

# Combinatorial enumeration of lattice paths by flaws with respect to a linear boundary of rational slope

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## Abstract

Let  $a, b$  be fixed positive coprime integers. For a positive integer  $g$ , write  $W_k(g)$  for the set of lattice paths from the startpoint  $(0, 0)$  to the endpoint  $(ga, gb)$  with steps restricted to  $\{(1, 0), (0, 1)\}$ , having exactly  $k$  flaws (lattice points lying above the linear boundary connecting the startpoint to the endpoint). We determine  $|W_k(g)|$  for all  $k$  and  $g$ . The enumeration of lattice paths with respect to a linear boundary while accounting for flaws has a long and rich history, dating back at least to the 1949 results of Chung and Feller. The only previously known values of  $|W_k(g)|$  are the extremal cases  $k = 0$  and  $k = g(a + b) - 1$ , determined by Bizley in 1954. Our main combinatorial result is that a certain subset of  $W_k(g)$  is in bijection with  $W_{k+1}(g)$ . One consequence is that the value  $|W_k(g)|$  is constant over each successive set of  $a + b$  values of  $k$ . This in turn allows us to derive a recursion for  $|W_k(g)|$  whose base case is given by Bizley's result for  $k = 0$ . We solve this recursion to obtain a closed form expression for  $|W_k(g)|$  for all  $k$  and  $g$ . Our methods are purely combinatorial.

**Mathematics Subject Classifications:** 05A15, 05E05

## 1 Introduction

The lattice path shown in Figure 1.1 contains exactly five lattice points that lie above the linear boundary joining the startpoint  $(0, 0)$  to the endpoint  $(8, 6)$ .

Throughout,  $a, b$  are fixed positive coprime integers and  $g$  is a positive integer. Our objective is to count the number of lattice paths from the startpoint  $(0, 0)$  to the endpoint  $(ga, gb)$  with steps restricted to  $\{(1, 0), (0, 1)\}$ , having exactly  $k$  lattice points lying above the linear boundary joining the startpoint to the endpoint.

Let  $p$  be a path. The *boundary* of  $p$  is the line joining its startpoint to its endpoint. The path  $p$  *contains* the lattice point  $(x + i, y + j)$  (equivalently,  $(x + i, y + j)$  is a *point of*  $p$ ) if  $p$  starts at  $(x, y)$ , and the first  $i + j \geq 0$  steps of  $p$  consist of  $i$  of the  $(1, 0)$  steps and

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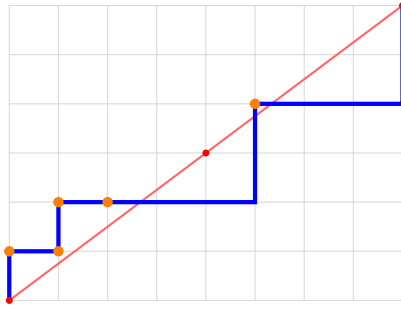


Figure 1.1: Lattice path from  $(0,0)$  to  $(8,6)$  with five lattice points above the line connecting  $(0,0)$  to  $(8,6)$ .

$j$  of the  $(0,1)$  steps (in any order). We consider the points of  $p$  to be ordered according to increasing values of  $i + j$ . A point of  $p$  is a *flaw* if it lies strictly above the boundary of  $p$ . For example, the path in Figure 1.1 has the five flaws  $(0,1)$ ,  $(1,1)$ ,  $(1,2)$ ,  $(2,2)$ ,  $(5,4)$  denoted in orange.

**Definition 1.1** (Sets  $W(g)$  and  $W_k(g)$ ). Let  $W(g)$  be the set of all paths from  $(0,0)$  to  $(ga, gb)$ , and let  $W_k(g)$  be the subset of such paths having exactly  $k$  flaws.  $\perp$

The possible values for the number  $k$  of flaws of a path are those satisfying  $0 \leq k < g(a+b)$ . Straightforward counting shows that  $|W(g)| = \binom{ga+gb}{ga}$ . The central objective of this paper is to find an explicit formula for  $|W_k(g)|$  for all  $g, k$  satisfying  $0 \leq k < g(a+b)$ . The extremal values  $|W_0(g)|$  and  $|W_{g(a+b)-1}(g)|$  were found by Bizley [9] in 1954 (see Theorem 1.10 below). Until now, the value of  $|W_k(g)|$  was unknown for all other  $k$ .

When  $g = 1$ , the values  $|W_k(g)|$  and  $|W_{g(a+b)-1-k}(g)|$  are equal: in this case, a path cannot contain lattice points on the boundary other than its startpoint and endpoint, so rotation of the path through  $180^\circ$  bijectively maps the set  $W_k(g)$  to the set  $W_{g(a+b)-1-k}(g)$ . However, in the case  $g > 1$ , a path can contain such lattice points and, because points on the boundary are not counted as flaws, rotation of the path through  $180^\circ$  does not map the set  $W_k(g)$  to the set  $W_{g(a+b)-1-k}(g)$ . In fact, the values  $|W_k(g)|$  and  $|W_{g(a+b)-1-k}(g)|$  are not equal in general.

Table 1.1 displays the numerical value of  $|W_k(4)|$  for  $(a, b) = (3, 2)$ , obtained by computer evaluation. We note two apparent properties suggested by these values:

- P1 (**Constant on blocks**). The value  $|W_k(g)|$  is constant on each of  $g$  distinct “blocks” of  $a + b$  consecutive values of  $k$ .
- P2 (**Strictly decreasing**). The value  $|W_k(g)|$  is strictly decreasing between successive blocks.

We shall show that properties P1 and P2 both hold for all values of  $g, a, b$ .

Table 1.1, in addition to displaying the value of  $|W_k(4)|$  for  $(a, b) = (3, 2)$ , also displays the value of the difference  $|W_k(4)| - |W_{k+1}(4)|$ . These differences suggest a strategy for achieving our central objective: identify a subset  $S_k(g)$  of  $W_k(g)$  having cardinality

| $k$ | $ W_k(4) $ | $ W_k(4)  -  W_{k+1}(4) $ |
|-----|------------|---------------------------|
| 0   | 7229       | 0                         |
| 1   | 7229       | 0                         |
| 2   | 7229       | 0                         |
| 3   | 7229       | 0                         |
| 4   | 7229       | 754                       |
| 5   | 6475       | 0                         |
| 6   | 6475       | 0                         |
| 7   | 6475       | 0                         |
| 8   | 6475       | 0                         |
| 9   | 6475       | 437                       |
| 10  | 6038       | 0                         |
| 11  | 6038       | 0                         |
| 12  | 6038       | 0                         |
| 13  | 6038       | 0                         |
| 14  | 6038       | 586                       |
| 15  | 5452       | 0                         |
| 16  | 5452       | 0                         |
| 17  | 5452       | 0                         |
| 18  | 5452       | 0                         |
| 19  | 5452       |                           |

Table 1.1: Computer evaluation of  $|W_k(4)|$  for  $(a, b) = (3, 2)$ .

$|W_k(g)| - |W_{k+1}(g)|$ , and show that the sets  $W_k(g) \setminus S_k(g)$  and  $W_{k+1}(g)$  are in bijection. We achieve this in our main combinatorial result (Theorem 1.5). Properties P1 and P2 follow as consequences of this result.

We introduce some additional vocabulary before defining the subset  $S_k(g)$ .

**Definition 1.2** (Path concatenation). Let  $p_1$  and  $p_2$  be paths having arbitrary startpoints. The *path concatenation*  $p_1p_2$  is the path that starts at the startpoint of  $p_1$ , takes all the (ordered) steps of  $p_1$ , and then takes all the (ordered) steps of  $p_2$ .  $\lrcorner$

**Definition 1.3** (Boundary points). The *boundary points* of a path  $p \in W(g)$  are the points of  $p$  that lie on its boundary. Boundary points of  $p$  other than the startpoint  $(0, 0)$  and endpoint  $(ga, gb)$  are *interior boundary points* of  $p$ . The startpoint and interior boundary points are the *non-terminal* boundary points.  $\lrcorner$

The lattice points lying on the boundary joining  $(0, 0)$  to  $(ga, gb)$  are the  $g + 1$  points of the form  $(ja, jb)$  for  $0 \leq j \leq g$  (see Figure 1.2). The number of interior boundary points of a path  $p \in W(g)$  therefore lies in  $\{0, 1, \dots, g - 1\}$ .

Recall that the number  $k$  of flaws of a path in  $W(g)$  satisfies  $0 \leq k < g(a + b)$ . A path in  $W(g)$  containing  $g(a + b) - 1$  flaws has *max flaws*. Equivalently, the set of paths with max flaws is  $W_{g(a+b)-1}(g)$ .

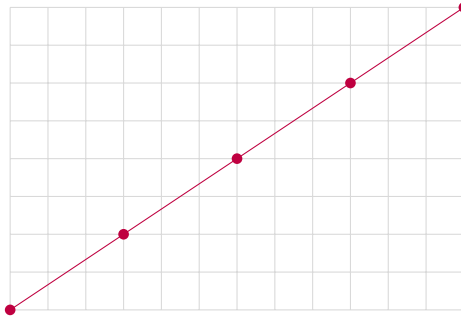


Figure 1.2: Let  $(a, b) = (3, 2)$ . The boundary of a path whose endpoint is  $(ga, gb) = (4 \cdot 3, 4 \cdot 2)$  contains  $g + 1 = 5$  lattice points (red vertices).

**Definition 1.4** (Subset  $\mathbf{S}_k(\mathbf{g})$ ). For  $0 \leq k < g(a + b) - 1$ , let  $S_k(g)$  be the set of *min-max paths*, namely the subset of  $W_k(g)$  containing all paths of the form  $p_1 p_2$ , where  $p_1 \in W_0(g - j)$  and  $p_2 \in W_k(j)$  for some  $j$  satisfying  $0 < j < g$ , and  $p_2$  has max flaws. We write  $S(g) := \bigcup_k S_k(g)$ .  $\lrcorner$

See Figure 1.3 for two example min-max paths in  $S(4)$ . A min-max path is, for some  $j$ , the concatenation of a path  $p_1$  from  $(0, 0)$  to  $((g - j)a, (g - j)b)$  having no flaws with a path  $p_2$  from  $(0, 0)$  to  $(ja, jb)$  having max flaws. The condition that  $p_2$  has max flaws implies that  $S_k(g)$  is empty unless  $k = j(a + b) - 1$  for some  $j$  satisfying  $0 < j < g$  (and in particular  $S_0(g)$  is empty). So we have

$$S_k(g) = \emptyset \quad \text{for } k \not\equiv -1 \pmod{a + b}, \quad (1.1)$$

and, for each  $j$  satisfying  $0 < j < g$ ,

$$S_{j(a+b)-1}(g) = \left\{ p_1 p_2 : p_1 \in W_0(g - j) \text{ and } p_2 \in W_{j(a+b)-1}(j) \right\}. \quad (1.2)$$

Note that the path  $p_2$  in Definition 1.4 does not contain an interior boundary point (because it has max flaws), but the path  $p_1$  might (see Figure 1.3).

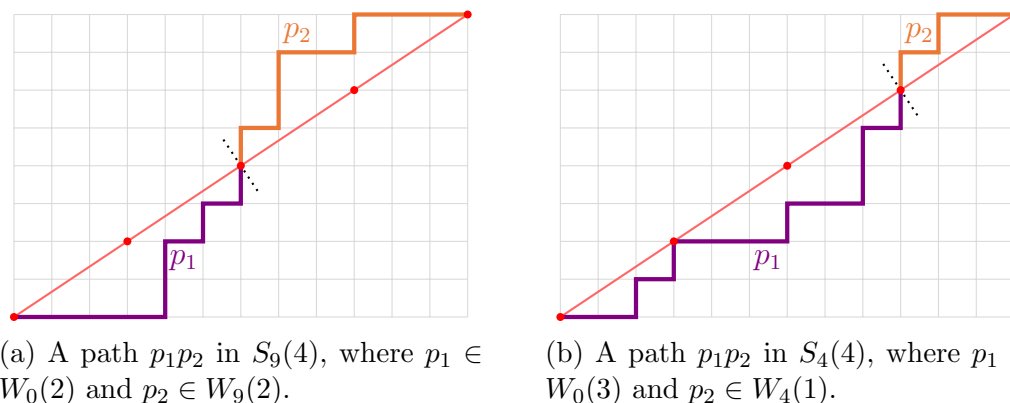


Figure 1.3: Two example min-max paths in  $S(4)$  for  $(a, b) = (3, 2)$ .

## 1.1 Main combinatorial result and consequences

**Theorem 1.5** (Main combinatorial result). *Let  $g, k$  satisfy  $0 \leq k < g(a+b) - 1$ . Then*

$$|W_k(g) \setminus S_k(g)| = |W_{k+1}(g)|.$$

We shall prove Theorem 1.5 in Section 3 combinatorially. Define

$$\mu_j(g) := |W_{j(a+b)}(g)| \quad \text{for each } j \text{ satisfying } 0 \leq j < g. \quad (1.3)$$

The following result is a first consequence of Theorem 1.5 and establishes property P1.

**Corollary 1.6** (Constant on blocks). *Let  $g$  be a positive integer. Then*

$$|W_k(g)| = \mu_j(g) \quad \text{for all } j, k \text{ satisfying } 0 \leq j < g \text{ and } j(a+b) \leq k < (j+1)(a+b).$$

*Proof.* The result follows directly from Theorem 1.5 and (1.1).  $\square$

We now observe two further consequences of Theorem 1.5.

**Corollary 1.7** (Recurrence relation). *Let  $g$  be a positive integer. Then*

$$\mu_{j-1}(g) - \mu_0(g-j)\mu_{j-1}(j) = \mu_j(g) \quad \text{for each } j \text{ satisfying } 0 < j < g.$$

*Proof.* Let  $j$  satisfy  $0 < j < g$  and let  $k = j(a+b) - 1$ . Since  $S_k(g)$  is a subset of  $W_k(g)$ , we have by Theorem 1.5 that

$$|W_k(g)| - |S_k(g)| = |W_{k+1}(g)|. \quad (1.4)$$

We know from Corollary 1.6 that  $|W_k(g)| = \mu_{j-1}(g)$  and  $|W_{k+1}(g)| = \mu_j(g)$ , and from (1.2) and Corollary 1.6 that  $|S_k(g)| = |S_{j(a+b)-1}(g)| = \mu_0(g-j)\mu_{j-1}(j)$ . Substitute these values into (1.4) to obtain the result.  $\square$

The next corollary establishes property P2.

**Corollary 1.8** (Strictly decreasing). *Let  $g$  be a positive integer. Then  $\mu_0(g) > \mu_1(g) > \cdots > \mu_{g-1}(g)$ .*

*Proof.* This follows from Corollary 1.7, noting that  $\mu_0(g-j)\mu_{j-1}(j) > 0$  for  $0 < j < g$  by the definition of  $\mu_j(g)$  given in (1.3).  $\square$

## 1.2 The value of $\mu_j(g)$ and the main enumerative result

Recall that our central objective is to find an explicit formula for  $|W_k(g)|$  for all  $g, k$  satisfying  $0 \leq k < g(a+b)$ . By Corollary 1.6, it is sufficient to determine the values  $\mu_j(g)$ .

The recurrence relation of Corollary 1.7 for  $\mu_j(g)$  has a unique solution for each  $j, g$  satisfying  $0 \leq j < g$ , provided the initial values  $\mu_0(g)$  (contained in the top row of Table 1.2) are known for all  $g$ . The required initial values  $\mu_0(g) = |W_0(g)|$  were given by Bizley [9] in 1954. We shall express these values in Corollary 1.11 in terms of a

|            |            |            |            |            |          |
|------------|------------|------------|------------|------------|----------|
| $\mu_0(1)$ | $\mu_0(2)$ | $\mu_0(3)$ | $\mu_0(4)$ | $\mu_0(5)$ | $\cdots$ |
|            | $\mu_1(2)$ | $\mu_1(3)$ | $\mu_1(4)$ | $\mu_1(5)$ | $\cdots$ |
|            |            | $\mu_2(3)$ | $\mu_2(4)$ | $\mu_2(5)$ | $\cdots$ |
|            |            |            | $\mu_3(4)$ | $\mu_3(5)$ | $\cdots$ |
|            |            |            |            | $\mu_4(5)$ | $\cdots$ |
|            |            |            |            |            | $\cdots$ |

Table 1.2: All values  $\mu_j(g)$  can be derived using only the values in the top row of the table, but can be more easily derived by also using the values in the coloured diagonal.

quantity  $H_g$ , introduced below. A derivation of  $\mu_j(g)$  using only the values of  $\mu_0(g)$  is given in [16, Chapter 4].

However, the values  $\mu_j(g)$  can be derived more easily by additionally making use of the values  $\mu_{g-1}(g)$  (contained in the coloured diagonal of Table 1.2). These additional values can be obtained using Corollary 1.6 from the values  $|W_{g(a+b)-1}(g)|$  given by Bizley [9], and can be expressed in terms of another quantity  $E_g$ . This is the approach we shall use to establish the value of  $\mu_j(g)$  in Theorem 1.12.

We now define the quantities  $H_g$  and  $E_g$  as sums over all integer partitions of  $g$ . Recall that a weakly increasing sequence of positive integers  $\lambda$  whose entries sum to  $g$  is a *partition of  $g$* ; we write  $\lambda \vdash g$  to indicate this. Each entry of  $\lambda$  is called a *part*. We use the notation  $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$  to mean that  $\lambda$  has  $m_i$  parts equal to  $i$ , so  $g = \sum_{i \geq 1} i m_i$ . For example, the partition  $(1, 1, 2, 3)$  of 7 is also written  $\langle 1^2 2^1 3^1 \rangle \vdash 7$ .

For  $i > 0$ , let

$$c_i := \frac{1}{i(a+b)} \binom{i(a+b)}{ia}. \quad (1.5)$$

For a partition  $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle \vdash g$ , let its *length* be  $\ell(\lambda) := \sum_{i \geq 1} m_i$ , and let

$$c_\lambda := \prod_{i \geq 1} \frac{c_i^{m_i}}{m_i!}. \quad (1.6)$$

Now let

$$H_g := \sum_{\lambda \vdash g} c_\lambda, \quad (1.7)$$

$$E_g := \sum_{\lambda \vdash g} (-1)^{g-\ell(\lambda)} c_\lambda, \quad (1.8)$$

and for convenience let

$$E_0 := 1 \quad \text{and} \quad H_0 := 1. \quad (1.9)$$

*Remark 1.9.* For each positive integer  $g$ , the quantities  $H_g$  and  $E_g$  are in fact specializations of (one part) *complete* and *elementary* symmetric functions  $h_g$  and  $e_g$ , respectively. The standard relationship between the *power sums*  $p_i$  and the complete and elementary

symmetric functions are

$$h_g = \sum_{\lambda \vdash g} \frac{p_\lambda}{z_\lambda} \quad \text{and} \quad e_g = \sum_{\lambda \vdash g} (-1)^{g-\ell(\lambda)} \frac{p_\lambda}{z_\lambda},$$

where for a partition  $\lambda = \langle 1^{m_1} 2^{m_2} \dots \rangle$  of  $g$ , we write  $z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!$  and  $p_\lambda := \prod_{i \geq 1} p_i^{m_i}$ . Then the quantities  $H_g$  and  $E_g$  in (1.7) and (1.8) are obtained from the specialization given by  $p_i = ic_i$ . The identity

$$\sum_{i=0}^g (-1)^i E_i H_{g-i} = 0 \tag{1.10}$$

results from another well-known relationship between the complete and elementary symmetric functions. See [1] or the classic reference [26] for details on the above and other relevant background on symmetric functions.  $\square$

**Theorem 1.10** (Bizley [9]). *We have that*

$$\begin{aligned} |W_0(g)| &= H_g, \\ |W_{g(a+b)-1}(g)| &= (-1)^{g+1} E_g. \end{aligned}$$

Using Corollary 1.6, we then obtain the value of  $\mu_0(g)$  and of  $\mu_{g-1}(g)$ .

**Corollary 1.11** (Value of  $\mu_0(g)$  and  $\mu_{g-1}(g)$ ). *We have that*

$$\mu_0(g) = H_g, \tag{1.11}$$

$$\mu_{g-1}(g) = (-1)^{g+1} E_g. \tag{1.12}$$

We can now give a closed form expression for the value of  $\mu_j(g)$ , which we remark is a truncated version of the sum from (1.10). This is our main enumerative result, and is the central consequence of Theorem 1.5.

**Theorem 1.12** (Path enumeration formula). *We have*

$$\mu_j(g) = \sum_{k=0}^j (-1)^k E_k H_{g-k} \quad \text{for } 0 \leq j < g.$$

*Proof.* The proof is by induction on  $j$ . The base case  $j = 0$  is given by (1.9) and (1.11). Assume that the formula holds for all cases up to  $j - 1$ , where  $0 < j < g$ . It follows from Corollary 1.7 and (1.11) and (1.12) that

$$\mu_j(g) = \mu_{j-1}(g) + (-1)^j E_j H_{g-j}.$$

The inductive hypothesis then gives

$$\mu_j(g) = \sum_{k=0}^{j-1} (-1)^k E_k H_{g-k} + (-1)^j E_j H_{g-j}$$

$$= \sum_{k=0}^j (-1)^k E_k H_{g-k}, \quad (1.13)$$

so case  $j$  holds and the induction is complete.  $\square$

*Remark 1.13.* The expression (1.13) can be written more compactly as a specialization of a Schur function via the identity  $\sum_{k=0}^j (-1)^k e_k h_{g-k} = (-1)^j s_{\langle 1^j(g-j)^1 \rangle}$ , where  $s_{\langle 1^j(g-j)^1 \rangle}$  is the Schur function indexed by the hook partition  $\langle 1^j(g-j)^1 \rangle$ . See [26, Section I.3, Example 9].  $\perp$

### 1.3 Background on rational Catalan combinatorics

In the case  $a = b = 1$ , the set  $W_0(g)$  comprises all lattice paths from  $(0, 0)$  to  $(g, g)$  having no flaws. Such paths are known as *Dyck paths*, and a classical result states that they are counted by the  $g^{\text{th}}$  *Catalan number*

$$C_g := \frac{1}{g+1} \binom{2g}{g}. \quad (1.14)$$

The *rational Catalan numbers* [4] are given by

$$C_{a,b} := \frac{1}{a+b} \binom{a+b}{a},$$

where  $a, b$  are restricted to be coprime so that  $C_{a,b}$  is always an integer. The quantity  $C_{a,b}$  is equal to  $c_1$  as defined in (1.5). The specialization  $C_{g,g+1}$  equals the Catalan number  $C_g$ . For a positive integer  $b$ , the specialization  $C_{g,gb+1}$  gives the *Fuss–Catalan number*

$$F_g^b := \frac{1}{gb+1} \binom{(b+1)g}{g}. \quad (1.15)$$

Dyck paths are one of the many combinatorial objects counted by the Catalan numbers; at least 213 other combinatorial objects are also counted by these numbers [27], including binary trees and noncrossing partitions. The Fuss–Catalan numbers count common generalizations of Catalan objects [28], including lattice paths from  $(0, 0)$  to  $(g, gb)$  having no flaws with respect to the linear boundary of integer slope  $b$  (see [15] for an argument expressed using ballot sequences, or [20] for a combinatorial proof using a construction akin to the *reflection principle*);  $b$ -ary trees, which are of much interest in computer science (see for example [6] and its references); and certain generalizations of noncrossing partitions.

The term *rational Catalan combinatorics* is traditionally used to describe the study of objects counted by the rational Catalan numbers  $C_{a,b}$  for general coprime  $a, b$ . This includes the enumeration of *rational Dyck paths*, namely lattice paths from  $(0, 0)$  to  $(a, b)$  having no flaws with respect to the linear boundary of rational slope  $b/a$ ; rational associahedra and noncrossing matchings [4]; and quantities associated with rational parking functions [3]. Fundamental statistics such as the area and  $\text{dinv}$  statistics on Dyck paths



have also been generalized to the rational case [17]. Rational Dyck paths have been connected to many different objects through a new framework involving the *signature* of coprime integers  $a, b$  [11].

Lattice paths from  $(0, 0)$  to  $(ga, gb)$  having no flaws can be considerably more difficult to count in the case  $g > 1$  (when the co-ordinates of the endpoint are not coprime) than in the case  $g = 1$  (when they are coprime): for example, the expression  $H_g$  for  $|W_0(g)|$  in Theorem 1.10 is complicated to evaluate in the case  $g > 1$ , but simplifies to  $C_{a,b}$  in the case  $g = 1$ . Though such paths are not necessarily counted by rational Catalan numbers, some authors consider these more general path-counting problems to be part of rational Catalan combinatorics whereas others use the term *rectangular Catalan combinatorics* [7]. Similar complications in moving from the case  $g = 1$  to the case  $g > 1$  occur elsewhere, for example when relating results between rational Dyck paths and simultaneous core partitions (see [2] and [18]), and in the study of parking functions and triangular partitions [5, 8] involving a substantial generalization of Bizley’s Theorem 1.10 to symmetric functions.

## 1.4 Counting paths that cross a linear boundary

The novelty of our work is in the combination of rational Catalan combinatorics with the counting of paths that contain flaws. To give further context, we briefly review the literature on the enumeration of lattice paths that cross a linear boundary.

The path enumeration setting we consider involves paths with step set  $\{(1, 0), (0, 1)\}$  in the two-dimensional lattice  $\mathbb{Z}^2$ ; a boundary line joining the startpoint  $(0, 0)$  of a path to its endpoint  $(ga, gb)$ , where  $g, a, b$  are positive integers such that  $a$  and  $b$  are coprime; and  $k$  flaws. Previous authors have defined a flaw differently from us, namely as a certain type of step (usually a  $(0, 1)$  step) of the path that lies above the boundary. Each of the references [12, 23, 24, 29, 30] adopts this step-based definition of flaw and a notion of the “wrong” side of the boundary, although the precise definition is not identical in all five references.

In the more general case that we consider here, where the value of  $b/a$  need not necessarily be an integer, the definition of a flaw as a step is no longer appropriate since some steps can lie only partially above the boundary (see Figure 1.4). Our definition of a flaw, as a lattice point of the path that lies above the boundary, does not have this ambiguity.

Note that these two definitions of flaws are genuinely different: Figure 1.5 shows that even in the case  $a = b = 1$  there is no simple relationship between the number of  $(0, 1)$  steps lying above the boundary and the number of lattice points lying above the boundary. In Figure 1.6 we further illustrate the difference between these two definitions of flaws when  $g = 2$  and  $a = b = 1$ , which is the smallest non-trivial case of Corollary 1.6.

We review previous results relating to these two definitions of flaws in Sections 1.4.1 and 1.4.2.

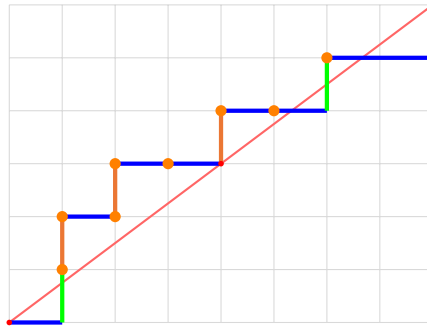
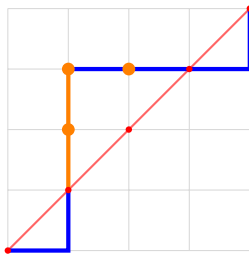
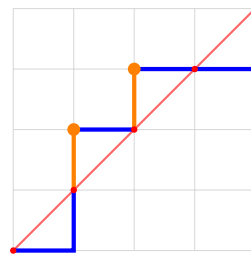


Figure 1.4: A path with a boundary whose slope is not an integer. This shows that both  $(0, 1)$  steps and  $(1, 0)$  steps can lie partially above and below the boundary simultaneously, whereas lattice points cannot.



(a) A path in which two  $(0, 1)$  steps and three lattice points lie above the boundary.



(b) A path in which two  $(0, 1)$  steps and two lattice points lie above the boundary.

Figure 1.5: Even in the case  $a = b = 1$ , there is no simple relationship between the number of  $(0, 1)$  steps lying above the boundary and the number of lattice points lying above the boundary.

#### 1.4.1 Boundaries of integer slope with $(0, 1)$ steps as flaws

In this part of the review, we take flaws to be  $(0, 1)$  steps of a path from  $(0, 0)$  to  $(g, gb)$  that lie above the boundary of integer slope  $b$ .

Firstly consider paths from  $(0, 0)$  to  $(g, g)$ . The number of such paths having  $k = 0$  flaws (no  $(0, 1)$  steps lying above the boundary, which is equivalent to having no lattice points above the boundary) is given by the Catalan number  $C_g$  defined in (1.14). Chung and Feller's influential 1949 work [13] showed that, remarkably, the same count applies for all  $k$ .

**Theorem 1.14** (Chung–Feller [13, Theorem 2A]). *Let  $k$  satisfy  $0 \leq k \leq g$ . Then the number of paths from  $(0, 0)$  to  $(g, g)$  having  $k$  of the  $(0, 1)$  steps lying above the boundary is  $C_g$ .*

Theorem 1.14 can be proven using bijective methods [10]. Huq generalized Theorem 1.14 to paths from  $(0, 0)$  to  $(g, gb)$  for each positive integer  $b$ .

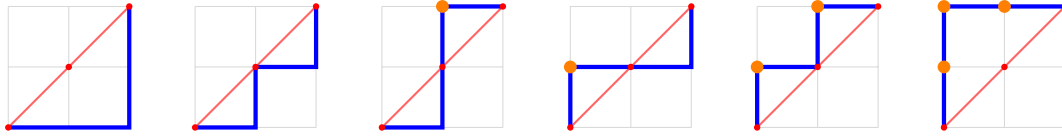


Figure 1.6: Set  $a = b = 1$  and  $g = 2$ . The six lattice paths from  $(0,0)$  to  $(2,2)$  are distributed as  $(0, 0, 1, 1, 2, 3)$  according to the number of lattice points above the boundary. Here  $W_0(2) = W_1(2) = \mu_0(2) = 2$  and  $W_2(2) = W_3(2) = \mu_1(2) = 1$  (illustrating Corollaries 1.6 and 1.8). The six lattice paths are distributed as  $(0, 0, 1, 1, 2, 2)$  according to the number of vertical steps above the boundary (illustrating the Chung–Feller result given in Theorem 1.14).

**Theorem 1.15** (Huq [24, Corollary 5.1.2]). *Let  $k$  satisfy  $0 \leq k \leq gb$ . Then the number of paths from  $(0,0)$  to  $(g, gb)$  having  $k$  of the  $(0,1)$  steps lying above the boundary is  $F_g^b$ , the Fuss–Catalan number defined in (1.15).*

Further variations on Theorem 1.14 have been found [23, 25, 29].

#### 1.4.2 Boundaries of rational slope with lattice points as flaws

In this part of the review, we take flaws to be lattice points of a path from  $(0,0)$  to  $(ga, gb)$  that lie strictly above the boundary. The number of these flaws is the measure  $k$  used for  $W_k(g)$  in Definition 1.1.

In 1950, Grossman [22] conjectured an explicit formula for the number of paths from  $(0,0)$  to  $(ga, gb)$  that may touch, but never rise above, the boundary  $ay = bx$ . In our terminology, such paths have no flaws and so are counted by  $|W_0(g)|$ . In 1954, Bizley [9, Eq. (10)] proved Grossman’s formula using generating functions. Bizley [9, Eq. (8)] also obtained an explicit formula for the number of paths that lie wholly below, and do not touch, the boundary  $ay = bx$  at an interior boundary point. In other words, such paths have neither flaws nor interior boundary points. Since the set of such paths is in bijection with the set of paths having max flaws (via rotation), this second result of Bizley’s gives the value  $|W_{g(a+b)-1}(g)|$ . The values  $|W_0(g)|$  and  $|W_{g(a+b)-1}(g)|$  are stated in Theorem 1.10.

One of the methods used by Bizley to prove Theorem 1.10 involves cyclic rotation of paths. Such methods, in particular the *Cycle Lemma*, have been frequently used to count lattice paths and have been popularized by many authors (see for example [14, 15]). Although we were able to use arguments similar to the Cycle Lemma to obtain results in certain special cases, we were not able to use it to obtain our general result.

## 2 Special cases and example of the main enumerative result

In this section we examine various special cases of the path enumeration formula of Theorem 1.12, and give an example of its application.

We firstly state the special case  $g = 1$ .

**Theorem 2.1** (Special case  $g = 1$ ). *We have*

$$|W_k(1)| = \frac{1}{a+b} \binom{a+b}{a} \quad \text{for all } k \text{ satisfying } 0 \leq k < a+b.$$

*Proof.* Let  $k$  satisfy  $0 \leq k < a+b$ . By Corollaries 1.6 and 1.11 with  $g = 1$ , we have

$$|W_k(1)| = \mu_0(1) = H_1. \quad (2.1)$$

Now by (1.7) and (1.6) we have

$$H_1 = c_{\langle 1^1 \rangle} = c_1.$$

Substitute into (2.1) and use (1.5) to give the result.  $\square$

Theorem 2.1 shows that  $|W_k(1)|$  is independent of  $k$  when  $g = 1$ . We highlight this special case because our general construction used to prove Theorem 1.5 simplifies greatly in the case  $g = 1$ . We re-examine this special case in Section 3.5.

We next give the special case  $g = 2$ .

**Theorem 2.2** (Special case  $g = 2$ ). *We have*

$$|W_k(2)| = \begin{cases} \frac{1}{2(a+b)} \left[ \binom{2a+2b}{2a} + \frac{1}{a+b} \binom{a+b}{a}^2 \right] & \text{for all } k \text{ satisfying } 0 \leq k < a+b, \\ \frac{1}{2(a+b)} \left[ \binom{2a+2b}{2a} - \frac{1}{a+b} \binom{a+b}{a}^2 \right] & \text{for all } k \text{ satisfying } a+b \leq k < 2(a+b). \end{cases}$$

*Proof.* By Corollary 1.6 with  $g = 2$ ,

$$\begin{aligned} |W_k(2)| &= \begin{cases} \mu_0(2) & \text{for } 0 \leq k < a+b, \\ \mu_1(2) & \text{for } a+b \leq k < 2(a+b), \end{cases} \\ &= \begin{cases} H_2 & \text{for } 0 \leq k < a+b, \\ -E_2 & \text{for } a+b \leq k < 2(a+b) \end{cases} \end{aligned} \quad (2.2)$$

using Corollary 1.11. Now by (1.7) and (1.6) we have

$$H_2 = c_{\langle 2^1 \rangle} + c_{\langle 1^2 \rangle} = c_2 + \frac{1}{2}c_1^2,$$

and by (1.8) and (1.6) we have

$$-E_2 = c_{\langle 2^1 \rangle} - c_{\langle 1^2 \rangle} = c_2 - \frac{1}{2}c_1^2.$$

Substitute into (2.2) and use (1.5) to give the result.  $\square$

We now consider the special case  $a = 1$ , when the slope of the boundary is the positive integer  $b$  and the startpoint and endpoint are  $(0, 0)$  and  $(g, gb)$ , respectively. Although Theorem 1.12 already provides an expression for  $\mu_j(g)$  in this case, we now derive an alternative formula involving Fuss–Catalan numbers that appears to be simpler.

**Theorem 2.3** (Alternative formula for  $a = 1$ ). *Let  $a = 1$ . Then for all  $j$  satisfying  $0 \leq j < g$ ,*

$$|W_{j(b+1)}(g)| = |W_{j(b+1)+1}(g)| = \cdots = |W_{j(b+1)+b}(g)| = \mu_j(g) = \sum_{\substack{i_1 + \cdots + i_{b+1} = g-1 \\ i_{b+1} \leq g-1-j}} F_{i_1}^b \cdots F_{i_{b+1}}^b \quad (2.3)$$

where  $F_i^b$  is the Fuss–Catalan number defined in (1.15).

*Proof.* Fix the positive integer  $g$ . All equalities of (2.3) except the last hold by Corollary 1.6. We shall establish the last equality by induction on  $j$ . We shall describe the structure of the induction by reference to Table 1.2, numbering table rows from 0 and columns from 1.

Let  $P_j$  be the statement that

$$\mu_j(t) = \sum_{\substack{i_1 + \cdots + i_{b+1} = t-1 \\ i_{b+1} \leq t-1-j}} F_{i_1}^b \cdots F_{i_{b+1}}^b \quad \text{for all } t \text{ satisfying } j+1 \leq t \leq g.$$

In other words,  $P_j$  is the statement that the last equality of (2.3) holds for all entries in row  $j$  of Table 1.2 up to and including column  $g$ . We prove the theorem by showing by induction on  $j$  that  $P_j$  holds for  $0 \leq j < g$ : that is, for all entries in rows  $0, 1, \dots, g-1$  of Table 1.2 up to and including column  $g$ .

For the base case  $P_0$ , let  $t$  satisfy  $1 \leq t \leq g$ . We require that

$$\mu_0(t) = \sum_{i_1 + \cdots + i_{b+1} = t-1} F_{i_1}^b \cdots F_{i_{b+1}}^b.$$

This is equivalent to showing that  $\mu_0(t) = F_t^b$ , by making use of the identity  $F_t^b = \sum_{i_1 + \cdots + i_{b+1} = t-1} F_{i_1}^b \cdots F_{i_{b+1}}^b$ ; this identity can be obtained by applying Lagrange inversion [19, Theorem 1.2.4] to the functional equation  $F(x) = xF(x)^{b+1} + 1$  satisfied by the generating series  $F(x) = \sum_{i \geq 0} F_i^b x^i$  (see [21, p. 362] for the identity and functional equation). Since  $a = 1$ , the set  $W(t)$  comprises all paths from  $(0, 0)$  to  $(t, tb)$ . As discussed in Section 1.3, the number of such paths with zero flaws is  $F_t^b$  and so  $\mu_0(t) = F_t^b$  as required. This establishes the base case  $P_0$ .

Now let  $j$  satisfy  $0 < j < g$  and assume that the case  $P_{j-1}$  holds, so that

$$\mu_{j-1}(t) = \sum_{\substack{i_1 + \cdots + i_{b+1} = t-1 \\ i_{b+1} \leq t-j}} F_{i_1}^b \cdots F_{i_{b+1}}^b \quad \text{for all } t \text{ satisfying } j \leq t \leq g. \quad (2.4)$$

Let  $t$  satisfy  $j+1 \leq t \leq g$ . Then the recurrence relation of Corollary 1.7 gives

$$\mu_j(t) = \mu_{j-1}(t) - \mu_0(t-j) \mu_{j-1}(j) = \mu_{j-1}(t) - F_{t-j}^b \mu_{j-1}(j).$$

Apply the inductive hypothesis (2.4) to  $\mu_{j-1}(t)$  and  $\mu_{j-1}(j)$  to deduce that

$$\begin{aligned}\mu_j(t) &= \sum_{\substack{i_1+\dots+i_{b+1}=t-1 \\ i_{b+1} \leq t-j}} F_{i_1}^b \cdots F_{i_{b+1}}^b - F_{t-j}^b \sum_{\substack{i_1+\dots+i_{b+1}=j-1 \\ i_{b+1}=0}} F_{i_1}^b \cdots F_{i_{b+1}}^b \\ &= \sum_{\substack{i_1+\dots+i_{b+1}=t-1 \\ i_{b+1} \leq t-j}} F_{i_1}^b \cdots F_{i_{b+1}}^b - \sum_{i_1+\dots+i_b=j-1} F_{i_1}^b \cdots F_{i_b}^b F_{t-j}^b\end{aligned}$$

using that  $F_0^b = 1$ . This gives

$$\mu_j(t) = \sum_{\substack{i_1+\dots+i_{b+1}=t-1 \\ i_{b+1} \leq t-1-j}} F_{i_1}^b \cdots F_{i_{b+1}}^b,$$

by expanding the right hand side according to whether  $i_{b+1} \leq t-1-j$  or  $i_{b+1} = t-j$ . Therefore the case  $P_j$  holds, completing the induction.  $\square$

We next give an alternative formula for the special case  $a = b = 1$  (when the slope of the boundary is 1) by taking  $b = 1$  in Theorem 2.3 and noting that the Fuss–Catalan number  $F_i^1$  equals the Catalan number  $C_i$ , in order to demonstrate a further simplification.

**Corollary 2.4** (Alternative formula for  $\mathbf{a} = \mathbf{b} = \mathbf{1}$ ). *Let  $a = b = 1$ . Then*

$$|W_{2j}(g)| = |W_{2j+1}(g)| = \mu_j(g) = \sum_{k=j}^{g-1} C_k C_{g-1-k} \quad \text{for all } j \text{ satisfying } 0 \leq j < g,$$

where  $C_i$  is the Catalan number defined in (1.14).

The reader is invited to compare Theorem 2.3 with Theorem 1.15: both apply to a boundary of integer slope  $b$ , but Theorem 2.3 takes flaws to be points above the boundary whereas Theorem 1.15 takes flaws to be  $(0, 1)$  steps above the boundary. The count in Theorem 2.3 depends on the number of flaws whereas the count in Theorem 1.15 does not.

We now give an example of how to apply Theorem 1.12.

**Example 2.5** (Computation using the path enumeration formula). Let  $(a, b) = (3, 2)$  and  $g = 4$ . We illustrate the use of the path enumeration formula Theorem 1.12 to calculate the number  $|W_k(4)|$  of paths from  $(0, 0)$  to  $(12, 8)$  having  $k$  flaws, for each  $k$  satisfying  $0 \leq k < 20$ . By Corollary 1.6, it is sufficient to determine  $\mu_j(4)$  for each  $j = 0, 1, 2, 3$ .

We begin by listing the partitions of the integers 1, 2, 3, 4.

Partitions of 4 :  $\langle 4^1 \rangle, \langle 1^1 3^1 \rangle, \langle 2^2 \rangle, \langle 1^2 2^1 \rangle, \langle 1^4 \rangle,$   
Partitions of 3 :  $\langle 3^1 \rangle, \langle 1^1 2^1 \rangle, \langle 1^3 \rangle,$   
Partitions of 2 :  $\langle 2^1 \rangle, \langle 1^2 \rangle,$   
Partitions of 1 :  $\langle 1^1 \rangle.$

Using (1.5), we compute

$$c_1 = 2, \quad c_2 = 21, \quad c_3 = \frac{1001}{3}, \quad c_4 = \frac{12597}{2}.$$

Using (1.6), we then compute (for example)

$$c_{\langle 1^2 2^1 \rangle} = \left( \frac{c_1^2}{2!} \right) \left( \frac{c_2^1}{1!} \right) = 42, \quad c_{\langle 1^3 \rangle} = \left( \frac{c_1^3}{3!} \right) = \frac{4}{3}.$$

The full set of  $c_\lambda$  values for  $\lambda \vdash j$  where  $1 \leq j \leq 4$  is

$$\begin{aligned} c_{\langle 4^1 \rangle} &= \frac{12597}{2}, & c_{\langle 1^1 3^1 \rangle} &= \frac{2002}{3}, & c_{\langle 2^2 \rangle} &= \frac{441}{2}, & c_{\langle 1^2 2^1 \rangle} &= 42, & c_{\langle 1^4 \rangle} &= \frac{2}{3}, \\ c_{\langle 3^1 \rangle} &= \frac{1001}{3}, & c_{\langle 1^1 2^1 \rangle} &= 42, & c_{\langle 1^3 \rangle} &= \frac{4}{3}, \\ c_{\langle 2^1 \rangle} &= 21, & c_{\langle 1^2 \rangle} &= 2, \\ c_{\langle 1^1 \rangle} &= 2. \end{aligned}$$

Using (1.7) and (1.8), we next calculate (for example)

$$\begin{aligned} H_3 &= c_{\langle 3^1 \rangle} + c_{\langle 1^1 2^1 \rangle} + c_{\langle 1^3 \rangle} = \frac{1001}{3} + 42 + \frac{4}{3} = 377, \\ E_3 &= (-1)^{3-1} c_{\langle 3^1 \rangle} + (-1)^{3-2} c_{\langle 1^1 2^1 \rangle} + (-1)^{3-3} c_{\langle 1^3 \rangle} = \frac{1001}{3} - 42 + \frac{4}{3} = 293. \end{aligned}$$

The full set of  $H_k$  and  $E_k$  values is

$$\begin{aligned} H_4 &= 7229, & E_4 &= -5452, \\ H_3 &= 377, & E_3 &= 293, \\ H_2 &= 23, & E_2 &= -19, \\ H_1 &= 2, & E_1 &= 2, \\ E_0 &= 1. \end{aligned}$$

Using Theorem 1.12, we then determine that

$$\begin{aligned} \mu_0(4) &= E_0 H_4 = 1 \cdot 7229 = 7229 \\ \mu_1(4) &= E_0 H_4 - E_1 H_3 = 1 \cdot 7229 - 2 \cdot 377 = 6475 \\ \mu_2(4) &= E_0 H_4 - E_1 H_3 + E_2 H_2 = 1 \cdot 7229 - 2 \cdot 377 - 19 \cdot 23 = 6038 \\ \mu_3(4) &= E_0 H_4 - E_1 H_3 + E_2 H_2 - E_3 H_1 = 1 \cdot 7229 - 2 \cdot 377 - 19 \cdot 23 - 293 \cdot 2 = 5452. \end{aligned}$$

(Alternatively, we may use (1.12) for a more direct calculation of the last value  $\mu_3(4) = (-1)^{4+1} E_4 = 5452$ .)

Using Corollary 1.6, we may now determine the value of  $|W_k(4)|$  for each  $k$  satisfying  $0 \leq k < 20$ . The resulting values agree with the computer enumeration shown in Table 1.1.  $\square$

We remark, as noted by Bizley [9], that both  $H_g$  and  $E_g$  are necessarily integers because of the counting result Theorem 1.10, even though this is not readily apparent from the forms (1.6), (1.7), and (1.8). We further remark that although the quantity  $c_i$  defined in (1.5) is not necessarily an integer, it is not difficult to show that  $ic_i$  is an integer.

### 3 Proof of the main combinatorial result

For convenience, we restate our main combinatorial result here.

**Theorem 1.5.** *Let  $g, k$  satisfy  $0 \leq k < g(a + b) - 1$ . Then*

$$|W_k(g) \setminus S_k(g)| = |W_{k+1}(g)|.$$

#### 3.1 Proof outline

We shall prove our main combinatorial result by considering fixed  $g, k$  satisfying  $0 \leq k < g(a + b) - 1$  and constructing injective maps

$$\begin{aligned} \phi: W_k(g) \setminus S_k(g) &\rightarrow W_{k+1}(g), \\ \psi: W_{k+1}(g) &\rightarrow W_k(g) \setminus S_k(g). \end{aligned}$$

In fact, the map  $\psi$  we shall construct is the inverse of  $\phi$ , although we shall not require this fact in our proof. We partition the set  $W_k(g) \setminus S_k(g)$  into subsets  $X$  and  $Y$ , and partition (using a different rule) the set  $W_{k+1}(g)$  into subsets  $\mathcal{X}$  and  $\mathcal{Y}$ . We allow each of the partitioning subsets to be empty. Using these partitions, we then specify the action of  $\phi$  using injective submaps  $\phi^X$  and  $\phi^Y$ , and the action of  $\psi$  using injective submaps  $\psi^{\mathcal{X}}$  and  $\psi^{\mathcal{Y}}$  (see Figure 3.1).

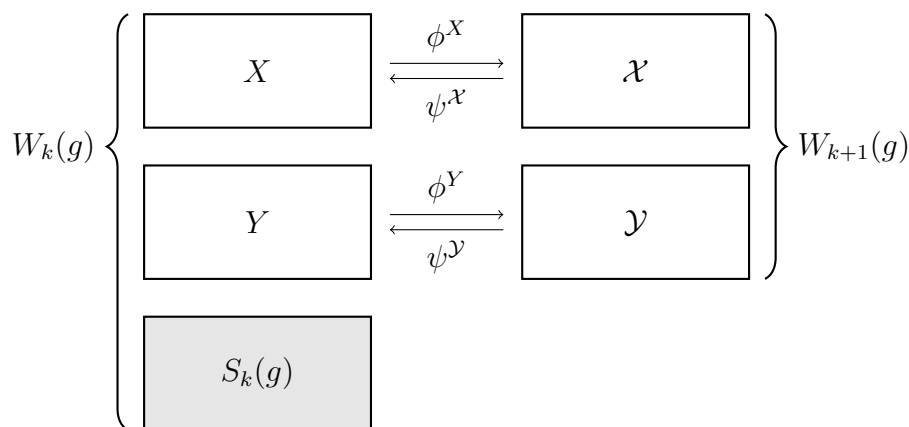


Figure 3.1: The map  $\phi: W_k(g) \setminus S_k(g) \rightarrow W_{k+1}(g)$  is defined piecewise using the maps  $\phi^X: X \rightarrow \mathcal{X}$  and  $\phi^Y: Y \rightarrow \mathcal{Y}$ . The map  $\psi: W_{k+1}(g) \rightarrow W_k(g) \setminus S_k(g)$  is likewise defined piecewise using the maps  $\psi^{\mathcal{X}}: \mathcal{X} \rightarrow X$  and  $\psi^{\mathcal{Y}}: \mathcal{Y} \rightarrow Y$ .

To prove Theorem 1.5, it suffices to

1. specify the partition of  $W_k(g) \setminus S_k(g)$  and of  $W_{k+1}(g)$  as illustrated in Figure 3.1, and
2. define the maps  $\phi$  and  $\psi$ , and show that they are both injective.



### 3.2 Partition of sets $W_k(g) \setminus S_k(g)$ and $W_{k+1}(g)$

We first introduce some additional terminology. Recall that the boundary of a path in  $W(g)$  is the line from  $(0, 0)$  to  $(ga, gb)$ .

**Definition 3.1** (Elevation). Let  $(i, j)$  be a point of a path in  $W(g)$ . The *elevation* of  $(i, j)$  is  $ja - ib$ .  $\lrcorner$

The elevation of a point of a path in  $W(g)$  is a measure of the directed distance from the point to the boundary. Points on the boundary have zero elevation; points above the boundary have positive elevation; points below the boundary have negative elevation.

**Definition 3.2** (Lowest points above, highest points below). Let  $p$  be a path. The *lowest points above the boundary (LPAs)* of  $p$  are those points of  $p$  (if any) attaining the smallest strictly positive elevation. The *highest points below the boundary (HPBs)* of  $p$  are defined analogously.  $\lrcorner$

See Figure 3.2a for an illustration of a path  $p$  with LPAs  $L, L', L''$  and HPBs  $H, H'$ . We note that the possible elevation values for an LPA are  $1, 2, \dots, \min(a, b)$ , and that the possible elevation values for an HPB are  $-1, -2, \dots, -\min(a, b)$ . Both the number of LPAs and the number of HPBs of a path in  $W(g)$  lie in  $\{0, 1, \dots, g\}$ .

We shall often consider a *subpath*  $p'$  of a path  $p$ , namely a consecutive sequence of steps of  $p$ . When viewed as a separate path in its own right, the boundary of  $p'$  need not coincide with the boundary of  $p$  (nor even have the same slope) and so its LPAs and HPBs need not necessarily be the same as those of  $p$  (see Figure 3.2). When we wish to view  $p'$  as a path in its own right, we shall refer to “the path  $p'$ ”; when we wish to view  $p'$  as a part of  $p$  we shall refer to “the subpath  $p'$ ”.

Definition 1.2 describes the combination of paths  $p_1$  and  $p_2$  to form the concatenated path  $p_1p_2$ . To reverse this process, we *split* the path  $p_1p_2$  at the endpoint of  $p_1$  into component paths  $p_1, p_2$ . We can similarly split a path at two distinct points to form component paths  $p_1, p_2, p_3$ . If the elevation of the startpoint and endpoint of  $p_i$  (viewed as a subpath) are equal, then  $p_i$  (viewed as a path) has a boundary with the same slope (the same values of  $a$  and  $b$ ) as the full path.

We make the following key observation about the change in the number of flaws when a path is split at an HPB or LPA and the resulting subpaths are interchanged.

**Key Observation 3.3.** Let  $p$  be a path containing exactly  $\beta$  interior boundary points and exactly  $\lambda$  LPAs. Suppose that  $p$  is split at an HPB  $H$  into  $p_1p_2$ , so that  $H$  is the endpoint of  $p_1$  and the startpoint of  $p_2$ . Then the rearranged path  $p_2p_1$  has exactly  $\beta + 1$  more flaws than  $p$ , namely all  $\beta$  interior boundary points of  $p$  together with the endpoint of  $p_2$ . If instead  $p$  is split at an LPA into  $p_1p_2$ , then the rearranged path  $p_2p_1$  has exactly  $\lambda$  fewer flaws than  $p$ , namely all  $\lambda$  LPAs of  $p$ .  $\lrcorner$

See Figure 3.3 for an illustration of Key Observation 3.3.

We now define the subsets  $X$  and  $Y$  of  $W_k(g) \setminus S_k(g)$  by reference to an arbitrary path  $p \in W_k(g) \setminus S_k(g)$ . Split  $p$  at its last non-terminal boundary point into  $qr$ , and regard  $q$

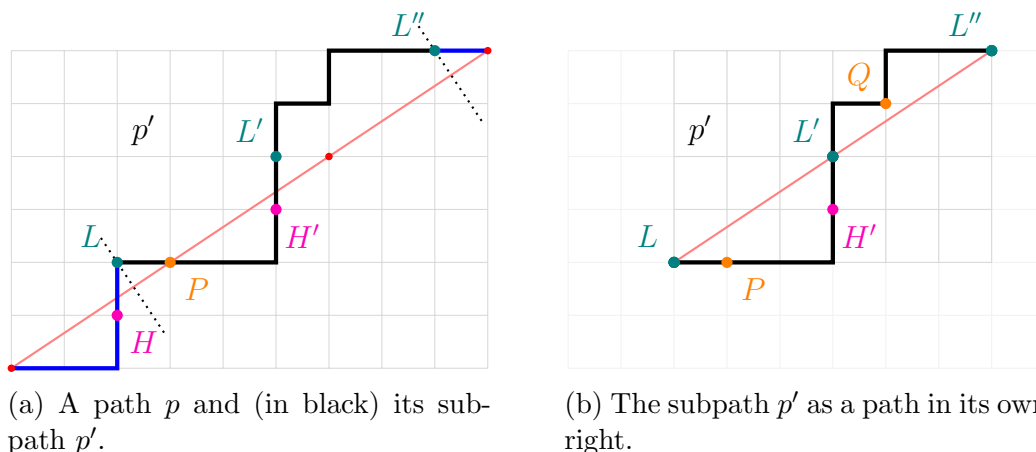


Figure 3.2: Let  $(a, b) = (3, 2)$ . The path  $p$  is a member of  $W_7(3) \setminus S_7(3)$  containing the interior boundary point  $P$  and the LPAs  $L, L', L''$  and HPBs  $H, H'$ . The subpath  $p'$  contains the same interior boundary point and LPAs as  $p$ , as well as the HPB  $H'$  of  $p$ . The path  $p'$  (on its own) has the same slope as  $p$ , but is a member of  $S_4(2) \subseteq W_4(2)$  containing the boundary points  $L, L', L''$ , the unique HPB  $P$ , and the unique LPA  $Q$ . We note that  $H'$  is not an HPB of the path  $p'$ .

and  $r$  as paths in their own right. If  $p$  has no interior boundary points, then  $p$  is split at its startpoint and  $q$  is empty. Since  $p \notin S_k(g)$ , either  $q$  has at least one flaw or  $r$  has non-max flaws; in the latter case,  $r$  has at least one HPB because  $p$  splits at its last non-terminal boundary point into  $qr$  and so  $r$  itself has no interior boundary points. Therefore exactly one of three cases holds:

**Case 1:**  $q$  has no flaws and  $r$  has non-max flaws. Then  $p \in X$ .

**Case 2:**  $q$  has at least one flaw and  $r$  has max flaws. Then  $p \in Y$ .

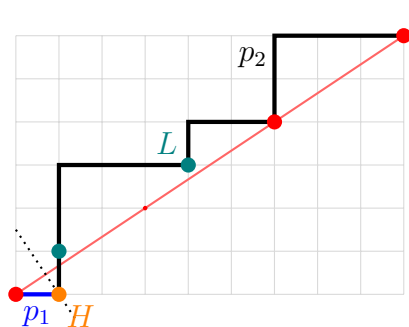
**Case 3:**  $q$  has at least one flaw and  $r$  has non-max flaws. If the LPAs of  $q$  are closer to the boundary of  $p$  than are the HPBs of  $r$ , then  $p \in Y$ . Otherwise  $p \in X$ .

See Figure 3.4 for an illustration of Case 3.

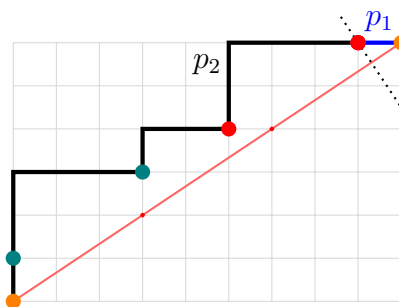
We now give a more concise definition of the subsets  $X$  and  $Y$ . Recall that  $k$  is fixed and satisfies  $0 \leq k < g(a + b) - 1$  throughout this section.

**Definition 3.4** (The subsets  $X$  and  $Y$ ). Let  $p \in W_k(g) \setminus S_k(g)$ . Split  $p$  at its last non-terminal boundary point into  $qr$ , and regard  $q$  and  $r$  as paths. The path  $p$  lies in  $Y$  provided:

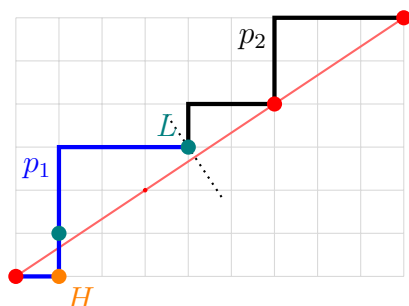
- (i)  $q$  has at least one flaw, and
- (ii) the elevation of the LPAs of  $q$  is smaller than the magnitude of the elevation of the HPBs of  $r$  (if any).



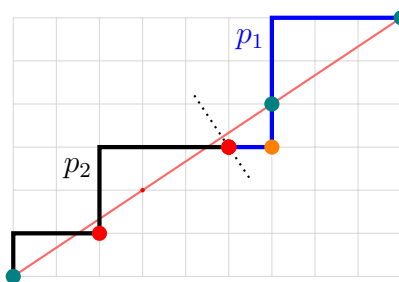
(a) Original path with 12 flaws.



(b) Rearranged path with 14 flaws.



(c) Original path with 12 flaws.



(d) Rearranged path with 10 flaws.

Figure 3.3: Rearrangement of the subpaths of a path with  $\beta = 1$  interior boundary points and  $\lambda = 2$  LPAs changes the number of flaws, according to Key Observation 3.3. Splitting the path at an HPB maps diagram (a) to diagram (b); splitting the same path at an LPA maps diagram (c) to diagram (d).

Otherwise,  $p$  lies in  $X$ . ┐

Note that  $Y$  is empty if  $k = 0$ . We now use Definition 3.4 to specify a canonical representation for a path in each of  $X$  and  $Y$  as a concatenation of paths.

**Definition 3.5** (Canonical representation of paths in  $\mathbf{X}$  and  $\mathbf{Y}$ ). Let  $p \in W_k(g) \setminus S_k(g)$ . Split  $p$  at its last non-terminal boundary point into  $p = qr$ .

**Case  $p \in \mathbf{X}$ :** the path  $r$  has at least one HPB. Split  $r$  at its last HPB into  $r = r_1r_2$ . The canonical representation of  $p$  is  $qr_1r_2$ .

**Case  $p \in \mathbf{Y}$ :** the path  $q$  has at least one LPA. Split  $q$  at its last LPA into  $q = q_1q_2$ . The canonical representation of  $p$  is  $q_1q_2r$ . ┐

We now define the subsets  $\mathcal{X}$  and  $\mathcal{Y}$  of  $W_{k+1}(g)$  by reference to an arbitrary path  $\mathbb{p} \in W_{k+1}(g)$ . Since  $\mathbb{p}$  has at least one flaw, it has at least one LPA. Throughout, we use regular typeface (for example  $p$ ) for a path originating in  $W_k(g)$  whereas we use

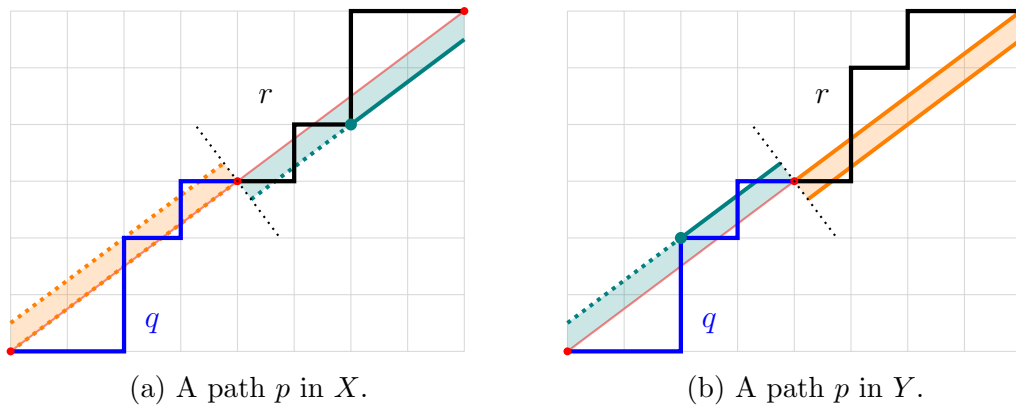


Figure 3.4: Let  $(a, b) = (4, 3)$  and split the path  $p$  into  $qr$  at its last non-terminal boundary point. In diagram (a), we have  $p \in W_6(2)$  and the green region is determined by the elevation of the HPBs of the subpath  $r$ ; this in turn determines an open orange “forbidden region” that the subpath  $q$  must avoid so that  $p \in X$ . In diagram (b), we have  $p \in W_7(2)$  and the green region is determined by the elevation of the LPAs of the subpath  $q$ ; this in turn determines a closed orange “forbidden region” that the subpath  $r$  must avoid so that  $p \in Y$ .

blackboard bold typeface (for example  $\mathbb{p}$ ) for a path originating in  $W_{k+1}(g)$ . This is intended to help distinguish the domain and codomain of the function  $\phi$  (namely  $W_k(g)$  and  $W_{k+1}(g)$ , respectively) from the domain and codomain of the function  $\psi$  (namely  $W_{k+1}(g)$  and  $W_k(g)$ , respectively).

**Definition 3.6** (The subsets  $\mathcal{X}$  and  $\mathcal{Y}$ ). Let  $\mathbb{p} \in W_{k+1}(g)$ . The path  $\mathbb{p}$  lies in  $\mathcal{Y}$  provided:

- (i)  $\mathbb{p}$  has at least two LPAs, and
- (ii) the subpath of  $\mathbb{p}$  lying between the last two LPAs of  $\mathbb{p}$  contains no boundary points of  $\mathbb{p}$ .

Otherwise,  $\mathbb{p}$  lies in  $\mathcal{X}$ . ┘

Note that  $\mathcal{Y}$  is empty if  $k = 0$ . See Figure 3.5 for an illustration of Definition 3.6.

We now use Definition 3.6 to specify a canonical path split for a path in each of  $\mathcal{X}$  and  $\mathcal{Y}$ .

**Definition 3.7** (Canonical representation of paths in  $\mathcal{X}$  and  $\mathcal{Y}$ ). Let  $\mathbb{p} \in W_{k+1}(g)$ .

**Case  $\mathbb{p} \in \mathcal{X}$ :** let  $L$  be the last LPA of  $\mathbb{p}$ , and let  $B$  be the boundary point of  $\mathbb{p}$  (possibly the startpoint of  $\mathbb{p}$ ) which immediately precedes  $L$ . Split  $\mathbb{p}$  at  $B$  and  $L$  into  $\mathbb{p} = \mathbb{q}\mathbb{r}_2\mathbb{r}_1$ . The canonical representation of  $\mathbb{p}$  is  $\mathbb{q}\mathbb{r}_2\mathbb{r}_1$ .

**Case  $\mathbb{p} \in \mathcal{Y}$ :** the path  $\mathbb{p}$  has at least two LPAs. Split  $\mathbb{p}$  at its last two LPAs into  $\mathbb{p} = \mathbb{q}_1\mathbb{r}\mathbb{q}_2$ . The canonical representation of  $\mathbb{p}$  is  $\mathbb{q}_1\mathbb{r}\mathbb{q}_2$ . ┘

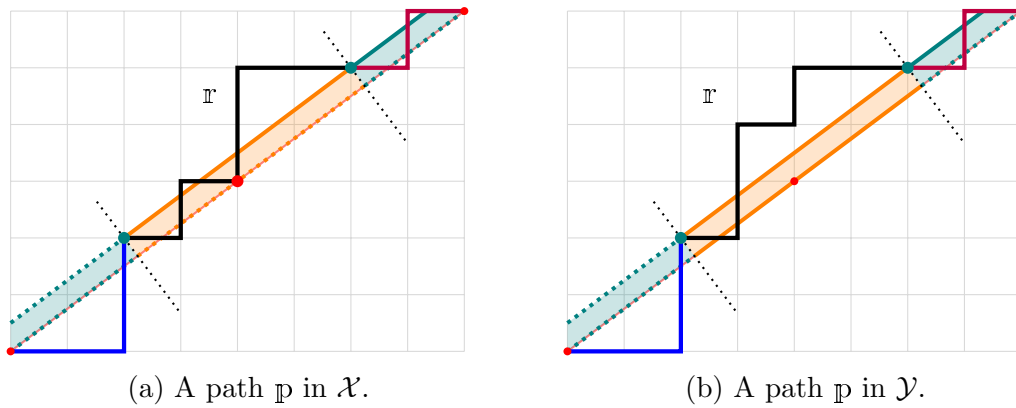


Figure 3.5: Let  $(a, b) = (4, 3)$ . In diagram (a), we have  $\mathbb{p} \in W_7(2)$  and the subpath  $\mathbb{r}$  between the (last) two LPAs of  $\mathbb{p}$  contains a boundary point of  $\mathbb{p}$ . In diagram (b), we have  $\mathbb{p} \in W_8(2)$  and the subpath  $\mathbb{r}$  between the (last) two LPAs of  $\mathbb{p}$  contains no boundary points of  $\mathbb{p}$ .

### 3.3 The actions of $\phi^X$ , $\phi^Y$ , $\psi^X$ and $\psi^Y$

We now define the maps  $\phi^X$ ,  $\phi^Y$ ,  $\psi^X$  and  $\psi^Y$ , whose domains and codomains are given in Figure 3.1. Illustrations of these maps are given in Figure 3.6.

#### 3.3.1 The actions of $\phi^X$ and $\phi^Y$

We now define the maps  $\phi^X$  and  $\phi^Y$ .

**Definition 3.8** (Actions of  $\phi^X$  and  $\phi^Y$ ). Let  $p \in W_k(g) \setminus S_k(g)$ .

**Case  $p \in X$ :** Write  $p = qr_1r_2$  according to Definition 3.5. Then  $\phi^X : X \rightarrow \mathcal{X}$  is given by

$$\phi^X(qr_1r_2) = qr_2r_1.$$

**Case  $p \in Y$ :** Write  $p = q_1q_2r$  according to Definition 3.5. Then  $\phi^Y : Y \rightarrow \mathcal{Y}$  is given by

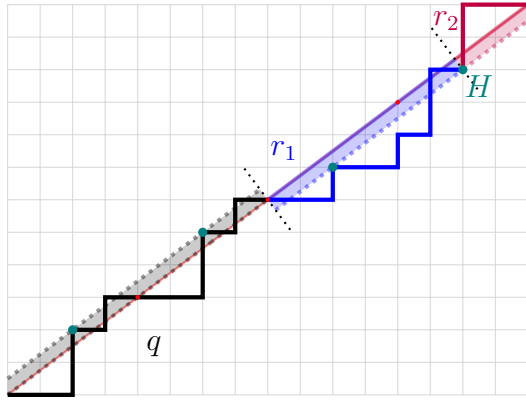
$$\phi^Y(q_1q_2r) = q_1rq_2. \quad \lrcorner$$

**Proposition 3.9.** *The map  $\phi^X$  is well defined.*

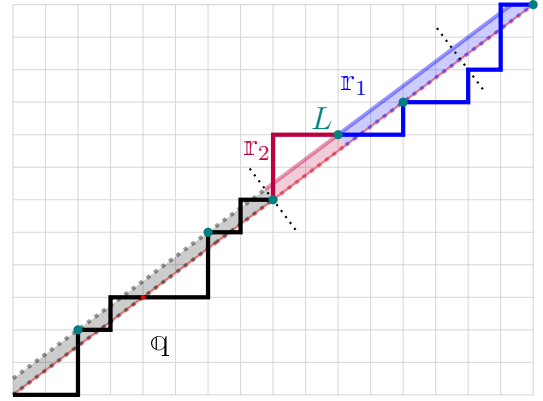
*Proof.* Let  $p \in X$ . We must check that  $\phi^X(p) = qr_2r_1$  belongs to  $\mathcal{X}$ . Let  $H$  be the startpoint of  $r_2$ . By Definition 3.5,  $H$  is the last HPB of the path  $r_1r_2$ . Since  $p$  is split at its last non-terminal boundary point into paths  $q$  and  $r_1r_2$ , we have:

1. the path  $r_1r_2$  has no interior boundary points.

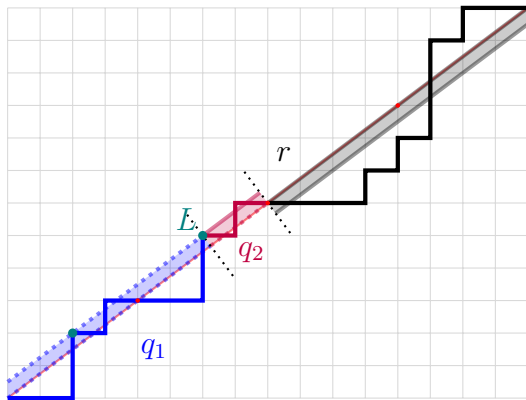
Since  $p \in X$ , by Definition 3.4 we have:



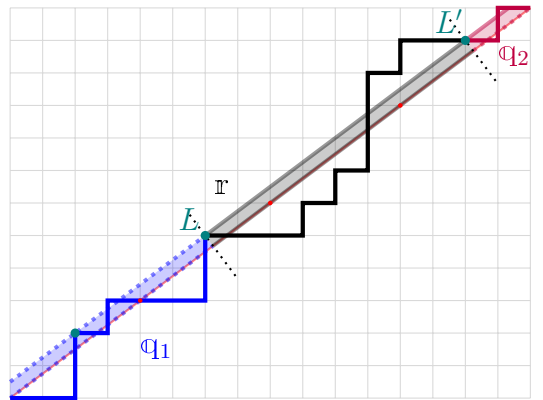
(a) Canonical representation of a path  $p$  in  $X$ .



(b) Canonical representation of a path  $p$  in  $X$ .



(c) Canonical representation of a path  $p$  in  $Y$ .



(d) Canonical representation of a path  $p$  in  $Y$ .

Figure 3.6: The maps  $\phi^X$  and  $\psi^X$  act on the paths in diagrams (a) and (b), respectively, and their images are (b) and (a), respectively. Similarly the maps  $\phi^Y$  and  $\psi^Y$  act on paths in diagram (c) and (d), respectively, and their images are (d) and (c), respectively. The LPAs and HPBs of the paths determine open or closed forbidden regions (denoted using dotted or solid lines, respectively) within which no points of the path can lie. The labelling of the points in this figure is consistent (when applicable) with that used in the proofs of Propositions 3.9, 3.11, 3.13 and 3.15.

2. the elevation of the LPAs of the path  $q$  (if any) is greater than or equal to the magnitude of the elevation of  $H$  in the path  $r_1 r_2$ .

It follows from statement 1 and Key Observation 3.3 that:

3. the path  $r_2 r_1$  has exactly one more flaw than does  $r_1 r_2$ .

It follows from statements 1 and 2 that:

4. the subpath  $r_2r_1$  contains exactly one of the LPAs of  $\phi^X(p)$  (namely the endpoint of  $r_2$ ).

It follows from statement 3 that  $\phi^X(p)$  contains exactly one more flaw than  $p$ . Furthermore, since the startpoint  $H$  of  $r_2$  is a boundary point of the path  $\phi^X(p) = qr_2r_1$ , statement 4 implies that  $\phi^X(p)$  cannot simultaneously satisfy both conditions (i) and (ii) of Definition 3.6. Therefore  $\phi^X(p) \in \mathcal{X}$ , as required.  $\square$

*Remark 3.10.* Continue with the notation from the proof of Proposition 3.9. We note for use in Section 3.4 that, since the startpoint  $H$  of  $r_2$  is the last HPB of the path  $r_1r_2$  and is a boundary point of the path  $\phi^X(p)$ , we have that  $H$  is the only boundary point of  $\phi^X(p) = qr_2r_1$  contained in the subpath  $r_2$ .  $\lrcorner$

**Proposition 3.11.** *The map  $\phi^Y$  is well defined.*

*Proof.* Let  $p \in Y$ . We must check that  $\phi^Y(p) = q_1rq_2$  belongs to  $\mathcal{Y}$ . Let  $L$  be the endpoint of  $q_1$ . By Definition 3.5,  $L$  is the last LPA of the path  $q_1q_2$ . Since  $p \in Y$ , by Definition 3.4 the elevation of the LPAs of the path  $q_1q_2$  (including  $L$ ) is smaller than the magnitude of the elevation of the HPBs of the path  $r$  (if any). Therefore:

1. the path  $\phi^Y(p)$  contains exactly one more flaw than  $p$ , namely the startpoint  $L'$  of  $q_2$ .
2. the points  $L$  and  $L'$  are the (distinct) last two LPAs of  $\phi^Y(p)$  (since the path  $r$  has no interior boundary points by Definition 3.5).
3. the subpath  $r$  contains no boundary points of  $\phi^Y(p)$ .

This shows by Definition 3.6 that  $q_1rq_2 \in \mathcal{Y}$ , as required.  $\square$

### 3.3.2 The actions of $\psi^{\mathcal{X}}$ and $\psi^{\mathcal{Y}}$

We now define the maps  $\psi^{\mathcal{X}}$  and  $\psi^{\mathcal{Y}}$ .

**Definition 3.12** (Actions of  $\psi^{\mathcal{X}}$  and  $\psi^{\mathcal{Y}}$ ). Let  $\mathbb{p} \in W_{k+1}(g)$ .

**Case  $\mathbb{p} \in \mathcal{X}$ :** Write  $\mathbb{p} = \mathbb{q}\mathbb{r}_2\mathbb{r}_1$  according to Definition 3.7. Then  $\psi^{\mathcal{X}} : \mathcal{X} \rightarrow X$  is given by

$$\psi^{\mathcal{X}}(\mathbb{q}\mathbb{r}_2\mathbb{r}_1) = \mathbb{q}\mathbb{r}_1\mathbb{r}_2.$$

**Case  $\mathbb{p} \in \mathcal{Y}$ :** Write  $\mathbb{p} = \mathbb{q}_1\mathbb{r}\mathbb{q}_2$  according to Definition 3.7. Then  $\psi^{\mathcal{Y}} : \mathcal{Y} \rightarrow Y$  is given by

$$\psi^{\mathcal{Y}}(\mathbb{q}_1\mathbb{r}\mathbb{q}_2) = \mathbb{q}_1\mathbb{q}_2\mathbb{r}. \quad \lrcorner$$

**Proposition 3.13.** *The map  $\psi^{\mathcal{X}}$  is well defined.*

*Proof.* Let  $\mathbb{p} \in \mathcal{X}$ . We must check that  $\psi^{\mathcal{X}}(\mathbb{p}) = \mathbb{q}\mathbb{r}_1\mathbb{r}_2$  belongs to  $X$ . Let  $L$  be the endpoint of the path  $\mathbb{r}_2$ . By Definition 3.7, we have:

1.  $L$  is the last LPA of  $\mathbb{p} = \mathbb{q}\mathbb{r}_2\mathbb{r}_1$ , and the startpoint of  $\mathbb{r}_2$  is the boundary point of  $\mathbb{p}$  which immediately precedes  $L$ .

By Definition 3.6, we have:

2. either  $\mathbb{p}$  has exactly one LPA, or the subpath of  $\mathbb{p}$  lying between the last two LPAs of  $\mathbb{p}$  contains a boundary point of  $\mathbb{p}$ .

It follows from statements 1 and 2 that:

3. the subpath  $\mathbb{r}_2\mathbb{r}_1$  of  $\mathbb{p}$  contains exactly one LPA of  $\mathbb{p}$ , namely the point  $L$ .

Statement 3 and Key Observation 3.3 imply that:

4. the path  $\psi^{\mathcal{X}}(\mathbb{p}) = \mathbb{q}\mathbb{r}_1\mathbb{r}_2$  splits at its last non-terminal boundary point into the paths  $\mathbb{q}$  and  $\mathbb{r}_1\mathbb{r}_2$ .
5. the path  $\psi^{\mathcal{X}}(\mathbb{p}) = \mathbb{q}\mathbb{r}_1\mathbb{r}_2$  has exactly one fewer flaw than  $\mathbb{p}$ .

The elevation of  $L$  in the path  $\mathbb{r}_2\mathbb{r}_1$  equals the magnitude of the elevation of the HPBs of the path  $\mathbb{r}_1\mathbb{r}_2$ . Since  $L$  is an LPA of  $\mathbb{p}$  by statement 1, this gives:

6. the elevation of the LPAs of the path  $\mathbb{q}$  (if any) is greater than or equal to the magnitude of the elevation of the HPBs of the path  $\mathbb{r}_1\mathbb{r}_2$ .

Statements 4, 5 and 6 show by Definition 3.4 that  $\psi^{\mathcal{X}}(\mathbb{p}) \in X$ . □

*Remark 3.14.* Continue with the notation from the proof of Proposition 3.13. We note for use in Section 3.4 that statement 1 implies the endpoint of  $\mathbb{r}_1$  is the last HPB of the path  $\mathbb{r}_1\mathbb{r}_2$ . ┘

**Proposition 3.15.** *The map  $\psi^{\mathcal{Y}}$  is well defined.*

*Proof.* Let  $\mathbb{p} \in \mathcal{Y}$ . We must check  $\psi^{\mathcal{Y}}(\mathbb{p}) = \mathbb{q}_1\mathbb{q}_2\mathbb{r}$  belongs to  $Y$ . Let  $L$  be the endpoint of the path  $\mathbb{q}_1$ , and let  $L'$  be the startpoint of the path  $\mathbb{q}_2$ . By Definition 3.7, we have:

1.  $L$  and  $L'$  are the last two LPAs of  $\mathbb{p} = \mathbb{q}_1\mathbb{r}\mathbb{q}_2$ .

It follows from statement 1 that

2. the path  $\psi^{\mathcal{Y}}(\mathbb{p}) = \mathbb{q}_1\mathbb{q}_2\mathbb{r}$  splits at its last non-terminal boundary point into the paths  $\mathbb{q}_1\mathbb{q}_2$  and  $\mathbb{r}$ .



The LPAs  $L$  and  $L'$  of  $\mathbb{p}$  combine to form a single point in  $\psi^{\mathcal{Y}}(\mathbb{p}) = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{r}$ , and so:

3. the subpaths  $\mathfrak{q}_1$  and  $\mathfrak{q}_2$  of  $\psi^{\mathcal{Y}}(\mathbb{p})$  collectively contain exactly one fewer flaw than the subpaths  $\mathfrak{q}_1$  and  $\mathfrak{q}_2$  of  $\mathbb{p}$ .
4. the path  $\mathfrak{q}_1\mathfrak{q}_2$  has at least one LPA, namely the point  $L = L'$ .

By statement 1 and Definition 3.6(ii), the subpath  $\mathfrak{r}$  contains no boundary points of  $\mathbb{p}$ . This, together with statement 4, implies:

5. the elevation of the LPAs of the path  $\mathfrak{q}_1\mathfrak{q}_2$  is smaller than the magnitude of the elevation of the HPBs of the path  $\mathfrak{r}$  (if any).

It follows from statement 5 that:

6. disregarding its startpoint and endpoint, the subpath  $\mathfrak{r}$  of  $\psi^{\mathcal{Y}}(\mathbb{p}) = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{r}$  contains the same number of flaws as the subpath  $\mathfrak{r}$  of  $\mathbb{p}$ .

By statements 3 and 6, the path  $\psi^{\mathcal{Y}}(\mathbb{p})$  contains exactly one fewer flaw than  $\mathbb{p}$ . By statement 4, the path  $\mathfrak{q}_1\mathfrak{q}_2$  has at least one flaw. Together with statements 2 and 5, this shows by Definition 3.4 that  $\psi^{\mathcal{Y}}(\mathbb{p}) \in Y$ .  $\square$

*Remark 3.16.* Continue with the notation from the proof of Proposition 3.15. We note for use in Section 3.4 that statement 1 implies the endpoint  $L$  of  $\mathfrak{q}_1$  is the last LPA of the path  $\mathfrak{q}_1\mathfrak{q}_2$ .  $\lrcorner$

### 3.4 The maps $\phi$ and $\psi$ are injective

We complete the proof of Theorem 1.5 by showing in turn that each of the maps  $\phi^X$ ,  $\phi^Y$ ,  $\psi^X$ ,  $\psi^Y$  is injective. We give the proof for  $\phi^X$  and  $\phi^Y$  in detail, and for  $\psi^X$  and  $\psi^Y$  in abbreviated form.

#### The map $\phi^X$ is injective:

Let  $p, p' \in X$ , and write  $p = qr_1r_2$  and  $p' = q'r'_1r'_2$  according to Definition 3.5. We suppose that  $\phi^X(p) = \phi^X(p')$ , and wish to show that  $p = p'$ .

By statement 4 in the proof of Proposition 3.9, the endpoint  $L$  of  $r_2$  is the last LPA of  $\phi^X(p) = qr_2r_1$ . By Remark 3.10, the startpoint  $H$  of  $r_2$  is the boundary point of  $\phi^X(p) = qr_2r_1$  immediately preceding  $L$ .

Therefore  $\phi^X(p) = qr_2r_1$  splits into  $qr_2$  and  $r_1$  at the last LPA  $L$  of  $\phi^X(p)$ , and the subpath  $qr_2$  splits into  $q$  and  $r_2$  at the boundary point of  $\phi^X(p)$  immediately preceding  $L$ . The corresponding statement holds for  $\phi^X(p')$ . Since  $\phi^X(p)$  and  $\phi^X(p')$  are equal by assumption, their LPAs and boundary points are identical. Therefore  $q = q'$  and  $r_2 = r'_2$  and  $r_1 = r'_1$  and so  $p = qr_1r_2 = q'r'_1r'_2 = p'$ , as required.

**The map  $\phi^Y$  is injective:**

Let  $p, p' \in Y$ , and write  $p = q_1 q_2 r$  and  $p' = q'_1 q'_2 r'$  according to Definition 3.5. We suppose that  $\phi^Y(p) = \phi^Y(p')$ , and wish to show that  $p = p'$ .

By statement 2 in the proof of Proposition 3.11, the endpoint  $L$  of  $q_1$  and the startpoint  $L'$  of  $q_2$  are the last two LPAs of  $\phi^Y(p) = q_1 r q_2$ , and so  $\phi^Y(p)$  splits at its last two LPAs into  $q_1$  and  $r$  and  $q_2$ . Likewise,  $\phi^Y(p')$  splits at its last two LPAs into  $q'_1$  and  $r'$  and  $q'_2$ . But  $\phi^Y(p)$  and  $\phi^Y(p')$  are equal by assumption, so their last two LPAs are identical. Therefore  $q_1 = q'_1$  and  $r = r'$  and  $q_2 = q'_2$  and so  $p = q_1 q_2 r = q'_1 q'_2 r' = p'$ , as required.

**The map  $\psi^{\mathcal{X}}$  is injective:**

Let  $\mathbb{p}, \mathbb{p}' \in \mathcal{X}$ , and write  $\mathbb{p} = \mathbb{q} \mathbb{r}_2 \mathbb{r}_1$  and  $\mathbb{p}' = \mathbb{q}' \mathbb{r}'_2 \mathbb{r}'_1$  according to Definition 3.7. We suppose that  $\psi^{\mathcal{X}}(\mathbb{p}) = \psi^{\mathcal{X}}(\mathbb{p}')$ , and wish to show that  $\mathbb{p} = \mathbb{p}'$ .

By statement 4 in the proof of Proposition 3.13, the path  $\psi^{\mathcal{X}}(\mathbb{p}) = \mathbb{q} \mathbb{r}_1 \mathbb{r}_2$  splits at its last non-terminal boundary point into  $\mathbb{q}$  and  $\mathbb{r}_1 \mathbb{r}_2$ . By Remark 3.14 the path  $\mathbb{r}_1 \mathbb{r}_2$  splits at its last HPB into  $\mathbb{r}_1$  and  $\mathbb{r}_2$ .

It follows that  $\mathbb{p} = \mathbb{q} \mathbb{r}_2 \mathbb{r}_1 = \mathbb{q}' \mathbb{r}'_2 \mathbb{r}'_1 = \mathbb{p}'$ , as required.

**The map  $\psi^{\mathcal{Y}}$  is injective:**

Let  $\mathbb{p}, \mathbb{p}' \in \mathcal{Y}$ , and write  $\mathbb{p} = \mathbb{q}_1 \mathbb{r} \mathbb{q}_2$  and  $\mathbb{p}' = \mathbb{q}'_1 \mathbb{r}' \mathbb{q}'_2$  according to Definition 3.7. We suppose that  $\psi^{\mathcal{Y}}(\mathbb{p}) = \psi^{\mathcal{Y}}(\mathbb{p}')$ , and wish to show that  $\mathbb{p} = \mathbb{p}'$ .

By statement 2 in the proof of Proposition 3.15, the path  $\psi^{\mathcal{Y}}(\mathbb{p}) = \mathbb{q}_1 \mathbb{q}_2 \mathbb{r}$  splits at its last non-terminal boundary point into  $\mathbb{q}_1 \mathbb{q}_2$  and  $\mathbb{r}$ . By Remark 3.16, the path  $\mathbb{q}_1 \mathbb{q}_2$  splits at its last LPA into  $\mathbb{q}_1$  and  $\mathbb{q}_2$ .

It follows that  $\mathbb{p} = \mathbb{q}_1 \mathbb{r} \mathbb{q}_2 = \mathbb{q}'_1 \mathbb{r}' \mathbb{q}'_2 = \mathbb{p}'$ , as required.

### 3.5 The special case $g = 1$ (Theorem 2.1)

We finally re-examine the special case  $g = 1$  (Theorem 2.1), involving paths from  $(0, 0)$  to  $(a, b)$ , to show how the proof of Theorem 1.5 described in Section 3 simplifies significantly. In doing so, we shall obtain a simple self-contained proof of Theorem 2.1.

Let  $k$  satisfy  $0 \leq k < a + b - 1$  and let  $p$  be a path in  $W_k(1)$ . Since

1. the path  $p$  cannot have any interior boundary points,

it follows that:

2. the subset  $S_k(1)$  of  $W_k(1)$  is empty by Definition 1.4,
3. the subset  $Y$  of  $W_k(1)$  is empty by Definition 3.4.

Since  $g = 1$ , the path  $p$  has at most one LPA and so

4. the subset  $\mathcal{Y}$  of  $W_{k+1}(1)$  is empty by Definition 3.6.

By reference to Figure 3.1, statements 2, 3 and 4 show that  $W_k(1) = X$  and  $W_{k+1}(1) = \mathcal{X}$ . We shall show that  $\phi^X$  and  $\psi^X$  are inverse maps, so that  $|W_k(1)| = |W_{k+1}(1)|$ . We may then conclude that  $|W_0(1)| = |W_1(1)| = \cdots = |W_{a+b-1}(1)|$ , which gives Theorem 2.1 because the total number of paths from  $(0, 0)$  to  $(a, b)$  is  $\binom{a+b}{a}$ .

It remains to show that  $\phi^X$  and  $\psi^X$  are inverse maps. Write  $p \in X$  as its canonical representation  $p = qr_1r_2$  according to Definition 3.5, where  $p$  has at least one HPB. By statement 1, we have that  $q$  is empty and so  $p = r_1r_2$ . Since  $g = 1$ , the path  $p$  has at most one HPB. Therefore by Definition 3.8 the map  $\phi^X$  splits  $p$  at its unique HPB into  $r_1r_2$  and replaces it by  $r_2r_1$ . That is,  $\phi^X$  cyclically permutes the steps of  $p$  by bringing the unique HPB to the origin.

Similarly, write  $\mathfrak{p} \in \mathcal{X}$  as its canonical representation  $\mathfrak{p} = \mathfrak{q}\mathfrak{r}_2\mathfrak{r}_1$  according to Definition 3.7, where  $\mathfrak{q}$  is empty by statement 1. By Definition 3.12, the map  $\psi^X$  splits  $\mathfrak{p}$  at its unique LPA into  $\mathfrak{r}_2\mathfrak{r}_1$  and replaces it by  $\mathfrak{r}_1\mathfrak{r}_2$ . That is,  $\psi^X$  cyclically permutes the steps of  $p$  by bringing the unique LPA to the origin.

Comparison of the descriptions of  $\phi^X$  and  $\psi^X$  shows that they are inverse maps, as required.

## 4 Conclusion

Our central objective was to find an explicit formula for  $|W_k(g)|$ , the number of simple lattice paths from  $(0, 0)$  to  $(ga, gb)$  having exactly  $k$  lattice points lying strictly above the linear boundary joining the startpoint to the endpoint. This is given by the closed form expression in Theorem 1.12, using the definition (1.3) of  $\mu_j(g)$ .

We conclude with two open problems for future study.

1. Evaluating  $|W_k(g)|$  via the path enumeration formula Theorem 1.12 involves a sum over integer partitions of  $g$ , and is therefore computationally intensive. In the special case  $a = 1$ , Theorem 2.3 provides an alternative expression to Theorem 1.12 that is computationally simpler. Is there a closed form expression for  $|W_k(g)|$  that is computationally simpler than Theorem 1.12 for other special cases of  $a, b$  (or in general)?
2. We established the path enumeration formula by solving the recurrence relation given in Corollary 1.7 and making use of the known values stated in Corollary 1.11. These known values are in turn predicated on Theorem 1.10, which was proved by Bizley using generating functions [9]. Is there a direct combinatorial proof of Theorem 1.10?

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