-1}, \\ L_3(x^r y^s z^t w^u) &= rux^{r-1}y^s z^t w^{u-1} - stx^r y^{s-1}z^{t-1}w^u. \end{aligned}$$

Proof. By (6.1), (6.2) and Definition 12.1. □

Next, we describe how L_1, L_2, L_3 act on the P-basis (7.1).

Lemma 12.6. *For $r, s, t, u \in \mathbb{N}$ we have*

$$\begin{aligned} L_1(x^{*r}y^{*s}z^{*t}w^{*u}) &= rsx^{*r-1}y^{*s-1}z^{*t}w^{*u} - tux^{*r}y^{*s}z^{*t-1}w^{*u-1}, \\ L_2(x^{*r}y^{*s}z^{*t}w^{*u}) &= rtx^{*r-1}y^{*s}z^{*t-1}w^{*u} - sux^{*r}y^{*s-1}z^{*t}w^{*u-1}, \\ L_3(x^{*r}y^{*s}z^{*t}w^{*u}) &= rux^{*r-1}y^{*s}z^{*t}w^{*u-1} - stx^{*r}y^{*s-1}z^{*t-1}w^{*u}. \end{aligned}$$

Proof. By Lemmas 8.2, 12.2. □

Proposition 12.7. *The following* (i)–(iii) *hold on P.*

(i) *L_1 commutes with*

$$A_2, \qquad A_3, \qquad A_2^*, \qquad A_3^*.$$

(ii) *L_2 commutes with*

$$A_3, \qquad A_1, \qquad A_3^*, \qquad A_1^*.$$

(iii) *L_3 commutes with*

$$A_1, \qquad A_2, \qquad A_1^*, \qquad A_2^*.$$

Proof. First we prove that L_1, A_2 commute. By Lemma 6.3, Proposition 6.5, and Definition 12.1,

$$\begin{aligned} [L_1, A_2] &= [D_x D_y - D_z D_w, M_x D_z + M_y D_w + M_z D_x + M_w D_y] \\ &= [D_x D_y, M_x D_z] + [D_x D_y, M_y D_w] - [D_z D_w, M_z D_x] - [D_z D_w, M_w D_y] \\ &= [D_x, M_x] D_y D_z + [D_y, M_y] D_x D_w - [D_z, M_z] D_x D_w - [D_w, M_w] D_y D_z \\ &= D_y D_z + D_x D_w - D_x D_w - D_y D_z \\ &= 0. \end{aligned}$$

Next we prove that L_1, A_2^* commute. We have

$$\begin{aligned} [L_1, A_2^*] &= [D_x D_y - D_z D_w, M_x D_x - M_y D_y + M_z D_z - M_w D_w] \\ &= [D_x D_y, M_x D_x] - [D_x D_y, M_y D_y] - [D_z D_w, M_z D_z] + [D_z D_w, M_w D_w] \\ &= [D_x, M_x] D_x D_y - [D_y, M_y] D_x D_y - [D_z, M_z] D_z D_w + [D_w, M_w] D_z D_w \\ &= D_x D_y - D_x D_y - D_z D_w + D_z D_w \\ &= 0. \end{aligned}$$

The remaining assertions are similarly proven. □

Lemma 12.8. *Each of L_1, L_2, L_3 is surjective but not injective.*

Proof. We first show that L_1 is surjective. To do this, we show that

$$x^r y^s z^t w^u \in L_1(P) \qquad\qquad r, s, t, u \in \mathbb{N}. \tag{12.1}$$

We will prove (12.1) by induction on u. First assume that $u = 0$. By Lemma 12.5,

$$x^r y^s z^t = \frac{L_1\big(x^{r+1} y^{s+1} z^t\big)}{(r+1)(s+1)} \in L_1(P).$$

Next assume that $u \geqslant 1$. By Lemma 12.5,

$$x^r y^s z^t w^u = \frac{L_1\big(x^{r+1} y^{s+1} z^t w^u\big) + tu x^{r+1} y^{s+1} z^{t-1} w^{u-1}}{(r+1)(s+1)}.$$

In the above fraction, the numerator term on the right is contained in $L_1(P)$ by induction. By these comments, $x^r y^s z^t w^u \in L_1(P)$. We have shown that L_1 is surjective. Observe that L_1 is not injective, because L_1 sends $P_0 \to 0$ and $P_1 \to 0$. We have proved our claims about L_1. The claims about L_2, L_3 are similarly proven. □

We make an observation for later use. For $N \in \mathbb{N}$,

$$\binom{N+3}{3} = \binom{N+1}{3} + (N+1)^2. \tag{12.2}$$

Lemma 12.9. *The following hold for $i \in \{1, 2, 3\}$ and $N \in \mathbb{N}$:*

(i) $L_i(P_N) = P_{N-2}$;

(ii) $\operatorname{Ker}(L_i) \cap P_N$ *has dimension* $(N+1)^2$.

Proof. (i) By Lemma 12.4(ii) and Lemma 12.8.
(ii) By linear algebra along with (12.2) and (i) above,

$$\begin{aligned} \dim\Big(\operatorname{Ker}(L_i) \cap P_N\Big) &= \dim P_N - \dim P_{N-2} \\ &= \binom{N+3}{3} - \binom{N+1}{3} = (N+1)^2. \end{aligned}$$

□

Definition 12.10. Define $R_1, R_2, R_3 \in \operatorname{End}(P)$ by

$$R_1 = M_x M_y - M_z M_w, \qquad R_2 = M_x M_z - M_w M_y, \qquad R_3 = M_x M_w - M_y M_z.$$

We clarify Definition 12.10. For $f \in P$,

$$R_1(f) = (xy - zw)f, \qquad R_2(f) = (xz - wy)f, \qquad R_3(f) = (xw - yz)f. \tag{12.3}$$

Lemma 12.11. *We have*

$$xy - zw = x^*y^* - z^*w^*, \qquad xz - wy = x^*z^* - w^*y^*, \qquad xw - yz = x^*w^* - y^*z^*.$$

In other words, the automorphism σ fixes each of

$$xy - zw, \qquad xz - wy, \qquad xw - yz.$$

Proof. Use Definition 7.1. □

Lemma 12.12. *We have*

$$R_1 = M_{x^*}M_{y^*} - M_{z^*}M_{w^*}, \qquad R_2 = M_{x^*}M_{z^*} - M_{w^*}M_{y^*}, \qquad R_3 = M_{x^*}M_{w^*} - M_{y^*}M_{z^*}.$$

Proof. By (12.3) and Lemma 12.11. □

Lemma 12.13. *For $i \in \{1, 2, 3\}$ the map R_i commutes with σ.*

Proof. Similar to the proof of Lemma 12.3. □

Lemma 12.14. *The following hold:*

(i) *for distinct $i, j \in \{1, 2, 3\}$ we have $[R_i, R_j] = 0$;*

(ii) *for $i \in \{1, 2, 3\}$ and $N \in \mathbb{N}$ we have $R_i(P_N) \subseteq P_{N+2}$.*

Proof. (i) By Definition 12.10 and since M_x, M_y, M_z, M_w mutually commute.
(ii) By Lemma 5.2 and (12.3). □

Next, we describe how R_1, R_2, R_3 act on the P-basis (5.1).

Lemma 12.15. *For $r, s, t, u \in \mathbb{N}$ we have*

$$\begin{aligned}
R_1(x^r y^s z^t w^u) &= x^{r+1}y^{s+1}z^t w^u - x^r y^s z^{t+1}w^{u+1}, \\
R_2(x^r y^s z^t w^u) &= x^{r+1}y^s z^{t+1}w^u - x^r y^{s+1}z^t w^{u+1}, \\
R_3(x^r y^s z^t w^u) &= x^{r+1}y^s z^t w^{u+1} - x^r y^{s+1}z^{t+1}w^u.
\end{aligned}$$

Proof. By (12.3). □

Next, we describe how R_1, R_2, R_3 act on the P-basis (7.1).

Lemma 12.16. *For $r, s, t, u \in \mathbb{N}$ we have*

$$\begin{aligned}
R_1(x^{*r} y^{*s} z^{*t} w^{*u}) &= x^{*r+1}y^{*s+1}z^{*t}w^{*u} - x^{*r}y^{*s}z^{*t+1}w^{*u+1}, \\
R_2(x^{*r} y^{*s} z^{*t} w^{*u}) &= x^{*r+1}y^{*s}z^{*t+1}w^{*u} - x^{*r}y^{*s+1}z^{*t}w^{*u+1}, \\
R_3(x^{*r} y^{*s} z^{*t} w^{*u}) &= x^{*r+1}y^{*s}z^{*t}w^{*u+1} - x^{*r}y^{*s+1}z^{*t+1}w^{*u}.
\end{aligned}$$

Proof. By Lemma 12.12. □

Proposition 12.17. *The following* (i)–(iii) *hold on* P.

(i) R_1 *commutes with*

$$A_2, \qquad A_3, \qquad A_2^*, \qquad A_3^*.$$

(ii) R_2 *commutes with*

$$A_3, \qquad A_1, \qquad A_3^*, \qquad A_1^*.$$

(iii) R_3 *commutes with*

$$A_1, \qquad A_2, \qquad A_1^*, \qquad A_2^*.$$

Proof. First we prove that R_1, A_2 commute. By Lemma 6.3, Proposition 6.5, and Definition 12.10,

$$\begin{aligned}
[A_2, R_1] &= [M_zD_x + M_wD_y + M_xD_z + M_yD_w, M_xM_y - M_zM_w] \\
&= [M_zD_x, M_xM_y] + [M_wD_y, M_xM_y] - [M_xD_z, M_zM_w] - [M_yD_w, M_zM_w] \\
&= M_yM_z[D_x, M_x] + M_xM_w[D_y, M_y] - M_xM_w[D_z, M_z] - M_yM_z[D_w, M_w] \\
&= M_yM_z + M_xM_w - M_xM_w - M_yM_z \\
&= 0.
\end{aligned}$$

Next we prove that R_1, A_2^* commute. We have

$$\begin{aligned}
[A_2^*, R_1] &= [M_xD_x - M_yD_y + M_zD_z - M_wD_w, M_xM_y - M_zM_w] \\
&= [M_xD_x, M_xM_y] - [M_yD_y, M_xM_y] - [M_z, D_z, M_zM_w] + [M_wD_w, M_zM_w] \\
&= M_xM_y[D_x, M_x] - M_xM_y[D_y, M_y] - M_zM_w[D_z, M_z] + M_zM_w[D_w, M_w] \\
&= M_xM_y - M_xM_y - M_zM_w + M_zM_w \\
&= 0.
\end{aligned}$$

The remaining assertions are similarly proven. □

Lemma 12.18. *Each of* R_1, R_2, R_3 *is injective but not surjective.*

Proof. The maps are injective, by (12.3) and since P is an integral domain. The maps are not surjective, because their images do not contain P_0, P_1 by Lemma 12.14(ii). □

Lemma 12.19. *For* $f, g \in P$ *we have*

$$\langle L_1 f, g\rangle = \langle f, R_1 g\rangle, \qquad \langle L_2 f, g\rangle = \langle f, R_2 g\rangle, \qquad \langle L_3 f, g\rangle = \langle f, R_3 g\rangle.$$

Proof. We first show that $\langle L_1 f, g\rangle = \langle f, R_1 g\rangle$. Using Lemmas 6.3(ii), 9.7 and Definitions 12.1, 12.10 we obtain

$$\begin{aligned}\langle L_1 f, g\rangle &= \langle D_x D_y f, g\rangle - \langle D_z D_w f, g\rangle \\ &= \langle D_y f, M_x g\rangle - \langle D_w f, M_z g\rangle \\ &= \langle f, M_y M_x g\rangle - \langle f, M_w M_z g\rangle \\ &= \langle f, M_x M_y g\rangle - \langle f, M_z M_w g\rangle \\ &= \langle f, R_1 g\rangle.\end{aligned}$$

The remaining assertions are similarly shown. □

Lemma 12.20. *For $f, g \in P$ we have*

$$\langle R_1 f, g\rangle = \langle f, L_1 g\rangle, \qquad \langle R_2 f, g\rangle = \langle f, L_2 g\rangle, \qquad \langle R_3 f, g\rangle = \langle f, L_3 g\rangle.$$

Proof. By Definition 9.1(iii) and Lemma 12.19. □

Proposition 12.21. *For $i \in \{1, 2, 3\}$ and $N \in \mathbb{N}$ the following sum is orthogonal and direct:*

$$P_N = R_i(P_{N-2}) + \mathrm{Ker}(L_i) \cap P_N. \tag{12.4}$$

Proof. Referring to (12.4), we consider the dimensions of the three terms. By Lemma 5.2, the dimension of P_N is equal to $\binom{N+3}{3}$. By Lemma 5.2 and since the map R_i is injective by Lemma 12.18, the dimension of $R_i(P_{N-2})$ is equal to $\binom{N+1}{3}$. By Lemma 12.9(ii) the dimension of $\mathrm{Ker}(L_i) \cap P_N$ is equal to $(N + 1)^2$. By these comments, for (12.4) the dimension of the left-hand side is equal to the sum of the dimensions of the two terms on the right-hand side. It remains to show that these two terms are orthogonal. For $f \in P_{N-2}$ and $g \in \mathrm{Ker}(L_i) \cap P_N$,

$$\langle R_i f, g\rangle = \langle f, L_i g\rangle = \langle f, 0\rangle = 0.$$

The result follows. □

Proposition 12.22. *For $i \in \{1, 2, 3\}$ and $N \in \mathbb{N}$ the following sum is direct:*

$$P_N = \sum_{\ell=0}^{\lfloor N/2\rfloor} R_i^{\ell}\Big(\mathrm{Ker}(L_i) \cap P_{N-2\ell}\Big). \tag{12.5}$$

Proof. By Proposition 12.21 and induction on N, together with the fact that R_i is injective. □

Shortly we will show that the sum (12.5) is orthogonal.

Proposition 12.23. *For $i \in \{1, 2, 3\}$ the following sum is direct:*

$$P = \sum_{N\in\mathbb{N}} \sum_{\ell\in\mathbb{N}} R_i^{\ell}\Big(\mathrm{Ker}(L_i) \cap P_N\Big). \tag{12.6}$$

Proof. In the direct sum $P = \sum_{N \in \mathbb{N}} P_N$ eliminate the summands using Proposition 12.22. Evaluate the result using a change of variables $N - 2\ell \to N$. $\square$

Shortly we will show that the sum (12.6) is orthogonal. We will also describe the summands from various points of view.

Lemma 12.24. *The following holds for $N, \ell \in \mathbb{N}$:*

(i) $R_1^\ell \Big(\mathrm{Ker}(L_1) \cap P_N\Big)$ *is invariant under each of*

$$A_2, \qquad A_3, \qquad A_2^*, \qquad A_3^*, \qquad \sigma.$$

(ii) $R_2^\ell \Big(\mathrm{Ker}(L_2) \cap P_N\Big)$ *is invariant under each of*

$$A_3, \qquad A_1, \qquad A_3^*, \qquad A_1^*, \qquad \sigma.$$

(iii) $R_3^\ell \Big(\mathrm{Ker}(L_3) \cap P_N\Big)$ *is invariant under each of*

$$A_1, \qquad A_2, \qquad A_1^*, \qquad A_2^*, \qquad \sigma.$$

Proof. (i) Each of L_1, R_1 commutes with each of A_2, A_3, A_2^*, A_3^* by Propositions 12.7, 12.17. Each of L_1, R_1 commutes with σ by Lemmas 12.3, 12.13. By Lemma 5.11, the subspace P_N is invariant under A_2, A_3, A_2^*, A_3^*. We mentioned at the end of Section 7 that $\sigma(P_N) = P_N$. The result follows.
(ii), (iii) Similar to the proof of (i). $\square$

13 More actions of $\mathfrak{sl}_2(\mathbb{C})$ on P

We continue to discuss the $\mathfrak{sl}_4(\mathbb{C})$-module $P = \mathbb{C}[x, y, z, w]$. Let $i \in \{1, 2, 3\}$. In this section, we use L_i, R_i to construct an $\mathfrak{sl}_2(\mathbb{C})$-action on P. We decompose P into an orthogonal direct sum of irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodules. These irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodules are infinite-dimensional. We investigate how the decomposition is related to the sum in (12.6).

Definition 13.1. Define a map $\Omega \in \mathrm{End}(P)$ such that for $N \in \mathbb{N}$, the subspace P_N is an eigenspace for Ω with eigenvalue N.

Lemma 13.2. *The map Ω commutes with σ and everything in $\mathfrak{sl}_4(\mathbb{C})$.*

Proof. For $N \in \mathbb{N}$ the subspace P_N is invariant under σ and everything in $\mathfrak{sl}_4(\mathbb{C})$. $\square$

Lemma 13.3. *We have*

$$\Omega = M_x D_x + M_y D_y + M_z D_z + M_w D_w,$$
$$\Omega = M_{x^*} D_{x^*} + M_{y^*} D_{y^*} + M_{z^*} D_{z^*} + M_{w^*} D_{w^*}.$$

Proof. To verify the first equation, apply each side to a P-basis vector from (5.1). To verify the second equation, apply each side to a P-basis vector from (7.1). ☐

Lemma 13.4. *We have*

$$\langle \Omega f, g\rangle = \langle f, \Omega g\rangle \qquad\qquad f, g \in P.$$

Proof. By Lemma 9.3 and Definition 13.1. ☐

Proposition 13.5. *For $i \in \{1,2,3\}$ we have*

$$[L_i, R_i] = \Omega + 2I, \qquad [\Omega, R_i] = 2R_i, \qquad [\Omega, L_i] = -2L_i.$$

Proof. We check that $[L_1, R_1] = \Omega + 2I$. Using Definitions 12.1, 12.10 we find

$$\begin{aligned}[L_1, R_1] &= [D_xD_y - D_zD_w, M_xM_y - M_zM_w]\\ &= [D_xD_y, M_xM_y] - [D_xD_y, M_zM_w] - [D_zD_w, M_xM_y] + [D_zD_w, M_zM_w].\end{aligned}$$

Using Lemma 6.3 we obtain

$$\begin{aligned}&[D_xD_y, M_xM_y] = M_xD_x + M_yD_y + I, \qquad && [D_xD_y, M_zM_w] = 0,\\ &[D_zD_w, M_xM_y] = 0, && [D_zD_w, M_zM_w] = M_zD_z + M_wD_w + I.\end{aligned}$$

By these comments and Lemma 13.3, we obtain $[L_1, R_1] = \Omega + 2I$. The remaining equations are similarly checked. ☐

Recall the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ from Example 3.1.

Lemma 13.6. *For $i \in \{1,2,3\}$ the vector space P becomes an $\mathfrak{sl}_2(\mathbb{C})$-module on which E, F, H act as follows:*

element φ	E	F	H
action of φ on P	$-L_i$	R_i	$-\Omega - 2I$

Proof. Compare the relations in Example 3.1 and Proposition 13.5. ☐

We recall the Casimir operator for $\mathfrak{sl}_2(\mathbb{C})$, see [8, p. 238] or [25, p. 118]. The Casimir operator C is the following element in the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$:

$$C = EF + FE + H^2/2.$$

The operator C looks as follows in terms of A, A^*:

$$C = \frac{4A^2 + 4A^{*2} - (AA^* - A^*A)^2}{8}.$$

The operator C generates the center of the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$. In particular, C commutes with each of E, F, H, A, A^*. On the $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{V}_N$,

$$C = \frac{N(N+2)}{2} I. \tag{13.1}$$

Proposition 13.7. *On the $\mathfrak{sl}_4(\mathbb{C})$-module P,*

$$\frac{(\Omega+2I)^2}{2} - L_1R_1 - R_1L_1 = \frac{4A_2^2 + 4A_3^{*2} - (A_2A_3^* - A_3^*A_2)^2}{8}$$
$$= \frac{4A_2^{*2} + 4A_3^2 - (A_2^*A_3 - A_3A_2^*)^2}{8}.$$

Call this common value C_1. Then C_1 commutes with

$$\Omega, \qquad L_1, \qquad R_1, \qquad A_2, \qquad A_2^*, \qquad A_3, \qquad A_3^*, \qquad \sigma.$$

Proof. The equations hold, because each of the three given expressions is equal to

$$2D_xD_yM_zM_w + 2D_zD_wM_xM_y - 2M_xM_yD_xD_y - 2M_zM_wD_zD_w + \Omega(\Omega+2I)/2.$$

This equality is checked using Lemma 6.3 along with Proposition 6.5 and Definitions 12.1, 12.10 and Lemma 13.3. The last assertion in the proposition statement follows from our comments about the Casimir operator and the fact that σ commutes with each of Ω, L_1, R_1. □

Proposition 13.8. *On the $\mathfrak{sl}_4(\mathbb{C})$-module P,*

$$\frac{(\Omega+2I)^2}{2} - L_2R_2 - R_2L_2 = \frac{4A_3^2 + 4A_1^{*2} - (A_3A_1^* - A_1^*A_3)^2}{8}$$
$$= \frac{4A_3^{*2} + 4A_1^2 - (A_3^*A_1 - A_1A_3^*)^2}{8}.$$

Call this common value C_2. Then C_2 commutes with

$$\Omega, \qquad L_2, \qquad R_2, \qquad A_3, \qquad A_3^*, \qquad A_1, \qquad A_1^*, \qquad \sigma.$$

Proof. Similar to the proof of Proposition 13.7. □

Proposition 13.9. *On the $\mathfrak{sl}_4(\mathbb{C})$-module P,*

$$\frac{(\Omega+2I)^2}{2} - L_3R_3 - R_3L_3 = \frac{4A_1^2 + 4A_2^{*2} - (A_1A_2^* - A_2^*A_1)^2}{8}$$
$$= \frac{4A_1^{*2} + 4A_2^2 - (A_1^*A_2 - A_2A_1^*)^2}{8}.$$

Call this common value C_3. Then C_3 commutes with

$$\Omega, \qquad L_3, \qquad R_3, \qquad A_1, \qquad A_1^*, \qquad A_2, \qquad A_2^*, \qquad \sigma.$$

Proof. Similar to the proof of Proposition 13.7. □

Lemma 13.10. *For $f, g \in P$ we have*

$$\langle C_1f, g\rangle = \langle f, C_1g\rangle, \qquad \langle C_2f, g\rangle = \langle f, C_2g\rangle, \qquad \langle C_3f, g\rangle = \langle f, C_3g\rangle.$$

Proof. The result for C_1 follows from Lemma 9.9 and the definition of C_1 in Proposition 13.7. The results for C_2, C_3 are similarly obtained. □

Lemma 13.11. *For $N \in \mathbb{N}$,*

$$C_1(P_N) \subseteq P_N, \qquad C_2(P_N) \subseteq P_N, \qquad C_3(P_N) \subseteq P_N.$$

Proof. Since each of C_1, C_2, C_3 commutes with Ω. □

Proposition 13.12. *The maps C_1, C_2, C_3 act as follows on the P-basis* (5.1)*. For $N \in \mathbb{N}$ and $(r, s, t, u) \in \mathcal{P}_N$,*

(i) *the vector*

$$C_1(x^r y^s z^t w^u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{r-1}y^{s-1}z^{t+1}w^{u+1}$	$2rs$
$x^r y^s z^t w^u$	$N(N+2)/2 - 2rs - 2tu$
$x^{r+1}y^{s+1}z^{t-1}w^{u-1}$	$2tu$

(ii) *the vector*

$$C_2(x^r y^s z^t w^u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{r-1}y^{s+1}z^{t-1}w^{u+1}$	$2rt$
$x^r y^s z^t w^u$	$N(N+2)/2 - 2rt - 2su$
$x^{r+1}y^{s-1}z^{t+1}w^{u-1}$	$2su$

(iii) *the vector*

$$C_3(x^r y^s z^t w^u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$x^{r-1}y^{s+1}z^{t+1}w^{u-1}$	$2ru$
$x^r y^s z^t w^u$	$N(N+2)/2 - 2ru - 2st$
$x^{r+1}y^{s-1}z^{t-1}w^{u+1}$	$2st$

To get the action of C_1, C_2, C_3 on the P-basis (7.1)*, replace x, y, z, w by x^*, y^*, z^*, w^* respectively in* (i)–(iii) *above.*

Proof. (i) By the definition of C_1 in Proposition 13.7, along with Proposition 5.10 or Lemma 12.5, Lemma 12.15, Definition 13.1.
(ii), (iii) Similar to the proof of (i) above.
The last assertion of the proposition statement holds because σ commutes with each of C_1, C_2, C_3. □

In Section 11 we discussed the finite-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules. We now discuss a more general type of $\mathfrak{sl}_2(\mathbb{C})$-module, said to be highest-weight. The following discussion is distilled from [25, Chapter VI]. Let W denote an $\mathfrak{sl}_2(\mathbb{C})$-module. A *highest-weight vector* in W is a nonzero $v \in W$ such that v is an eigenvector for H and $Ev = 0$. The $\mathfrak{sl}_2(\mathbb{C})$-module W is said to be *highest-weight* whenever W contains a highest-weight vector v such that W is spanned by $\{F^\ell v\}_{\ell \in \mathbb{N}}$. Assume that W is highest-weight. Then the previously mentioned vector v is unique up to multiplication by a nonzero scalar in $\mathbb{C}$. By the *highest-weight of* W, we mean the H-eigenvalue associated with v. A pair of irreducible highest-weight $\mathfrak{sl}_2(\mathbb{C})$-modules are isomorphic if and only if they have the same highest-weight [25, p. 109].

Lemma 13.13. *Pick $i \in \{1, 2, 3\}$ and consider the corresponding $\mathfrak{sl}_2(\mathbb{C})$-module structure on P from Lemma 13.6. Let $N \in \mathbb{N}$. Pick $0 \neq v \in \mathrm{Ker}(L_i) \cap P_N$ and define $v_\ell = R_i^\ell v/\ell!$ for $\ell \in \mathbb{N}$. Then:*

$$\begin{array}{lll} v_\ell \in P_{N+2\ell}, & \Omega v_\ell = (N + 2\ell)v_\ell & \ell \in \mathbb{N}, \\ R_i v_\ell = (\ell + 1)v_{\ell+1} & \ell \in \mathbb{N}, & \\ L_i v_\ell = (N + \ell + 1)v_{\ell-1} & \ell \geqslant 1, & L_i v_0 = 0. \end{array}$$

The vectors $\{v_\ell\}_{\ell \in \mathbb{N}}$ form a basis for an $\mathfrak{sl}_2(\mathbb{C})$-submodule of P. Denote this $\mathfrak{sl}_2(\mathbb{C})$-submodule by W. Then W is irreducible and highest-weight, with highest-weight $-N - 2$. On W,

$$C_i = \frac{N(N + 2)}{2} I.$$

Proof. Routine application of the $\mathfrak{sl}_2(\mathbb{C})$ representation theory, see [25, Chapter VI]. □

We refer to the $\mathfrak{sl}_2(\mathbb{C})$-module P in Lemma 13.13. In that lemma, we constructed some irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodules of P. In the next result, we show that every irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodule of P comes from the construction.

Lemma 13.14. *Pick $i \in \{1, 2, 3\}$ and consider the corresponding $\mathfrak{sl}_2(\mathbb{C})$-module structure on P from Lemma 13.6. Then the following hold.*

(i) *Each irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodule of P is highest-weight.*

(ii) *Let W denote an irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodule of P, with highest-weight ζ. Then ζ is an integer at most -2. Write $\zeta = -N - 2$. Then there exists $0 \neq v \in \mathrm{Ker}(L_i) \cap P_N$ such that $\{R_i^\ell v/\ell!\}_{\ell \in \mathbb{N}}$ is a basis for W.*

Proof. By Definition 13.1, Ω is diagonalizable on P with eigenvalues $0, 1, 2, \ldots$ Let W denote an irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodule of P. The action of Ω on W is diagonalizable with each eigenvalue contained in $\mathbb{N}$. By this and Lemma 13.6, H is diagonalizable on W with each eigenvalue an integer at most -2. Let ζ denote the maximal eigenvalue of H on W. Let $0 \neq v \in W$ denote an eigenvector for H with eigenvalue ζ. We have $Ev = 0$; otherwise $Ev \in W$ is an eigenvector for H with eigenvalue $\zeta + 2$, contradicting the maximality of ζ. By these comments, v is a highest-weight vector for W. Write $\zeta = -N - 2$ and note that $v \in \mathrm{Ker}(L_i) \cap P_N$. Using v we define vectors $\{v_\ell\}_{\ell \in \mathbb{N}}$ as in Lemma 13.13. By Lemma 13.13 the vectors $\{v_\ell\}_{\ell \in \mathbb{N}}$ form a basis for a highest-weight $\mathfrak{sl}_2(\mathbb{C})$-submodule of W, which must equal W by the irreducibility of W. □

Lemma 13.15. *Pick $i \in \{1, 2, 3\}$ and consider the corresponding $\mathfrak{sl}_2(\mathbb{C})$-module structure on P from Lemma 13.6. For $N \in \mathbb{N}$ the following are the same:*

(i) *the span of the irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodules of P that have highest-weight $-N-2$;*

(ii) $\sum_{\ell \in \mathbb{N}} R_i^\ell \big(\mathrm{Ker}(L_i) \cap P_N\big)$*;*

(iii) *the eigenspace of C_i on P with eigenvalue $N(N+2)/2$.*

Proof. Let $P[N], P'[N], P''[N]$ denote the subspaces in (i), (ii), (iii) respectively. By Lemmas 13.13, 13.14 we have $P[N] = P'[N]$ and on this common value $C_i = N(N+2)/2\, I$. By this and Proposition 12.23, we obtain $P[N] = P'[N] = P''[N]$. □

Corollary 13.16. *Pick $i \in \{1, 2, 3\}$ and consider the corresponding $\mathfrak{sl}_2(\mathbb{C})$-module structure on P from Lemma 13.6. For $N \in \mathbb{N}$ the subspace $\sum_{\ell \in \mathbb{N}} R_i^\ell \big(\mathrm{Ker}(L_i) \cap P_N\big)$ is an $\mathfrak{sl}_2(\mathbb{C})$-submodule of P.*

Proof. By Lemma 13.15(i),(ii). □

Lemma 13.17. *For $i \in \{1, 2, 3\}$ the following hold:*

(i) *the map C_i is diagonalizable on P;*

(ii) *the eigenvalues of C_i on P are*

$$N(N+2)/2 \qquad\qquad N \in \mathbb{N};$$

(iii) *the eigenspaces of C_i on P are mutually orthogonal.*

Proof. (i), (ii) By Proposition 12.23 and Lemma 13.15(ii),(iii).
(iii) By Lemma 13.10 and (ii) above. □

Recall the direct sum decompositions (12.5) and (12.6).

Proposition 13.18. *The following hold for $i \in \{1, 2, 3\}$:*

(i) *the summands in* (12.6) *are mutually orthogonal;*

(ii) *for* $N \in \mathbb{N}$ *the summands in* (12.5) *are mutually orthogonal.*

Proof. (i) By Lemma 9.3 along with Lemmas 13.15(ii),(iii) and 13.17(iii).
(ii) By (i) above. □

Pick $i \in \{1, 2, 3\}$ and consider the corresponding $\mathfrak{sl}_2(\mathbb{C})$-module structure on P from Lemma 13.6. Our next general goal is to show that the $\mathfrak{sl}_2(\mathbb{C})$-module P is an orthogonal direct sum of irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodules.

Lemma 13.19. *For* $i \in \{1, 2, 3\}$ *and* $N, \ell \in \mathbb{N}$ *the following holds on* $R_i^\ell\big(\mathrm{Ker}(L_i) \cap P_N\big)$*:*

$$L_i R_i = (\ell + 1)(N + \ell + 2)I, \qquad R_i L_i = \ell(N + \ell + 1)I.$$

Proof. Let $0 \neq v \in \mathrm{Ker}(L_i) \cap P_N$ and define $v_\ell = R_i^\ell v/\ell!$. By Lemma 13.13 we obtain

$$L_i R_i v_\ell = (\ell + 1)(N + \ell + 2)v_\ell, \qquad R_i L_i v_\ell = \ell(N + \ell + 1)v_\ell.$$

The result follows. □

Lemma 13.20. *Let* $i \in \{1, 2, 3\}$ *and* $N, \ell \in \mathbb{N}$*. Then for* $f, g \in R_i^\ell\big(\mathrm{Ker}(L_i) \cap P_N\big)$ *we have*

$$\langle R_i f, R_i g\rangle = (\ell + 1)(N + \ell + 2)\langle f, g\rangle, \qquad \langle L_i f, L_i g\rangle = \ell(N + \ell + 1)\langle f, g\rangle.$$

Proof. By Lemma 12.20 and Lemma 13.19,

$$\langle R_i f, R_i g\rangle = \langle f, L_i R_i g\rangle = (\ell + 1)(N + \ell + 2)\langle f, g\rangle.$$

By Lemma 12.19 and Lemma 13.19,

$$\langle L_i f, L_i g\rangle = \langle f, R_i L_i g\rangle = \ell(N + \ell + 1)\langle f, g\rangle.$$

□

Corollary 13.21. *Let* $i \in \{1, 2, 3\}$ *and* $N \in \mathbb{N}$*. Pick* $u, v \in \mathrm{Ker}(L_i) \cap P_N$ *such that* $\langle u, v\rangle = 0$*. Then* $\langle R_i^\ell u, R_i^\ell v\rangle = 0$ *for* $\ell \in \mathbb{N}$*.*

Proof. By Lemma 13.20 and induction on ℓ. □

Proposition 13.22. *Pick* $i \in \{1, 2, 3\}$ *and consider the corresponding* $\mathfrak{sl}_2(\mathbb{C})$*-module structure on* P *from Lemma 13.6. Then for* $N \in \mathbb{N}$ *the* $\mathfrak{sl}_2(\mathbb{C})$*-submodule*

$$\sum_{\ell \in \mathbb{N}} R_i^\ell\big(\mathrm{Ker}(L_i) \cap P_N\big)$$

is an orthogonal direct sum of irreducible $\mathfrak{sl}_2(\mathbb{C})$*-submodules.*

Proof. By Lemma 12.9(ii), the subspace $\mathrm{Ker}(L_i) \cap P_N$ has dimension $(N+1)^2$. Abbreviate $d = (N+1)^2$. Let $\{v_j\}_{j=1}^d$ denote an orthogonal basis for $\mathrm{Ker}(L_i) \cap P_N$. For $1 \leqslant j \leqslant d$ we use v_j and Lemma 13.13 to construct an irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodule W_j of P. By Lemma 13.13 and the construction,

$$\sum_{\ell \in \mathbb{N}} R_i^\ell \big(\mathrm{Ker}(L_i) \cap P_N\big) = \sum_{j=1}^{d} W_j. \tag{13.2}$$

By Corollary 13.21 the subspaces $\{W_j\}_{j=1}^d$ are mutually orthogonal. The result follows. □

Proposition 13.23. *Pick $i \in \{1,2,3\}$ and consider the corresponding $\mathfrak{sl}_2(\mathbb{C})$-module structure on P from Lemma 13.6. Then the $\mathfrak{sl}_2(\mathbb{C})$-module P is an orthogonal direct sum of irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodules.*

Proof. By Propositions 12.23, 13.18(i), 13.22. □

14 The Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, revisited

We continue to discuss the $\mathfrak{sl}_4(\mathbb{C})$-module $P = \mathbb{C}[x,y,z,w]$. Let $i \in \{1,2,3\}$. By Propositions 12.23, 13.18(i) we have an orthogonal direct sum

$$P = \sum_{N \in \mathbb{N}} \sum_{\ell \in \mathbb{N}} R_i^\ell \Big(\mathrm{Ker}(L_i) \cap P_N\Big). \tag{14.1}$$

In this section, we show how each summand in (14.1) becomes an irreducible module for the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

We mentioned $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ in Corollary 3.12 and Definition 3.13. We now have some more comments about $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. For $N, M \in \mathbb{N}$ the vector space $\mathbb{V}_N \otimes \mathbb{V}_M$ is an $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ module with the following action. Let $a, b \in \mathfrak{sl}_2(\mathbb{C})$ and consider the element (a,b) in $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. For $u \in \mathbb{V}_N$ and $v \in \mathbb{V}_M$, the element (a,b) sends

$$u \otimes v \mapsto (au) \otimes v + u \otimes (bv).$$

It is routine to check (or see [45, Section 3.8]) that the finite-dimensional irreducible modules for $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ are, up to isomorphism,

$$\mathbb{V}_N \otimes \mathbb{V}_M \qquad\qquad N, M \in \mathbb{N}. \tag{14.2}$$

Proposition 14.1. *The following hold for $i \in \{1,2,3\}$.*

(i) *Each summand in* (14.1) *is an irreducible submodule for the ith Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.*

(ii) *For $N \in \mathbb{N}$ and $\ell \in \mathbb{N}$ the corresponding summand in* (14.1) *is isomorphic to $\mathbb{V}_N \otimes \mathbb{V}_N$ as a module for the ith Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.*

Proof. Let j, k denote the elements in $\{1,2,3\}\backslash\{i\}$. Let $\mathcal{L}$ denote the Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ generated by A_j, A_k, A_j^*, A_k^*. By Corollary 3.12 the Lie algebra $\mathcal{L}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. By Definition 3.13, $\mathcal{L}$ is the ith Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. For notational convenience, throughout this proof we identify the Lie algebras $\mathcal{L}$ and $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ via the isomorphism in Lemma 3.11. The $\mathfrak{sl}_4(\mathbb{C})$-module P becomes an $\mathcal{L}$-module by restricting the $\mathfrak{sl}_4(\mathbb{C})$ action to $\mathcal{L}$. Let $N, \ell \in \mathbb{N}$ be given, and let W denote the corresponding summand in (14.1). We show that W is an irreducible $\mathcal{L}$-submodule of P that is isomorphic to $\mathbb{V}_N \otimes \mathbb{V}_N$. The subspace W is invariant under A_j, A_k, A_j^*, A_k^* by Lemma 12.24, so W is an $\mathcal{L}$-submodule of P. Let $\mathcal{W}$ denote an irreducible $\mathcal{L}$-submodule of W. By the discussion around (14.2), the $\mathcal{L}$-module $\mathcal{W}$ is isomorphic to $\mathbb{V}_r \otimes \mathbb{V}_s$ for some $r, s \in \mathbb{N}$. Viewing $\mathcal{W}$ as a module for the copy of $\mathfrak{sl}_2(\mathbb{C})$ generated by A_j, A_k^*, we find that $\mathcal{W}$ is a direct sum of $s+1$ irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodules, each isomorphic to $\mathbb{V}_r$. Viewing $\mathcal{W}$ as a module for the copy of $\mathfrak{sl}_2(\mathbb{C})$ generated by A_j^*, A_k, we find that $\mathcal{W}$ is a direct sum of $r+1$ irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodules, each isomorphic to $\mathbb{V}_s$. For both of these $\mathfrak{sl}_2(\mathbb{C})$-actions, the Casimir operator acts as C_i by Propositions 13.7–13.9, and by Lemma 13.15(ii),(iii) this operator acts on $\mathcal{W}$ as $N(N+2)/2$ times the identity. By these comments and the discussion around (13.1), we obtain $r = s = N$. Thus the $\mathcal{L}$-module $\mathcal{W}$ is isomorphic to $\mathbb{V}_N \otimes \mathbb{V}_N$. The dimension of $\mathbb{V}_N \otimes \mathbb{V}_N$ is $(N+1)^2$, and this is the dimension of W in view of Lemma 12.9(ii) and the injectivity of R_i. By these comments, $\mathcal{W} = W$. We have shown that W is an irreducible $\mathcal{L}$-submodule of P that is isomorphic to $\mathbb{V}_N \otimes \mathbb{V}_N$. □

Let $i \in \{1,2,3\}$ and $N \in \mathbb{N}$. By Propositions 12.22, 13.18(ii) we have an orthogonal direct sum

$$P_N = \sum_{\ell=0}^{\lfloor N/2 \rfloor} R_i^{\ell}\Big(\mathrm{Ker}(L_i) \cap P_{N-2\ell}\Big). \tag{14.3}$$

Proposition 14.2. *The following hold for $i \in \{1,2,3\}$ and $N \in \mathbb{N}$.*

(i) *Each summand in* (14.3) *is an irreducible submodule for the ith Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.*

(ii) *For $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ the ℓ-summand in* (14.3) *has dimension $(N-2\ell+1)^2$.*

(iii) *For $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ the ℓ-summand in* (14.3) *is isomorphic to $\mathbb{V}_{N-2\ell} \otimes \mathbb{V}_{N-2\ell}$ as a module for the ith Lie subalgebra of $\mathfrak{sl}_4(\mathbb{C})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.*

(iv) *For $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ the ℓ-summand in* (14.3) *is an eigenspace for the action of C_i on P_N; the eigenvalue is*

$$\frac{(N-2\ell)(N-2\ell+2)}{2}.$$

Proof. (i)–(iii) By Proposition 14.1.
(iv) By Lemma 13.15(ii),(iii). □

Let $i \in \{1,2,3\}$ and $N \in \mathbb{N}$. We finish this section with a summary of how C_i acts on P_N.

Corollary 14.3. *The following hold for $i \in \{1,2,3\}$ and $N \in \mathbb{N}$.*

(i) *The action of C_i on P_N is diagonalizable.*

(ii) *For the action of C_i on P_N the eigenvalues are*

$$\frac{(N-2\ell)(N-2\ell+2)}{2}, \qquad 0 \leqslant \ell \leqslant \lfloor N/2 \rfloor.$$

(iii) *For $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ the $(N-2\ell)(N-2\ell+2)/2$-eigenspace for C_i on P_N has dimension $(N-2\ell+1)^2$.*

(iv) *the eigenspaces of C_i on P_N are mutually orthogonal.*

Proof. By Propositions 13.18(ii) and 14.2(ii),(iv). □

15 Some bases for the vector space $R_i^\ell\big(\mathrm{Ker}(L_i) \cap P_N\big)$

We continue to discuss the $\mathfrak{sl}_4(\mathbb{C})$-module $P = \mathbb{C}[x,y,z,w]$. Pick $i \in \{1,2,3\}$ and recall the orthogonal direct sum (14.1). In this section, we find some bases for each summand.

Lemma 15.1. *For $i \in \{1,2,3\}$ and $N \in \mathbb{N}$ the subspace $\mathrm{Ker}(L_i) \cap P_N$ contains x^N and x^{*N}.*

Proof. We first consider x^N. We have $x^N \in P_N$ since x^N is homogeneous with total degree N. We have $x^N \in \mathrm{Ker}(L_i)$ by Definition 12.1 and since

$$D_y(x^N) = 0, \qquad D_z(x^N) = 0, \qquad D_w(x^N) = 0.$$

By these comments, $x^N \in \mathrm{Ker}(L_i) \cap P_N$. We have $x^{*N} \in \mathrm{Ker}(L_i) \cap P_N$ because $\sigma(x^N) = x^{*N}$ and $\mathrm{Ker}(L_i) \cap P_N$ is invariant under σ by Lemma 12.24. □

Definition 15.2. Let η denote an indeterminate. For $N \in \mathbb{N}$ we define some polynomials $\{f_n\}_{n=0}^{N+1}$ in $\mathbb{C}[\eta]$ such that $f_0 = 1$ and

$$\begin{aligned} \eta f_n &= n f_{n-1} + (N-n) f_{n+1} \qquad (0 \leqslant n \leqslant N-1), \\ \eta f_N &= N f_{N-1} + f_{N+1}, \end{aligned}$$

where $f_{-1} = 0$. The polynomial f_n has degree n for $0 \leqslant n \leqslant N+1$. The $\{f_n\}_{n=0}^{N+1}$ are Krawtchouk polynomials, see [32, Section 9.11], [42], [54, Section 6].

We refer to Definition 15.2. By [42, Lemma 4.8] we have

$$f_{N+1}(\eta) = \frac{(\eta - N)(\eta - N + 2)(\eta - N + 4)\cdots(\eta + N)}{N!}. \tag{15.1}$$

The polynomials $\{f_n\}_{n=0}^{N+1}$ are related to the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{V}_N$ in the following way. Above Definition 11.1 we discussed the basis $\{u_n\}_{n=0}^{N}$ for $\mathbb{V}_N$. By that discussion and $A = E + F$ we obtain

$$\begin{aligned} Au_n &= nu_{n-1} + (N-n)u_{n+1} \qquad (0 \leqslant n \leqslant N-1), \\ Au_N &= Nu_{N-1}, \end{aligned}$$

where $u_{-1} = 0$. Comparing this recurrence with the one in Definition 15.2, we find

$$f_n(A)u_0 = u_n \qquad (0 \leqslant n \leqslant N).$$

We also find that $N!f_{N+1}$ is the minimal polynomial of A on $\mathbb{V}_N$. See [42] for more information about the Krawtchouk polynomials and $\mathfrak{sl}_2(\mathbb{C})$.

Lemma 15.3. *For $i \in \{1, 2, 3\}$ and $N \in \mathbb{N}$ the following hold on P_N:*

$$A_i f_n(A_i) = nf_{n-1}(A_i) + (N-n)f_{n+1}(A_i) \qquad (0 \leqslant n \leqslant N), \tag{15.2}$$

$$f_{N+1}(A_i) = 0. \tag{15.3}$$

Proof. By Lemma 7.11 and (15.1) we obtain (15.3). From this and Definition 15.2 we obtain (15.2). □

Lemma 15.4. *For $N \in \mathbb{N}$ and $0 \leqslant n \leqslant N$ we have*

$$f_n(A_1)x^N = x^{N-n}y^n, \qquad f_n(A_2)x^N = x^{N-n}z^n, \qquad f_n(A_3)x^N = x^{N-n}w^n.$$

Proof. We first prove our assertions involving A_1. Define

$$\xi_n = x^{N-n}y^n \qquad (0 \leqslant n \leqslant N), \qquad \xi_{N+1} = 0.$$

By Proposition 5.10(i),

$$A_1\xi_n = n\xi_{n-1} + (N-n)\xi_{n+1} \qquad (0 \leqslant n \leqslant N). \tag{15.4}$$

Comparing (15.2), (15.4) we see that the sequences $\{f_n(A_1)x^N\}_{n=0}^{N}$ and $\{\xi_n\}_{n=0}^{N}$ satisfy the same recurrence. These sequences have the same initial condition, since $f_0(A_1)x^N = x^N = \xi_0$. Therefore $f_n(A_1)x^N = \xi_n$ for $0 \leqslant n \leqslant N$. We have proven our assertions involving A_1. The remaining assertions are similarly proven. □

Let $N \in \mathbb{N}$. In the next result, we give an orthogonal basis for $\mathrm{Ker}(L_1) \cap P_N$. Similar orthogonal bases exist for $\mathrm{Ker}(L_2) \cap P_N$ and $\mathrm{Ker}(L_3) \cap P_N$.

Proposition 15.5. *For $N \in \mathbb{N}$ the vectors*

$$v_{j,k} = f_j(A_2)f_k(A_3)x^N \qquad\qquad 0 \leqslant j,k \leqslant N \tag{15.5}$$

form an orthogonal basis for $\operatorname{Ker}(L_1) \cap P_N$*. Moreover for* $0 \leqslant j,k \leqslant N$*,*

$$A_2 v_{j,k} = j v_{j-1,k} + (N-j)v_{j+1,k}, \tag{15.6}$$

$$A_3 v_{j,k} = k v_{j,k-1} + (N-k)v_{j,k+1}, \tag{15.7}$$

$$A_2^* v_{j,k} = (N-2k)v_{j,k}, \tag{15.8}$$

$$A_3^* v_{j,k} = (N-2j)v_{j,k}, \tag{15.9}$$

$$\|v_{j,k}\|^2 = N!\binom{N}{j}^{-1}\binom{N}{k}^{-1}. \tag{15.10}$$

In the above lines, we interpret $v_{a,b} = 0$ unless $0 \leqslant a,b \leqslant N$.

Proof. By Lemma 12.24(i) the subspace $\operatorname{Ker}(L_1) \cap P_N$ is invariant under A_2 and A_3. By this and Lemma 15.1, the vectors (15.5) are contained in $\operatorname{Ker}(L_1) \cap P_N$. The vectors (15.5) satisfy the recurrence (15.6) by Lemma 15.3. The vectors (15.5) satisfy the recurrence (15.7) by Lemma 15.3 and $[A_2, A_3] = 0$. Concerning (15.8), we use Proposition 5.10(v), Lemma 15.4, and $[A_2, A_2^*] = 0$ to obtain

$$\begin{aligned} A_2^* v_{j,k} &= A_2^* f_j(A_2) f_k(A_3)x^N = f_j(A_2)A_2^* f_k(A_3)x^N = f_j(A_2)A_2^* x^{N-k}w^k \\ &= (N-2k)f_j(A_2)x^{N-k}w^k = (N-2k)f_j(A_2)f_k(A_3)x^N = (N-2k)v_{j,k}. \end{aligned}$$

Concerning (15.9), we use Proposition 5.10(vi), Lemma 15.4, and $[A_2, A_3] = 0 = [A_3^*, A_3]$ to obtain

$$\begin{aligned} A_3^* v_{j,k} &= A_3^* f_j(A_2) f_k(A_3)x^N = f_k(A_3)A_3^* f_j(A_2)x^N = f_k(A_3)A_3^* x^{N-j}z^j \\ &= (N-2j)f_k(A_3)x^{N-j}z^j = (N-2j)f_k(A_3)f_j(A_2)x^N = (N-2j)v_{j,k}. \end{aligned}$$

The vectors (15.5) are mutually orthogonal by (15.8), (15.9) and Lemma 9.9. Next we show (15.10). Assume for the moment that $j \geqslant 1$. By Lemma 9.9 and (15.6),

$$(N-j+1)\|v_{j,k}\|^2 = \langle A_2 v_{j-1,k}, v_{j,k}\rangle = \langle v_{j-1,k}, A_2 v_{j,k}\rangle = j\|v_{j-1,k}\|^2. \tag{15.11}$$

Assume for the moment that $k \geqslant 1$. By Lemma 9.9 and (15.7),

$$(N-k+1)\|v_{j,k}\|^2 = \langle A_3 v_{j,k-1}, v_{j,k}\rangle = \langle v_{j,k-1}, A_3 v_{j,k}\rangle = k\|v_{j,k-1}\|^2. \tag{15.12}$$

By (15.11), (15.12) and induction on $j+k$,

$$\|v_{j,k}\|^2 = \|v_{0,0}\|^2\binom{N}{j}^{-1}\binom{N}{k}^{-1}.$$

This and $\|v_{0,0}\|^2 = \|x^N\|^2 = N!$ yields (15.10). By (15.10) the vectors (15.5) are nonzero. The vectors (15.5) are linearly independent, because they are nonzero and mutually orthogonal. These vectors form a basis for $\operatorname{Ker}(L_1) \cap P_N$ in view of Proposition 12.9(ii). □

Proposition 15.6. *The following hold for* $N, \ell \in \mathbb{N}$*:*

(i) $R_1^\ell\big(\mathrm{Ker}(L_1) \cap P_N\big)$ *has an orthogonal basis*

$$R_1^\ell f_j(A_2) f_k(A_3) x^N \qquad\qquad 0 \leqslant j, k \leqslant N;$$

(ii) $R_2^\ell\big(\mathrm{Ker}(L_2) \cap P_N\big)$ *has an orthogonal basis*

$$R_2^\ell f_j(A_3) f_k(A_1) x^N \qquad\qquad 0 \leqslant j, k \leqslant N;$$

(iii) $R_3^\ell\big(\mathrm{Ker}(L_3) \cap P_N\big)$ *has an orthogonal basis*

$$R_3^\ell f_j(A_1) f_k(A_2) x^N \qquad\qquad 0 \leqslant j, k \leqslant N.$$

Proof. (i) By Lemma 12.18, Corollary 13.21, and Proposition 15.5.
(ii), (iii) Similar to the proof of (i) above. □

Proposition 15.7. *The following hold for* $N, \ell \in \mathbb{N}$*:*

(i) $R_1^\ell\big(\mathrm{Ker}(L_1) \cap P_N\big)$ *has a basis*

$$R_1^\ell A_2^j A_3^k x^N \qquad\qquad 0 \leqslant j, k \leqslant N;$$

(ii) $R_2^\ell\big(\mathrm{Ker}(L_2) \cap P_N\big)$ *has a basis*

$$R_2^\ell A_3^j A_1^k x^N \qquad\qquad 0 \leqslant j, k \leqslant N;$$

(iii) $R_3^\ell\big(\mathrm{Ker}(L_3) \cap P_N\big)$ *has a basis*

$$R_3^\ell A_1^j A_2^k x^N \qquad\qquad 0 \leqslant j, k \leqslant N.$$

Proof. By Proposition 15.6 and since the polynomial f_n has degree n for $0 \leqslant n \leqslant N$. □

Corollary 15.8. *The following hold:*

(i) P *has a basis*

$$R_1^\ell A_2^j A_3^k x^N \qquad\qquad N, \ell \in \mathbb{N}, \qquad\qquad 0 \leqslant j, k \leqslant N;$$

(ii) P *has a basis*

$$R_2^\ell A_3^j A_1^k x^N \qquad\qquad N, \ell \in \mathbb{N}, \qquad\qquad 0 \leqslant j, k \leqslant N;$$

(iii) P *has a basis*

$$R_3^\ell A_1^j A_2^k x^N \qquad\qquad N, \ell \in \mathbb{N}, \qquad\qquad 0 \leqslant j, k \leqslant N.$$

Proof. By Proposition 15.7 and since the sum (14.1) is direct. □

Proposition 15.9. *Lemmas 15.3, 15.4 and Propositions 15.5, 15.6, 15.7 and Corollary 15.8 all remain valid if we replace* x, y, z, w *by* x^*, y^*, z^*, w^* *respectively and swap* $A_i \leftrightarrow A_i^*$ *for* $i \in \{1, 2, 3\}$*.*

Proof. Apply σ throughout the listed results, and use Propositions 7.12, 9.13 along with Lemmas 12.3, 12.13. □

16 Some bases for the vector space P_N

We continue to discuss the $\mathfrak{sl}_4(\mathbb{C})$-module $P = \mathbb{C}[x, y, z, w]$. Pick $i \in \{1, 2, 3\}$ and $N \in \mathbb{N}$. Recall the direct sum decomposition (14.3). In Proposition 15.7, we obtained a basis for each summand. In this section, we use these bases to obtain some bases for P_N.

Proposition 16.1. *The following hold for $N \in \mathbb{N}$:*

(i) *P_N has a basis*

$$R_1^\ell A_2^j A_3^k x^{N-2\ell} \qquad 0 \leqslant \ell \leqslant \lfloor N/2 \rfloor, \qquad 0 \leqslant j, k \leqslant N - 2\ell;$$

(ii) *P_N has a basis*

$$R_2^\ell A_3^j A_1^k x^{N-2\ell} \qquad 0 \leqslant \ell \leqslant \lfloor N/2 \rfloor, \qquad 0 \leqslant j, k \leqslant N - 2\ell;$$

(iii) *P_N has a basis*

$$R_3^\ell A_1^j A_2^k x^{N-2\ell} \qquad 0 \leqslant \ell \leqslant \lfloor N/2 \rfloor, \qquad 0 \leqslant j, k \leqslant N - 2\ell.$$

Proof. By Proposition 15.7 and since the sum (14.3) is direct. □

Proposition 16.2. *Proposition 16.1 remains valid if we replace x by x^* and A_i by A_i^* for $i \in \{1, 2, 3\}$.*

Proof. Apply σ throughout Proposition 16.1, and use Proposition 7.12 along with Lemma 12.13 and the fact that $\sigma(P_N) = P_N$. □

17 The hypercube $H(N, 2)$

We turn our attention to graph theory. For us, a graph is understood to be finite and undirected, without loops or multiple edges. For the rest of this paper, fix $N \in \mathbb{N}$. We define a graph $H(N, 2)$ as follows. The vertex set X consists of the N-tuples of elements taken from the set $\{1, -1\}$. Vertices $x, y \in X$ are adjacent whenever they differ in exactly one coordinate. The graph $H(N, 2)$ is called the *N-cube* or a *hypercube* or a *binary Hamming graph*. The graph $H(N, 2)$ is distance-regular in the sense of [7, Chapter 1]. Background information about $H(N, 2)$ can be found in [1,2,7,19,23]. Going forward, we will assume that the reader is generally familiar with $H(N, 2)$. In the next few paragraphs, we recall from [23] some features of $H(N, 2)$ for later use.

We have $|X| = 2^N$. For $x, y \in X$ let $\partial(x, y)$ denote the path-length distance between x, y. Then $\partial(x, y)$ is equal to the number of coordinates at which x, y differ. The diameter of $H(N, 2)$ is N. The graph $H(N, 2)$ is a bipartite antipodal 2-cover. The intersection numbers c_i, b_i of $H(N, 2)$ satisfy

$$c_i = i, \qquad b_i = N - i \qquad (0 \leqslant i \leqslant N).$$

The valencies k_i of $H(N, 2)$ satisfy

$$k_i = \binom{N}{i} \qquad (0 \leqslant i \leqslant N).$$

Definition 17.1. Let V denote the vector space with basis X. We call V the *standard module* associated with $H(N, 2)$.

Definition 17.2. We endow V with a Hermitian form $\langle \, , \, \rangle$ with respect to which the basis X is orthonormal.

Let us abbreviate $\Gamma = H(N, 2)$. For $x \in X$ let the set $\Gamma(x)$ consist of the vertices in X that are adjacent to x. Note that $|\Gamma(x)| = N$.

Definition 17.3. Define $\mathsf{A} \in \mathrm{End}(V)$ such that

$$\mathsf{A}x = \sum_{\xi \in \Gamma(x)} \xi, \qquad\qquad x \in X. \tag{17.1}$$

We call A the *adjacency map* for $H(N, 2)$.

Lemma 17.4. *For $x, y \in X$ we have*

$$\langle \mathsf{A}x, y \rangle = \langle x, \mathsf{A}y \rangle = \begin{cases} 1 & \text{if } \partial(x, y) = 1; \\ 0, & \text{if } \partial(x, y) \neq 1. \end{cases}$$

Proof. By Definition 17.3. □

By [7, p. 45] and [23, Lemma 3.5], the map A is diagonalizable with eigenvalues

$$\theta_i = N - 2i \qquad\qquad (0 \leqslant i \leqslant N). \tag{17.2}$$

For $0 \leqslant i \leqslant N$ let E_i denote the primitive idempotent of A associated with θ_i. By [23, Lemma 3.5] the eigenspace $\mathsf{E}_i V$ has dimension $m_i = \binom{N}{i}$.

Lemma 17.5. *For $0 \leqslant i \leqslant N$ and $x, y \in X$,*

$$\langle \mathsf{E}_i x, y \rangle = \langle x, \mathsf{E}_i y \rangle = \langle \mathsf{E}_i y, x \rangle = \langle y, \mathsf{E}_i x \rangle.$$

Proof. By (2.1) and Lemma 17.4, and since the eigenvalues (17.2) are real. □

The ordering $\{\mathsf{E}_i\}_{i=0}^N$ is Q-polynomial in the sense of [23, Section 12]. For this ordering the corresponding dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^N$ is defined in [23, Definition 3.6]. By [23, Lemma 3.7] we have

$$\theta_i^* = N - 2i \qquad\qquad (0 \leqslant i \leqslant N). \tag{17.3}$$

By [7, p. 194], the above Q-polynomial structure is formally self-dual in the sense of [7, Section 2.3].

Definition 17.6. We define the vector space $V^{\otimes 3} = V \otimes V \otimes V$ and the set

$$X^{\otimes 3} = \{x \otimes y \otimes z | x, y, z \in X\}.$$

Observe that $X^{\otimes 3}$ is a basis for $V^{\otimes 3}$.

Lemma 17.7. *The following hold.*

(i) *There exists a unique Hermitian form* $\langle\,,\,\rangle$ *on* $V^{\otimes 3}$ *with respect to which the basis* $X^{\otimes 3}$ *is orthonormal.*

(ii) *For* $u, v, w, u', v', w' \in V$ *we have*

$$\langle u \otimes v \otimes w, u' \otimes v' \otimes w' \rangle = \langle u, u' \rangle \langle v, v' \rangle \langle w, w' \rangle.$$

Proof. Item (i) is clear. Item (ii) is routinely checked. □

Let G denote the automorphism group of $H(N, 2)$. By [7, Theorem 9.2.1] the group G is isomorphic to the wreath product of the symmetric groups S_N and S_2. The elements of S_N permute the vertex coordinates $\{1, 2, \ldots, N\}$ and the elements of S_2 permute the set $\{1, -1\}$. By [2, p. 207] the graph $H(N, 2)$ is distance-transitive in the sense of [2, p. 189].

Next, we describe how V becomes a G-module. Pick $v \in V$ and write $v = \sum_{x \in X} v_x x$ ($v_x \in \mathbb{C}$). For all $g \in G$,

$$g(v) = \sum_{x \in X} v_x g(x).$$

Lemma 17.8. *For* $g \in G$ *the following hold on* V*:*

(i) $g\mathsf{A} = \mathsf{A}g$*;*

(ii) $g\mathsf{E}_i = \mathsf{E}_i g$ $(0 \leqslant i \leqslant N)$.

Proof. (i) Because g respects adjacency in $H(N, 2)$.
(ii) By (i) and since E_i is a polynomial in A. □

Next, we describe how $V^{\otimes 3}$ becomes a G-module. For $g \in G$ and $u, v, w \in V$,

$$g(u \otimes v \otimes w) = g(u) \otimes g(v) \otimes g(w). \qquad (17.4)$$

Definition 17.9. Define the subspace

$$\mathrm{Fix}(G) = \{v \in V^{\otimes 3} | g(v) = v \ \forall g \in G\}.$$

Our next general goal is to obtain a basis for $\mathrm{Fix}(G)$. To reach this goal, we consider the action of G on the set $X^{\otimes 3}$. We will describe the partition of $X^{\otimes 3}$ into G-orbits.

Recall the set of profiles $\mathcal{P}_N$ from Definition 5.1.

Definition 17.10. For $x \otimes y \otimes z \in X^{\otimes 3}$ we define its profile as follows. Write

$$x = (x_1, x_2, \dots, x_N), \qquad y = (y_1, y_2, \dots, y_N), \qquad z = (z_1, z_2, \dots, z_N).$$

Define

$$\begin{aligned} r &= \big|\{i|1 \leqslant i \leqslant N,\ x_i = y_i = z_i\}\big|, \\ s &= \big|\{i|1 \leqslant i \leqslant N,\ x_i \neq y_i = z_i\}\big|, \\ t &= \big|\{i|1 \leqslant i \leqslant N,\ y_i \neq z_i = x_i\}\big|, \\ u &= \big|\{i|1 \leqslant i \leqslant N,\ z_i \neq x_i = y_i\}\big|. \end{aligned}$$

Note that $(r, s, t, u) \in \mathcal{P}_N$. We call (r, s, t, u) the *profile of* $x \otimes y \otimes z$.

Lemma 17.11. *For a profile* $(r, s, t, u) \in \mathcal{P}_N$ *the number of elements in* $X^{\otimes 3}$ *with this profile is equal to*

$$\frac{N! 2^N}{r! s! t! u!}. \tag{17.5}$$

Proof. By combinatorial counting. □

Lemma 17.12. *A pair of elements in* $X^{\otimes 3}$ *are in the same* G*-orbit if and only if they have the same profile.*

Proof. This is routinely checked. □

As an aside, we interpret the profile concept using the distance function ∂.

Lemma 17.13. *Let* $x \otimes y \otimes z \in X^{\otimes 3}$ *with profile* (r, s, t, u)*. Then*

$$\partial(x, y) = s + t, \qquad \partial(y, z) = t + u, \qquad \partial(z, x) = u + s.$$

Moreover

$$r = \frac{2N - \partial(x, y) - \partial(y, z) - \partial(z, x)}{2}, \qquad s = \frac{\partial(z, x) + \partial(x, y) - \partial(y, z)}{2},$$
$$t = \frac{\partial(x, y) + \partial(y, z) - \partial(z, x)}{2}, \qquad u = \frac{\partial(y, z) + \partial(z, x) - \partial(x, y)}{2}.$$

Proof. By Definition 17.10. □

Definition 17.14. Let the set $\mathcal{P}''_N$ consist of the 3-tuples of integers (h, i, j) such that

$$\begin{aligned} &0 \leqslant h, i, j \leqslant N, \qquad && h + i + j \text{ is even}, \qquad && h + i + j \leqslant 2N, \\ &h \leqslant i + j, && i \leqslant j + h, && j \leqslant h + i. \end{aligned}$$

Lemma 17.15. *There exists a bijection $\mathcal{P}_N \to \mathcal{P}''_N$ that sends*

$$(r, s, t, u) \mapsto (t + u, u + s, s + t).$$

The inverse bijection $\mathcal{P}''_N \to \mathcal{P}_N$ sends

$$(h, i, j) \mapsto \left(\frac{2N - h - i - j}{2}, \frac{i + j - h}{2}, \frac{j + h - i}{2}, \frac{h + i - j}{2}\right).$$

Proof. This is readily checked. □

Lemma 17.16. *For $0 \leqslant h, i, j \leqslant N$ the following* (i)–(iii) *are equivalent:*

(i) *there exists $x \otimes y \otimes z \in X^{\otimes 3}$ such that*

$$h = \partial(y, z), \qquad i = \partial(z, x), \qquad j = \partial(x, y);$$

(ii) *there exists $(r, s, t, u) \in \mathcal{P}_N$ such that*

$$h = t + u, \qquad i = u + s, \qquad j = s + t;$$

(iii) $(h, i, j) \in \mathcal{P}''_N$.

Assume that (i)–(iii) *hold. Then (r, s, t, u) is the profile of $x \otimes y \otimes z$.*

Proof. (i) ⇔ (ii) By Lemmas 17.11 17.13.
(ii) ⇔ (iii) By Lemma 17.15.
The last assertion follows from Lemmas 17.13, 17.15. □

Remark 17.17. For $H(N, 2)$ and $0 \leqslant h, i, j \leqslant N$ there are some parameters called the intersection number $p^h_{i,j}$ [23, p. 401] and Krein parameter $q^h_{i,j}$ [23, p. 402]. We use these parameters in a minimal way, but for the sake of completeness let us discuss them briefly. We have $p^h_{i,j} = p^h_{j,i}$ [23, p. 401] and $q^h_{i,j} = q^h_{j,i}$ [23, p. 402]. We have $p^h_{i,j} = q^h_{i,j}$ [41, Lemma 22]. This common value is nonzero if and only if $(h, i, j) \in \mathcal{P}''_N$; see [41, Corollary 28]. By [41, Proposition 12], for $(h, i, j) \in \mathcal{P}''_N$ we have

$$k_h p^h_{i,j} = k_i p^i_{j,h} = k_j p^j_{h,i} = m_h q^h_{i,j} = m_i q^i_{j,h} = m_j q^j_{h,i} = \frac{N!}{r!s!t!u!},$$

where

$$h = t + u, \qquad i = u + s, \qquad j = s + t.$$

Definition 17.18. For a profile $(r, s, t, u) \in \mathcal{P}_N$ we define a vector

$$B(r, s, t, u) = \sum_{x \otimes y \otimes z} x \otimes y \otimes z,$$

where the sum is over the elements $x \otimes y \otimes z$ in $X^{\otimes 3}$ with profile (r, s, t, u).

Example 17.19. We have

$$B(N,0,0,0) = \sum_{x \in X} x \otimes x \otimes x.$$

We have a remark about notation.

Remark 17.20. Let $0 \leqslant h, i, j \leqslant N$. In [57, Definition 9.9] we defined a vector

$$P_{h,i,j} = \sum_{x \otimes y \otimes z} x \otimes y \otimes z,$$

where the sum is over the elements $x \otimes y \otimes z$ in $X^{\otimes 3}$ such that

$$h = \partial(y,z), \qquad i = \partial(z,x), \qquad j = \partial(x,y).$$

By Lemma 17.16, $P_{h,i,j} \neq 0$ if and only if $(h,i,j) \in \mathcal{P}''_N$. In this case

$$P_{h,i,j} = B(r,s,t,u),$$

where

$$h = t + u, \qquad i = u + s, \qquad j = s + t.$$

Lemma 17.21. *The vectors*

$$B(r,s,t,u) \qquad (r,s,t,u) \in \mathcal{P}_N$$

are mutually orthogonal and

$$\|B(r,s,t,u)\|^2 = \frac{N!2^N}{r!s!t!u!} \qquad (r,s,t,u) \in \mathcal{P}_N.$$

Proof. The vectors $X^{\otimes 3}$ are orthonormal with respect to $\langle\, ,\, \rangle$. The result follows in view of Lemma 17.11 and Definition 17.18. □

Proposition 17.22. *The vectors*

$$B(r,s,t,u) \qquad (r,s,t,u) \in \mathcal{P}_N \tag{17.6}$$

form a basis for $\mathrm{Fix}(G)$.

Proof. The vectors (17.6) are linearly independent by Lemma 17.21. These vectors span $\mathrm{Fix}(G)$ by Lemma 17.12 and Definition 17.18. □

Corollary 17.23. *We have*

$$\dim \mathrm{Fix}(G) = \binom{N+3}{3}.$$

Proof. By (5.2) and Proposition 17.22. □

Next, we describe the basis for Fix(G) that is dual to the one in Proposition 17.22, with respect to the Hermitian form $\langle\, ,\, \rangle$.

Definition 17.24. For a profile $(r,s,t,u) \in \mathcal{P}_N$ we define a vector

$$\tilde{B}(r,s,t,u) = \frac{r!s!t!u!}{N!2^N} B(r,s,t,u).$$

Proposition 17.25. *The vectors*

$$\tilde{B}(r,s,t,u) \qquad\qquad (r,s,t,u) \in \mathcal{P}_N$$

form a basis for Fix(G)*.*

Proof. By Proposition 17.22 and Definition 17.24. □

Lemma 17.26. *The* Fix(G)*-basis*

$$B(r,s,t,u) \qquad\qquad (r,s,t,u) \in \mathcal{P}_N$$

and the Fix(G)*-basis*

$$\tilde{B}(r,s,t,u) \qquad\qquad (r,s,t,u) \in \mathcal{P}_N$$

are dual with respect to $\langle\, ,\, \rangle$*.*

Proof. By Lemma 17.21 and Definition 17.24. □

Recall the vector space P_N from Lemma 5.2.

Lemma 17.27. *There exists a vector space isomorphism* $\ddagger : P_N \to \mathrm{Fix}(G)$ *that sends*

$$x^r y^s z^t w^u \mapsto \left(N!2^N\right)^{1/2} \tilde{B}(r,s,t,u) \qquad\qquad (r,s,t,u) \in \mathcal{P}_N.$$

Proof. By Lemma 5.2 and Proposition 17.25. □

Theorem 17.28. *The Hermitian forms on* P_N *and* Fix(G) *are related as follows:*

$$\langle f,g \rangle = \langle f^{\ddagger}, g^{\ddagger} \rangle \qquad\qquad f,g \in P_N.$$

Proof. By Definitions 9.2, 17.24 and Lemmas 17.26, 17.27. □

Our next goal is to turn Fix(G) into an $\mathfrak{sl}_4(\mathbb{C})$-module.

Definition 17.29. (See [57, Definition 6.2].) We define $A^{(1)}, A^{(2)}, A^{(3)} \in \text{End}(V^{\otimes 3})$ as follows. For $x \otimes y \otimes z \in X^{\otimes 3}$,

$$A^{(1)}(x \otimes y \otimes z) = \sum_{\xi \in \Gamma(x)} \xi \otimes y \otimes z,$$
$$A^{(2)}(x \otimes y \otimes z) = \sum_{\xi \in \Gamma(y)} x \otimes \xi \otimes z,$$
$$A^{(3)}(x \otimes y \otimes z) = \sum_{\xi \in \Gamma(z)} x \otimes y \otimes \xi.$$

In the next result, we clarify Definition 17.29.

Lemma 17.30. *For $u, v, w \in V$ we have*

$$A^{(1)}(u \otimes v \otimes w) = \mathsf{A}u \otimes v \otimes w,$$
$$A^{(2)}(u \otimes v \otimes w) = u \otimes \mathsf{A}v \otimes w,$$
$$A^{(3)}(u \otimes v \otimes w) = u \otimes v \otimes \mathsf{A}w,$$

where A is the adjacency map from Definition 17.3.

Proof. By Definitions 17.3, 17.29. □

Definition 17.31. (See [57, Definition 7.1].) We define $A^{*(1)}, A^{*(2)}, A^{*(3)} \in \text{End}(V^{\otimes 3})$ as follows. For $x \otimes y \otimes z \in X^{\otimes 3}$,

$$A^{*(1)}(x \otimes y \otimes z) = x \otimes y \otimes z \, \theta^*_{\partial(y,z)},$$
$$A^{*(2)}(x \otimes y \otimes z) = x \otimes y \otimes z \, \theta^*_{\partial(z,x)},$$
$$A^{*(3)}(x \otimes y \otimes z) = x \otimes y \otimes z \, \theta^*_{\partial(x,y)}.$$

Proposition 17.32. *For a profile $(r, s, t, u) \in \mathcal{P}_N$ the following* (i)–(vi) *hold:*

(i) *the vector*

$$A^{(1)} \tilde{B}(r, s, t, u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$\tilde{B}(r-1, s+1, t, u)$	r
$\tilde{B}(r+1, s-1, t, u)$	s
$\tilde{B}(r, s, t-1, u+1)$	t
$\tilde{B}(r, s, t+1, u-1)$	u

(ii) *the vector*

$$A^{(2)}\tilde{B}(r,s,t,u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$\tilde{B}(r-1,s,t+1,u)$	r
$\tilde{B}(r,s-1,t,u+1)$	s
$\tilde{B}(r+1,s,t-1,u)$	t
$\tilde{B}(r,s+1,t,u-1)$	u

(iii) *the vector*

$$A^{(3)}\tilde{B}(r,s,t,u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$\tilde{B}(r-1,s,t,u+1)$	r
$\tilde{B}(r,s-1,t+1,u)$	s
$\tilde{B}(r,s+1,t-1,u)$	t
$\tilde{B}(r+1,s,t,u-1)$	u

(iv) $A^{*(1)}\tilde{B}(r,s,t,u) = (r+s-t-u)\tilde{B}(r,s,t,u)$;

(v) $A^{*(2)}\tilde{B}(r,s,t,u) = (r-s+t-u)\tilde{B}(r,s,t,u)$;

(vi) $A^{*(3)}\tilde{B}(r,s,t,u) = (r-s-t+u)\tilde{B}(r,s,t,u)$.

Proof. (i)–(iii) By combinatorial counting.
(iv)–(vi) By (17.3) and Lemma 17.13 along with Definitions 17.18, 17.24, 17.31. □

Lemma 17.33. *The subspace* Fix(G) *is invariant under the maps*

$$A^{(i)}, \quad A^{*(i)} \qquad\qquad i \in \{1,2,3\}.$$

Proof. By Propositions 17.25, 17.32. □

Theorem 17.34. *The subspace* Fix(G) *becomes an* $\mathfrak{sl}_4(\mathbb{C})$*-module on which*

$$A_i = A^{(i)}, \qquad\qquad A_i^* = A^{*(i)} \qquad\qquad i \in \{1,2,3\}.$$

Moreover, the map $\ddagger : P_N \to \mathrm{Fix}(G)$ *is an isomorphism of* $\mathfrak{sl}_4(\mathbb{C})$*-modules.*

Proof. Compare Propositions 5.10, 17.32 using Lemma 17.27. □

Our next general goal is to explain what the $\mathfrak{sl}_4(\mathbb{C})$-module isomorphism $\ddagger : P_N \to \mathrm{Fix}(G)$ does to the P_N-basis given in Lemma 7.6.

Definition 17.35. (See [57, Definition 9.14].) For $0 \leqslant h, i, j \leqslant N$ we define a vector

$$Q_{h,i,j} = 2^N \sum_{x \in X} \mathsf{E}_h x \otimes \mathsf{E}_i x \otimes \mathsf{E}_j x.$$

Lemma 17.36. *For $0 \leqslant h, i, j \leqslant N$ the following hold:*

(i) $\|Q_{h,i,j}\|^2 = \|P_{h,i,j}\|^2$, *where $P_{h,i,j}$ is from Remark 17.20;*

(ii) $Q_{h,i,j} \neq 0$ *if and only if* $(h, i, j) \in \mathcal{P}''_N$.

Proof. (i) By [57, Lemmas 9.11, 9.16] and Remark 17.17.
(ii) By Remark 17.20 and (i) above. □

Definition 17.37. For a profile $(r, s, t, u) \in \mathcal{P}_N$ we define a vector

$$B^*(r, s, t, u) = Q_{h,i,j},$$

where

$$h = t + u, \qquad i = u + s, \qquad j = s + t. \tag{17.7}$$

Lemma 17.38. *The vectors*

$$B^*(r, s, t, u) \qquad (r, s, t, u) \in \mathcal{P}_N \tag{17.8}$$

are mutually orthogonal and

$$\|B^*(r, s, t, u)\|^2 = \frac{N!2^N}{r!s!t!u!} \qquad (r, s, t, u) \in \mathcal{P}_N. \tag{17.9}$$

Proof. The vectors (17.8) are mutually orthogonal by [57, Lemma 9.16] and Definition 17.37. For $(r, s, t, u) \in \mathcal{P}_N$ and (h, i, j) from (17.7), the following holds by Definition 17.37, Lemma 17.36(i), Remark 17.20, Lemma 17.21:

$$\|B^*(r, s, t, u)\|^2 = \|Q_{h,i,j}\|^2 = \|P_{h,i,j}\|^2 = \|B(r, s, t, u)\|^2 = \frac{N!2^N}{r!s!t!u!}.$$

□

Proposition 17.39. *The vectors*

$$B^*(r, s, t, u) \qquad (r, s, t, u) \in \mathcal{P}_N \tag{17.10}$$

form a basis for $\mathrm{Fix}(G)$.

Proof. We first show that the vectors (17.10) are contained in $\mathrm{Fix}(G)$. Pick $g \in G$ and $(r, s, t, u) \in \mathcal{P}_N$. Let (h, i, j) satisfy (17.7). Using Lemma 17.8(ii) and (17.4) and Definitions 17.35, 17.37 we obtain

$$\begin{aligned} g\big(B^*(r,s,t,u)\big) &= g\big(Q_{h,i,j}\big) \\ &= g\left(2^N \sum_{x\in X} \mathsf{E}_h x \otimes \mathsf{E}_i x \otimes \mathsf{E}_j x\right) \\ &= 2^N \sum_{x\in X} g(\mathsf{E}_h x) \otimes g(\mathsf{E}_i x) \otimes g(\mathsf{E}_j x) \\ &= 2^N \sum_{x\in X} \mathsf{E}_h g(x) \otimes \mathsf{E}_i g(x) \otimes \mathsf{E}_j g(x) \\ &= 2^N \sum_{y\in X} \mathsf{E}_h y \otimes \mathsf{E}_i y \otimes \mathsf{E}_j y \\ &= B^*(r,s,t,u). \end{aligned}$$

Therefore $B^*(r, s, t, u) \in \mathrm{Fix}(G)$. We have shown that the vectors (17.10) are contained in $\mathrm{Fix}(G)$. The vectors (17.10) are linearly independent by Lemma 17.38. The result follows in view of (5.2) and Corollary 17.23. □

Next, we describe the basis for $\mathrm{Fix}(G)$ that is dual to the one in Proposition 17.39, with respect to the Hermitian form $\langle\, ,\, \rangle$.

Definition 17.40. For a profile $(r, s, t, u) \in \mathcal{P}_N$ we define a vector

$$\tilde{B}^*(r,s,t,u) = \frac{r!s!t!u!}{N!2^N} B^*(r,s,t,u).$$

Proposition 17.41. *The vectors*

$$\tilde{B}^*(r,s,t,u) \qquad\qquad (r,s,t,u) \in \mathcal{P}_N$$

form a basis for $\mathrm{Fix}(G)$*.*

Proof. By Proposition 17.39 and Definition 17.40. □

Lemma 17.42. *The* $\mathrm{Fix}(G)$*-basis*

$$B^*(r,s,t,u) \qquad\qquad (r,s,t,u) \in \mathcal{P}_N$$

and the $\mathrm{Fix}(G)$*-basis*

$$\tilde{B}^*(r,s,t,u) \qquad\qquad (r,s,t,u) \in \mathcal{P}_N$$

are dual with respect to $\langle\, ,\, \rangle$*.*

Proof. By Lemma 17.38 and Definition 17.40. □

The following result is a variation on [57, Lemma 9.18].

Lemma 17.43. *We have*

$$B(N,0,0,0) = 2^{-N} \sum_{(r,s,t,u)\in \mathcal{P}_N} B^*(r,s,t,u).$$

Proof. We have

$$\begin{aligned}
B(N,0,0,0) &= \sum_{x\in X} x\otimes x\otimes x && \text{by Example 17.19}\\
&= \sum_{x\in X}\left(\sum_{h=0}^{N} \mathsf{E}_h x\right)\otimes\left(\sum_{i=0}^{N} \mathsf{E}_i x\right)\otimes\left(\sum_{j=0}^{N} \mathsf{E}_j x\right) && \text{by } I=\textstyle\sum_{\ell=0}^{N}\mathsf{E}_\ell\\
&= \sum_{x\in X}\sum_{h=0}^{N}\sum_{i=0}^{N}\sum_{j=0}^{N} \mathsf{E}_h x\otimes \mathsf{E}_i x\otimes \mathsf{E}_j x\\
&= \sum_{h=0}^{N}\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{x\in X} \mathsf{E}_h x\otimes \mathsf{E}_i x\otimes \mathsf{E}_j x\\
&= 2^{-N}\sum_{h=0}^{N}\sum_{i=0}^{N}\sum_{j=0}^{N} Q_{h,i,j} && \text{by Definition 17.35}\\
&= 2^{-N}\sum_{(h,i,j)\in \mathcal{P}''_N} Q_{h,i,j} && \text{by Lemma 17.36(ii)}\\
&= 2^{-N}\sum_{(r,s,t,u)\in \mathcal{P}_N} B^*(r,s,t,u) && \text{by Lem. 17.15 and Def. 17.37.}
\end{aligned}$$

□

Lemma 17.44. *For a profile* $(r,s,t,u)\in\mathcal{P}_N$ *the following* (i)–(iii) *hold:*

(i) $A^{(1)}B^*(r,s,t,u) = (r+s-t-u)B^*(r,s,t,u)$;

(ii) $A^{(2)}B^*(r,s,t,u) = (r-s+t-u)B^*(r,s,t,u)$;

(iii) $A^{(3)}B^*(r,s,t,u) = (r-s-t+u)B^*(r,s,t,u)$.

Proof. Use (17.2) and Lemma 17.30 along with Definitions 17.35, 17.37. □

Recall the map $\ddagger : P_N \to \mathrm{Fix}(G)$ from Lemma 17.27.

Proposition 17.45. *The map* $\ddagger : P_N \to \mathrm{Fix}(G)$ *sends*

$$x^{*r}y^{*s}z^{*t}w^{*u} \mapsto \left(N!2^N\right)^{1/2}\tilde{B}^*(r,s,t,u) \qquad (r,s,t,u)\in\mathcal{P}_N.$$

Proof. For each profile $(r,s,t,u) \in \mathcal{P}_N$ we define a vector

$$\Delta(r,s,t,u) = \left(x^{*r}y^{*s}z^{*t}w^{*u}\right)^{\ddagger} - \left(N!2^N\right)^{1/2}\tilde{B}^*(r,s,t,u).$$

We show that $\Delta(r,s,t,u) = 0$. By Proposition 7.8(i)–(iii), Theorem 17.34, Definition 17.40, and Lemma 17.44,

$$\begin{aligned}
A^{(1)}\Delta(r,s,t,u) &= (r+s-t-u)\Delta(r,s,t,u),\\
A^{(2)}\Delta(r,s,t,u) &= (r-s+t-u)\Delta(r,s,t,u),\\
A^{(3)}\Delta(r,s,t,u) &= (r-s-t+u)\Delta(r,s,t,u).
\end{aligned}$$

Therefore, $\Delta(r,s,t,u)$ is contained in an $\mathbb{H}$-weight space of the $\mathfrak{sl}_4(\mathbb{C})$-module $\mathrm{Fix}(G)$. The vector space $\mathrm{Fix}(G)$ is the direct sum of its $\mathbb{H}$-weight spaces. Therefore, the nonzero vectors among

$$\Delta(r,s,t,u) \qquad\qquad (r,s,t,u) \in \mathcal{P}_N$$

are linearly independent. To finish the proof, it suffices to show that

$$0 = \sum_{(r,s,t,u)\in\mathcal{P}_N} \frac{\Delta(r,s,t,u)}{r!s!t!u!}. \tag{17.11}$$

We have

$$\begin{aligned}
\sum_{(r,s,t,u)\in\mathcal{P}_N} \frac{\left(x^{*r}y^{*s}z^{*t}w^{*u}\right)^{\ddagger}}{r!s!t!u!} &= \left(\sum_{(r,s,t,u)\in\mathcal{P}_N} \frac{x^{*r}y^{*s}z^{*t}w^{*u}}{r!s!t!u!}\right)^{\ddagger}\\
&= \frac{2^N}{N!}\left(x^N\right)^{\ddagger} && \text{by Lemma 9.12}\\
&= \frac{2^N}{N!}\left(N!2^N\right)^{1/2}\tilde{B}(N,0,0,0) && \text{by Lemma 17.27}\\
&= \frac{\left(N!2^N\right)^{1/2}}{N!}B(N,0,0,0) && \text{by Definition 17.24.}
\end{aligned}$$

We also have

$$\begin{aligned}
\sum_{(r,s,t,u)\in\mathcal{P}_N} \frac{\tilde{B}^*(r,s,t,u)}{r!s!t!u!} &= \frac{1}{N!2^N}\sum_{(r,s,t,u)\in\mathcal{P}_N} B^*(r,s,t,u) && \text{by Definition 17.40}\\
&= \frac{B(N,0,0,0)}{N!} && \text{by Lemma 17.43.}
\end{aligned}$$

We may now argue

$$\sum_{(r,s,t,u)\in\mathcal{P}_N} \frac{\Delta(r,s,t,u)}{r!s!t!u!} = \sum_{(r,s,t,u)\in\mathcal{P}_N} \frac{\left(x^{*r}y^{*s}z^{*t}w^{*u}\right)^{\ddagger}}{r!s!t!u!} - \left(N!2^N\right)^{1/2}\sum_{(r,s,t,u)\in\mathcal{P}_N} \frac{\tilde{B}^*(r,s,t,u)}{r!s!t!u!}$$

$$= \frac{\left(N!2^N\right)^{1/2}}{N!} B(N,0,0,0) - \frac{\left(N!2^N\right)^{1/2}}{N!} B(N,0,0,0)$$
$$= 0.$$

We have shown (17.11), and the result follows. □

In the next four results, we give some comments and tie up some loose ends.

Proposition 17.46. *For a profile* $(r,s,t,u) \in \mathcal{P}_N$ *the following* (i)–(vi) *hold:*

(i) $A^{(1)}\tilde{B}^*(r,s,t,u) = (r+s-t-u)\tilde{B}^*(r,s,t,u)$;

(ii) $A^{(2)}\tilde{B}^*(r,s,t,u) = (r-s+t-u)\tilde{B}^*(r,s,t,u)$;

(iii) $A^{(3)}\tilde{B}^*(r,s,t,u) = (r-s-t+u)\tilde{B}^*(r,s,t,u)$;

(iv) *the vector*

$$A^{*(1)}\tilde{B}^*(r,s,t,u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$\tilde{B}^*(r-1,s+1,t,u)$	r
$\tilde{B}^*(r+1,s-1,t,u)$	s
$\tilde{B}^*(r,s,t-1,u+1)$	t
$\tilde{B}^*(r,s,t+1,u-1)$	u

(v) *the vector*

$$A^{*(2)}\tilde{B}^*(r,s,t,u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$\tilde{B}^*(r-1,s,t+1,u)$	r
$\tilde{B}^*(r,s-1,t,u+1)$	s
$\tilde{B}^*(r+1,s,t-1,u)$	t
$\tilde{B}^*(r,s+1,t,u-1)$	u

(vi) *the vector*

$$A^{*(3)}\tilde{B}^*(r,s,t,u)$$

is a linear combination with the following terms and coefficients:

Term	Coefficient
$\tilde{B}^*(r-1,s,t,u+1)$	r
$\tilde{B}^*(r,s-1,t+1,u)$	s
$\tilde{B}^*(r,s+1,t-1,u)$	t
$\tilde{B}^*(r+1,s,t,u-1)$	u

Proof. Apply the map ‡ to everything in Proposition 7.8, and evaluate the result using Theorem 17.34 and Proposition 17.45. □

Lemma 17.47. *We have*

$$B^*(N,0,0,0) = 2^{-N} \sum_{(r,s,t,u)\in \mathcal{P}_N} B(r,s,t,u).$$

Proof. Apply the map ‡ to everything in Lemma 9.5. In the resulting equation, evaluate the left-hand side using Definition 17.40 and Proposition 17.45. Evaluate the right-hand side using Definition 17.24 and Lemma 17.27. The result follows. □

Lemma 17.48. *For the* $\mathfrak{sl}_4(\mathbb{C})$*-module* $\mathrm{Fix}(G)$*, each* $\mathbb{H}$*-weight space has dimension one and each* $\mathbb{H}^*$*-weight space has dimension one.*

Proof. By Lemmas 5.13, 7.9 and since the $\mathfrak{sl}_4(\mathbb{C})$-modules P_N, $\mathrm{Fix}(G)$ are isomorphic. □

Lemma 17.49. *For the* $\mathfrak{sl}_4(\mathbb{C})$*-module* $\mathrm{Fix}(G)$*, the following holds for* $i \in \{1,2,3\}$*:*

$$\langle A_i u, v\rangle = \langle u, A_i v\rangle, \qquad \langle A^*_i u, v\rangle = \langle u, A^*_i v\rangle \qquad u,v \in \mathrm{Fix}(G).$$

Proof. Apply the map ‡ to everything in Lemma 9.9, and evaluate the result using Theorem 17.28 and the fact that ‡ is an isomorphism of $\mathfrak{sl}_4(\mathbb{C})$-modules. □

We comment about notation. Earlier in this section, we discussed the vectors

$$B(r,s,t,u), \qquad B^*(r,s,t,u) \qquad (r,s,t,u) \in \mathcal{P}_N. \tag{17.12}$$

Up to notation, the vectors (17.12) are the same as the vectors

$$P_{h,i,j}, \qquad Q_{h,i,j} \qquad (h,i,j) \in \mathcal{P}''_N. \tag{17.13}$$

In the next section, we will adopt a point of view in which the notation (17.13) is more convenient than the notation (17.12). To prepare for the next section, we restate a few results using the notation (17.13).

Corollary 17.50. *The following* (i)–(iv) *hold.*

(i) *The vectors*

$$P_{h,i,j} \qquad (h,i,j) \in \mathcal{P}''_N$$

form an orthogonal basis for $\mathrm{Fix}(G)$*.*

(ii) *For* $(h,i,j) \in \mathcal{P}''_N$*,*

$$A^{*(1)}\big(P_{h,i,j}\big) = \theta^*_h P_{h,i,j}, \qquad A^{*(2)}\big(P_{h,i,j}\big) = \theta^*_i P_{h,i,j},$$
$$A^{*(3)}\big(P_{h,i,j}\big) = \theta^*_j P_{h,i,j}.$$

(iii) *The vectors*

$$Q_{h,i,j} \qquad (h,i,j) \in \mathcal{P}''_N$$

form an orthogonal basis for $\mathrm{Fix}(G)$.

(iv) *For* $(h,i,j) \in \mathcal{P}''_N$,

$$A^{(1)}(Q_{h,i,j}) = \theta_h Q_{h,i,j}, \qquad A^{(2)}(Q_{h,i,j}) = \theta_i Q_{h,i,j},$$
$$A^{(3)}(Q_{h,i,j}) = \theta_j Q_{h,i,j}.$$

Proof. (i) By Lemma 17.15, Remark 17.20, Lemma 17.21, and Proposition 17.22.
(ii) By Remark 17.20 and Definition 17.31.
(iii) By Lemma 17.15, Definition 17.37, Lemma 17.38, and Proposition 17.39.
(iv) By Lemma 17.30 and Definition 17.35. □

18 The subconstituent algebra of $H(N,2)$

We continue to discuss the hypercube $H(N,2)$. In [23] the subconstituent algebra of $H(N,2)$ is described in detail. In this section, we explain what the subconstituent algebra of $H(N,2)$ has to do with our results from the previous sections.

We now review some concepts and notation about $\Gamma = H(N,2)$.

Definition 18.1. For $x, y \in X$ we define a map $e_{x,y} \in \mathrm{End}(V)$ that sends $y \mapsto x$ and all other vertices to 0. Note that $\{e_{x,y}\}_{x,y\in X}$ form a basis for $\mathrm{End}(V)$.

Definition 18.2. We endow $\mathrm{End}(V)$ with a Hermitian form $\langle\, , \rangle$ with respect to which the basis $\{e_{x,y}\}_{x,y\in X}$ is orthonormal.

For $x \in X$ and $0 \leqslant i \leqslant N$, define the set $\Gamma_i(x) = \{y \in X | \partial(x,y) = i\}$. For $0 \leqslant i \leqslant N$ define $\mathsf{A}_i \in \mathrm{End}(V)$ such that

$$\mathsf{A}_i x = \sum_{\xi \in \Gamma_i(x)} \xi, \qquad x \in X.$$

By [56, Section 3] we have $\mathsf{A}_i = \binom{N}{i} f_i(\mathsf{A})$, where f_i is from Definition 15.2. For the rest of this section, fix $\varkappa \in X$. Define $\mathsf{A}^* = \mathsf{A}^*(\varkappa) \in \mathrm{End}(V)$ such that

$$\mathsf{A}^* x = \theta^*_{\partial(x,\varkappa)} x, \qquad x \in X. \tag{18.1}$$

By construction, the map A^* is diagonalizable with eigenvalues $\{\theta^*_i\}_{i=0}^N$. By [23, Theorem 4.2] we have

$$[\mathsf{A}, [\mathsf{A}, \mathsf{A}^*]] = 4\mathsf{A}^*, \qquad [\mathsf{A}^*, [\mathsf{A}^*, \mathsf{A}]] = 4\mathsf{A}.$$

Let $T = T(\varkappa)$ denote the subalgebra of $\mathrm{End}(V)$ generated by A, A^*. By [23, Corollary 14.15] we have $\dim T = \binom{N+3}{3}$. We call T the *subconstituent algebra* (or *Terwilliger algebra*) of $H(N,2)$ with respect to $\varkappa$; see [23, Definition 2.1].

We mention some bases for the vector space T. For $0 \leqslant i \leqslant N$ define $\mathsf{E}^*_i = \mathsf{E}^*_i(\varkappa) \in \mathrm{End}(V)$ such that

$$\mathsf{E}^*_i x = \begin{cases} x, & \text{if } \partial(x,\varkappa) = i; \\ 0, & \text{if } \partial(x,\varkappa) \neq i, \end{cases} \qquad x \in X.$$

By construction, E^*_i is the primitive idempotent of A^* for the eigenvalue θ^*_i. For $0 \leqslant i \leqslant N$ define $\mathsf{A}^*_i = \mathsf{A}^*_i(\varkappa) \in \mathrm{End}(V)$ such that

$$\mathsf{A}^*_i x = 2^N \langle \mathsf{E}_i \varkappa, x\rangle x, \qquad x \in X.$$

By [56, Section 11] we have $\mathsf{A}^*_i = \binom{N}{i} f_i(\mathsf{A}^*)$. By [23, p. 403] and Remark 17.17, the following hold for $0 \leqslant h,i,j \leqslant N$:

$$\begin{aligned} \mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j \neq 0 \quad &\text{iff} \quad (h,i,j) \in \mathcal{P}''_N; \\ \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j \neq 0 \quad &\text{iff} \quad (h,i,j) \in \mathcal{P}''_N. \end{aligned}$$

Lemma 18.3. *The following is an orthogonal basis for the vector space T:*

$$\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j \qquad (h,i,j) \in \mathcal{P}''_N. \tag{18.2}$$

Moreover, the following is an orthogonal basis for the vector space T:

$$\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j \qquad (h,i,j) \in \mathcal{P}''_N. \tag{18.3}$$

Proof. By [56, Lemma 8.1] and Remark 17.17 and

$$\dim T = \binom{N+3}{3} = |\mathcal{P}_N| = |\mathcal{P}''_N|.$$

□

By [23, p. 404] the algebra T is semisimple.

Next, we review the Wedderburn decomposition of T. By [23, Corollary 14.12] the center of T is generated by

$$\phi = \frac{4\mathsf{A}^2 + 4\mathsf{A}^{*2} - (\mathsf{A}\mathsf{A}^* - \mathsf{A}^*\mathsf{A})^2}{8}. \tag{18.4}$$

By [23, Theorem 6.3 and Lemma 14.6], the map ϕ is diagonalizable with eigenvalues

$$\frac{(N-2\ell)(N-2\ell+2)}{2} \qquad 0 \leqslant \ell \leqslant \lfloor N/2 \rfloor.$$

For $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ let $\phi_\ell \in \mathrm{End}(V)$ denote the primitive idempotent of ϕ associated with the eigenvalue

$$\frac{(N-2\ell)(N-2\ell+2)}{2}.$$

By [23, Theorem 14.10] the following is a basis for the center of T:

$$\phi_\ell, \qquad\qquad 0 \leqslant \ell \leqslant \lfloor N/2 \rfloor.$$

By [23, Corollary 14.9],

$$\phi = \sum_{\ell=0}^{\lfloor N/2 \rfloor} \frac{(N-2\ell)(N-2\ell+2)}{2} \phi_\ell. \tag{18.5}$$

By [23, Theorem 14.14], for $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ the subspace $\phi_\ell T$ is a minimal 2-sided ideal of T with dimension $(N-2\ell+1)^2$. By [23, Theorems 14.10, 14.14] we have

$$T = \sum_{\ell=0}^{\lfloor N/2 \rfloor} \phi_\ell T \qquad\qquad \text{(orthogonal direct sum)}. \tag{18.6}$$

This is the Wedderburn decomposition of T.

Next, we define some maps in $\mathrm{End}(T)$.

Definition 18.4. We define $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \mathcal{A}^{(3)} \in \mathrm{End}(T)$ such that for $(h,i,j) \in \mathcal{P}''_N$,

$$\mathcal{A}^{(1)}\big(\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j\big) = \theta_h \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j,$$
$$\mathcal{A}^{(2)}\big(\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j\big) = \theta_i \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j,$$
$$\mathcal{A}^{(3)}\big(\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j\big) = \theta_j \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j.$$

The maps $\mathcal{A}^{(2)}, \mathcal{A}^{(3)}$ have the following interpretation.

Lemma 18.5. *For $B \in T$,*

$$\mathcal{A}^{(2)}(B) = \mathsf{A}B, \qquad\qquad \mathcal{A}^{(3)}(B) = B\mathsf{A}.$$

Proof. By the second assertion in Lemma 18.3, along with Definition 18.4. □

Definition 18.6. We define $\mathcal{A}^{*(1)}, \mathcal{A}^{*(2)}, \mathcal{A}^{*(3)} \in \mathrm{End}(T)$ such that for $(h,i,j) \in \mathcal{P}''_N$,

$$\mathcal{A}^{*(1)}\big(\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j\big) = \theta^*_h \mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j,$$
$$\mathcal{A}^{*(2)}\big(\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j\big) = \theta^*_j \mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j,$$
$$\mathcal{A}^{*(3)}\big(\mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j\big) = \theta^*_i \mathsf{E}^*_i \mathsf{A}_h \mathsf{E}^*_j.$$

The maps $\mathcal{A}^{*(2)}, \mathcal{A}^{*(3)}$ have the following interpretation.

Lemma 18.7. *For $B \in T$,*

$$\mathcal{A}^{*(2)}(B) = B\mathsf{A}^*, \qquad \mathcal{A}^{*(3)}(B) = \mathsf{A}^* B.$$

Proof. By the first assertion in Lemma 18.3, along with Definition 18.6. □

We are going to show that the vector space T becomes an $\mathfrak{sl}_4(\mathbb{C})$-module on which $A_i = \mathcal{A}^{(i)}$ and $A_i^* = \mathcal{A}^{*(i)}$ for $i \in \{1, 2, 3\}$. We are also going to show that the $\mathfrak{sl}_4(\mathbb{C})$-modules T and P_N are isomorphic. In addition, we will interpret the Wedderburn decomposition of T in terms of the decomposition of P_N given in (14.3) with $i = 1$.

Recall from Definition 17.6 the vector space $V^{\otimes 3}$ and the set $X^{\otimes 3}$.

Definition 18.8. We define a $\mathbb{C}$-linear map $\varepsilon : V^{\otimes 3} \to \mathrm{End}(V)$ as follows. For $x \otimes y \otimes z \in X^{\otimes 3}$,

$$\varepsilon(x \otimes y \otimes z) = \begin{cases} 2^{N/2} e_{y,z}, & \text{if } x = \varkappa; \\ 0, & \text{if } x \neq \varkappa. \end{cases}$$

Lemma 18.9. *For $x \otimes y \otimes z \in X^{\otimes 3}$ the map $\varepsilon(x \otimes y \otimes z)$ sends*

$$\psi \mapsto 2^{N/2} \langle x, \varkappa \rangle \langle z, \psi \rangle y, \qquad \psi \in X.$$

Proof. For $\psi \in X$ we have

$$\varepsilon(x \otimes y \otimes z)(\psi) = 2^{N/2} \delta_{x,\varkappa} e_{y,z}(\psi) = 2^{N/2} \delta_{x,\varkappa} \delta_{\psi,z} y = 2^{N/2} \langle x, \varkappa \rangle \langle z, \psi \rangle y.$$

□

Lemma 18.10. *For $u, v, w \in V$ the map $\varepsilon(u \otimes v \otimes w)$ sends*

$$\psi \mapsto 2^{N/2} \langle u, \varkappa \rangle \langle w, \overline{\psi} \rangle v, \qquad \psi \in V.$$

Proof. By Lemma 18.9 and $\mathbb{C}$-linearity in each of the arguments u, v, w, ψ. □

Recall the vectors $P_{h,i,j}$ from Remark 17.20.

Lemma 18.11. *For $(h, i, j) \in \mathcal{P}''_N$,*

$$\varepsilon(P_{h,i,j}) = 2^{N/2} \mathsf{E}_j^* \mathsf{A}_h \mathsf{E}_i^*. \tag{18.7}$$

Proof. To verify (18.7), we apply each side to a vertex $\psi \in X$. First assume that $\partial(\psi, \varkappa) = i$. Then each side of (18.7) sends

$$\psi \mapsto 2^{N/2} \sum_{y \in \Gamma_j(\varkappa) \cap \Gamma_h(\psi)} y.$$

Next assume that $\partial(\psi, \varkappa) \neq i$. Then each side of (18.7) sends $\psi \mapsto 0$. □

We bring in some notation. For $u, v \in V$ we define a vector $u \circ v \in V$ as follows. Write

$$u = \sum_{x \in X} u_x x, \qquad v = \sum_{x \in X} v_x x, \qquad u_x, v_x \in \mathbb{C}.$$

Define

$$u \circ v = \sum_{x \in X} u_x v_x x.$$

Note that $u_x = \langle u, x \rangle$ and $v_x = \langle v, x \rangle$ for $x \in X$. Therefore,

$$u \circ v = \sum_{x \in X} \langle u, x \rangle \langle v, x \rangle x.$$

The following result is well known. We give a short proof for the sake of completeness.

Lemma 18.12. (See [56, Lemma 9.3].) *For $0 \leqslant h \leqslant N$ we have*

$$\mathsf{A}^*_h v = 2^N \mathsf{E}_h \varkappa \circ v, \qquad v \in V.$$

Proof. For $v \in V$ we have

$$\begin{aligned} \mathsf{A}^*_h v = \mathsf{A}^*_h \sum_{x \in X} \langle v, x \rangle x &= \sum_{x \in X} \langle v, x \rangle \mathsf{A}^*_h x = 2^N \sum_{x \in X} \langle v, x \rangle \langle \mathsf{E}_h \varkappa, x \rangle x \\ &= 2^N \sum_{x \in X} \langle \mathsf{E}_h \varkappa, x \rangle \langle v, x \rangle x = 2^N \mathsf{E}_h \varkappa \circ v. \end{aligned}$$

□

Recall the vectors $Q_{h,i,j}$ from Definition 17.35.

Lemma 18.13. *For $(h, i, j) \in \mathcal{P}''_N$,*

$$\varepsilon(Q_{h,i,j}) = 2^{N/2} \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j. \tag{18.8}$$

Proof. Let $\psi \in X$. Using in order Definition 17.35, Lemma 18.10, Lemma 17.5, Lemma 18.12 we obtain

$$\begin{aligned} \varepsilon(Q_{h,i,j})(\psi) &= 2^N \sum_{x \in X} \varepsilon\big(\mathsf{E}_h x \otimes \mathsf{E}_i x \otimes \mathsf{E}_j x\big)(\psi) \\ &= 2^{3N/2} \sum_{x \in X} \langle \mathsf{E}_h x, \varkappa \rangle \langle \mathsf{E}_j x, \overline{\psi} \rangle \mathsf{E}_i x \\ &= 2^{3N/2} \sum_{x \in X} \langle \mathsf{E}_h x, \varkappa \rangle \langle \mathsf{E}_j x, \psi \rangle \mathsf{E}_i x \\ &= 2^{3N/2} \mathsf{E}_i \sum_{x \in X} \langle \mathsf{E}_h \varkappa, x \rangle \langle \mathsf{E}_j \psi, x \rangle x \\ &= 2^{3N/2} \mathsf{E}_i \big(\mathsf{E}_h \varkappa \circ \mathsf{E}_j \psi\big) \\ &= 2^{N/2} \mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j (\psi). \end{aligned}$$

The result follows. □

Lemma 18.14. *The restriction* $\varepsilon|_{\mathrm{Fix}(G)}$ *of* ε *to* $\mathrm{Fix}(G)$ *gives a bijection* $\varepsilon|_{\mathrm{Fix}(G)} : \mathrm{Fix}(G) \to T$.

Proof. By Corollary 17.50(iii) and Lemmas 18.3, 18.13. □

Lemma 18.15. *For* $k \in \{1, 2, 3\}$ *the following diagrams commute:*

$$\begin{CD} \mathrm{Fix}(G) @>\varepsilon>> T \\ @VA^{(k)}VV @VV\mathcal{A}^{(k)}V \\ \mathrm{Fix}(G) @>>\varepsilon> T \end{CD} \qquad\qquad \begin{CD} \mathrm{Fix}(G) @>\varepsilon>> T \\ @VA^{*(k)}VV @VV\mathcal{A}^{*(k)}V \\ \mathrm{Fix}(G) @>>\varepsilon> T \end{CD}$$

Proof. Let $(h, i, j) \in \mathcal{P}''_N$. To verify the first diagram, chase $Q_{h,i,j}$ around the diagram using Corollary 17.50(iv), Definition 18.4, and Lemma 18.13. To verify the second diagram, chase $P_{h,i,j}$ around the diagram using Corollary 17.50(ii), Definition 18.6, and Lemma 18.11. □

Theorem 18.16. *The vector space* T *becomes an* $\mathfrak{sl}_4(\mathbb{C})$*-module on which*

$$A_i = \mathcal{A}^{(i)}, \qquad A_i^* = \mathcal{A}^{*(i)} \qquad i \in \{1, 2, 3\}.$$

Moreover, the map $\varepsilon|_{\mathrm{Fix}(G)} : \mathrm{Fix}(G) \to T$ *is an isomorphism of* $\mathfrak{sl}_4(\mathbb{C})$*-modules.*

Proof. By Theorem 17.34 and Lemmas 18.14, 18.15. □

Next, we compare the Hermitian forms on $\mathrm{Fix}(G)$ and T.

Lemma 18.17. *For* $(h, i, j) \in \mathcal{P}''_N$,

$$\|P_{h,i,j}\|^2 = 2^N \|\mathsf{E}_j^* \mathsf{A}_h \mathsf{E}_i^*\|^2, \qquad \|Q_{h,i,j}\|^2 = 2^N \|\mathsf{E}_i \mathsf{A}_h^* \mathsf{E}_j\|^2.$$

Proof. In these equations, the left-hand side is computed using Remark 17.20, Lemma 17.21, and Lemma 17.36(i), while the right-hand side is computed using Remark 17.17 and [56, Corollary 8.2]. □

Theorem 18.18. *The Hermitian forms on* $\mathrm{Fix}(G)$ *and* T *are related as follows:*

$$\langle u, v \rangle = \langle \varepsilon(u), \varepsilon(v) \rangle \qquad u, v \in \mathrm{Fix}(G).$$

Proof. Without loss of generality, we may assume that u, v are in the basis for $\mathrm{Fix}(G)$ from Corollary 17.50(i). First assume that $u \neq v$. Then $\langle u, v \rangle = 0$ by Corollary 17.50(i), and $\langle \varepsilon(u), \varepsilon(v) \rangle = 0$ by Lemmas 18.3, 18.11. Next assume that $u = v$, and write $u = v = P_{h,i,j}$. Using Lemmas 18.11, 18.17 we obtain

$$\|\varepsilon(u)\|^2 = \|2^{N/2} \mathsf{E}_j^* \mathsf{A}_h \mathsf{E}_i^*\|^2 = 2^N \|\mathsf{E}_j^* \mathsf{A}_h \mathsf{E}_i^*\|^2 = \|P_{h,i,j}\|^2 = \|u\|^2.$$

The result follows. □

Recall the map $\ddagger : P_N \to \text{Fix}(G)$ from Lemma 17.27.

Definition 18.19. We define a $\mathbb{C}$-linear map $\vartheta : P_N \to T$ to be the composition

$$\vartheta : \quad P_N \xrightarrow[\ddagger]{} \text{Fix}(G) \xrightarrow[\varepsilon]{} T.$$

In the next result, we clarify how the map ϑ acts on P_N.

Theorem 18.20. *For $(r, s, t, u) \in \mathcal{P}_N$ the map ϑ sends*

$$x^r y^s z^t w^u \mapsto \frac{r!s!t!u!}{(N!)^{1/2}} \mathsf{E}_j^* \mathsf{A}_h \mathsf{E}_i^*, \tag{18.9}$$

$$x^{*r} y^{*s} z^{*t} w^{*u} \mapsto \frac{r!s!t!u!}{(N!)^{1/2}} \mathsf{E}_i \mathsf{A}_h^* \mathsf{E}_j, \tag{18.10}$$

where

$$h = t + u, \qquad i = u + s, \qquad j = s + t.$$

Proof. The action (18.9) is obtained from Remark 17.20, Definition 17.24, and Lemmas 17.27, 18.11. The action (18.10) is obtained from Definitions 17.37, 17.40, Proposition 17.45, and Lemma 18.13. □

Theorem 18.21. *The map $\vartheta : P_N \to T$ is an isomorphism of $\mathfrak{sl}_4(\mathbb{C})$-modules.*

Proof. By Definition 18.19 and Theorems 17.34, 18.16. □

Theorem 18.22. *The Hermitian forms on P_N and T are related as follows:*

$$\langle f, g \rangle = \langle \vartheta(f), \vartheta(g) \rangle \qquad f, g \in P_N.$$

Proof. By Theorems 17.28, 18.18, and Definition 18.19. □

Definition 18.23. Let † denote the antiautomorphism of $\text{End}(V)$ that sends $e_{x,y} \leftrightarrow e_{y,x}$ for all $x, y \in X$. We have $\dagger^2 = \text{id}$. We call † the *transpose map*.

Lemma 18.24. *The map † fixes each of A, A^*. Moreover, T is invariant under †. The restriction of † to T gives an antiautomorphism of the algebra T.*

Proof. The first assertion is a routine consequence of (17.1) and (18.1). The second assertion holds because T is generated by A, A^*. The third assertion holds because the map † is invertible. □

Lemma 18.25. *The map † fixes each of $\mathsf{A}_i, \mathsf{E}_i, \mathsf{A}_i^*, \mathsf{E}_i^*$ for $0 \leqslant i \leqslant N$.*

Proof. Each of $\mathsf{A}_i, \mathsf{E}_i$ is a polynomial in A. Each of $\mathsf{A}_i^*, \mathsf{E}_i^*$ is a polynomial in A^*. The result follows from these comments and Lemma 18.24. □

Lemma 18.26. *There exists an automorphism of the algebra T that swaps $\mathsf{A} \leftrightarrow \mathsf{A}^*$. This map swaps $\mathsf{A}_i \leftrightarrow \mathsf{A}^*_i$ for $0 \leqslant i \leqslant N$ and $\mathsf{E}_i \leftrightarrow \mathsf{E}^*_i$ for $0 \leqslant i \leqslant N$.*

Proof. The automorphism is given in [15, Section 9] and [15, Theorem 6.4]. □

Definition 18.27. We define a $\mathbb{C}$-linear map $S : T \to T$ to be the composition

$$S: \quad T \xrightarrow[\mathsf{A} \leftrightarrow \mathsf{A}^*]{} T \xrightarrow[\dagger]{} T.$$

Note that S is an antiautomorphism of T such that $S^2 = \mathrm{id}$.

Lemma 18.28. *The map S swaps*

$$\mathsf{E}_i \mathsf{A}^*_h \mathsf{E}_j \leftrightarrow \mathsf{E}^*_j \mathsf{A}_h \mathsf{E}^*_i$$

for $(h, i, j) \in \mathcal{P}''_N$.

Proof. By Lemmas 18.24–18.26 and Definition 18.27. □

Recall the automorphism σ of P from Proposition 7.12.

Proposition 18.29. *The following diagram commutes:*

$$\begin{array}{ccc} P_N & \xrightarrow{\vartheta} & T \\ \sigma \big\downarrow & & \big\downarrow S \\ P_N & \xrightarrow[\vartheta]{} & T \end{array}$$

Proof. For $(r, s, t, u) \in \mathcal{P}_N$ chase $x^r y^s z^t w^u$ around the diagram, using Theorem 18.20, Lemma 18.28 and the comment at the end of Section 7. □

Next, we consider the decomposition of P_N given in (14.3) with $i = 1$. We compare this decomposition with the Wedderburn decomposition of T given in (18.6).

Theorem 18.30. *For $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ the map ϑ sends*

$$R_1^\ell \Big(\mathrm{Ker}(L_1) \cap P_{N-2\ell} \Big) \mapsto \phi_\ell T.$$

Proof. During this proof, we will refer to the decomposition of P_N given in (14.3). Throughout the proof, we assume that $i = 1$ in (14.3). In Proposition 13.7 we defined $C_1 \in \mathrm{End}(P)$ such that on P,

$$C_1 = \frac{4A_2^2 + 4A_3^{*2} - (A_2 A_3^* - A_3^* A_2)^2}{8}. \tag{18.11}$$

We have $C_1(P_N) \subseteq P_N$ by Lemma 13.11. By Proposition 14.2(iv), for $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ the ℓ-summand in (14.3) is an eigenspace for the action of C_1 on P_N; the eigenvalue is $(N-2\ell)(N-2\ell+2)/2$. We now bring in the $\mathfrak{sl}_4(\mathbb{C})$-module T. Define $C^{(1)} \in \operatorname{End}(T)$ by

$$C^{(1)} = \frac{4\big(\mathcal{A}^{(2)}\big)^2 + 4\big(\mathcal{A}^{*(3)}\big)^2 - \big(\mathcal{A}^{(2)}\mathcal{A}^{*(3)} - \mathcal{A}^{*(3)}\mathcal{A}^{(2)}\big)^2}{8}, \tag{18.12}$$

where $\mathcal{A}^{(2)}$, $\mathcal{A}^{*(3)}$ are from Definitions 18.4, 18.6. Comparing (18.11), (18.12) and using Theorem 18.21, we see that the following diagram commutes:

$$\begin{array}{ccc} P_N & \xrightarrow{\ \vartheta\ } & T \\ {\scriptstyle C_1}\big\downarrow & & \big\downarrow{\scriptstyle C^{(1)}} \\ P_N & \xrightarrow[\ \vartheta\ ]{} & T \end{array}$$

Recall the central element $\phi \in T$ from (18.4). By Lemmas 18.5, 18.7 and (18.12),

$$C^{(1)}(B) = \phi B, \qquad\qquad B \in T. \tag{18.13}$$

Consider the Wedderburn decomposition of T from (18.6). We claim that for $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ the ℓ-summand in (18.6) is an eigenspace for $C^{(1)}$ with eigenvalue $(N-2\ell)(N-2\ell+2)/2$. To prove the claim, let ℓ be given. The ℓ-summand in (18.6) is equal to $\phi_\ell T$. Using (18.5), (18.13) we find that for $B \in T$,

$$C^{(1)}(\phi_\ell B) = \phi\phi_\ell B = \frac{(N-2\ell)(N-2\ell+2)}{2}\phi_\ell B.$$

The claim in proven. Our discussion shows that for $0 \leqslant \ell \leqslant \lfloor N/2 \rfloor$ the map ϑ sends the ℓ-summand in (14.3) to the ℓ-summand in (18.6). The result follows.

□

19 Directions for future research

In this section, we give some suggestions for future research.

Problem 19.1. Recall the polynomial algebra $P = \mathbb{C}[x, y, z, w]$. Define

$$x^{\downarrow} = \frac{x-y-z-w}{2}, \qquad y^{\downarrow} = \frac{-x+y-z-w}{2},$$
$$z^{\downarrow} = \frac{-x-y+z-w}{2}, \qquad w^{\downarrow} = \frac{-x-y-z+w}{2}.$$

The vectors $x^{\downarrow}, y^{\downarrow}, z^{\downarrow}, w^{\downarrow}$ form a basis for P_1. Consequently, the following vectors form a basis for P:

$$x^{\downarrow r} y^{\downarrow s} z^{\downarrow t} w^{\downarrow u}, \qquad\qquad r, s, t, u \in \mathbb{N}. \tag{19.1}$$

This basis is an eigenbasis for the automorphism σ of P from Proposition 7.12. To see this, note that

$$x^{\downarrow} = -\frac{x^* - y^* - z^* - w^*}{2}, \qquad y^{\downarrow} = \frac{-x^* + y^* - z^* - w^*}{2},$$
$$z^{\downarrow} = \frac{-x^* - y^* + z^* - w^*}{2}, \qquad w^{\downarrow} = \frac{-x^* - y^* - z^* + w^*}{2},$$

where x^*, y^*, z^*, w^* are from Definition 7.1. Therefore, σ sends

$$x^{\downarrow} \mapsto -x^{\downarrow}, \qquad y^{\downarrow} \mapsto y^{\downarrow}, \qquad z^{\downarrow} \mapsto z^{\downarrow}, \qquad w^{\downarrow} \mapsto w^{\downarrow}.$$

It would be interesting to explore how $\mathfrak{sl}_4(\mathbb{C})$ acts on the basis (19.1).

Problem 19.2. We refer to Problem 19.1. The vectors $x^{\downarrow}, y^{\downarrow}, z^{\downarrow}, w^{\downarrow}$ are common eigenvectors for the following three elements of $\mathfrak{sl}_4(\mathbb{C})$:

$$A_1^{\downarrow} = \frac{[[A_3^*, A_1], A_2^*]}{4} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$A_2^{\downarrow} = \frac{[[A_1^*, A_2], A_3^*]}{4} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$A_3^{\downarrow} = \frac{[[A_2^*, A_3], A_1^*]}{4} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Specifically,

(i) $A_1^{\downarrow}$ sends

$$x^{\downarrow} \mapsto x^{\downarrow}, \qquad y^{\downarrow} \mapsto y^{\downarrow}, \qquad z^{\downarrow} \mapsto -z^{\downarrow}, \qquad w^{\downarrow} \mapsto -w^{\downarrow};$$

(ii) $A_2^{\downarrow}$ sends

$$x^{\downarrow} \mapsto x^{\downarrow}, \qquad y^{\downarrow} \mapsto -y^{\downarrow}, \qquad z^{\downarrow} \mapsto z^{\downarrow}, \qquad w^{\downarrow} \mapsto -w^{\downarrow};$$

(iii) $A_3^{\downarrow}$ sends

$$x^{\downarrow} \mapsto x^{\downarrow}, \qquad y^{\downarrow} \mapsto -y^{\downarrow}, \qquad z^{\downarrow} \mapsto -z^{\downarrow}, \qquad w^{\downarrow} \mapsto w^{\downarrow}.$$

The elements $A_1^{\downarrow}, A_2^{\downarrow}, A_3^{\downarrow}$ form a basis for a Cartan subalgebra $\mathbb{H}^{\downarrow}$ of $\mathfrak{sl}_4(\mathbb{C})$. It would be interesting to explore how $\mathbb{H}$, $\mathbb{H}^*$, $\mathbb{H}^{\downarrow}$ are related.

Problem 19.3. In the present paper we treated the graph $H(N, 2)$ in an S_3-symmetric way. As we mentioned in Section 17, $H(N, 2)$ is a Q-polynomial distance-regular graph that has diameter N and is a bipartite antipodal 2-cover. Let Γ denote any Q-polynomial distance-regular graph that has diameter N and is a bipartite antipodal 2-cover. Such a graph is called 2-homogeneous; see [12, 14]. To avoid trivialities, let us assume that Γ is not isomorphic to $H(N, 2)$. The intersection numbers of Γ are determined by N and a certain scalar parameter q; see [12, Theorem 35]. We seek an S_3-symmetric treatment of Γ that is analogous to the present paper. Such a treatment would amount to a q-analog of the treatment in the present paper.

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